PSL UNIVERSITÉ PARIS

HABILITATION À DIRIGER DES RECHERCHES

DE L'UNIVERSITÉ PSL

Présentée à l'Université Paris-Dauphine

Optimal control, Shape optimisation
&
Applications to population dynamics

Présentation des travaux par

Idriss Mazari-Fouquer

Le 10 Novembre 2025

Discipline

Mathématiques et applications

Composition du jury :

Dorin BUCUR Professeur, Université Savoie Mont-Blanc	*Rapporteur*
Inwon KIM Professor, University of California Los Angeles	*Rapporteur*
Yuan LOU Professor, Shanghai Jiaotong University	*Rapporteur*
Guillaume CARLIER Professeur, Paris Dauphine Université PSL	*Coordinateur*
Pierre CARDALIAGUET Professeur, Paris Dauphine Université PSL	*Examinateur*
Benoît PERTHAME Professeur émérite, Sorbonne Université, Académie des Sciences	*Examinateur*
Cristina TROMBETTI Professor, Università degli Studi di Napoli Federico II	*Examinatrice*

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Mémoire présenté en vue de l'obtention de
l'Habilitation à diriger des recherches
Soutenu le 10 Novembre 2025 par:

Idriss Mazari-Fouquer

Après avis des rapporteurs:
Dorin Bucur (Université Savoie Mont Blanc)
Inwon Kim (University of California Los Angeles)
Yuan Lou (Shanghai Jiao-Tong)

En présence des rapporteurs et de:
Pierre Cardaliaguet (Paris Dauphine Université PSL)-*Examinateur*
Guillaume Carlier (Paris Dauphine Université PSL)-*Coordinateur*
Benoît Perthame (Paris Sorbonne Université, Académie des Sciences)-*Examinateur*
Cristina Trombetti (Universitá de Napoli)-*Examinatrice*

Author address:

CEREMADE, Paris Dauphine Université PSL, Place du Maréchal de Lattre de Tassigny, 75016 Paris.

Bien entendu, ces études n'aboutissaient à aucun résultat ou, du moins, le résultat était-il si trouble et insensé qu'il ne menait à rien. Non sans mélancolie, Jonathan en convenait lorsqu'il nous montrait ces signes énigmatiques. "Impossible, nous disait-il, de déceler leur sens."

Thomas Mann, *Docteur Faustus*

Contents

Abstract

Résumé Français

Nous présentons dans ce mémoire une synthèse de nos différentes activités de recherche dans le domaine du contrôle optimal, de l'optimisation de formes, et de leurs applications à la dynamique des populations. De manière plus spécifique, nous nous intéressons dans ce manuscrit à plusieurs aspects de ces questions :

- Dans un premier temps, nous nous penchons sur les propriétés ponctuelles des contrôles optimaux. Cette question est motivée par de nombreux facteurs, en particulier par les algorithmes d'approximation numérique mis en œuvre dans la résolution de ces problèmes.
- Nous nous penchons ensuite sur les propriétés géométriques des contrôles optimaux, en insistant sur les cas où l'on observe l'émergence de structures complexes.
- Une autre section de ce mémoire est dédiée à la question de la stabilité quantitative des contrôles optimaux ; la motivation est ici double : en plus de fournir des renseignements fins sur les problèmes sous-jacents, de telles estimations ont des applications nombreuses, y compris pour l'approximation numérique des problèmes de contrôle optimal
- Nous présentons ensuite plusieurs modèles de type "Théorie des jeux" pour la gestion optimale des pêcheries, en distinguant les cas où les pêcheurs présents sont en nombre fini (ce qui mène à des questions de type "équilibres de Nash") ou infini (ce qui mène à des modèles de type "jeux à champ moyen"). Une grande importance est accordée à l'étude de phénomènes tels que la tragédie des communs.
- Enfin, dans un dernier chapitre, nous présentons deux axes de recherche plus récents consacrés à des problèmes d'optimisation vectoriels motivés ou par la physique ou par la biologie.

English Abstract

In this manuscript, we present an overview of our research activity within the fields of shape optimisation and optimal control and their applications to population dynamics. More specifically, we focus on the following aspects of these questions:

- First, we analyse pointwise properties of optimal controls. This question has several motivations, stemming in particular from the study of the numerical methods put forth to solve such problems.
- We then focus on the study of geometric properties of the optimal controls, with a special emphasis on cases where complex structures emerge.

- Another section of this memoir is devoted to the quantitative stability of optimal controls. There are at least two reasons for such a study: beyond shedding a new light on the fine properties of the underlying optimal control problems, such estimates prove very useful in a variety of domains, including in the numerical approximation of optimal controls.
- We then present several game theoretical models for the optimal management of fisheries, distinguishing between cases where only a finite number of fishermen are present (leading to "Nash equilibria" type questions), or where there are infinitely many of them (leading to "Mean Field Games" models). We devote a large part of our study to the understanding of phenomena such as the tragedy of the commons.
- Finally, we present two more recent research directions, devoted to the understanding of vectorial shape optimisation problems arising either in physics or in biology.

Remerciements & Acknowledgements

Je tiens tout d'abord à remercier Guillaume (C.) d'avoir accepté d'encadrer ce mémoire d'habilitation et, plus généralement, pour toutes les discussions (mathématiques et autres) qu'il a eues avec moi. Naturellement, je suis très reconnaissant aux rapporteurs de ce manuscrit, et en premier lieu à Dorin, qui a accepté cette tâche ingrate; un grand merci, ainsi que pour toutes nos conversations mathématiques! *Likewise, I would like to very warmly thank Yuan: I can only say that I am very grateful to him. Finally, I would like to extend my deepest thanks to Inwon for taking the time to review this manuscript; thank you so much!*

Par ailleurs, je suis–pour sa participation à ce jury mais pour mille autres choses encore-très reconnaissant à Pierre, qu'il s'agisse de discussions mathématiques ou de questions d'enseignement. De même, Benoît, depuis mes débuts en thèse, a été une présence régulière et d'un grand soutien, et c'est bien évidemment un honneur de l'avoir dans ce jury. *Finally, I am honored that Cristina agreed to sit on this committee; many thanks for this, as well as your unparalleled hospitality in Naples.*

Je tiens à remercier Yannick et Grégoire, qui ont guidé mes premiers pas en recherche et avec qui j'ai la chance de pouvoir continuer à travailler; je mesure chaque jour un peu plus la chance que j'ai eue de vous avoir comme encadrants, et de vous avoir comme collaborateurs.

Among my collaborators and people I had the pleasure and honour of interacting with, I would like to extend special thanks to Grégoire, Paolo, Léo, Jimmy, Raphael, Lorenzo & Ziad, who agreed to have a go at this habilitation memoir[1], and which all all gave me many comments and suggestions on how to improve this manuscript. Likewise, I would like to warmly thank Antoine, Emmanuel, Antonin, Adrian, Domènec, Ulisse, Elisa, Beni, Andrea, Greta, Sydney, Gisella, Kevin, Ana, Phillip... for collaborating with me, bringing me to new topics and broadening my mathematical horizons.

J'ai ces dernières années eu la chance de co-organiser le GT Calva, qui a beaucoup contribué à élargir mon paysage mathématiques, et je suis très heureux d'avoir pu travailler aux côtés de Camille, Maxime, Jean-François, Élise, Paul, Joao, Antoine et Charles.

La grande majorité des travaux présentés ici ont été réalisés au sein du CEREMADE, et je tiens à remercier les collègues que j'ai plaisir d'y côtoyer: outre l'environnement scientifiquement merveilleux dont il s'agit, je suis reconnaissant à toutes et tous pour l'atmosphère particulièrement agréable qu'ils et elles y créent. Je tiens à remercier tout particulièrement Isabelle, Anne-Laure & César, Abdou & Marko (et avant eux Gilles & Thomas) pour le rôle central qui est le leur, pour leur patience et leur bienveillance. Non moins remarquable fut la patience de Guillaume

[1]Special thanks to Borjan for spotting a spelling mistake in Mann's first name.

(L.) (et celle de Guillaume (B.)), qui, malgré ma présence régulière dans leur bureau ne m'en ont jamais chassé. S'agissant de Guillaume L., je tiens à lui adresser une pensée particulière pour son soutien et nos nombreuses discussions (ainsi que pour m'avoir encouragé à harmoniser une bibliographie peut-être trop hétéroclite en termes de présentation). Je dois également mentionner les co-bureaux que j'ai eu la chance d'avoir, Olga, François (pour nos passions musicales communes), Zhenjie et Lucas–surtout pour son côté fruité. Que soient également remerciés David, pour ses jeux et toutes nos conversations scientifiques, Régis et Irène, qui ont fait preuve d'une endurance remarquable quand je les dérangeais pour des questions mathématiques, Mathieu pour les discussions scientifiques et son travail de directeur, Ivar, Maxime, Jean, Maria, Pierre L., Anna, Yating, Clément (T. et C.), Éric, Paul (P. et G.), Julien (G., S. et C.) , Émeric, Nejla, Marc, Yannick, Antoine, Charles, Jacques, Patrick, Vincent (notamment pour l'accueil qu'il m'a en tant que directeur réservé lors de mon arrivée), Ellie... Enfin, toute ma reconnaissance aux personnes que j'ai eu la chance de rencontrer au cours de ces quelques années: Cyril, Luca, Chiara, Tim, Chiu-Yen, Bo, Carlo, Gloria, Alba, Vincenzo, Elvise, Simone, Giampiero, Apostolos (the book is coming, I promise!), Matthieu, Elisa, Noemi, Ayman, Aymeric, Luis, Nicolas... et tant d'autres que j'oublie sans doute–j'espère que vous m'en excuserez. Une pensée particulière pour Ilias: on va finir par bosser ensemble!

Je tiens à remercier, le plus chaleureusement possible, Bernard & Jean, qui m'ont tant appris au cours des années, ainsi que Julia, Juliette, Clément, Florent, Benoît, Florestan, Lucie et Ziad.

S'agissant des remerciements plus personnels, le bilinguisme est de rigueur: Ich möchte die ganze Kanzler Familie danken für die Unterstützung die sie mir gegeben haben. Wien wird immer eine Stadt sein, wo Ich Ruhe und Wärme finde: Danke Gerti und Stefan (und Honey)! Il va sans dire que j'adresse tous mes remerciements et toute ma reconnaissance à la famille Mazari, mon père, ma mère, mon frère, Coline et Minette pour leur soutien constant.

Abschließend, Ich muss natürlich Laura für Alles danken.

Overview of the scientific activities

This memoir serves as a presentation of my research activity, specifically focusing on the themes and problematics that developed since my PhD defence in July 2020.

Main themes of research

(1) **Optimal control problems in population dynamics–scalar problems** This first line of research that stems naturally from my PhD. The prototypical problem is: how should on distribute resources in a domain to optimise certain criteria? While this is a generic question, let us mention that the focus is on pointwise characterisations (*i.e.* bang-bang property of optimiser) as well as on the global geometric properties of the optimal distributions (*e.g.* symmetry in time, symmetry in space, asymptotic behaviour with respect to the parameters). The relevant contributions are presented in Chapters 1–2, and are in collaboration with B. Bogosel, G. Nadin, Y. Privat & D. Ruiz-Balet.

(2) **Quantitative stability estimates and regularity properties of optimal controls** This second line of research developed during my postdoctoral studies and deals with similar optimal control problems, with a different outlook. Namely, we are interested in establishing quantitative inequalities for optimal control problems and in deriving regularity properties in the following sense: if the optimal control m^* is the characteristic function of a set , what can be said about the regularity of that set? These informations are then used in the study of numerical approximations of the problem. These contributions were developed with A. Chambolle, Y. Privat, E. Trélat & D. Ruiz-Balet. The relevant contributions are presented in Chapter 3.

(3) **(Mean Field) Game models in mathematical biology** This third line of research emerged more recently and is focused on the development of models for the management of fisheries: considering a population of fishes that is being harvested, and assuming that the fishermen are engaged in a competition to harvest as much fish as possible, is it possible to quantify, or to understand qualitatively, the influence of this competition on the fishes' population? We study this question from three different perspectives, that of Nash equilibria, of the long-time behaviour of MFG system and, finally, of traveling waves in MFG systems. These works were done in collaboration with Z. Kobeissi & D. Ruiz-Balet and are presented in Chapter 4.

(4) **Vectorial optimisation problems** In the last part of this memoir, I present in a synthetic way some recent contributions to the optimisation of systems of PDEs. These contributions include the optimal control of cooperative systems in mathematical biology (where the contributions presented were obtained in collaboration with L. Girardin) and the shape optimisation of vectorial operators (this line of work corresponds to joint work with A. Henrot and Y. Privat regarding the Faber-Krahn inequality for the Stokes operator). We refer to Chapter 5.

List of publications

Publications in international peer-reviewed journals

(1) B. Bogosel, I. Mazari-Fouquer, G. Nadin, Optimisation of space-time periodic eigenvalues.
Accepted for publication, *Annali della Scuola Normale Superiore, Classe di Scienze*, 2025.
(2) Z. Kobeissi, I. Mazari-Fouquer, D. Ruiz-Balet, Mean-field games for harvesting problems: Uniqueness, long-time behaviour and weak KAM theory.
Accepted for publication, *Journal of Differential Equations*, 2025.
(3) A. Chambolle, I. Mazari-Fouquer, Y. Privat, Stability of optimal shapes and convergence of thresholding algorithms in linear and spectral optimal control problems.
Accepted for publication, *Mathematische Annalen*, 2025.
(4) I. Mazari-Fouquer, Y. Privat, E. Trélat, Large-time optimal observation domain for linear parabolic systems.
Accepted for publication, *Annales de l'Institut Henri Poincaré-C-Analyse Non-linéaire*, 2025.
(5) I. Mazari-Fouquer, Another look at qualitative properties of eigenvalues using effective Hamiltonians.
Accepted for publication, *Compte-rendus de l'Académie des Sciences*, 2025.
(6) Z. Kobeissi, I. Mazari-Fouquer, D. Ruiz-Balet, The tragedy of commons: a Mean-Field Games approach to the reversal of travelling waves.
Nonlinearity, 2024.
DOI:10.1088/1361-6544/ad7b97
(7) A. Henrot, I. Mazari-Fouquer, Y. Privat, Is the Faber-Krahn inequality true for the Stokes operator?
Calculus of Variations and PDEs, 2024.
DOI: 10.1007/s00526-024-02820-7
(8) I. Mazari-Fouquer, Maximising the biomass with respect to the carrying capacity: some qualitative results.
Journal of Differential Equations, 2024.
DOI: 10.1016/j.jde.2024.02.007
(9) I. Mazari-Fouquer, Existence of optimal shapes in parabolic bilinear optimal control problems.
Archive for Rational Mechanics and Analysis, 2024.
DOI: 10.1007/s00205-024-01958-0

(10) F. Auricchio, M. Marino, I. Mazari-Fouquer, U. Stefanelli, Analysis of a combined filtered/phase-field approach to topology optimization in elasticity.
Applied Mathematics and Optimization, 2024.
DOI: 10.1007/s00245-024-10104-x
(11) P. Baumann, I. Mazari-Fouquer, K. Sturm, The topological state derivative: an optimal control perspective on topology optimisation.
Journal of Geometric Analysis, Special Issue: Differential Geometric PDE Control, Shape Optimization and Applications, 2023,
DOI:10.1007/s12220-023-01295-w
(12) I. Mazari, The bang-bang property in some parabolic bilinear optimal control problems via two-scale asymptotic expansions,
Journal of Functional Analysis, 2023,
DOI:10.1016/j.jfa.2023.109855
(13) I. Mazari, Y. Privat, Qualitative analysis of optimisation problems with respect to non-constant Robin coefficients.
Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, accepted for publication, 2022.
(14) E. Davoli, I. Mazari, U. Stefanelli, Spectral optimization of inhomogeneous plates.
SIAM Journal on Control and Optimization, accepted for publication, 2022.
ArXiv:2107.11207
(15) I. Mazari, D. Ruiz-Balet, Spatial ecology, optimal control and game theoretical fishing problems.
Journal of Mathematical Biology, 2022.
DOI:10.1007/s00285-022-01829-w
(16) I. Mazari, D. Ruiz-Balet. Quantitative stability for eigenvalues of Schrödinger operator, Quantitative bathtub principle & Application to the turnpike property for a bilinear optimal control problem.
SIAM Journal on Mathematical Analysis 54(3):3848–3883, 2022.
DOI:10.1137/21M1393121
(17) I. Mazari, A note on the rearrangement of functions in time and on the parabolic Talenti inequality.
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DOI:10.1007/s11565-022-00392-y
(18) I. Mazari, D. Ruiz-Balet, E. Zuazua. Constrained controls of gene-flow models.
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(19) I. Mazari, Some comparison results and a partial bang-bang property for two-phases problems in balls.
Mathematics in Engineering, Special Issue *Calculus of Variations and Nonlinear Analysis: Advances and Applications*, 2023.
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DOI:10.1080/03605302.2021.2007533

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Archive for Rational Mechanics and Analysis, 2021.
DOI:10.1007/s00205-021-01726-4

(22) I. Mazari. Quantitative estimates for parabolic optimal control problems under L^∞ and L^1 constraints.
Nonlinear Analysis, 215, 2022.
DOI:10.1016/j.na.2021.112649

(23) A. Isis Toledo Marrero, I. Mazari, G. Nadin, Optimisation of the total population size with respect to the initial condition for semilinear parabolic equations: Two-scale expansions and symmetrisations.
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DOI:10.1088/1361-6544/ac23b9

(24) I. Mazari, D. Ruiz-Balet. A fragmentation phenomenon for a non-energetic optimal control problem: optimisation of the total population size in logistic diffusive models.
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DOI:10.1137/20M132818X

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DOI:10.1051/cocv/2020052

(26) I. Mazari. A quantitative inequality for the first eigenvalue of a Schrödinger operator.
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DOI:10.1016/j.jde.2020.06.057

(27) I. Mazari, G. Nadin, Y. Privat. Optimal location of resources maximizing the total population size in logistic models.
Journal de Mathématiques Pures et Appliquées 134 (2020), pp. 1-35.
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Discrete and continuous dynamical systems, Series B 24 (2019), pp. 6693-6724.
DOI:10.3934/dcdsb.2019163

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(29) L. Girardin, I. Mazari-Fouquer, Generalized principal eigenvalues of space-time periodic, weakly coupled, cooperative, parabolic systems.
Accepted for publication, *Mémoires de la Société Mathématique de France*, 2024.

Proceedings

(30) I. Mazari, G. Nadin, Y. Privat. Optimal control of resources for species survival.

Proceedings in Applied Mathematics and Mechanics 18 (Special Issue: 89th Annual Meeting of GAMM) (2018).
DOI:0.1002/pamm.201800086.

Submitted articles

(31) S. Cantrell, C. Cosner, A. Lam, I. Mazari-Fouquer, Mean Field Games and Ideal Free Distribution, 2025.

(32) I. Mazari-Fouquer, G. Nadin, Localising optimality conditions for the linear optimal control of semilinear equations via concentration results for oscillating solutions of linear parabolic equations. Submitted, 2022.

Advisorship

Since my PhD, I have taken part in several (co)supervisions of students at different stages and I have hosted and collaborated with young researchers.

Supervision of post-doctoral students:

- Domènec Ruiz-i-Balet, CEREMADE (2024-2025; 50%, co-advisor: Antonin Chambolle, CNRS, CEREMADE) on the topic “PDEs and Artificial Intelligence”.
- Raphaël Prunier, CEREMADE (2023-2025), on the topic “Qualitative properties and regularity issues for optimal control problems arising in population dynamics”. Funded by the PSL Young Researcher Starting Grant I obtained in 2023.

Supervision of PhD students:

- Greta Lamonaca, CEREMADE & IDP (50%; co-director: Grégoire Nadin, CNRS, IDP) on the topic “Mean Field Game Problems for fisheries”. Funded by an FSMP doctoral grant.

Supervision of Master internships:

- Greta Lamonaca (2024), on the topic “Mean Field Game Problems for fisheries”. Internship funded by the PSL Young Researcher Starting Grant I obtained in 2023.

Collaborations with PhD students:

- Paolo Acampora, Universita Federico II, Napoli (advisors: C. Nitsch, C. Trombetti),
- Lorenzo Ferreri, Scuola Normale Superiore, Pisa (advisor: B. Velichkov),
- Andrea Gentile, Universita Federico II, Napoli (advisor: C. Nitsch).

Introduction

In this introduction, we present the structure of this memoir, which means we briefly review each chapter and the type of questions they deal with. The overarching theme of the works I chose to present here is PDE constrained optimisation, the latter expression referring either to *shape optimisation* (meaning we want to optimise the domain on which the PDE is set) or to *optimal control* (meaning that we are trying to optimise the coefficients of the underlying differential operator). Most of the examples we will discuss come from three fields:

(1) Population dynamics,
(2) Physics,
(3) Game theory (in interaction with population dynamics).

Regarding ongoing and future research Each chapter concludes with a list of several research projects related to the questions of the chapter at hand, the level of detail depending on the degree of advancement of these projects.

Chapter 1: Pointwise properties of optimal control problems & applications to population dynamics

In the first chapter, we focus on generic optimal control problems for scalar equations. Namely, we consider the "general" optimal control problem

$$\max_m \int_\Omega j(u_m) \text{ subject to } Lu_m = \Phi(m, u_m) \text{ in a domain } \Omega$$

where:

- L is a (possibly non-linear) differential operator (either parabolic or elliptic) endowed with Neumann boundary conditions,
- $m \in L^\infty(\Omega)$ is the control, possibly restricted to belonging to a certain admissible class,
- Φ denotes the coupling between the control m and the state u_m,
- j is a given cost-function.

From an applications' perspective, we like to think of u_m as a population density, and the control m, depending on the situation, can be thought of either as describing the environment (for instance, interpreting it as the distribution of the resources available to the population) or as an influx of populations. Regardless of the case we are in, some natural constraints to impose on m are the following:

- m should be uniformly bounded, meaning we should enforce a constraint of the type $0 \leq m \leq 1$ a.e. in Ω.
- The global ability to control should be limited, for instance imposing $\int_\Omega m = m_0$.

Under these two constraints, one of the main questions is: is it true that any optimal control m^* saturates the constraint $0 \leq m^* \leq 1$, *i.e.* is it true that $m^* \in \{0;1\}$ a.e.? In that case we say that the *bang-bang property* is satisfied. This question is motivated by several reasons laid out in Chapter 1, some of the main ones being:

(1) First, this is a natural question in the field of mathematical biology and population dynamics: the equations being non-linear, and the optimal control problems not being "standard", there is no hope to fully characterise the solutions of a problem such as "What is the best way to spread resources in a domain so as to maximise the total population size?", and we must start somewhere.
(2) Second, this is an important consideration when dealing with numerical approximations of these controls, and the theoretical study of their convergence.
(3) Third (and this is linked to the second point), such properties are usually the first step when deriving local quantitative stability estimates.

Chapter 1 is structured in three parts:

(1) The first one deals with a general introduction, presenting in more detail the type of questions we have in mind and paying considerable attention to the reasons we investigate this bang-bang property, and delineating what we can hope for in terms of general results.
(2) The second part deals with the bilinear case, the prototypical example being

$$\max_{0\leq m\leq 1 \text{ a.e., } \int_\Omega m=m_0} \int_\Omega j(\theta_m) \text{ subject to } \begin{cases} -\Delta\theta_m - \theta_m(m-\theta_m)=0 & \text{in } \Omega\,, \\ \partial_\nu \theta_m = 0 & \text{on } \partial\Omega\,, \\ \theta_m \geq 0\,, \theta_m \neq 0 \end{cases}. \tag{0.1}$$

θ_m represents the density of a population having access to resources m and the optimal control problem amounts to the following question: how should one spread resources inside a domain to best benefit the population? We prove a general result, that states that for any increasing j (without any convexity or concavity assumption) the optimal controls always satisfy the bang-bang property. We spend some time sketching out this proof and discussing the parabolic counterparts of this result, as it is foundational in several later chapters.
(3) The third and final part of the chapter describes the case of linear control problems, the prototypical example being the parabolic model

$$\max_{0\leq m\leq 1 \text{ a.e., } \int_\Omega m(t,\cdot)=m_0 \text{ for a.e.} t} \int_\Omega j(y_m(T,\cdot)) \tag{0.2}$$
$$\text{subject to } \begin{cases} \partial_t y_m - \Delta y_m = m + f(y_m) & \text{in } (0;T)\times\Omega\,, \\ y_m(0,\cdot)=0\,, \\ \partial_\nu y_m = 0 & \text{on } (0;T)\times\partial\Omega\,, \end{cases}$$

which corresponds to the optimal control of a population through an influx of new individuals. The results in that direction are much less systematic, but shed a new light on possible approaches to the analysis of optimality conditions in optimal control problem.

The relevant references are [**MNP20**, **MNTM21**, **MNP22a**, **MNP22a**, **MFN22**, **Maz23a**, **MF24a**].

Chapter 2: Geometric properties for optimal control problems & applications to population dynamics

In the second chapter, we focus on global geometric properties of optimisers in optimal control problem. To be more specific, we want to answer the following question: where should we act in the domain in an optimal way? Although some robust tools, mostly stemming from rearrangements or other isoperimetric-type properties, exist, they often fail to provide conclusive answers to questions coming from population dynamics. In Chapter 2, we investigate two main problems where the answers are either unexpected (from a geometric perspective) or surprisingly hard to study. To be more specific (throughout, m still denotes the control) :

(1) We will first review some well-known properties for the optimisation of elliptic eigenvalue problems, typically

$$\min_{0\leq m\leq 1 \text{ a.e.}, \ \int_\Omega m=m_0} \lambda(m) \text{ where } \lambda(m) \text{ is the lowest eigenvalue of } -\mu\Delta - m. \tag{0.3}$$

This problem is the standard one in optimal control, and it appears naturally in population dynamics: if m describes the density of resources available to a population, then the lower $\lambda(m)$, the higher the chances of the population at surviving. The moral of the story is the following: one should act in very "simple" ways: when the domain Ω is a square and the boundary conditions are of Neumann type, the optimal control should be non-increasing in every direction; when Ω is a ball and the boundary conditions are of Dirichlet type one should act in a centred ball etc. No new results regarding this problem are given.

(2) We then investigate

$$\max_{0\leq m\leq 1 \text{ a.e.}, \ \int_\Omega m=m_0} \int_\Omega j(\theta_m) \text{ subject to } \begin{cases} -\mu\Delta\theta_m - \theta_m(m-\theta_m)=0 & \text{in } \Omega\,, \\ \partial_\nu\theta_m = 0 & \text{on } \partial\Omega\,, \\ \theta_m\geq 0\,, \theta_m\neq 0 \end{cases}. \tag{0.4}$$

in order to showcase how rich of a behaviour such problems can exhibit. Recall that from Chapter 1, when $j'>0$, we know that any solution to (0.4) is a characteristic function, $m^*=\mathbb{1}_{E^*}$. We will then prove the following types of results: To be more specific:

(a) First, when the diffusivity μ is large enough, optimisers m^* behave like optimisers of (0.3) (thus the behaviour is "simple"). This is independent of the precise shape of j, as long as $j'>0$.

(b) Second, we want to prove that the situation is more complex as $\mu\to 0$. To do so, we will consider two distinct j's:

- When $j:x\mapsto x^3$, the situation is similar to (0.4). This is due to the energetic nature of the problem (in a sense made precise in Chapter 2).
- On the other hand, for the more biologically relevant case $j: x\mapsto x$, which corresponds to maximising the total population

size, we prove that, as $\mu \to 0$, the BV norm of maximisers or, alternatively, the perimeter of the optimal sets, goes to $+\infty$. In the one-dimensional case, this means that the number of connected components diverges, leading to what we dub a "fragmentation" phenomenon.

(3) We then move on to another spectral optimisation problem, this time a periodic parabolic problem, that is of crucial importance when studying the dynamics of a population in a time-periodic environment. Here, the main objective is the lowest eigenvalue $\Lambda(m)$ of the operator $\tau\partial_t - \mu\Delta - m$ endowed with time-space periodic boundary conditions, which is associated with the equation

$$\begin{cases} \tau\partial_t\varphi_m - \Delta\varphi_m - m\varphi_m = \Lambda(m)\varphi_m & \text{in } (0;T)\times\mathbb{T}\,, \\ \varphi_m(0,\cdot) = \varphi_m(T,\cdot)\,, \\ \varphi_m \geq 0\,, \not\equiv 0. \end{cases}$$

As (0.3), this has a natural interpretation in terms of population dynamics: the lower m, the higher the chances of a population accessing resources m to survive. This leads to the main problem of that part,

$$\min_{0\leq m\leq 1 \text{ a.e.},\ \iint_{(0;T)\times\mathbb{T}} m = m_0} \Lambda(m).$$

For this problem, we are interested in symmetry and monotonicity of optimisers. The lack of symmetry of the underlying operator prohibits using rearrangements techniques. We will state and partially prove the following type of results:

- First, in general, all optimisers of Λ are symmetric in time and space.
- Second, when considering the case $m(t,x) = c(t)V(x)$ where V is fixed and c is the optimisation variable, we will prove that, in the regimes $\tau = \mu \to \infty$ and $\mu = \tau^2 \to 0$, the optimal c should be symmetric decreasing.
- Finally, we will provide some numerical simulations that lend credence to the fact that optimisers should always be symmetric decreasing.

The relevant references are [**MRB21**, **MNP22a**, **Maz22a**, **BMFN25**].

Chapter 3: Quantitative estimates in optimal control problems

In the third chapter, the emphasis is on quantitative estimates. Namely, considering an optimal control of the form (0.1), and setting $J(m) = \int_\Omega j(\theta_m)$, we want to investigate estimates of the type

$$J(m) - J(m^*) \leq -C\|m - m^*\|^2$$

where m^* is a maximiser (assumed to be unique for simplicity) and for a certain norm. Chapter 3 is divided into several parts:

(1) The first one is devoted to a discussion of the type of norm we should use in quantitative estimates, using a linear optimal control problem (the so-called "bathtub principle") as a reference case.
(2) The next objective is to present the systematic approach we developed in several successive contributions [**MRB22a**, **Maz22b**, **CMF24**, **MFPT25**,

CMFSP24] to derive quantitative inequalities. We can briefly summarise these as follows:

   (a) Deriving quantitative estimates for the linearised criterion (the so-called "quantitative bathtub principle").
   (b) The study of quantitative estimates for smooth deformations of optimisers (this corresponds to the study of shape Hessian, to use the terminology of shape optimisation). This relies on the introduction of new diagonalisation basis for these shape hessians.
   (c) Finally, the use of a selection principle to show that these two steps suffice to derive global estimates.

(3) We will then state (and give more or less details depending on the case) quantitative inequalities for the following criteria:
   (a) First, eigenvalue minimisation problems: letting $\lambda(m)$ denote the lowest eigenvalue of $-\Delta - m$ endowed with Dirichlet boundary conditions, we explain how to derive a quantitative inequality for the problem
   $$\min_m \lambda(m)$$
   under L^∞ and L^1 constraints on m.
   (b) Second, linear control problems, where the problem for which we seek a quantitative inequality is (once again under L^∞ and L^1 constraints)
   $$\max_m \int_\Omega j(u_m) \text{ subject to } \begin{cases} -\Delta u_m = m & \text{in } \Omega\,, \\ u_m \in W_0^{1,2}(\Omega). \end{cases}$$
   (c) Finally, for parabolic optimal control problems of the form
   $$\max_m \int_\Omega j(u_m(T,\cdot)) \text{ subject to } \begin{cases} \partial_t u_m - \Delta u_m = m & \text{in } (0;T)\times\Omega\,, \\ u_m(t,\cdot) \in W_0^{1,2}(\Omega)\,, \\ u_m(0,\cdot) = 0 & \text{in } \Omega. \end{cases}$$

(4) In the last part of the chapter, we will focus on applications of quantitative estimates to several problems:
   (a) The first one has to do with the turnpike phenomenon in optimal control problem, where quantitative inequalities provide such a turnpike phenomenon for bilinear optimal control problems.
   (b) The second one has to do with the convergence of thresholding schemes in optimal control problems; this is natural, as quantitative inequalities should give the natural notion of "non-degenerate optimiser" in this infinite dimensional context.
   (c) The third one has to do with the optimal location of sensors in the theory of controllability, and allows to gain finer estimates on the longtime behaviour of optimal captors for the control of parabolic equations.

The relevant references are [**MRB22a**, **Maz22b**, **CMF24**, **CMFSP24**, **MFPT25**].

Chapter 4: (Mean Field) Games models in the optimal management of fisheries

Chapter 4 presents several game theoretical models for the optimal management of fisheries, the common underlying question being: assuming a population of fish is being harvested by fishermen that are engaged in a competition, how does this competition influence the fishes' population dynamics? Can we investigate instances of the tragedy of the commons? The latter expression refers to a situation where the selfishness of agents (the fishermen) ends up depleting the common resources, while also proving detrimental at the level of each individual player. We will investigate three main situations:

(1) First, we will be focusing on a static case, with a finite number of players. The focus will be on the existence and qualitative properties of Nash equilibria. To be more specific, given a population fishes θ living inside a domain Ω, accessing resources $K = K(x) \in L^\infty(\Omega)$ assume it is being harvested by two players, each fishing according to a strategy $\alpha_i \in L^\infty(\Omega)$ ($i = 1, 2$). Given that each player is trying to find a strategy that maximises her or his harvest, is it possible, in this competitive setting, to find a Nash equilibrium, that is, a couple of strategies (α_1^*, α_2^*) such that

$$J_1(\alpha_1^*, \alpha_2^*) = \max_{0\leq\alpha_1\leq 1\,,\int_\Omega \alpha_1\leq m_1} \int_\Omega \alpha_1 \theta_{\alpha_1,\alpha_2^*}\,,$$

$$J_2(\alpha_2^*, \alpha_2^*) = \max_{0\leq\alpha_2\leq 1\,,\int_\Omega \alpha_2\leq m_2} \int_\Omega \alpha_2 \theta_{\alpha_1^*,\alpha_2}\,.$$

The existence of a Nash equilibrium is not guaranteed in general, and the cases where we obtain it are studied using the tools of Chapter 1.

(2) Second, we will be investigating the case of infinitely many players, leading to a first Mean Field Game models. Without going too much into details at this stage (and in particular dropping a particular part of the equation that consists of a convolution kernel), this leads to a model of the following type:

$$\begin{cases} -\partial_t u^T - \frac{\|\nabla_x u^T\|^2}{2} = \theta^T & \text{in } (0,T)\times\mathbb{T}^d\,, \\ u^T(T,\cdot)\equiv 0, & \\ \partial_t m^T - \nabla\cdot\left(m^T\nabla_x u^T\right) = 0 & \text{in } (0,T)\times\mathbb{T}^d\,, \\ m^T(0,\cdot) = m_0, & \\ \partial_t\theta^T - \mu\Delta\theta^T = \theta^T(K-\theta^T) - m\theta^T & \text{in } (0,T)\times\mathbb{T}^d\,, \\ \theta^T(0,\cdot) = \theta_0, & \end{cases}$$

where
- $T > 0$ is the time horizon the game is played in,
- θ^T is the fishes' density, m^T describes the density of agents, initially distributed according to m_0, and u^T is the so-called "value function" that describes, starting at a point x at time t, what the optimal outcome of the game is.

The emphasis will be on the uniqueness and long-time behaviour of solutions, as well as on the computation of explicit solutions to the so-called ergodic system, which will lead us to another instance of the tragedy of the commons.

(3) Finally, we will switch to a different setting by investigating the tragedy of the commons through the lens of travelling waves, the formulation under scrutiny being: is it possible that an invasive species could go extinct solely

due to the selfishness of fishermen harvesting it? To be more precise, we will be considering a bistable non-linearity, that is, a non-linearity $f : [0;1] \to \mathbb{R}$ having exactly three roots: 0 and 1, both of which are stable ($f'(0), f'(1) < 0$), and a third, instable one $\eta \in (0;1)$. Assuming that $\int_0^1 f > 0$, it has been known for decades that:

(a) There exists a unique speed $c > 0$, and a unique (up to a translation) profile U such that $u : (t,x) \mapsto U(x-ct)$ is a solution of
$$\partial_t u - \Delta u = f(u)\,, u(-\infty) = 1\,, u(+\infty) = 0.$$
(b) Furthermore, this "travelling wave" is stable in the sense that, for any initial condition θ_0 such that $\theta_0(-\infty) = 1\,, \theta_0(+\infty) = 0$, asymptotically, the solution to
$$\begin{cases} \partial_t \theta - \Delta\theta = f(\theta)\,, \\ \theta(0,\cdot) = \theta_0 \end{cases}$$
"looks like" $U(x-ct)$.

Our question is then: can adding fishermen lead to a reversal of this travelling waves, meaning that a species that would survive in the absence of fishermen could go extinct? We will focus on the constructions of reversed travelling waves, as well as on the tragedy of the commons. A special emphasis will be given on the analysis of scenarii where coordination between fishermen prevents the extinction of fishes, while benefiting each fisherman at an individual level.

The relevant contributions are [**KMFRB24b, KMFRB24a, MRB22b**].

Chapter 5: Some vectorial optimisation problems

The last chapter deals with two categories of vectorial optimisation problems and corresponds to two recent developing research lines; as such, it will be much shorter and less systematic than the previous ones:

(1) First, we will be considering systems of interacting species in the cooperative case. There, the main optimisation problems under consideration will be the optimisation of the coupling matrix, in particular in mutation models. This will serve to motivate some open problems and justify their relevance.
(2) Second, we will be discussing the Faber-Krahn inequality for the Stokes operator, focusing on the optimality of a particular configuration. This is also a vectorial problem, the main difficulty of which comes from the incompressibility condition embedded in the Stokes operator.

The relevant contributions are [**GM21, HMFP24**].

General structure of each chapter

Each chapter is more or less self-contained, and the structure of each chapter goes as follows:

(1) First, an overview of the problem at hand with, depending on the chapter, a discussion of the main difficulties and of the related literature.
(2) Second, a statement of the results, and the relevant comments (in particular, the comparison with existing results).

(3) Third, we give some sketches of proof when we deem them particularly relevant for understanding the results.
(4) Finally, each chapter contains a discussion of open problems; depending on the chapter, this discussion might be more (in the case of ongoing research) or less (in the case of research we plan to tackle) detailed.

CHAPTER 1

Existence and pointwise properties of optimal controls

This chapter presents the contributions [**MNP20**, **MNTM21**, **MNP22a**, **MFN22**, **Maz23a**, **MF24a**], which all deal with the analysis of pointwise properties of optimal controls in elliptic or parabolic equations.

1.1. Introduction

1.1.1. Overview of the problem and scope. In this chapter we present a series of contributions all devoted to the understanding of a simple question: consider a smooth bounded domain $\Omega \subset \mathbb{R}^d$, a differential operator L (elliptic or parabolic depending on the context), possibly non-linear, and a function $m \in L^\infty(\Omega)$ called the control. Let $\Phi = \Phi(m, u)$ denote the coupling between the control and the state; in this chapter, we shall focus mostly on two cases:

- The case $\Phi(m, u) = mu$: this is called the *bilinear* case.
- The case $\Phi(m, u) = m$: this is called the *linear* case.

Motivated by applications to population dynamics (see Section 1.2 below for a short discussion and Section 1.3) we assume that the controls m belong to an admissible class of the form

$$\mathcal{M} = \left\{ m \in L^\infty(\Omega) : \, 0 \leq m \leq 1 \text{ a.e.}, \, \int_\Omega m = m_0 \right\}. \tag{1.1}$$

Assume that for any admissible control $m \in \mathcal{M}$ the partial differential equation

$$Lu_m = \Phi(m, u_m)$$

endowed with boundary conditions is uniquely solvable and consider the problem

$$\sup_{m \in \mathcal{M}} \left(J(m) := \int_\Omega j(u_m) \right). \tag{1.2}$$

The main question under consideration in this chapter is the following:

Is it true that optimisers m^* saturate the constraints, i.e. *is it true that $m^* \in \{0, 1\}$ a.e. in Ω (this is the so-called* bang-bang *property)? What happens if they do not?*

Although we discuss some motivations in Section 1.2, as well as in Sections 1.3–1.4, let us briefly justify the relevance of this question in applications:

1. This question is a basic one in optimal control: is it true that optimisers of a constrained problem saturate the constraints?
2. From the point of view of population dynamics, one should think of u_m as a population density following a reaction-diffusion of the type

$$-\Delta u_m = f(u_m) + \Phi(m, u_m).$$

When Φ is bilinear ($\Phi(m,u)=mu$), the control m can be interpreted as the distribution of resources available to the population and the question is recast as: how should one spread resources inside a domain? In the linear case ($\Phi(m,u)=m$), m should be thought of as an influx of population, and the question would be: how should one add new individuals to ensure the best possible outcome for a population? As, in general, it is impossible to fully characterise the solutions of such optimisation problems, the bang-bang property is a first valuable information.

(3) If the bang-bang property is satisfied, any optimiser m^* writes $m^*=\mathbb{1}_{E^*}$ for some measurable subset E^* of Ω. This allows to use the tools developed in the context of shape optimisation to tackle questions such as stability of optimal controls.

(4) This bang-bang property is also fundamental when putting forth efficient numerical algorithms, the convergence of which is partially the subject of Chapter 3.

The main contributions of this chapter are the following:

(1) We first give a complete answer to this question in the case of bilinear optimal control problems, by showing that, for any monotone optimal control problem (for the sake of simplicity, simply assume that j is increasing), the bang-bang property is satisfied. This holds for elliptic and parabolic control problems. We refer to Section 1.3.

(2) We then provide an analysis of optimality conditions in the case of linear controls; as we will see, this case is much more delicate. To give an example, if the underlying PDE writes $-\Delta u_m=f(u_m)+m$, the interplay between the non-linearity f and the cost functional j becomes crucial, while it is immaterial when dealing with bilinear optimal control problems. We refer to Section 1.4.

As the chapter is long, we now present a plan to help the reader navigate it.

1.1.2. Plan of the Chapter. The chapter is structured as follows:

(1) In Section 1.2, we briefly introduce the two prototypical questions under study and we state simplified versions of our results.

(2) In Section 1.3, we focus on the case of bilinear optimal control problems, with a particular emphasis on the elliptic case. We give fewer details regarding the parabolic one. We also comment on possible generalisations of the results.

(3) In Section 1.4, we focus on the case of linear control problems.

(4) In the conclusion of the chapter, we discuss some open questions and research lines related to these queries.

1.2. A succinct presentation of the two basic problems

In this paragraph, we present the two prototypical examples we have in mind, we explain why it is a relevant problem and we discuss some preliminary facts about the validity (or lack thereof) of the bang-bang property. Throughout, the admissible class of controls is

$$\mathcal{M}:=\{m\in L^\infty(\Omega)\,:\,0\leq m\leq 1\,,\int_\Omega m=m_0\}.$$

In this chapter, and although we deal with fairly general equations and criteria, two examples stemming from population dynamics are important to keep in mind from the applied perspective. We briefly describe them, the relevant mathematical facts and detailed description of the literature are postponed to Sections 1.3 and 1.4 respectively.

Bilinear control problems: optimal control of resources

The first problem, a bilinear one, is the optimal control of resources available to a population. Namely, we consider a population density θ_m living in a domain Ω, subject to a random dispersal, to a Malthusian death term $-\theta_m^2$ and that can access resources distributed according to a function $m \in L^\infty(\Omega)$, leading to a linear growth term $m\theta_m$. This leads to the equation

$$\begin{cases} -\Delta\theta_m - \theta_m(m-\theta_m) = 0 & \text{in } \Omega\,, \\ \partial_\nu \theta_m = 0 & \text{on } \partial\Omega\,, \\ \theta_m \geq 0\,, \not\equiv 0. \end{cases}$$

Motivated by problems raised by Lou [169] we study problems of the form

$$\max_{m\in\mathcal{M}} \int_\Omega j(\theta_m),$$

with a particular emphasis on the problem of the *total population size*, that is,

$$\max_{m\in\mathcal{M}} \int_\Omega \theta_m.$$

This amounts to answering the question "how should one spread resources inside the domain to optimise the population size?". This problem will be a common thread of this chapter and of the following one as, despite its apparent simplicity, it has a very rich mathematical structure.

Our first main theorem of the chapter, Theorem 1.7, roughly states that whenever j is increasing, any solution m^* of this optimisation problem is bang-bang. It should be noted that it is a general result that does not rely on the exact shape of the non-linearity (meaning that it hols for non-linear elliptic equation of the form $-\Delta\theta_m = f(x,\theta_m) + m\theta_m$, as long as the PDE is well posed in a certain sense).

Linear control problems: optimal control of invasive species

The second optimal control problem is a linear one, which we present in the parabolic case (although one could carry similar work in the elliptic setting). This time, we consider a population density y_m subject to a bistable evolution. In other words, we consider a bistable non-linearity f, say for instance $f(y) = y(y-\eta)(1-y)$, where $\eta \in (0;1)$. This non-linearity models the Allee effect: when the population is below a certain threshold η, it tends to die out, while when it is above that threshold, it tends to survive. The density y typically represents the fraction of a certain subgroup within a global population, and this model has proved particularly useful when modelling invasive phenomena [6, 135]. In the context of control of invasive, disease-carrying species (typically, mosquitoes), a recent and efficient approach consists in releasing some genetically engineered mosquitoes within the global population, to eventually replace the population [6, 7, 8]. The control m can then be described as the release of mosquitoes, and the optimal control problem typically writes, for a time horizon T,

$$\max_{m,,\forall t m(t,\cdot)\in\mathcal{M}} \int_\Omega y_m(T,\cdot) \text{ subject to } \begin{cases}\partial_t y_m - \Delta y_m = f(y_m)+m & \text{in } (0;T)\times\Omega\,,\\ \partial_\nu y_m(t,\cdot)=0 & \text{on } (0;T)\times\partial\Omega\,,\\ y_m(0,\cdot)=y_i & \text{in } \Omega.\end{cases}.$$

One of our main results, Theorem 1.17, explicits the dependence of the bang-bang property on the interplay between the convexity of f, and that of j.

1.3. Optimisation of the potential: bilinear optimal control problems

In this first section, we present the contributions [MF24a, Maz23a, MNP22a], all devoted to bilinear optimal control problems. This part is structured as follows:

(1) We first present the general setting and state our main results.
(2) We then discuss at length several aspects of the problem, in particular the applications of and difficulties tied to the bang-bang property, as well as the related literature.
(3) We finally offer some elements of proof.

1.3.1. Bilinear optimal control problems I: the elliptic case.

Throughout, Ω is a bounded, smooth domain in $\mathbb{R}^d$.

Admissible class and underlying PDE

We work with the admissible class

$$\mathcal{M} := \left\{ m \in L^\infty(\Omega) :\, 0\leq m \leq 1 \text{ a.e.}, \int_\Omega m = m_0 \right\} \tag{1.3}$$

where $m_0 \in (0;|\Omega|)$.

Remark 1.1. We could in fact work with other classes of constraints defined as rearrangement classes, but this would simply muddy the message. We refer to Section 1.3.4.6 below.

Regarding the PDE, we consider any non-linearity $f=f(\theta)$ such that, for any admissible $m\in\mathcal{M}$, the PDE

$$\begin{cases}-\Delta\theta_m = f(\theta_m)+m\theta_m & \text{in } \Omega\,,\\ \partial_\nu\theta_m = 0 & \text{on } \partial\Omega\,,\\ \theta_m\geq 0\,,\theta_m\not\equiv 0\end{cases} \tag{1.4}$$

is well posed in the following sense:

(1) θ_m exists and is unique,
(2) $\theta_m \in L^\infty(\Omega)$,
(3) θ_m is linearly stable, in the sense that for any m, the first eigenvalue $\Lambda(m)$ of the operator

$$u \mapsto -\Delta u - f'(\theta_m)u - mu$$

endowed with Neumann boundary conditions is positive:

$$\Lambda(m) > 0. \tag{1.5}$$

This covers a fairly general class of non-linearities f, several of which are reviewed in Section 1.3.4.6. It should be noted that f could also depend on x. Incidentally, for all intents and purposes, we would also consider much more general operators than the Laplacian, typically $-\nabla\cdot(A(x)\nabla\cdot)+\langle X(x),\nabla\rangle$ but for the sake of simplicity we stick to the simple case of the Laplacian and refer again to Section 1.3.4.6. For the time being, we use as an example the following non-linearity: $f : u \mapsto -u^2$, so that our main equation becomes

$$\begin{cases} -\Delta\theta_m = \theta_m(m - \theta_m) & \text{in } \Omega\,, \\ \theta_m \geq 0\,, \theta_m \not\equiv 0\,, \\ \partial_\nu \theta_m = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Equation (1.6) is the standard *logistic-diffusive equation*; it models the equilibrium distribution of a population (with density θ_m) subject to three main phenomena:

(1) First, a random dispersal within the domain, accounted for by the Laplacian term.
(2) Second, a *per capita* growth rate $m \in L^\infty(\Omega)$.
(3) Finally, the Malthusian death term $-\theta_m^2$.

This equation has been a tenet of spatial ecology since the inception of PDE-based population dynamics. We refer to the seminal works of Kolmogorov, Petrovski & Piskounov [**147**], Fisher [**99**]. As regards its qualitative analysis, a foundational paper is the study of Skellam [**211**], which propelled the qualitative study of spatially heterogeneous population dynamics, with a particular emphasis on the influence of such heterogeneities on the outcome of population evolution. Regarding the well posedness and stability of (1.6), the first systematic study was carried out by Cantrell & Cosner [**62**]. We summarise their findings here:

THEOREM 1.2. (1.6) *has a solution if, and only if, the first eigenvalue* $\lambda(m)$ *of* $-\Delta - m$ *endowed with Neumann boundary conditions is negative. Furthermore, if a solution exists, it is unique and linearly stable. Finally, it is globally attractive for the initial value problem*

$$\begin{cases} \partial_t\theta - \Delta\theta = \theta(m - \theta) & \textit{in } (0;+\infty)\times\Omega\,, \\ \partial_\nu\theta = 0 & \textit{on } (0;+\infty)\times\Omega\,, \\ \theta(0,\cdot) = \theta_0(\cdot) \geq 0\,, \not\equiv 0, \end{cases}$$

for any non-negative, non-identically zero initial condition θ_0*.*

REMARK 1.3. Theorem 1.2 naturally leads to considering an optimal control with a natural interpretation, namely, solve

$$\min_{m\in\mathcal{M}} \lambda(m).$$

As the lower $\lambda(m)$ is, the more unstable the steady-state $z \equiv 0$, this optimisation problem amounts to finding the best resources distribution to ensure the survival of a species. We go back to this interpretation in the following paragraph, as spectral optimisation problems are usually a good touching ground for bilinear control problems.

The main optimisation problems

We consider a $\mathscr{C}^2$ function j (again, one could assume some dependence of j on x; we refer to Section 1.3.4.6) and we investigate the optimal control problem

$$\max_{m\in\mathcal{M}} \int_\Omega j(\theta_m). \tag{1.7}$$

Our goal is to provide a bang-bang property in the following sense:

Definition 1.4. A bang-bang function m is an element of $\mathcal{M}$ that satisfies $m \in \{0;1\}$ a.e. in Ω. The problem (1.7) is said to satisfy the bang-bang property if any solution m^* of (1.7) is bang-bang.

Remark 1.5. The existence of an optimiser for (1.7) is an easy consequence of the direct method in the calculus of variations.

Remark 1.6. Bang-bang functions are in fact the extreme points of the convex, weakly $L^\infty-*$ compact set $\mathcal{M}$. In that sense, investigating the bang-bang property for (1.7) amounts to investigating whether optimisers of an optimal control problem are extreme points of the admissible set.

Let us also single out, for further reference (in particular when discussing the usual strategies used to prove the bang-bang property), the optimisation problem stemming from Remark 1.3, which amounts to the optimisation of the survival ability associated with a resources distribution $m \in L^\infty(\Omega)$: letting $\lambda(m)$ be the lowest eigenvalue of $-\Delta - m$ endowed with Neumann boundary conditions, which can also be defined as

$$\lambda(m) := \min_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 - \int_\Omega m u^2}{\int_\Omega u^2}, \tag{1.8}$$

the optimal survival ability problem is

$$\min_{m \in \mathcal{M}} \lambda(m). \tag{1.9}$$

It should be noted that this is a famous, and quite well-understood problem. We also refer to Section 1.3.4.

Our main result

Our first main result is taken from [**MNP22a**]:

Theorem 1.7 ([**MNP22a**]). *Assume that $j' > 0$ on $(0;+\infty)$. Any solution m^* of (1.7) is bang-bang: there exists a measurable subset $E^* \subset \Omega$ such that*

$$m^* = \mathbb{1}_{E^*}.$$

Remark 1.8 (A few comments).

- This theorem came after a series of partial contributions to such questions, which we review in Section 1.3.4.1, in particular those of Nagahara & Yanagida [**187**].
- The interest of this theorem, as stated earlier, is not purely optimisation-oriented. Rather, as far as we are concerned, its main point is that it allows to use the machinery developed in the context of shape optimisation to derive finer properties. We refer to Chapter 3 for the study of quantitative inequalities in such optimal control problems, and to the conclusion of the present chapter for our research program, which will focus on the study of the so-called optimal set E^*. Furthermore, this property allows to put forth and study efficient numerical algorithms. We refer again to Chapter 3, and also to Section 1.3.4.4 for a presentation of these methods.
- Observe the difference between our problem and the more standard energetic problem, by which we mean problems where the functional to be optimised is the energy of the PDE defining the constraint of the problem. A typical example is given by (2.1). We defer a detailed discussion of those aspects to subsection 1.3.4.3.

- We claim that our result is to some extent sharp, in the sense that when j is not increasing the bang-bang property is in general not true. We refer to Section 1.3.4.2. We also note that monotonicity can be expected to play a major role here given the Buttazzo-Dal Maso theorem that we partly review in Section 1.3.4.3.

1.3.2. Bilinear optimal control problems II: parabolic models.

1.3.2.1. *Statement of the main result.* We now present related results for parabolic equations. This paragraph corresponds to [**Maz23a**, **MF24a**]. We introduce a time horizon $T > 0$ and we let $g = g(u)$ be a $\mathscr{C}^2$ function[1] that satisfies

$$\lim_{u \to \infty} g(u) < 0.$$

We work with an initial condition u_0 satisfying

$$u_0 \geq 0\,, u_0 \not\equiv 0$$

and with homogeneous Neumann boundary conditions. Likewise, we work only with the heat equation, although our methods would apply to more general diffusion processes. By the maximum principle and standard considerations in parabolic equations, for any $m \in L^\infty((0;T) \times \Omega)$, there exists a unique solution u_m to

$$\begin{cases} \partial_t u_m - \Delta u_m = m u_m + u_m g(u_m) & \text{in } (0;T) \times \Omega\,, \\ u_m(0, \cdot) = u_0 & \text{in } \Omega\,, \\ \partial_\nu u_m = 0 & \text{on } (0;T) \times \partial\Omega, \end{cases} \tag{1.10}$$

that further satisfies

$$\forall t > 0\,, \min_{\overline{\Omega}} u_m > 0.$$

Surprisingly, the type of constraints we enforce on the admissible controls (*i.e.* whether they depend only on space, or on time and space) play a major role. For the sake of readability, we work with two fairly simple classes.

(1) Time independent controls: we let $V_0 \in (0; |\Omega|)$ and we define

$$\mathcal{M}_{\text{ind}} := \left\{ m \in L^\infty(\Omega) :\, 0 \leq m \leq 1 \text{ a.e.}, \int_\Omega m = V_0 \right\}. \tag{1.11}$$

(2) Time-dependent controls: we let $V_1 \in (0; T|\Omega|)$ and we define

$$\mathcal{M}_{\text{dep}} := \left\{ m \in L^\infty((0;T) \times \Omega) :\, 0 \leq m \leq 1 \text{ a.e.}, \iint_{(0;T)\times\Omega} m = V_1 \right\}. \tag{1.12}$$

We let $j = j(u)$ and $j_T = j_T(u)$ be two cost functions. The two optimal control problems under consideration are

$$\max_{m \in \mathcal{M}_{\text{ind}}} J(m) := \iint_{(0;T)\times\Omega} j(u_m) + \int_\Omega j_T\left(u_m(T, \cdot)\right). \tag{1.13}$$

and

$$\max_{m \in \mathcal{M}_{\text{dep}}} J(m) := \iint_{(0;T)\times\Omega} j(u_m) + \int_\Omega j_T\left(u_m(T, \cdot)\right). \tag{1.14}$$

[1] Like in the elliptic case, g could depend on t and x as well but this would simply make things notationally heavier.

We assume the following monotonicity conditions on $j\,, j_T$:

$$j'\,, j_T' \geq 0 \text{ on } (0;+\infty) \text{ and one of these inequalities is strict.} \tag{1.15}$$

The main theorem is the following; it summarises [**MF24a**, **Maz23a**]

THEOREM 1.9. *We assume* (1.15) *holds.*

(1) [**Maz23a**] *Assume* $d=1$. *Any solution* m^* *of* (1.13) *is bang-bang:* $m^* \in \{0;1\}$ *a.e. in* $\Omega = (0;2\pi)$.
(2) [**MF24a**] *In any dimension, any solution* m^* *of* (1.14) *is bang-bang:* $m^* \in \{0;1\}$ *a.e. in* $(0;T)\times\Omega$.

A legitimate question might be whether the one-dimensional restriction in the time independent case is indeed necessary. We do not believe it, but this seems to come at much greater technical cost. We give fewer details here than in the elliptic case and merely sketch the proofs.

1.3.3. Some applications of the techniques to other problems. Although we do not detail those here, let us mention that we used the methodology developed in the rest of the chapter to other relevant cases. More specifically:

(1) With Y. Privat, we studied in [**MP24**] the optimisation of Robin boundary coefficients, that is, we investigated qualitative properties for problems of the form

$$\min_{\beta\in L^\infty(\partial\Omega)\,,0\leq\beta\leq 1\,,\int_{\partial\Omega}\beta=\beta_0} \int_{\Omega \text{ or } \partial\Omega} j(u_\beta) \text{ subject to } \begin{cases} -\Delta u_\beta = f(u_\beta) & \text{in } \Omega\,,\\ \partial_\nu u_\beta + \beta u_\beta = 0 & \text{on } \partial\Omega. \end{cases}$$

This work fits in a growing series of contributions devoted to the understanding of the optimisation of boundary coefficients [**47**, **46**, **115**, **133**, **152**, **160**, **89**, **90**]. Our results are similar in terms of bang-bang property, and we also provide a detailed analysis of several cases where the bang-bang property fails.

(2) In [**MF24b**], we considered the optimisation of the carrying capacity in logistic-diffusive models, meaning that we studied problems of the form

$$\sup_{K\in L^\infty(\Omega)\,,0\leq K\leq 1\,,\int_\Omega K=K_0} \int_\Omega j(v_K) \text{ subject to } \begin{cases} -\Delta v_K - v_K\left(r(x) - \frac{v_K}{K}\right) = 0 & \text{in } \Omega\,,\\ \partial_\nu v_K = 0 & \text{on } \partial\Omega\,,\\ v_K \geq 0\,, \not\equiv 0. \end{cases}$$

The focus of [**MF24b**] was on the existence and qualitative properties of the optimal carrying capacity K; these properties were in parts derived using the methodology presented in this chapter.

1.3.4. State of the art and comments. Some comments, several of which have strong ties to our current research interests, are in order regarding Theorem 1.7. We mostly make these comments in the case of elliptic problems, as the same hold *mutatis mutandis* for parabolic optimal control problems.

1.3.4.1. *Some motivations coming from mathematical biology.* We started considering this type of properties while working on a problem raised by Lou [**170**, **169**] in the 2000's. Namely, *what is the influence of spatial heterogeneity on the total* *biomass of a population?* In other words, what can be said about the optimisation problems

$$\sup_{m\in\mathcal{M}(\Omega)} / \inf \int_\Omega \theta_m \text{ subject to } \begin{cases} -\Delta\theta_m = \theta_m(m-\theta_m) & \text{in } \Omega\,, \\ \partial_\nu\theta_m = 0 & \text{on } \partial\Omega\,, ? \\ \theta_m \geq 0\,, \theta_m \not\equiv 0 \end{cases} \tag{1.16}$$

Lou proved that the constant resources distribution minimises the population size (see (1.17) below) and showed that, when one only imposes an L^1 bound on the resources distribution in the one-dimensional case sequences of maximisers accumulating at a dirac point were doing better than characteristic functions of intervals. These works started a line of research focusing on several properties of the ratio $\int_\Omega \theta_m / \int_\Omega m$ (see in particular the works of Bai, He & Li [**17**] and Inoue & Kuto [**138**] (we also refer to [**162, 121**]) , and on whether the bang-bang property held or not for the maximisation problem. A first partial answer came when Nagahara & Yanagida [**187**] proved the following weaker result: if m is such that $\{0 < m < 1\}$ contains an open ball, then m can not be a maximiser of the population size. This was followed by our first work with Nadin & Privat [**MNP20**], where we showed that the bang-bang property holds whenever the domain is "small" enough (or, alternatively, when the diffusivity is large enough). We provided a final answer in [**MNP22a**].

1.3.4.2. *Regarding the sharpness of the assumptions: bilinearity and monotonicity.* We now comment on the sharpness of our result, by which we mean the following: both the bilinearity and the monotonicity are necessary.

(1) Regarding the bilinearity of the control: the first comment is that our result might seem at first sight surprising, given that it does not rely on any convexity assumption on j. In particular, Theorem 1.7 applies to $j(u) = \sqrt{u}$. The heart of the proof is from the fact that the monotonicity of j in u entails an "almost-convexity" of J in the sense that, roughly speaking, its second-derivative d^2J is positive (except on a finite-dimensional subspace) on the tangent cone to any non-bang-bang admissible control m; this might sound a bit technical, but it will be clarified in the course of the proof. Suffices to say it relies on the Hopf-Cole transform.

    Observe that this theorem is trivially false in the case of linear control problems. As an example, consider the following simple problem:

$$\max_{m\in\mathcal{M}(\Omega)} S(m) := \int_\Omega \sqrt{y_m} \text{ subject to } \begin{cases} -\Delta y_m + y_m = m & \text{in } \Omega\,, \\ \partial_\nu y_m = 0 & \text{on } \partial\Omega. \end{cases}$$

    As the map $x \mapsto \sqrt{x}$ is concave, and $m \mapsto y_m$ is linear, the function S is (strictly) concave in m. Furthermore, it is readily seen that $m \equiv ⨍_\Omega \bar{m}$ is a critical point of S, which leads to the following conclusion: the unique maximiser of S is constant.

(2) Regarding the monotonicity of the functional: the monotonicity assumption of the functional is both necessary and to be expected in the context of the bang-bang property. Regarding the fact that it is necessary, go back

to the study of the total population size, and consider the problem

$$\max_{m\in\mathcal{M}(\Omega)} R(m) := \left(-\int_\Omega \theta_m\right) \text{ with } \begin{cases} -\Delta\theta_m = \theta_m(m-\theta_m) & \text{in } \Omega\,, \\ \theta_m \geq 0\,, \theta_m \not\equiv 0\,, \\ \partial_\nu \theta_m = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

Following Lou [169], it suffices to divide the PDE on θ_m by θ_m and to integrate by parts to obtain

$$\int_\Omega \theta_m = \int_\Omega m + \int_\Omega \frac{|\nabla\theta_m|^2}{\theta_m^2} \geq \int_\Omega m,$$

and the last inequality is strict unless m is constant. Consequently, we deduce that $m \equiv \fint_\Omega \bar{m}$ is the unique solution of (1.17). A natural question would be whether, when j is decreasing in u, one can expect optimisers to not be bang-bang. This is wrong in general, or at least seems to strongly depend on boundary conditions. Indeed, consider the problem

$$\min_{m\in\mathcal{M}(\mathbb{B})} \int_\mathbb{B} y_m \text{ subject to } \begin{cases} -\Delta y_m - \varepsilon m y_m = 1 & \text{in } \mathbb{B}\,, \\ y_m = 0 & \text{on } \partial\mathbb{B}, \end{cases} \tag{1.18}$$

where $\mathbb{B}$ is the unit ball and $\varepsilon > 0$ is chosen small enough to ensure the well posedness of the equation. It is easy to see that in the case $\bar{m} = \mathbb{1}_{\mathbb{B}'}$ the unique minimiser, for $\varepsilon > 0$ small enough, is the characteristic function $m_\# = \mathbb{1}_\mathbb{A}$ where $\mathbb{A} = \{r_0 \leq \|x\| \leq 1\}$ where r_0 is chosen to satisfy the proper volume constraint. This is a constraint of Talenti inequalities [215] and, while we do not detail the proof here, we simply point out that this simple example shows that no general result can be expected in general.

Remark 1.10 (Monotonicity and the Buttazzo-Dal Maso Theorem [56]). Let us make a final tangential comment regarding the fact that monotonicity should be expected in this context by recalling the Buttazzo-Dal Maso theorem. Consider a general shape optimisation problem of the form

$$\sup_{\Omega\subset D\,,|\Omega|\leq V_0} \mathcal{J}(\Omega) := \int_\Omega j(u_\Omega) \text{ subject to } \begin{cases} -\Delta u_\Omega = f(x,u_\Omega) & \text{in } \Omega\,, \\ u_\Omega \in W_0^{1,2}(\Omega), \end{cases}$$

where D is a fixed sub-domain of $\mathbb{R}^d$. The Buttazzo-Dal Maso theorem gives the existence of an optimal shape[2] Ω^* provided the functional is continuous in the right[3] topology and, most importantly, under the monotonicity assumption

$$\Omega_1 \subset \Omega_2 \Rightarrow \mathcal{J}(\Omega_1) \leq \mathcal{J}(\Omega_2).$$

The link with the optimal control problem (1.7) is the following: consider a fixed bound domain Ω and a volume constraint $V_0 \in (0;|\Omega|)$. Let, for any $F \subset \Omega$ with volume m_0, $m_F := \mathbb{1}_F$. Fix $\varepsilon > 0$ and define $u_{E,\varepsilon}$ as the solution of

$$\begin{cases} -\Delta u_{E,\varepsilon} + \frac{1}{\varepsilon}(1-m_E)u_{E,\varepsilon} = f(x,u_{E,\varepsilon}) & \text{in } \Omega\,, \\ u_{E,\varepsilon} \in W_0^{1,2}(\Omega), \end{cases}$$

[2] within the class of quasi-open sets.

[3] namely, the convergence of the resolvents of the Dirichlet Laplace operator.

Throughout, we still assume f is chosen so that this equation is always well posed. It is expected that, as $\varepsilon \to 0$, $u_{E,\varepsilon}$ converges to the solution of $-\Delta u_E = f(x, u_E)\,, u_E \in W_0^{1,2}(E)$, so that a good approximation of the shape optimisation problem

$$\sup_{E\subset\Omega\,,|E|=m_0} \int_E j(u_E)$$

should be the optimal control problem

$$\sup_{m\in\mathcal{M}} \int_\Omega j(u_m).$$

Furthermore, under the assumption $j' > 0$ we will show that the function J is increasing in m in the sense that for any $m \leq m'$ $J(m) \leq J(m')$. Thus, as the regime in which we have existence for the shape optimisation problem should correspond to the regime for which we have the bang-bang property in the optimal control problem, it is natural to expect such monotonicity assumptions. Nevertheless, and although the tools of Buttazzo and Dal Maso were adapted to the optimal control setting for the study of variational problems with L^p constraints [**57**], they fail to provide the bang-bang property–the reason has to do with the topology imposed on shapes: the natural topology for optimal control problems is not compatible with the one required in the study of shape optimisation problems.

1.3.4.3. *Usual strategies to derive the bang-bang property.* There are, as far as we are aware, two main ways to prove the bang-bang property. To illustrate them, we go back to (2.1), which consists in minimising the lowest eigenvalue of $-\Delta - m$ with Neumann boundary conditions. We claim that any solution of (2.1) is bang-bang, which can be proved in the following ways:

(1) <u>An energetic approach:</u> the first approach relies on the fact that as an infimum of linear functionals in m (see (1.8)), λ is concave in m and it is easy to see it is in fact strictly concave. Thus any optimiser is an extreme point of $\mathcal{M}$, that is, a bang-bang function. This approach can be used to prove that, with the notations of Theorem 1.7, any solution of

$$\max_{m\in\mathcal{M}} \int_\Omega \theta_m^3$$

is bang-bang. Indeed, one can show that θ_m is characterised as the unique minimiser of

$$E(m,\cdot) : W^{1,2}(\Omega) \ni u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{3}\int_\Omega u^3 - \frac{1}{2}\int_\Omega mu^2$$

so that, through elementary manipulations and using the weak formulation of (1.6),

$$\underset{m\in\mathcal{M}}{\operatorname{argmax}} \int_\Omega \theta_m^3 = \underset{m\in\mathcal{M}}{\operatorname{argmin}} \max_{u\in W_0^{1,2}(\Omega)} E(m,u).$$

Using this formulation, and the fact that $m \mapsto \min_{u\in W^{1,2}(\Omega)} E(m,u)$ is strictly concave in m and the same argument gives the conclusion. When j is merely increasing however, this is not a viable approach.

(2) <u>A unique continuation approach:</u> we can also use unique continuation-like properties. Indeed, the first order optimality conditions for (2.1) read as follows: if an optimiser m^* is not bang-bang, then the associated eigenfunction u_{m^*} is constant on the measurable set $\{0 < m^* < 1\}$. Here,

observe that if $\lambda(m^*) \geq 0$, since $-\Delta u_{m^*} = (\lambda(m^*) + u_{m^*})u_{m^*}$, we directly reach a contradiction. In general, the question would then become: can the solution of an elliptic equation have level sets with positive measure? This is a deep and technical question, but several results (stemming from [**134**, Chapter XXVIII]) can help. Nevertheless, for more general optimal control problems, neither of these approaches is usable, and we refer to Remark 1.12 for more details.

1.3.4.4. *The role of the bang-bang property in numerical schemes.* As we briefly mentioned, the bang-bang property is the foundation of one of the most widely used algorithm in the simulation of optimal control problems, the thresholding algorithm. The thresholding algorithm is rooted in the seminal works of Bence, Merriman & Osher [**173**], which aimed at approximating the mean curvature flow using the fundamental solution of the heat equation. The basic idea was to generate, for an initial condition $\Omega_0 = \Omega$, a sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ of shapes in the following way: letting $h > 0$ be a small enough parameter and $G_h := (4\pi h)^{-\frac{d}{2}} e^{-\frac{\|x\|^2}{4h}}$, let $\Omega_{k+1} := \{G_h \star \mathbb{1}_{\Omega_k} > \lambda\}$ where λ is a fixed (thresholding) parameter. Up to proper rescaling, this is a good approximation of the mean curvature flow, which has the advantage of being computationally efficient. On the other hand, in optimal control problems, following the works of Céa, Gioan & Michel [**74**] and of Osher & Santosa [**191**], a projected gradient/fixed-point approach was used in a systematic manner in a series of works investigating the simulation of optimal control problems, in particular in mathematical biology [**221, MRB21, MNP20, 150, 141, 140, ?**]. Interestingly, one can also trace such ideas back to the works of Céa, Gioan & Michel [**74**]. The core idea is the following: assume that the set of constraints $\mathcal{M}(\Omega)$ is defined by a reference $\bar{m} = \mathbb{1}_E$. The optimality conditions for (1.7) typically take the form (provided the optimiser is a bang-bang function $m^* = \mathbb{1}_{E^*}$)

$$E^* = \{\phi_{E^*} > c^*\}$$

for some Lagrange multiplier c^*. The function ϕ_{E^*} is called the switch function of the problem, and solves a PDE of the form

$$\mathcal{L}_m \phi_m = \ell_m$$

where $\mathcal{L}_m$ is a linear (possibly non symmetric) operator and ℓ_m is a function. The projected gradient algorithm consists in fixing an initial guess m_0, and letting, for any $m \in \mathcal{M}(\Omega)$, G_m be the fundamental solution of the operator $\mathcal{L}_m$, in setting

$$\phi_{k+1} := \left(G_{m_k} \star \ell_{m_k}\right), m_{k+1} = \mathbb{1}_{\{\phi_{k+1} > c_{k+1}\}}$$

where c_{k+1} is chosen to ensure that m_{k+1} satisfies the right integral constraint (we will see some explicit examples in the following sections). An important question is the well posedness of this algorithm and its behaviour. It is reasonable to assume that in order for this algorithm to be well-defined, the bang-bang property needs to be satisfied. In fact, the implications of a lack of bang-bang property motivated some of our works on optimality conditions for linear control problems[**MFN22, MNTM21**] (see also Section 1.4 below). Regarding the convergence of this algorithm, although its BMO counterpart is well-understood, many questions remain open; we have, nevertheless, tackled several limit cases with Chambolle and Privat [**CMF24**]. We refer to Chapter 3.

1.3.4.5. *From optimal control to free boundary problems.* Very much related to the previous point, as explained in [**CMF24**], is the regularity of the optimal set. Indeed, assume that $\bar{m} = \mathbb{1}_E$ and that the bang-bang property is satisfied. In other words, letting m^* be a maximiser, we can write $m^* = \mathbb{1}_{E^*}$. A natural question is: *how regular is the set* E^*? To be more precise, the validity of the bang-bang property *allows to recast optimal control problems into free boundary problems*, for which a variety of tools are available. Without getting too much into specifics at this stage, let us observe that (following the terminology of Monneau & Weiss) the ensuing problems fall into the category of (non-minimising) *unstable free boundary problems.* With Ferreri & Prunier, we have recently undertaken a systematic study of such regularity properties, and this is an active line of our research. We refer to the conclusion of the chapter.

1.3.4.6. *Regarding the constraints and the differential operator.* As we mentioned, Theorem 1.7 is stated for the admissible class $\mathcal{M}$ defined in (1.3). It is however possible to replace $\mathcal{M}$ with something more general, a rearrangement class. Essentially, for any two $f\,, g \in L^\infty(\Omega)$, we say that f is dominated by g and we write

$$f \preceq g$$

if, and only if,

$$\forall r \in (0;|\Omega|)\,, \sup_{E\subset\Omega\,,|E|=r} \int_E f \leq \sup_{F\subset\Omega\,,|F|=r} \int_F g.$$

We can then define, for a given $m_{\text{ref}} \in L^\infty(\Omega)\,, m_{\text{ref}} \geq 0$, the admissible class

$$\mathcal{M}(m_{\text{ref}}) := \{m \in L^\infty(\Omega) : \, m \preceq m_{\text{ref}}\}\,. \tag{1.19}$$

When $m_{\text{ref}} = \mathbb{1}_E$ for a given set E of volume m_0, we recover the class $\mathcal{M}$. Following Migliaccio [**174**], $\mathcal{M}(m_{\text{ref}})$ is convex and compact for the weak $L^\infty - *$ topology. The set of extreme points of $\mathcal{M}(m_{\text{ref}})$ is given by

$$\mathcal{K}(m_{\text{ref}}) = \left\{ m \in L^\infty(\Omega) \,: \forall r \in (0;|\Omega|)\,, \sup_{E\subset\Omega\,,|E|=r} \int_E m = \sup_{E\subset\Omega\,,|E|=r} \int_E m_{\text{ref}} \right\}. \tag{1.20}$$

Retaining the notations and assumptions of Theorem 1.7, it is possible, using the same proof, to show that any solution of

$$\max_{m\in\mathcal{M}(m_{\text{ref}})} \int_\Omega j(\theta_m)$$

is in fact an element of $\mathcal{K}(m_{\text{ref}})$.

Furthermore, we would like to mention that we could work with more general operators, different boundary conditions and other non-linearities f. Namely, we claim that, in order to prove that, with the assumption $j' > 0$ on $(0;+\infty)$, any solution of

$$\max_{m\in\mathcal{M}} \int_\Omega j(u_m)$$

$$\text{subject to } \begin{cases} -\nabla\cdot(A(x)\nabla u_m) + \langle X(x), \nabla u_m\rangle + cu_m = f(x,u_m) + mu_m & \text{in } \Omega\,, \\ Bu_m = 0 & \text{on } \partial\Omega \end{cases}$$

is a bang-bang function, only a few elements are necessary:

(1) First of all, that the equation is well posed, and that $u_m \in L^\infty(\Omega)$. This requires that $A(\cdot)$ be uniformly elliptic, that A be smooth, as well as some higher integrability on c and X. More importantly, we need u_m to be linearly stable in the sense that the lowest eigenvalue of
$$-\nabla\cdot(A(x)\nabla u)+\langle X(x),\nabla u\rangle+cu-\partial_u f(x,u_m)-mu$$
endowed with B-boundary conditions be positive. If f is smooth enough and
$$\partial_u f(x,u)<0$$
for any $u>0$, as well as
$$\lim_{x\to\infty}\inf_{x\in\overline{\Omega}}\frac{f(x,u)}{u}=-\infty,$$
then this non-linearity works.

(2) The second thing is a strong maximum principle, by which we mean $\min_\Omega u_m>0$. This requires working with Robin or Neumann boundary conditions.

We claim that these are sufficient assumptions to guarantee the validity of the bang-bang property.

1.3.5. Main elements of proof of Theorem 1.7. We give some elements of proof of Theorem 1.7. For the details, we refer to [**MNP22a**]. The bottom-line is to use second-order optimality conditions to show that monotone, bilinear problem are "convex enough".

Simplifying assumptions For the sake of readability, we work under the following assumptions:

(1) Rather than working with Neumann boundary conditions, we work in the torus $\mathbb{T}^d$.
(2) In order to illustrate the type of structure we need, we write $f(u)=-u^2$.

Hopf-Cole transform, first order optimality conditions As for any $m\in\mathcal{M}$ we have $\theta_m>0$ we can write $\theta_m=e^{\varphi_m}$ with
$$-\Delta\varphi_m-|\nabla\varphi_m|^2-m=\tilde{f}(\varphi_m)\text{ in }\mathbb{T}^d \tag{1.21}$$
with $\tilde{f}(\varphi)=f(e^\varphi)/e^\varphi$. Standard considerations on the linear stability of the steady state θ_m guarantee the Gateaux-differentiability of the solution mapping $m\mapsto\varphi_m$. Letting, for any $m\in\mathcal{M}$ and any admissible (in the sense of convex analysis) perturbation h at m, $\dot\varphi_m$ (resp. $\ddot\varphi_m$) denote the first (resp. second) order Gateaux derivative of $m\mapsto\varphi_m$ at m in the direction h, we observe that $\dot\varphi_m$, $\ddot\varphi_m$ solve
$$\begin{cases}-\Delta\dot\varphi_m-2\langle\nabla\dot\varphi_m,\nabla\varphi_m\rangle-\dot\varphi_m\tilde{f}'(\varphi_m)=h & \text{in }\mathbb{T}^d\,,\\ -\Delta\ddot\varphi_m-2\langle\nabla\ddot\varphi_m,\nabla\varphi_m\rangle-\ddot\varphi_m\tilde{f}'(\varphi_m)=2|\nabla\dot\varphi_m|^2+(\dot\varphi_m)^2\tilde{f}''(\varphi_m) & \text{in }\mathbb{T}^d.\end{cases} \tag{1.22}$$

Introducing the operator $L_m:u\mapsto\Delta u-2\langle\nabla\varphi_m,\nabla u\rangle-u\tilde{f}'(\varphi_m)$, (1.22) rewrites in a compact form as
$$\begin{cases}L_m\dot\varphi_m=h & \text{in }\mathbb{T}^d\,,\\ L_m\ddot\varphi_m=2|\nabla\dot\varphi_m|^2+(\dot\varphi_m)^2\tilde{f}''(\varphi_m) & \text{in }\mathbb{T}^d\,,\end{cases} \tag{1.23}$$

Now, consider the criterion

$$J(m)=\int_{\mathbb{T}^d} j(u_m)=\int_{\mathbb{T}^d} \tilde{j}(\varphi_m) \text{ where } \tilde{j}(\varphi)=j(e^{\varphi}).$$

The first-order derivative of J at m in the direction h reads

$$\dot{J}(m)[h]=\int_{\mathbb{T}^d} \dot{\varphi}_m \tilde{j}'(\varphi_m).$$

To write it in a more tractable form, we introduce the switch function η_m as the (unique) solution of

$$L_m^*\eta_m=\tilde{j}'(\varphi_m) \text{ in } \mathbb{T}^d\,, \tag{1.24}$$

where L_m^* is the adjoint of the differential operator L_m. In particular,

$$\dot{J}(m)[h]=\int_{\mathbb{T}^d} h\eta_m. \tag{1.25}$$

Now assume that there exists a solution m^* of (1.7) that is not bang-bang or, in other words, that the set $\omega_0=\{0<m^*<1\}$ has positive measure. In particular, we can fix $\varepsilon>0$ such that

$$\omega_\varepsilon^*:=\{\varepsilon<m^*<1-\varepsilon\} \tag{1.26}$$

satisfies

$$|\omega_\varepsilon^*|>0. \tag{1.27}$$

Since for any $h\in L^\infty(\mathbb{T}^d)$ supported in ω_ε^* with $\int_{\mathbb{T}^d} h=0$ and any $t\in(-1;1)$ with $|t|$ small enough there holds $m^*+th\in\mathcal{M}(\mathbb{T}^d)$ we deduce that for any such h we have

$$\int_{\mathbb{T}^d} h\eta_m=0.$$

In particular we obtain:

LEMMA 1.11. *If m^* is optimal then, for the set ω_ε^* introduced in* (1.26) *and satisfying* (1.27) *we have $\eta_m\equiv\mu$ in ω_ε^* for some constant $\mu\in\mathbb{R}$.*

REMARK 1.12. When trying to derive the bang-bang property, a typical strategy is to use Lemma 1.11 to conclude that any optimiser is bang-bang. Indeed, for a general optimal control problem, the fact that a maximiser is not bang-bang usually entails that the switch function has a level-set of positive measure. As this switch function solves a PDE, this can be discarded using a unique continuation principle (see [**134**, Chapter XXVIII]). This approach works, for instance, for the minimisation of eigenvalues with respect to a potential. Here, however, the structure of the equation on η_m prevents the use of such tools as Carleman estimates and, thus, to rely on unique continuation.

At this stage, the monotonicity of j did not play any role. We have the following result:

LEMMA 1.13. *For any $m\in\mathcal{M}(\mathbb{T}^d)$, $\min_{\mathbb{T}^d}\eta_m>0$.*

This lemma is where the assumption $j'>0$ is necessary.

SKETCH OF PROOF OF LEMMA 1.13. A very quick proof goes as follows: observe that with the choice $f(u) = -u^2$ we see that $\tilde{f}$ is decreasing in φ. In particular, toying around with (1.21)–(1.22), the maximum principle yields that, for any $h \in L^\infty(\mathbb{T}^d)\,, h \geq 0\,, h \not\equiv 0$, $\min_{\mathbb{T}^d} \dot{\varphi}_m > 0$. In particular, for such an h, $\dot{J}(m)[h] = \int_{\mathbb{T}^d} \dot{\varphi}_m \tilde{j}'(\varphi_m) > 0$. Thus, for any $h \in L^\infty(\mathbb{T}^d)\,, h \geq 0\,, h \not\equiv 0$, $\int_{\mathbb{T}^d} h\eta_m > 0$, whence $\eta_m > 0$ in $\mathbb{T}^d$. In general, one does not need to assume that $\tilde{f}$ is decreasing, but finer maximum principle type arguments are then required.

□

Second-order optimality conditions We turn to the second-order derivatives of the criterion. Namely, with the notations of (1.22)–(1.23), we obtain that the second-order derivative of J at m in an admissible direction h writes

$$\ddot{J}(m)[h,h] = \int_{\mathbb{T}^d} \ddot{\varphi}_m \tilde{j}'(\varphi_m) + \int_{\mathbb{T}^d} (\dot{\varphi}_m)^2 \, \tilde{j}''(\varphi_m). \tag{1.28}$$

Using the equation on η_m (see (1.24)), (1.28) rewrites

$$\ddot{J}(m)[h,h] = \int_{\mathbb{T}^d} \eta_m \left|\nabla \dot{\varphi}_m\right|^2 + \int_{\mathbb{T}^d} (\dot{\varphi}_m)^2 \, W_m \tag{1.29}$$

where the potential W_m is defined as

$$W_m = \eta_m \tilde{f}''(\varphi_m) + \tilde{j}''(\varphi_m).$$

Using elliptic regularity, one sees that $W_m \in L^\infty(\mathbb{T}^d)$. Lemma 1.13 and a straightforward compactness argument in m imply the existence of two constants $\alpha > 0\,, \beta \in \mathbb{R}$ such that

$$\ddot{J}(m)[h,h] \geq \alpha \int_{\mathbb{T}^d} \left|\nabla \dot{\varphi}_m\right|^2 - \beta \int_{\mathbb{T}^d} (\dot{\varphi}_m)^2\,. \tag{1.30}$$

Now, we can move to the second order optimality conditions: let m^* be a non bang-bang optimiser. Let ω_ε^* be defined by (1.26) and satisfy (1.27). Introduce the space

$$T(m^*) := \left\{ h \in L^2(\mathbb{T}^d) : \, h \text{ supported in } \omega_\varepsilon^*\,, \int_{\mathbb{T}^d} h = 0 \right\}. \tag{1.31}$$

As for any $h \in T(m^*)$ we already saw that

$$\dot{J}(m^*)[h] = 0$$

we must have

$$\ddot{J}(m^*)[h,h] \geq 0.$$

In particular, if m^* is a non bang-bang solution of (1.7) we deduce from (1.30) that

$$\forall h \in T(m^*)\,, \alpha \int_{\mathbb{T}^d} \left|\nabla \dot{\varphi}_m\right|^2 \leq \beta \int_{\mathbb{T}^d} (\dot{\varphi}_m)^2\,. \tag{1.32}$$

REMARK 1.14. *A priori*, these optimality conditions only hold for $h \in T(m^*) \cap L^\infty(\Omega)$. Nevertheless an easy approximation argument allows to derive the same conditions for any $h \in T(m^*)$.

Recall that $\dot{\varphi}_m$ solves a linear elliptic PDE (1.22). We reach a contradiction using the following result:

PROPOSITION 1.15. *There exists* $h \in T(m^*)$ *such that* $\alpha \int_{\mathbb{T}^d} \left|\nabla \dot{\varphi}_m\right|^2 > \beta \int_{\mathbb{T}^d} (\dot{\varphi}_m)^2\,.$

Sketch of proof of Proposition 1.15. There are two approaches to proving Proposition 1.15.

(1) First approach: Fourier decomposition: in this first approach, we go back from the Hopf-Cole world of φ_m to the elliptic one of θ_m. Indeed, consider $\dot\theta_m := \dot\varphi_m \cdot \theta_m$, which solves

$$-\Delta\dot\theta_m - (c+m)\dot\theta_m - f'(\theta_m)\dot\theta_m = h\theta_m,$$

and, since $\min_{\mathbb{T}^d} u_m > 0$, it suffices to find h such that, for a sufficiently large $M > 0$,

$$\int_{\mathbb{T}^d} |\nabla\dot\theta_m|^2 > M\int_{\mathbb{T}^d}\left(\dot\theta_m\right)^2. \tag{1.33}$$

Letting $\mathscr{L}$ be the differential operator $-\Delta-(c+m+f'(\theta_m))$ the equation on $\dot\theta_m$ rewrites

$$\mathscr{L}\dot\theta_m = h\theta_m.$$

Let $\{\xi_k,\psi_k\}_{k\in\mathbb{N}}$ be the set of eigenvalues and eigenfunctions associated with $\mathscr{L}$ (as $\mathscr{L}$ is symmetric and compact, this is a simple application of the spectral theorem. Now, from (1.27), $L^2(\omega_\varepsilon^*)$ is infinite dimensional. Introduce the linear forms

$$I_0 : L^2(\omega_\varepsilon^*) \ni h \mapsto \int_{\omega_\varepsilon^*} h$$

$$\text{and, for any } k\in\mathbb{N}\,, R_k : L^2(\omega_\varepsilon^*) \ni h \mapsto \int_{\omega_\varepsilon^*} hu_m\psi_k.$$

Fix $N\in\mathbb{N}$ and pick $h_N \in \cap_{k=0,\dots,N}\ker(R_k)\cap\ker(I_0)\setminus\{0\}$. By construction, extending h_N by zero outside of ω_ε^*, there exists a sequence $\{a_{k,N}\}_{k\geq N}$ such that

$$hu_m = \sum_{k\geq N} a_{k,N}\psi_k$$

and, up to normalisation, we choose

$$\sum_{k\geq N} a_{k,N}^2 = 1.$$

Letting $\dot\theta_N$ be the associated solution, we have the explicit expression

$$\dot\theta_N = \sum_{k\geq N}\frac{a_{k,N}}{\xi_k}\psi_k.$$

This implies

$$\int_{\mathbb{T}^d}|\nabla\dot\theta_N|^2 = \sum_{k\geq N}\frac{a_{k,N}^2}{\xi_k} > \xi_N\sum_{k\geq N}\frac{a_{k,N}^2}{\xi_k^2} = \xi_N\int_{\mathbb{T}^d}\dot\theta_N^2.$$

This provides the conclusion.

(2) The general case: a more down to earth approach is the following: assume Proposition 1.15 does not hold. in particular, for any $h\in T(m^*)$ there holds

$$\int_{\mathbb{T}^d}|\nabla\dot\varphi_m|^2 \lesssim \int_{\mathbb{T}^d}(\dot\varphi_m)^2. \tag{1.34}$$

Now, let $S : T(m^*) \ni h \mapsto \dot{\varphi}_m$. S is injective, and is thus a bijection onto its image $\Im(S)$. As $T(m^*)$ is infinite dimensional, so is $\Im(S)$. From (1.34) and the compactness of Sobolev embeddings, any sequence in $\Im(S)$ that is bounded in $L^2(\mathbb{T}^d)$ is strongly converging in $L^2(\mathbb{T}^d)$. However, as $\Im(S)$ is infinite dimensional, we can use the Gram-Schmidt procedure to create an orthonormal sequence $\{\bar{\varphi}_k\}_{k\in\mathbb{N}} \in \Im(S)^{\mathbb{N}}$. As it is infinite, the Parseval inequality implies that $\{\bar{\varphi}_k\}_{k\in\mathbb{N}}$ converges weakly to zero, a contradiction with the strong L^2 convergence.

□

Extensions to other settings We conclude with some remarks.

(1) When working with general f or with general boundary conditions, one needs to use the theory of principal eigenvalues to derive a suitable form of the maximum principle that implies the strong positivity of Lemma 1.13.
(2) The dependency of f or j on x plays no role in the proofs.
(3) One can consider non-symmetric equations *i.e.* one can consider operators of the form $-\nabla(A\nabla\cdot)+\langle X,\nabla\rangle-f(\cdot)-m$ rather than simply $-\Delta-f(\cdot)-m$, in which case the second approach presented needs to be used as one can not guarantee the existence of a Fourier basis for the linearised operator.

1.3.6. Elements of proof of Theorem 1.9.

1.3.6.1. *Statement of the main result.* The basic computations are identical in the time independent and in the time-dependent case.

Hopf-Cole transform and optimality conditions

For the sake of simplicity, assume $u_0 = e^{\varphi_0} > 0$. From the strong maximum principle we can write $u_m = e^{\varphi_m}$ where φ_m solves

$$\begin{cases} \partial_t\varphi_m - \Delta\varphi_m - |\nabla\varphi_m|^2 = m + \tilde{g}(\varphi_m) & \text{in } (0;T)\times\Omega\,, \\ \partial_\nu\varphi_m = 0 & \text{on } \partial\Omega\,, \\ \varphi_m(0,\cdot) = \varphi_0 & \text{in } \Omega \end{cases} \tag{1.35}$$

for some non-linearity $\tilde{g}$. We introduce the adjoint state as the unique solution η_m to the backwards parabolic problem

$$\begin{cases} -\partial_t\eta_m - \Delta\eta_m + 2\nabla\cdot(\eta_m\nabla\varphi_m) = \partial_u\tilde{j}(\varphi_m) & \text{in } (0;T)\times\Omega\,, \\ \eta_m(T,\cdot) = \partial_u\tilde{j}_T(\varphi_m(T,\cdot))\,, & \\ \partial_\nu\eta_m = 0 & \text{on } (0;T)\times\Omega \end{cases} \tag{1.36}$$

where $\tilde{j}(\varphi) = j(e^{\varphi})\,, \tilde{j}_T(\varphi) = j(e^{\varphi})$. From the same reasoning as in the elliptic case, we deduce the following facts:

(1) First, for any $\varepsilon > 0$

$$\inf_{[0;T-\varepsilon]\times\overline{\Omega}} \eta_m > 0.$$

For the sake of readability, we can assume that

$$\min_{[0;T]\times\Omega} \eta_m > 0. \tag{1.37}$$

(2) Second, using this information and finer parabolic regularity there exist two constants $A > 0\,, B \in \mathbb{R}$ such that, for any $m \in \mathcal{M}_{\text{ind or dep}}$, for any admissible perturbation h at m, we have

$$\ddot{J}(m)[h,h] \geq A \iint_{(0;T)\times\Omega} |\nabla \dot{\varphi}_m|^2 - B \int_{(0;T)\times\Omega} (\dot{\varphi}_m)^2\,, \tag{1.38}$$

where $\dot{\varphi}_m$ solves the linear parabolic equation

$$\begin{cases} \partial_t \dot{\varphi}_m - \Delta \dot{\varphi}_m - 2\langle \nabla \dot{\varphi}_m, \nabla \varphi_m\rangle + V_m \varphi_m = h & \text{in } (0;T)\times\Omega\,, \\ \partial_\nu \dot{\varphi}_m = 0 & \text{on } (0;T)\times\partial\Omega\,, \\ \dot{\varphi}_m(0,\cdot) = 0 & \text{in } \Omega. \end{cases} \tag{1.39}$$

The potential V_m depends on φ_m, and its finer regularity (in t and x) is crucial, but is ultimately obtained as a consequence of tedious parabolic regularity results.

The situation, while similar to the elliptic case, is complexified by the fact that the second-order derivative no longer depends on the entire $W^{1,2}((0;T)\times\Omega)$ norm of the linearisation $\dot{\varphi}_m$, but just on its $L^2(0,T;W^{1,2}(\Omega))$ norm; this prohibits using the contradiction approach (Approach 2, the general case) that was given in the case of elliptic problems. This difficulty is circumvented in the following ways:

(1) In the time independent case: surprisingly, this case is the most technical and only seems to work either in the one-dimensional case or in the torus in the multi-dimensional case. Here, we argue by contradiction and assume that we are given a non-bang-bang maximiser m^*. As in the elliptic case, the contradiction can be reached provided, a certain measurable subset ω^* of Ω, with $|\omega^*| > 0$, being given, we can find $h \in L^2(\Omega)$ supported in ω^* such that the solution $\dot{\varphi}_m$ of (1.39) satisfies

$$\iint_{(0;T)\times\Omega} |\nabla \dot{\varphi}_m|^2 \gg \iint_{(0;T)\times\Omega} (\dot{\varphi}_m)^2\,, \int_\Omega h^2 = 1.$$

When Ω is the one-dimensional torus $\mathbb{R}/2\pi\mathbb{Z}$, we choose, as in the elliptic case, an integer N and a perturbation h_N supported in ω^* that writes

$$h_N = \sum_{k=N}^{\infty} a_{k,N}\sin(kx) + b_{k,N}\cos(kx).$$

The bulk of [**Maz23a**] is devoted to proving that, if we set $\dot{\varphi}_N$ the solution of (1.39), the function $\dot{u}_N := \dot{\varphi}_N e^{\varphi_m}$ asymptotically satisfies

$$\dot{u}_m \approx_{N\to\infty} \sum_{k=N}^{\infty} e^{-k^2 t}\left(a_{k,N}\sin(kx) + b_{k,N}\cos(kx)\right).$$

(2) In the time-dependent case: in the time-dependent case, barring some technical difficulties, the main idea is to reduce the problem to the study of an initial-value problem. Namely, letting $\omega_\varepsilon^* := \{\varepsilon < m^* < 1-\varepsilon\}$ be of positive measure, standard results in measure theory ensure that a.e. $t \in (0;T)$ is a Lebesgue point of $\mathbb{1}_{\omega_\varepsilon^*}$ seen as a function in $L^1(0,T;L^1(\Omega))$. In particular, relying on the theory of solutions to parabolic equations with measure data developed by Stampachia, Boccardo, Gallouët [**32, 212**] (we refer to the monograph of Ponce [**195**] for the elliptic case) and working a bit to recover suitable approximation topologies, we show that we can

reach a contradiction provided the following fact can be showed: ω^* being a measurable subset of positive measure of Ω, find a non-zero $h \in L^2(\Omega)$ supported in ω^* such that the solution to the initial value problem

$$\begin{cases} \partial_t \psi_h - \Delta \psi_h - 2\langle \nabla \varphi_m, \nabla \psi_h \rangle + V_m \psi_h = 0 & \text{in } (0;T) \times \Omega\,, \\ \partial_\nu \psi_h = 0 & \text{on } (0;T) \times \Omega\,, \\ \psi_h(0,\cdot) = h & \text{in } \Omega \end{cases} \tag{1.40}$$

satisfies

$$\iint_{(0;T)\times\Omega} |\nabla \psi_m|^2 \gg \iint_{(0;T)\times\Omega} (\psi_m)^2\,.$$

As in the elliptic case, the key is to choose h as a sum of high enough Fourier modes. The analysis, in this case, works perfectly fine in any dimension as the dependency on the perturbation is much simpler (it can be considered as an initial condition).

1.4. Linear control problems

The last works we present in this chapter are the two articles [**MNTM21, MFN22**]. The underlying motivation comes from the optimal control of invasive species, a topic of particular relevance in the last few years [**8, 6, 31, 38, 135, 184**]. In this setting, the natural problem to consider is not a bilinear optimal control problem with monostable non-linearities, but rather a linear control problem with a bistable non-linearity. To be more specific, consider a function $f : [0;1] \to \mathbb{R}$ with

$$f(0) = f(\eta) = f(1) = 0\,, f'(0)\,, f'(1) < 0\,, f'(\eta) > 0,$$

where $\eta \in (0;1)$ is the unique intermediate root of f. Such non-linearities model a situation where, if a population initially has a density (pointwise) lower than η, it will die out while, if enough individuals are present, the population will eventually survive–this is the Allee effect. The underlying reaction-diffusion equation reads

$$\partial_t u - \Delta u = f(u).$$

Another interpretation of the population density u that needs to be kept in mind when discussing possible applications is that u actually represents the local fraction N_1/N of a sub-population N_1 of a global population N. In particular, such equations can be used as follows: letting the global population be a population of mosquitoes divided into two groups, one having the ability to carry a disease and to reproduce and the other being made of disease-free or sterile mosquitoes, one tries to release enough of the second (better) class of mosquitoes within the population to ensure it will eventually be prevailing.

1.4.1. Optimal control of the initial condition. While the dominance of this second category of mosquitoes can be quantified in a variety of way, we settle in [**MNTM21**] on the following formulation:

$$\max_{y \in \mathcal{Y}} \int_\Omega j(u_y(T,\cdot)) \text{ subject to } \begin{cases} \partial_t u_y - \Delta u_y = f(u_y) & \text{in } (0;T) \times \Omega\,, \\ \partial_\nu u_y = 0 & \text{on } (0;T) \times \partial\Omega\,, \\ u_y(0,\cdot) = y \geq 0, \end{cases} \tag{1.41}$$

where as usual

$$\mathcal{Y} = \left\{ y \in L^\infty(\Omega) :\ 0 \leq y \leq 1\,, \int_\Omega y = Y_0 \right\}.$$

Note that here we are optimising with respect to the initial condition; in [**MFN22**] we focus on internal controls; we refer to the next subsection for this work.

As we observed in (1.17), in general, it is not possible to expect the bang-bang property to hold for such linear control problems, and this leads to several difficulties when considering the numerical simulations of the optimisers. The contributions [**MNTM21, MFN22**] had two goals. First of all, provide a finer understanding of these questions: since it is easy to see that if we forget for a minute that f is bistable and if we assume that it is convex and that j is convex then any optimal y^* is bang-bang, can we "localise" this result *i.e.* is it true that on the set $\{f''(u_{y^*})\geq 0\}$ the optimal y^* is bang-bang? Second, this turns out to be an important question for numerical simulations. Before we discuss more in details the related works, let us state our main result:

THEOREM 1.16. *Assume $\Omega=\mathbb{T}$ is the one-dimensional torus. Let y^* be a solution of (1.43) and p_{y^*} be the adjoint state. Assume $\omega^*:=\{0<y^*<1\}$ has non-empty interior. Then*

$$p_{y^*}(0,\cdot)f''(y^*)\leq 0 \tag{1.42}$$

on the interior of ω^*. In particular, if $\partial_u j>0$, $f''(y^*)\leq 0$ on the interior of ω^*.

Motivation and application: numerical simulations As we alluded to, a motivation for [**MNTM21**] was the numerical method introduced by Nadin & Toledo Marrero [**185**]. To be more precise, introduce, for a given admissible control y, the adjoint state p_y as the unique solution to

$$\begin{cases}-\partial_t p_y-\Delta p_y-f'(u_y)p_y=0 & \text{in } (0;T)\times\Omega\,,\\ \partial_\nu p_y=0 & \text{on } (0;T)\times\Omega\,,\\ p_y(T,\cdot)=j'(u_y) & \text{in } \Omega.\end{cases} \tag{1.43}$$

The optimality conditions for (1.43) read as follows: there exists a constant $c^*>0$ such that

(1) $\{p_{y^*}(0,\cdot)>c^*\}\subset\{y^*=1\}$,
(2) $\{p_{y^*}(0,\cdot)<c^*\}\subset\{y^*=0\}$,
(3) $\omega^*=\{0<y^*<1\}\subset\{p_{y^*}(0,\cdot)=c^*\}$.

Recall that for problems that satisfy the bang-bang property, the thresholding algorithm works efficiently; here, however, as we can expect (1.43) to not satisfy this property, one needs to take more care, but essentially also wants to apply a fixed-point method. Should the set ω^* have positive measure, one would get, on this set

$$f'(y^*)=\frac{-\partial_t p_{y^*}(0,\cdot)}{p_{y^*}(0,\cdot)}.$$

In a very simplified manner (in particular there are several aspects we do not touch on here that would require some care), the algorithm proposed in [**185**, Section 5] reads as follows:

(1) Start from an initial guess y_0, compute the direct and adjoint states u_{y_0}, p_{y_0}.
(2) For any $k\in\mathbb{N}$ let $p_k:=p_{y_k}(0,\cdot)$. Compute c_k as

$$c_k:=\sup\{c\in\mathbb{R}\,,|\{p_k>c_k\}|\leq Y_0\}.$$

(3) Set $y_{k+1} = 1$ on $\{p_k > c_k\}$, $y_{k+1} = 0$ on $\{p_k < c_k\}$. Define y_{k+1} as a root of the equation
$$f'(y_{k+1}) = -\frac{\partial_t p_{y_k}(0,\cdot)}{c_k}$$
on ω^*.

We sweep some details under the rug, to highlight the fact that to put this algorithm forth one needs to solve an equation of the form
$$f'(y) = \alpha.$$
The problem is that this equation is in general posed: when f is bistable, this equation has exactly zero, one or two solutions. Our result (1.42) provides a root selection principle: whenever two roots are possible, one should choose the one located in the concavity zone of f. While [**185**] took all possible roots of the equation and compared the performance of each of these, in [**MNTM21**], using Theorem 1.16, we picked the "concave" root, and compared our algorithm with other classical algorithms (simulated annealing, interior point etc) and observed that it provided similar results to simulated annealing while being much cheaper. We refer to [**MNTM21**, Section 4] for more details on the implementation.

Related works Let us point to the work of Halim & El Smaily [**116**] where the same question is investigated, but from the point of view of advection: namely, how does the value of the optimal control problem behave under the addition of an advection term? Furthermore, we would like to briefly mention that another interpretation of (1.43) has to do with the following central question in spatial ecology: under which conditions on the initial condition can a population evolving according to a bistable nonlinearity survive? Stated in such generality, there is no hope of answering this question, but a particular formulation has been investigated by several authors, namely: consider the case $\Omega = \mathbb{R}$ and as an initial condition the function $\psi_{L,r} := \mathbb{1}_{(-L-r;-r)\cup(r;L+r)}$. Under which conditions on L and r does the associated solution converge to 1 as $t \to \infty$? This question is related to the phenomenon of threshold (*i.e.* focusing on the case $r = 0$, and simply investigating the influence of L), for which we refer to the works of Kanel [**139**], Zlatos [**223**] and Polácik [**194**]. Nadin [**184**] used an optimal control approach to provide more insight into this problem.

1.4.2. Optimality conditions for the internal optimal control of parabolic equations. There is, from a theoretical perspective, one main problem with Theorem 1.16; namely, the assumption that ω^* has a non-empty interior. As we discussed in the case of bilinear optimal control problems, such regularity assumptions are usually very difficult to prove. We tried, with Nadin, to go beyond this case and to establish that (1.42) holds whenever ω^* is measurable, with positive measure. Unfortunately, we were not able to give a complete answer–this is due to fine regularity issues in parabolic equations. Nevertheless, we were able to derive a similar result for the case of an internal control problem. In other words, we consider in [**MFN22**] the following optimal control problem

$$\max_{y\in\mathcal{Y}_T}\int_\Omega f(v_y) \text{ subject to } \begin{cases}\partial_t v_y - \Delta v_y = f(v_y) + y & \text{in } (0;T)\times\Omega\,,\\ v_y(0,,\cdot) = v_0 & \text{in } \Omega\,,\\ \partial_\nu v_y = 0 & \text{on } (0;T)\times\partial\Omega.\end{cases} \tag{1.44}$$

In this equation, the admissible class is given by
(1.45)
$$\mathcal{Y}_T := \left\{ y \in L^\infty((0;T)\times\Omega) : 0 \leq y \leq 1 \text{ a.e., and for a.e. } t\,, \int_\Omega y(t,\cdot) = V_0(t) \right\}$$
where $V_0 : (0;T) \times (0;|\Omega|)$ is a given volume constraint.

An imprecise formulation of the main results of [**MFN22**] is the following:

THEOREM 1.17. *Let j be $\mathscr{C}^2$ and let y^* be a solution of* (1.44)*. Assume that $\omega^* := \{0 < y^* < 1\}$ has positive measure. Then, q_{y^*} denoting the adjoint state*
$$\begin{cases} -\partial_t q_{y^*} - \Delta q_{y^*} = f'(v_{y^*}) q_{y^*} & \textit{in } (0;T)\times\Omega\,, \\ \partial_\nu q_{y^*} = 0 & \textit{on } (0;T)\times\partial\Omega\,, \\ q_{y^*}(T,\cdot) = j'(v_{y^*}) & \textit{in } \Omega \end{cases}$$
there holds
$$q_{y^*} f''(v_{y^*}) \leq 0 \textit{ a.e. in } \omega^*.$$

1.4.3. Elements of proof for Theorems 1.16–1.17. We now present some elements of proof.

Optimal control of the initial condition: Theorem 1.16 As in [**MNTM21**] we present the case $\Omega = \mathbb{T}$ *i.e.* in the case of the one-dimensional torus–the same reasoning would hold in higher dimensional domains.

In this case, letting y^* be a solution of (1.43), it is straightforward to see that, if $\omega^* := \{0 < y^* < 1\}$ has positive measure, then for any $h \in L^\infty(\Omega)$ supported in ω^*, the second-order Gateaux derivative of the criterion must be non-positive, that is,
$$\iint_{(0;T)\times\Omega} (\dot{u}_y)^2 f''(u_{y^*}) p_{y^*} + \int_\Omega \partial^2_{uu} j(u_{y^*}(T,\cdot))(\dot{u}_{y^*})^2 \geq 0 \tag{1.46}$$
where $\dot{u}_{y^*}$ solves
$$\begin{cases} \partial_t \dot{u}_{y^*} - \Delta \dot{u}_{y^*} = f'(u_{y^*}) \dot{u}_{y^*} & \text{in } (0;T)\times\mathbb{T}\,, \\ \dot{u}_{y^*}(0,\cdot) = h & \text{in } \mathbb{T}. \end{cases} \tag{1.47}$$
The idea is now to choose an initial perturbation h in such a way that $(\dot{u}_y)^2$ is concentrated at 0 and on the interior ω^*_{int} of the set ω^*. To this end, let θ be a smooth cut-off function supported in ω^*_{int} and define, for any $k \in \mathbb{N}$,
$$h_k(x) := \theta(x) \left(\cos(kx) - \frac{1}{\int_{\omega^*_{int}} \theta} \int_{\omega^*_{int}} \cos(k\cdot)\theta(\cdot) \right).$$
Assume for the sake of readability that $\frac{1}{\int_{\omega^*_{int}} \theta} \int_{\omega^*_{int}} \cos(k\cdot)\theta(\cdot) = 0$ for any k. The situation is thus the following: we must study the behaviour of a sequence $\{\dot{u}_k\}_{k\in\mathbb{N}}$ defined as the solution of
$$\begin{cases} \partial_t \dot{u}_K - \Delta \dot{u}_k = V \dot{u}_k & \text{in } (0;T)\times\mathbb{T}\,, \\ \dot{u}_K(0,\cdot) = \cos(k\cdot)\theta(\cdot) & \text{in } \mathbb{T}. \end{cases} \tag{1.48}$$
Here, $V = f'(u_{y^*})$ is a quite regular potential. It is natural, given the theory of multi-scale expansion [**5, 4**] to expect that
$$\dot{u}_k(t,x) \approx e^{-k^2 t} \cos(kx)\theta(x). \tag{1.49}$$

The bulk of [**MNTM21**] is to prove that indeed this approximation holds in strong enough topologies, so that we can write the following: if $p_{y^*}(0,\cdot)f''(y^*)>0$ on a subset of positive measure of ω^*_{int}, choose θ smooth, positive, compactly supported in ω^*_{int} so that $\int_{\mathbb{T}} p_{y^*}(0,\cdot)f''(y^*)\theta>0$. Then we can show rigorously

$$\iint_{(0;T)\times\Omega}(\dot u_k)^2 f''(u_{y^*})p_{y^*}+\int_\Omega \partial^2_{uu}j(u_{y^*}(T,\cdot))(\dot u_k)^2 \approx \iint_{(0;T)\times\mathbb{T}} e^{-2k^2t}f''(u_{y^*})p_{y^*}\cos(kx)^2\theta(x)dtdx.$$

As $\{\cos(k\cdot)^2\}_{k\in\mathbb{N}}$ converges weakly to $\frac12$, the Laplace method gives

$$\iint_{(0;T)\times\Omega}(\dot u_k)^2 f''(u_{y^*})p_{y^*}+\int_\Omega \partial^2_{uu}j(u_{y^*}(T,\cdot))(\dot u_k)^2 \sim \frac{1}{2k^2}\int_{\mathbb{T}} f''(y^*)p_{y^*}(0,\cdot)\theta>0,$$

a contradiction. Alternatively, we can say that the main proof of [**MNTM21**] relies on the fact that, for the weak topology on measures, we have, in the sense of measures

$$\frac{\dot u_k^2}{\iint_{(0;T)\times\Omega}(\dot u_k)^2} \underset{k\to\infty}{\rightharpoonup} \frac{1}{\int_\Omega\theta}\delta_{t=0}\otimes\theta(\cdot). \tag{1.50}$$

Internal control problems We now move to some elements of proof for Theorem 1.17. The basic observation is the same: assume that $\omega^*:=\{0<y^*<1\}$ has positive measure. Then, for any perturbation h supported in ω^* and such that, for a.e. $t\in(0;T)$, $\int_\Omega h(t,\cdot)=0$, there holds

$$\iint_{(0;T)\times\Omega}(\dot v_{y^*})^2 q_{y^*}f''(v_{y^*})+\int_\Omega(\dot v_{y^*})^2\partial^2_{vv}j(v_{y^*}(T,\cdot))\geq 0 \tag{1.51}$$

where

$$\begin{cases}\partial_t\dot v_{y^*}-\Delta\dot v_{y^*}=f'(v_{y^*})\dot v_{y^*}+h & \text{in } (0;T)\times\Omega\,,\\ \dot v_{y^*}(0,\cdot)=0 & \text{in } \mathbb{T}.\end{cases} \tag{1.52}$$

Argue by contradiction that ω^* has positive measure and that $q_{y^*}f''(v_{y^*})>0$ in a subset of positive measure ω^*_+ of ω^*. Disregarding for the sake of presentation some tedious regularity issues, assume that you can find a fixed $\omega^*_0\subset\Omega$ and $t_0\in(0;T)\,,\varepsilon>0$ such that $(t_0-\varepsilon;t_0+\varepsilon)\times\omega^*_0\subset\omega^*_+$ (in the general case, one would simply rely on the same type of construction that was used in [**MF24a**]). Let h_0 be supported in ω^*_0 with $\int_\Omega h_0=0$. Letting $h_\delta:(t,x)\mapsto\frac{1}{2\delta}\mathbb{1}_{(t_0-\delta;t_0+\delta)}h_0(x)$ and passing to the limit $\delta\to0$ in (1.51) we deduce that, for any such h_0, there holds

$$\iint_{(0;T)\times\Omega}(\dot w_{y^*})^2 q_{y^*}f''(v_{y^*})+\int_\Omega(\dot w_{y^*})^2\partial^2_{vv}j(v_{y^*}(T,\cdot))\geq 0 \tag{1.53}$$

with

$$\begin{cases}\partial_t\dot w_{y^*}-\Delta\dot w_{y^*}=f'(v_{y^*})\dot w_{y^*} & \text{in } (0;T)\times\Omega\,,\\ \dot w_{y^*}(t_0,\cdot)=h_0 & \text{in } \mathbb{T}.\end{cases} \tag{1.54}$$

We are back to an initial value problem. Now, let $K \in \mathbb{N}$ and choose an initial condition h_K supported in ω_0^* with $\int_\Omega h_K = 0$ and that writes

$$h_K = \sum_{j \geq K} a_{j,K} \psi_j \,, \sum_{j \geq K} a_{j,K}^2 = 1$$

where the $\{\xi_j, \psi_j\}_{j \in N}$ are the eigen-elements of the Laplace operator in Ω. The existence of such an h_K is guaranteed by the same arguments used in [**MNP22a**]. Technically, it is (once again) reasonable to expect that, if we let $\dot{w}_K$ be the solution of (1.54) associated with h_K, $\dot{w}_K$ writes

$$\dot{w}_K \approx \sum_{j \geq K} a_{j,K} e^{-\xi_j t} \psi_j .$$

The main result of [**MNP22a**] is the following: letting $\nu_K := \frac{\dot{w}_K^2}{\iint_{(0;T)\times\Omega} (\dot{w}_K)^2}$, there exists a probability measure ν_∞ that decomposes as

$$\nu_\infty = \delta_{t=t_0} \otimes \bar{\nu}$$

for some probability measure $\bar{\nu}$ on Ω supported in ω^* such that

$$\nu_K \underset{K\to\infty}{\rightharpoonup} \nu_\infty .$$

Passing to the limit in (1.53) this gives

$$\int_\Omega q_{y^*}(t_0, \cdot) f''(v_{y^*}(t_0, \cdot)) d\bar{\nu} \geq 0.$$

Observe that the reason this approach can not give information on the optimisation with respect to the initial condition is that, while v_{y^*} is continuous on $(0;T) \times \Omega$, it is not continuous at $t = 0$ so that one can not pass to the limit. Doing so would require showing that $\bar{\nu}$ has a density with respect to the Lebesgue measure, and whether or not this is true is not clear at the moment.

1.5. Research plan

We conclude this chapter with two ongoing and future research directions directly related to the problems studied here. Namely, assume that, given an optimal control problem, the optimal control m^* is bang-bang, so that we might rewrite it $m^* = \mathbb{1}_{E^*}$. The central question becomes:

How regular is the set E^*?

This question is both very delicate, as it belongs to the class of free boundary problems, and important as, as we shall see in the next chapter, it is usually a starting point when studying the stability of optimisers and, in turn, the convergence properties of numerical methods. Although these are ongoing works with L. Ferreri, J. Lamboley, M. Nahon & R. Prunier we can describe briefly some short term goals and objectives, as well as long term objectives.

1.5.1. Unstable free boundary problems for bilinear optimal control problems. We begin with the case of bilinear optimal control problems; for the sake of readability, let us work on the problem

$$\max_{m \in \mathcal{M}} \int_\Omega j(\theta_m) \text{ subject to } \begin{cases} -\mu\Delta\theta_m = m\theta_m + f(\theta_m) & \text{in } \Omega\,, \\ \partial_\nu \theta_m = 0 & \text{on } \partial\Omega\,, \\ \theta_m \geq 0\,, \theta_m \not\equiv 0 \end{cases} \tag{1.55}$$

where

$$\mathcal{M} := \left\{ m \in L^\infty(\Omega) : \ 0 \leq m \leq 1 \, , \int_\Omega m = m_0 \right\} , \tag{1.56}$$

f is, as usual, chosen to ensure the well posedness and the stability of the solution θ_m, and $\mu > 0$ is a diffusivity parameter. We have seen (Theorem 1.7) that, provided j is increasing on $(0; +\infty)$ and f satisfies some natural assumption, any solution m^* writes

$$m^* = \mathbb{1}_{E^*}$$

and that the set E^* is the level set of a certain function η_{m^*} (see (1.24))

$$E^* = \{\eta_{m^*} > c\}.$$

Overall, letting $\theta_m = e^{\varphi_m}$, this yields the following system on $(\varphi_{m^*}, \eta_{m^*})$:

$$\begin{cases} -\mu\Delta\varphi_{m^*} - \mu|\nabla\varphi_{m^*}|^2 = \mathbb{1}_{\{\eta_{m^*} > c\}} + \tilde{f}(\varphi_{m^*}) & \text{in } \Omega \, , \\ -\mu\Delta\eta_{m^*} + 2\mu\nabla \cdot (\eta_{m^*}\nabla\varphi_{m^*}) = \tilde{j}'(\varphi_{m^*}) & \text{in } \Omega \, , \\ \partial_\nu \eta_{m^*} = \partial_\nu \varphi_{m^*} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.57}$$

where the functions $\tilde{f}, \tilde{j}$ are related to f and j respectively. The system (1.57) can be interpreted as a system of first-order optimality conditions, and looks like a free-boundary problem with a non-standard coupling–we simply mention that the fact that m^* is maximising also gives second-order optimality conditions on $\{\eta_{m^*} > c\}$, which is of crucial importance. The natural question becomes: given (1.57), what can be said about the regularity:

(1) of η_{m^*}?
(2) of $\{\eta_{m^*} > c\}$?

Thus, the problem of regularity resembles a free boundary problem (of unstable type, as we explain later on). As far as we are aware, this type of question has only been settled in a specific case, that of the composite membrane problem.

The composite membrane problem The composite membrane problem is one of the oldest optimal control problems, and amounts to solving the minimisation problem

$$\min_{m \in \mathcal{M}} \lambda(m) \text{ where } \lambda(m) = \min_{u \in W_0^{1,2}(\Omega) \, , \int_\Omega u^2 = 1} \int_\Omega |\nabla u|^2 - \int_\Omega m u^2 \tag{1.58}$$

where $\mathcal{M}$ is defined in Eq. (1.56). As we already observed in this chapter, as an infimum of linear functionals, λ is concave and, in fact, any minimiser m^* is bang-bang. Furthermore, the optimality conditions for (1.58) read

$$m^* = \mathbb{1}_{\{u_{m^*} > c^*\}}$$

for some c^*, where u_{m^*} is the associated eigenfunction. Naturally, when Ω is a ball, standard rearrangement techniques [**145**] show that $m^* = \mathbb{1}_{\mathbb{B}^*}$ where $\mathbb{B}^*$ is a concentric ball with the same volume. In general domains, although some symmetry or regularity properties can be obtained (typically, when Ω is convex, any optimal set E^* such that an optimiser of (1.58) writes $m^* = \mathbb{1}_{E^*}$ is convex), there is no explicit characterisation of the solutions. Although we will come back to these geometric properties in the following chapter, let us for the moment point to [**77**].

Coming back to optimality conditions, if we let u_{m^*} be the eigenfunction associated with an optimiser m^*, we obtain the equation

$$\begin{cases} -\Delta u_{m^*} = (\lambda(m^*) + \mathbb{1}_{\{u_{m^*} > c^*\}}) u_{m^*} & \text{in } \Omega\,, \\ u_{m^*} \in W_0^{1,2}(\Omega)\,, \\ \int_\Omega (u_{m^*})^2 = 1. \end{cases} \tag{1.59}$$

Does this system imply any regularity on the function u_{m^*} and on the set $\{u_{m^*} > c^*\}$? This is an old question in free boundary problems that was finally settled by Chanillo, Kenig & To [79] using deep ideas from the study of free boundary problems [210, 205, 78, 15]. Of particular importance to us in their study is their reliance on the techniques introduced by Monneau & Weiss [177] in the study of such unstable free boundary problems. Their results [79] provide analyticity of the free boundary in dimension 2 (some weaker estimates on the dimension of the singular set in higher dimensions).

It is thus reasonable to wonder whether the same type of regularity holds here. In the following paragraphs, we explore the similarities and the differences between these two problems.

Is it reasonable to expect the same type of behaviour here? Let us start with the following fact: why should we expect a similar behaviour? To this end, let us go back to (1.55) and write $\theta_{m,\mu}$ instead of θ_m to insist on the role of diffusivity. It is in fact possible, using the tools of [MF24b], to show that, for any j satisfying $j' > 0$ on $(0; +\infty)$, there exists $\mu^* > 0$ such that, for any $\mu \geq \mu^*$, m^* solves (1.55) if, and only if, m^* solves the energy minimisation

$$\min_{m \in \mathcal{M}} \min_{\varphi \in W^{1,2}(\Omega)\,, \int_\Omega \varphi = 0} \frac{1}{2} \int_\Omega |\nabla \varphi|^2 - \int_\Omega m\varphi, \tag{1.60}$$

for which the regularity theory, although not written, is the same as for the composite membrane problem (1.58). Thus, at least in the large diffusivity setting, the problem should be solved. However, and this will be a core point in the next chapter, energy minimisation and general bilinear optimisation problems are completely different from the geometric point of view, especially in the low diffusivity regime.

More in details: the core problem In an ongoing work with L. Ferreri & R. Prunier, we were able to show the $\mathscr{C}^{1,1}$ regularity of the function η_{m^*}. The regularity of the free boundary remains elusive at the moment. In our case, the underlying free boundary equation to study writes

$$-\Delta \eta = f \mathbb{1}_{\{\eta > 0\}} - g \mathbb{1}_{\{\eta > 0\}}$$

and the difficulty is that both f and g are positive. In contrast, in the cases studied for instance in [79], the underlying problem always satisfies $-\Delta \eta \geq 0$, which provides a measure of non-degeneracy.

More precisely, the main problem when trying to adapt the tools coming from the composite membrane problem is the (non-)degeneracy of the blow-ups, that is, of the zoomed-in equation on $\eta_r : x \mapsto \frac{\eta(r(\cdot - x_0))}{r^2}$, close to points x_0 of the free boundary where the gradient of η_{m^*} vanishes. Usually, degeneracy is ruled out by a direct variational argument or by a super-harmonicity argument (related to the condition $-\Delta \eta \geq 0$ presented above), but this is not practicable here as the functional that needs to be optimised need not bear any resemblance to the natural energy associated with the PDE.

Another possibility to rule out degeneracies at the blown-up scale would be the derivation of *a priori* perimeter estimates. Although several tools exist to handle such cases [**45**] they do not seem applicable here, as they also deal with problems that are more variational in nature.

Remark: some non-energetic optimal control problems and stable free boundary problems There has recently been a surge in interest in the study of free boundary problems with particular structure, which either focused on non-minimising cases [**148**] or on general optimal control problems [**55**]. Let us briefly discuss the latter case, which bears the closest resemblance to our setting. Buttazzo, Casado-Diaz & Maestre studied the following problem:

$$\max_{m\in\mathcal{M}}\int_\Omega j(v_m) \text{ subject to } \begin{cases}-\Delta v_m+mv_m=f(v_m)\,,\\ \partial_\nu v_m=0.\end{cases} \tag{1.61}$$

They show that any solution m^* of this problem is BV so that, if m^* is a characteristic function, the underlying set has finite perimeter. There are two obstructions to applying their technique here:

(1) The first one is that, despite their similarities, (1.55) and (1.61) behave very differently, as even in the case $j=\mathrm{Id}$ the optimal m in (1.61) is a constant.

(2) At a more analytic level, if we look at a solution of (1.61) and if we pick an optimiser m^*, it is easy to see that there exists a function ζ_{m^*} and an increasing map g such that

$$m^*=g(\zeta_{m^*})$$

and

$$-\Delta\zeta_{m^*}+\zeta_{m^*}g(\zeta_{m^*})=F$$

for some smooth enough map F. The point here is that since g is increasing, one can put forth the theory of renormalised L^1 solutions to show that any solution of such an equation satisfies

$$\zeta_{m^*}g(\zeta_{m^*})\in BV(\Omega),$$

which is exactly what Buttazzo, Casado-Diaz & Maestre do. On the other hand, for problems of the form (1.55), the underlying free boundary equation rather writes

$$-\Delta\eta_{m^*}-\eta_{m^*}g(\eta_{m^*})=F$$

for some increasing g, and all the estimates fail. This is not a surprise, and, from a pure free boundary perspective, this is merely the reflection of the divide between (according to the terminology of Monneau & Weiss) stable and unstable free boundary problems.

This regularity problem will be a primary focus of our research in the years to come, as it is in fact an essential tool in understanding finer properties of optimal bang-bang controls.

1.5.2. Unstable free boundary problems for fourth order operators. To conclude this chapter, we want to mention another regularity problem that seems to be of fundamental importance and that we have started investigating; this problem rather has to do with the regularity of linear optimal controls. Surprisingly, it seems that the model case is the optimisation of potentials in a fourth-order problem. To give a concrete example, consider the optimisation problem

$$\max_{m\in\mathcal{M}} \int_\Omega y_m^2 \text{ subject to } \begin{cases} -\Delta y_m = m & \text{in } \Omega\,, \\ y_m \in W_0^{1,2}(\Omega). \end{cases} \tag{1.62}$$

Here, $\mathcal{M}$ is defined in (1.56). That any optimiser m^* is bang-bang follows from a simple convexity argument. As for the regularity observe that, if we introduce the adjoint state p_{m^*} as the solution of

$$\begin{cases} -\Delta p_{m^*} = y_{m^*} & \text{in } \Omega\,, \\ p_{m^*} \in W_0^{1,2}(\Omega) \end{cases}$$

then we can characterise

$$m^* = \{p_{m^*} > c\}$$

for some c. As p_{m^*} solves $\Delta^2 p_{m^*} = m^*$ a good first step to understand the regularity will be the investigation of the regularity properties of the problem

$$\min_{m\in\mathcal{M}} \min_{p\in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)} \frac{1}{2}\int_\Omega (\Delta p)^2 - \int_\Omega mp. \tag{1.63}$$

This problem fits in a growing line of research aimed at understanding higher order free boundary problems and the available pool of techniques is growing rapidly [**91**, **114**, **180**] but we expect the same type of difficulties that arise when going from the stable to unstable free boundary problem.

CHAPTER 2

Geometric and asymptotic properties of optimal controls

In this chapter, we focus more on the global qualitative properties of optimal controls. We will be focusing on the articles [**BMFN25**, **MRB21**, **MNP22a**, **Maz22a**, **MF25**].

2.1. Introduction

While the previous chapter focused on pointwise properties of optimal control problems (specifically on the bang-bang property, or lack thereof), we want to understand in more detail the *geometric properties* of the optimal control problems, in particular *symmetry* properties. In this chapter, we will be focusing on two main problems:

(1) The first one is the problem already encountered in Chapter 1, where one aims at optimising resources distributions, where we will exemplify what we refer to as the concentration/fragmentation phenomenon in optimal resources management. For these problems, the main difficulty is the lack of an energetic formulation of the problem.
(2) The second one is an optimal control problem related to periodic parabolic eigenvalues. In that problem, the lack of symmetry of the underlying operator is the main difficulty that needs to be overcome in the study of geometric properties of minimisers.

Nevertheless, in order to provide context, we will be led to discussing in detail another problem, that of the optimal survival ability in elliptic models. In the next paragraph, Section 2.1.1, we present the three main problems (optimal survival ability, optimal resources management and optimisation of parabolic eigenvalues) in an efficient way, summarising briefly our findings.

2.1.1. Paradigmatic problems. We now present the three main problems under consideration.

The reference case: optimisation of symmetric elliptic eigenvalue problems

The problem that can be used as a test case for optimal control problems in population dynamics is the following one:

$$\min_{m\in\mathcal{M}}\lambda(m,\mu)\text{ where }\lambda(m,\mu)=\min_{u\in W^{1,2}_0(\Omega)\,,\int_\Omega u^2=1}\frac{\mu}{2}\int_\Omega|\nabla u|^2-\int_\Omega mu^2. \tag{2.1}$$

and the admissible class is defined as

$$\mathcal{M}:=\left\{m\in L^\infty(\Omega):\ 0\leq m\leq 1\text{ a.e., }\int_\Omega m=m_0\right\}. \tag{2.2}$$

We do not present any new result on the composite membrane problem (2.1), but we will be using it as a reference point for the qualitative properties under scrutiny as it is a remarkably well-behaved problem. We underline that most of its good properties are due to the fact that the problem is energetic and that the underlying operator $-\mu\Delta-m$ is symmetric. Furthermore, its relevance in population dynamics (see Remark 1.3) makes it a tenet of optimal control in mathematical ecology.

Although we defer a discussion of the precise properties of (2.1) to section 2.1.2, we highlight the following facts:

(1) Any optimiser m^* of (2.1) is bang-bang *i.e.* $m^* = \mathbb{1}_{E^*}$.
(2) The geometry of the set E^* is easy to understand in several cases: it is convex and symmetric when Ω is convex, it is a ball when Ω is a ball etc.
(3) These geometric properties are independent from the diffusivity rate $\mu > 0$.
(4) A key method in all these proofs is rearrangement inequalities, whether these refer to Talenti comparison principles or Pólya-Szegö inequalities.

The main techniques used to derive such properties are presented in Section 2.1.2, where we also discuss whether they can be applied to the other problems we have in mind.

First problem: optimisation of resources distribution in logistic diffusive models The first problem for which we present new contributions is the usual total population size functional, namely, for a given diffusivity $\mu > 0$ and a given cost function j that satisfies $j' > 0$ in $(0; +\infty)$,

$$\max_{m\in\mathcal{M}} \int_\Omega j\left(\theta_{m,\mu}\right) \text{ subject to } \begin{cases} -\mu\Delta\theta_{m,\mu} - \theta_{m,\mu}\left(m - \theta_{m,\mu}\right) = 0 & \text{in } \Omega\,, \\ \partial_\nu\theta_{m,\mu} = 0 & \text{on } \partial\Omega\,, \\ \theta_{m,\mu} \geq 0\,, \theta_{m,\mu} \not\equiv 0. \end{cases} \tag{2.3}$$

The admissible set is given by (2.2). Broadly speaking, our contributions are as follows (we refer to section 2.2 for a precise discussion of the results):

(1) As we saw in Chapter 1, any optimiser m^* is bang-bang: $m^* = \mathbb{1}_{E^*}$.
(2) Building on considerations first derived in [**MNP20**] and that we can refine using [**MF24b**] we show that whenever the diffusivity μ is large enough, E^* behaves like the minimisers of (2.1). In particular, the optimisers have a very simple geometry when Ω itself has simple geometry (and we refer to this as "concentration" of resources).
(3) On the contrary, we show that, when Ω is a torus, the low-diffusivity regime is quite different and the specific non-linearity j under consideration plays a major role. We investigate two main cases:
    - The first one is $j : x \mapsto x^3$, which can be related to the minimisation of the energy associated with the PDE satisfied by $\theta_{m,\mu}$. In that case, we can show that rearrangement inequalities apply, and that the optimisers are also concentrated.
    - The second one is $j : x \mapsto x$, which amounts to maximising the total population size. In this case, based on our contributions of [**MRB21**] we prove that the perimeters of the optimisers go to $+\infty$ as $\mu \to 0$. Our techniques were later refined by Heo & Kim [**131**] to apply to general domains Ω's. Finally, in [**MNP22a**], we quantify the blow-up rate of the perimeter.

This problem is presented in Section 2.2.

Second problem: parabolic eigenvalue problems. The last problem of the chapter deals with time-periodic eigenvalues. Namely, for any time-periodic function $m : [0;T] \times \Omega$, for any $\tau\,,\mu > 0$, consider the first time-periodic eigenvalue $\Lambda(m,\tau,\mu)$ of the operator

$$\tau\partial_t - \mu\Delta - m$$

endowed with homogeneous Neumann boundary conditions in space, and periodicity conditions in time. The associated eigen-equation is

$$\begin{cases} \tau\partial_t\varphi_{m,\mu} - \mu\Delta\varphi_{m,\mu} - m\varphi_{m,\mu} = \Lambda(\tau,m,\mu)\varphi_{m,\mu} & \text{in } (0;T)\times\Omega\,, \\ \partial_\nu\varphi_{m,\mu} = 0 & \text{on } (0;T)\times\partial\Omega\,, \\ \varphi_{m,\mu}(0,\cdot) = \varphi_{m,\mu}(T,\cdot) & \text{in } \Omega\,, \\ \varphi_{m,\mu} \geq 0\,, \varphi_{m,\mu} \not\equiv 0 \end{cases} \tag{2.4}$$

and the optimisation problem is

$$\min_{m\in\mathcal{M}_T} \Lambda(m,\tau,\mu). \tag{2.5}$$

In this last problem, the set of admissible designs is

$$\mathcal{M}_T := \left\{ m \in L^\infty((0;T)\times\Omega) :\ 0 \leq m \leq 1 \text{ a.e.}, \iint_{(0;T)\times\Omega} m = m_0 \right\} \tag{2.6}$$

for some $m_0 \in (0;T|\Omega|)$.

We might summarise our results on (2.5) as follows (we refer to Section 2.3 for precise statements):

(1) Any optimiser of (2.5) is bang-bang, which leads to the question of whether optimisers are symmetric or even symmetric decreasing in time. Here and throughout, by symmetry in time we mean "symmetric with respect to $\frac{T}{2}$". Similarly, by symmetric monotonic or symmetric decreasing we mean "symmetric monotonic" or "symmetric decreasing" with respect to $\frac{T}{2}$. The conjecture that optimisers of (2.5) or symmetric or symmetric decreasing in time is not that reasonable at first sight, as the lack of symmetry of the operator on the one hand, and the fact that it is merely a transport equation in the time variable, do not make it clear that an isoperimetric property should in fact be satisfied. This is nevertheless justified by recent contributions of Liu, Lou & Song [**167**] that will be presented in due time.
(2) In [**BMFN25**] we show that optimisers of (2.5) are always symmetric in time.
(3) Furthermore, in [**BMFN25**], we study precisely two scaling regimes for a simplified version of the problem, where we work with $m = c(t)V(x)$ for a fixed V and optimise the activation function c: in the limit cases $\tau = \mu \to \infty$ and $\mu = \tau^2 \to 0$, we prove that the optimal c^* is asymptotically symmetric decreasing in time. This opens up several new conjectures that will be discussed in the research plan.

These results are presented in Section 2.3.

2.1.2. Classical tools to derive geometric properties in energetic optimisation: a study of elliptic eigenvalue optimisation. As announced, we begin with a short review of (2.1), which is our reference problem. The behaviour of (2.1) is quite well-understood, although it is deceitfully simple. In some specific cases, it is possible to fully characterise the optimisers and, in general the optimal sets E^* have at least bounded perimeter. Overall, a general paradigm is: *in energetic or spectral optimisation, optimal sets are regular and behave in certain regimes like perimeter-minimising sets.*

Let us briefly review how we proceed in such situations.

Rearrangement inequalities and their applications to spectral optimisation

The isoperimetric inequality and its analytic counterpart(s), rearrangement inequalities (*e.g.* Talenti inequality) are the basic tools when the geometry of the optimal domain is "simple" – say, a ball for Dirichlet boundary conditions or a square for Neumann boundary conditions. We refer once again to [**145**] for a general reference and we briefly explain the logic behind rearrangements in the case of Dirichlet boundary conditions, and how these apply to (2.1) to prove the following fact: letting $\Omega = \mathbb{B}(0;1)$ and letting $\mathbb{B}^*$ be the centred ball of volume m_0, there holds

$$\forall m \in \mathcal{M}\,, \forall \mu > 0\,, \lambda(\mathbb{1}_{\mathbb{B}^*}, \mu) \leq \lambda(m, \mu). \tag{2.7}$$

Let us present a proof that relies on Talenti inequalities. We briefly recall the notion of Schwarz symmetrisation:

(1) For any measurable set $\Omega \subset \mathbb{R}^d$ of finite measure, let its Schwarz rearrangement Ω^* be the centred ball with the same volume.
(2) For any function $f \in L^\infty(\Omega)$, introduce the Schwarz rearrangement f^* of f as the unique function defined on Ω^* that satisfies
   (a) f is radially symmetric and non-increasing.
   (b) f and f^* are equimeasurable in the sense that

$$\forall s \in \mathbb{R}\,, |\{f \geq s\}| = |\{f^* \geq s\}|.$$

Several inequalities, which are the analytic translation of the isoperimetric inequality and are actually equivalent to it, play a crucial role. The one we focus on is the classical Talenti inequality. Introduced by Talenti in [**215**], it reads as follows: for any smooth domain Ω, any $w \in L^\infty(\Omega)\,, w \leq 0$ and any $f \in L^\infty(\Omega)\,, f \geq 0$, let $u\,, v$ be the solutions of

$$\begin{cases} -\Delta u - wu = f & \text{in } \Omega\,, \\ u \in W^{1,2}_0(\Omega) \end{cases} \text{ and } \begin{cases} -\Delta v - w^* u = f^* & \text{in } \Omega^*\,, \\ v \in W^{1,2}_0(\Omega^*). \end{cases}$$

Then there holds

$$\forall r \in (0; |\Omega|)\,, \sup_{|E|=r} \int_E u \leq \sup_{|F|=r} \int_F v. \tag{2.8}$$

The condition $w \leq 0$ is just here to ensure the well-posedness of the equation. A straightforward proof of (2.7) based on (2.8) is to use the Perron-Frobenius approach to $\lambda(m, \mu)$. Indeed, set $w_m := (m-1)$, observe that $w_m^* = w_{m^*}$ and let T_m be the resolvent of $-\Delta - w_m$. The Talenti inequality merely states that, seen as an operator of $L^2(\Omega)$,

$$\|T_m\|_{\text{op}} \leq \|T_{m^*}\|_{\text{op}}$$

and, iterating, we deduce that

$$\frac{1}{\lambda(m,\mu)+1} = \lim_{k\to\infty} \|T_m^k\|_{\mathrm{op}}^{\frac{1}{k}} \leq \lim_{k\to\infty} \|T_{m^*}^k\|^{\frac{1}{k}} = \frac{1}{\lambda(m^*,\mu)+1},$$

which completes the proof. Here, we sweep under the rug the precise details that allow to go from $\|T_m^k\|_{\mathrm{op}} \leq \|T_m^k\|_{\mathrm{op}}$ to $\|T_m^k\|_{\mathrm{op}} \leq \|T_{m^*}^k\|_{\mathrm{op}}$. This approach also holds for the optimisation of the Dirichlet energy.

It turns out that similar approaches (relying rather on Pólya-Szegö inequalities than Talenti, which are much more delicate in the Neumann case) are also very powerful to handle the case of Neumann (or periodic) boundary conditions; we refer to [**10**, **9**, **11**, **12**, **13**, **22**, **23**, **117**, **118**, **150**, **152**, **181**, **182**, **218**] for applications of symmetrisation in spectral optimisation and more recent results on Talenti inequalities.

Thus, at least for simple geometries, say, the ball, the geometry of the optimisers is remarkably simple: it is also radially symmetric and decreasing. In general, the situation is not so favorable: although when the domain Ω is convex and symmetric with respect to a hyperplane any solution of (2.1) is the characteristic function of a convex set (that is also symmetric with respect to that hyperplane), it was observed in [**77**] by Chanillo, Grieser, Imai, Kurata & Ohnishi that symmetry could be broken, and that the optimiser, when Ω is an annulus, is not necessarily radially symmetric. Nevertheless, it is expected that the minimisers satisfy isoperimetric-like properties and can thus be well characterised geometrically.

Other techniques: rearrangement inequalities

The most traditional ways to use rearrangements relies on two fundamental inequalities, the Hardy-Littlewood and the Polyá-Szegó inequalities. Nevertheless, these approaches do not really work for non-symmetric operators, and we thus skip them for the time being. For general references on rearrangements, we refer to the monographs of Kawohl [**142**] and of Kesavan [**145**].

A comment on symmetric decreasing rearrangement and its periodic version Although the Schwarz rearrangement we introduced above is well suited to handle Dirichlet boundary conditions, in several of the problems we consider, we rather expect to deal with Neumann boundary conditions. In this case, the natural geometry, rather than the ball, is the unit square $\Omega = [0;1]^d$ (or the torus $\mathbb{T}^d$, in which case one rather works with periodic boundary conditions). In these conditions, the notion of rearrangement needs to be modified and one can on the concept of decreasing rearrangement introduced by Kawohl [**142**] and studied in details by Berestycki & Lachand-Robert [**24**]. However, the main point of interest, to us, is a variation on this notion of decreasing rearrangement that was particularly useful to Berestycki, Hamel & Roques in studying spectral optimisation problems stemming from population dynamics [**22**]. For this reason, we focus on this *periodic decreasing rearrangement*. We work in the d-dimensional torus $\mathbb{T}^d = [-\pi;\pi]^d$: Define, for any $i \in \{1,\dots,d\}$ and for any function $f = f(x_1,\dots,x_d)$, $f^{\sharp_i}$ as the unique function such that:

(1) For a.e. $x_1,\dots,x_{i-1}, x_{i+1},\dots,x_d$, $x_i \mapsto f(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_d)$ is symmetric with respect to $x_i = 0$ and non-increasing.
(2) For any $s \in \mathbb{R}$,

$$\begin{aligned}|\{x_i : f(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_d) \geq t\}| \\ = \left|\{x_i : f^{\sharp_i}(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_d) \geq t\}\right|.\end{aligned}$$

Then, fixing an enumeration $i_1, \dots, i_d$ of $\{1, \dots, d\}$ we define

$$f^\sharp := \left(\left(f^{\sharp_{i_1}}\right)^{\sharp_{i_2}} \dots\right)^{\sharp_{i_d}}.$$

In general, the function that is obtained depends on the choice $i_1, \dots, i_d$, but this symmetrisation nevertheless enjoys enough properties similar to the Schwarz rearrangement to make it usable in spectral optimisation; typically, the Polyá-Szegö inequality

$$\forall f \in W^{1,2}(\mathbb{T}^d), \int_{\mathbb{T}^d} |\nabla f^\sharp|^2 \leq \int_{\mathbb{T}^d} |\nabla f|^2 \tag{2.9}$$

and the Hardy-Littlewood inequality

$$\forall f\,, g \in L^\infty(\mathbb{T}^d)\,, \int_{\mathbb{T}^d} fg \leq \int_{\mathbb{T}^d} f^\sharp g^\sharp \tag{2.10}$$

are satisfied.

This can be used to show that, $\mathcal{M}$ still being defined by (2.2) with $\Omega = [0;1]^d$, there exist solutions to

$$\min_{m \in \mathcal{M}} \min_{u \in W^{1,2}(\mathbb{T}^d)\,, \int_\Omega u^2 = 1} \int_\Omega |\nabla u|^2 - \int_\Omega mu^2$$

that are decreasing in every direction. The question of whether any optimiser satisfies this monotonicity (*i.e.* whether for any optimal m^* we have $(m^*)^\sharp = m^*$) is, as far as we are aware, open, apart from the two dimensional case, where we settled it in [**MNP20**].

At any rate, the bottom line is: whenever rearrangements are usable, the optimisers tend to have "nice" properties: typically, using periodic rearrangements, we can obtain that the supports of minimisers are connected and even enjoy nice perimeter properties.

Can this approach accommodate non-symmetric operators? Let us skip ahead and consider the optimisation of space-time periodic eigenvalue, that is, Problem (2.5). We claimed, in the previous paragraph, that rearrangement approaches mostly work for the optimisation of the eigenvalues of symmetric operators, which is problematic if we want to adapt such proofs to parabolic optimisation: indeed, the operator $\partial_t - \Delta$, even if we see it as a degenerate elliptic operator by writing that in a certain sense

$$\partial_t - \Delta = \lim_{\varepsilon \to 0} (\partial_t - \varepsilon \partial_{tt}^2 - \Delta), \tag{2.11}$$

is not symmetric. While we defer a more detailed discussion to Section 2.3, we mention two related sets of results:

(1) First, in a series of contributions, Hamel, Nadirashvili & Russ put forth a unifying approach relying on rearrangements to provide comparison results for general operators of the form

$$-\nabla \cdot (A \nabla \cdot) + \langle X, \nabla \rangle + m$$

but their results lead to rearranging the vector field X as well, which is a problem in parabolic problems.

(2) Second, Nadin, in [**182**], proved that *in the one-dimensional case* it is possible to obtain a comparison result. Namely, he showed that for any constant A, if we let $\zeta(A,m)$ be the first eigenvalue of the operator

$$-\frac{d^2}{dx^2}-A\frac{d}{dx}-A^2m$$

endowed with periodic boundary conditions, and if we let $m^\sharp$ be the periodic decreasing rearrangement of m, then

$$\zeta(A,m^\sharp)\leq \zeta(A,m). \tag{2.12}$$

Unfortunately, these techniques can not be applied to the optimisation of parabolic eigenvalues: the result of Nadin only holds in the one-dimensional case, and the results of Hamel, Nadirashvili & Russ, entail modifying the drift part of the equation, effectively replacing the time derivative ∂_t with something akin to $\partial_t \mathbb{1}_{[0;T/2)}-\partial_t\mathbb{1}_{[T:2;T]}$, which would break the parabolicity of the problem.

A Modica approach to the good behaviour of optimisers in energetic problems for small diffusivities

Let us be more specific regarding this "isoperimetric" property of optimisers, which can be made rigorous in the low-diffusivity limit $\mu\to 0$. To do so, we state below, in an informal manner, a proposition that is a straightforward adaptation of the arguments introduced by Modica [**176**] in the study of phase transition.

Proposition 2.1 (Informal). *Let, for any $\mu>0$, m_μ^* be a solution of* (2.1) *and E_μ^* be such that $m_\mu^*=\mathbb{1}_{E_\mu^*}$. Any L^1-closure point, as $\mu\to 0$, of the sequence $\{E_\mu^*\}_{\mu\to 0}$ is a solution of*

$$\min_{E\,,|E|=V_0}\operatorname{Per}(E).$$

In particular, we do not expect the set E to fragment (*i.e.* to split into several connected components) or to have a wildly oscillating boundaries.

Unfortunately, and as we shall see in Section 2.2, such techniques do not apply in the case of non-energetic optimisation. In fact, *for the total population size, optimal sets tend to maximise the perimeter as $\mu\to 0^+$.*

2.2. Concentration and fragmentation in spatial ecology

The goal of this section is to illustrate what we refer to as "concentration/fragmentation" phenomena in spatial ecology, namely, situations where the dispersal of resources is more favorable to the population than their concentration. As it turns out, and although most of the references we will be using focus on the case of the total population size, we can state some results for some general functionals. In this section, we will once again be working with the reaction-diffusion equation

$$\begin{cases}-\mu\Delta\theta_{m,\mu}=\theta_{m,\mu}\left(m-\theta_{m,\mu}\right) & \text{in } \Omega\,,\\ \partial_\nu\theta_{m,\mu}=0 & \text{on } \partial\Omega\,,\\ \theta_{m,\mu}\geq 0\,,\not\equiv 0 & \text{in } \Omega.\end{cases} \tag{2.13}$$

In contrast to the first chapter, we emphasise the dependency on the diffusivity μ. We let $j:[0;\infty)\to\mathbb{R}$ be a $\mathscr{C}^2$, increasing function that satisfies

$$j'>0 \text{ on } (0;+\infty) \tag{2.14}$$

and we consider the optimal control problem

$$\max_{m\in\mathcal{M}} J(m,\mu) := \int_\Omega j\left(\theta_{m,\mu}\right). \tag{2.15}$$

For the definition of $\mathcal{M}$ we refer to (2.2). Under assumption (2.14), Theorem 1.7 guarantees that any solution m_μ^* of (2.15) is bang-bang: $m_\mu^* = \mathbb{1}_{E_\mu^*}$. We use the following loose terminology:

(1) We say that (2.15) has a "concentration property" if the set E_μ^* has nice geometric properties (typically, it is invariant by rearrangements)–essentially, this means that the optimisers look like the optimisers for spectral optimisation problems.
(2) On the other hand, we say that (2.15) has a "fragmentation property" if the set E_μ^* has a very high perimeter, or a very high number of connected components.

As it turns out, and perhaps surprisingly, a critical role is played by the monotonicity of J with respect to the diffusivity $\mu > 0$. To be more specific, here is what we are going to show:

(1) First, for any problem of the form (2.15), under assumption (2.14), concentration holds for large diffusivities and, in fact, the optimisers are stationary in μ.
(2) Second, when $\mu \to 0$ and when $j = \mathrm{Id}$, the BV-norms of the optimisers blow-up in a quantified way (meaning we have fragmentation) while, on the other hand, for $j : x \mapsto x^3$, the optimisers $m_\mu^* = \mathbb{1}_{E_\mu^*}$ are such that E_μ^* tend to minimise the perimeter (meaning we have concentration). Naturally, in this second part, the analysis is much more case oriented.

2.2.1. The large diffusivity regime for general non-energetic problems: concentration phenomena. In this section, we work with Neumann boundary conditions, although all the results stated here also hold, *mutatis mutandis*, for periodic boundary conditions. As we hinted at earlier, in the large diffusivity regime $\mu \to \infty$ we expect solutions to (2.15) to be well-behaved. In order to make this statement precise, we introduce the reference problem (that the solutions of (2.15) should solve for μ large enough):

$$\min_{m\in\mathcal{M}} \min_{\phi\in W^{1,2}(\mathbb{T}^d),\int_{\mathbb{T}^d}\phi=0} \frac{1}{2}\int_{\mathbb{T}^d} |\nabla\phi|^2 - \int_{\mathbb{T}^d} m\phi. \tag{2.16}$$

Using the aforementioned rearrangement techniques, it is possible to show the following facts:

(1) When we work in the case of periodic boundary conditions, that is, with $\Omega = \mathbb{T}^d$, optimisers are symmetric non-increasing in any direction.
(2) In the one-dimensional case, with $\mathbb{T} = (0;1)$, there exist exactly two minimisers, $m^* = \mathbb{1}_{(0;m_0)}$ and $m_* = \mathbb{1}_{(1-m_0;1)}$.
(3) In more general geometries, it is possible to derive regularity properties adapting the methods of Chanillo, Kenig & To [**79**] (see the work in preparation with Ferreri & Prunier).

At any rate, (2.16) can be considered the simplest optimal control problem, and behaves similarly to (2.1).

The main result here is the following:

THEOREM 2.2. *Let j satisfy* (2.14). *Let for any $\mu > 0$ m_μ^* be a solution of* (2.15). *As $\mu \to \infty$, $\{m_\mu^*\}_{\mu\to\infty}$ converges up to a subsequence, strongly in $L^1(\Omega)$, to a solution m_∞ of* (2.16).

REMARK 2.3. We obtained this theorem in the one-dimensional case for $j = \mathrm{Id}$ in [**MNP20**]. Theorem 2.2, which never appeared as such in any publication, can in fact be refined in the one-dimensional case to obtain a stationarity result, meaning that for some $\mu^* > 0$, for any $\mu \geq \mu^*$, m_μ^* solves (2.16).

SKETCH OF PROOF OF THEOREM 2.2. The core of the proof relies on the asymptotic expansion of the solution $\theta_{m,\mu}$ with respect to the diffusivity μ. More specifically, we show that

$$\theta_{m,\mu} = m_0 + \frac{\eta_m}{\mu} + \underset{\mu\to\infty}{O}\left(\frac{1}{\mu^2}\right),$$

in sufficiently regular norms, for some η_m. Such expansions are quite standard and were for instance used by He & Ni [**122, 123, 124**] in their study of Lotka-Volterra systems, as well as in our contribution [**MNP20**]. Let us briefly explain how to derive such an expansion: first of all, let us derive it at a formal level. By the weak formulation of (2.13) and the maximum principle (which implies $\|\theta_{m,\mu}\|_{L^\infty(\mathbb{T}^d)} \leq \|m\|_{L^\infty(\mathbb{T}^d)}$ we obtain

$$\mu\int_{\mathbb{T}^d} |\nabla\theta_{m,\mu}|^2 \leq 1$$

whence we can expect $\theta_{m,\mu}$ to converge to a constant. Furthermore, since, for any $\mu > 0$,

$$\int_{\mathbb{T}^d} \theta_{m,\mu}(m - \theta_{m,\mu}) = 0 \tag{2.17}$$

we deduce that this constant can only be m_0. Looking, formally, for the following order term η_m, we see that η_m should solve

$$-\Delta\eta_m = m_0(m - m_0)$$

supplemented with Neumann boundary conditions. As this equation is underdetermined, we need to specify the value of $\int_{\mathbb{T}^d} \eta_m$. Going back once again to (2.17) we obtain

$$m_0\int_{\mathbb{T}^d} \eta_m = \int_{\mathbb{T}^d} \eta_m(m - m_0) = \frac{1}{m_0}\int_{\mathbb{T}^d} |\nabla\eta_m|^2$$

and we are thus led to introduce η_m as the (unique) solution to

$$\begin{cases} -\Delta\eta_m - m(m - m_0) = 0 & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} \eta_m = \frac{1}{m_0^2}\int_{\mathbb{T}^d} |\nabla\eta_m|^2. \end{cases} \tag{2.18}$$

It then suffices to show that

$$\left\|\theta_{m,\mu} - m_0 - \frac{\eta_m}{\mu}\right\|_{W^{1,2}(\mathbb{T}^d)} \leq C\frac{\int_{\mathbb{T}^d} \eta_m}{\mu^2}. \tag{2.19}$$

This follows from standard elliptic estimates, and we refer for instance to [**MNP20**].

Thus, we can write, for some $\xi \in (m_0; \|m\|_{L^\infty(\Omega)} - m_0)$,

$$J(m,\mu) = \int_{\mathbb{T}^d} j(m_0) + \int_{\mathbb{T}^d} j'(m_0)(\theta_{m,\mu} - m_0)$$

$$
\begin{aligned}
&+\int_{\mathbb{T}^d}\frac{j''(\xi)}{2}\left(\theta_{m,\mu}-m_0\right)^2\\
&=\int_{\mathbb{T}^d}j(m_0)+\frac{j'(m_0)}{\mu}\int_{\mathbb{T}^d}\eta_m+j'(m_0)\int_{\mathbb{T}^d}\left(\theta_{m,\mu}-m_0-\frac{\eta_m}{\mu}\right)\\
&+\int_{\mathbb{T}^d}\frac{j''(\xi)}{2}\left(\theta_{m,\mu}-m_0\right)^2\\
&=j(m_0)|\mathbb{T}^d|+\frac{j'(m_0)}{\mu}\int_{\mathbb{T}^d}\eta_m+\underset{\mu\to\infty}{O}\left(\frac{\int_{\mathbb{T}^d}\eta_m}{\mu^2}\right)\\
&+\underset{\mu\to\infty}{O}\left(\frac{\int_{\mathbb{T}^d}\eta_m}{\mu^2}\right)\\
&=j(m_0)|\mathbb{T}^d|+\frac{j'(m_0)}{\mu}\int_{\mathbb{T}^d}\eta_m+\int_{\mathbb{T}^d}\eta_m\underset{\mu\to\infty}{O}\left(\frac{1}{\mu^2}\right).
\end{aligned}
$$

Now, let for any $\mu>0$ m_μ^* be a solution of (2.15). It follows that for any $m\in\mathcal{M}$ there holds

$$J(m_\mu^*,\mu)\geq J(m,\mu).$$

In particular, there exists $A>0$ such that for any $m\in\mathcal{M}$,

$$\int_{\mathbb{T}^d}\eta_m-\frac{A}{\mu}\int_{\mathbb{T}^d}\eta_m\leq\int_{\mathbb{T}^d}\eta_{m_\mu^*}+\frac{A}{\mu}\int_{\mathbb{T}^d}\eta^*_{m,\mu}.$$

Let m_∞ be a weak $L^\infty-*$ closure point of $\{m_\mu\}_{\mu\to\infty}$. We deduce, passing to the limit $\mu\to\infty$ in the previous estimate, that for any $m\in\mathcal{M}$ there holds

$$\int_{\mathbb{T}^d}\eta_{m_\infty}\geq\int_{\mathbb{T}^d}\eta_m.$$

Thus m_∞ solves (2.16). However, any solution of (2.16) is an extreme point of $\mathcal{M}$, whence the weak convergence is strong in any $L^p(\mathbb{T}^d)$. This concludes the proof. $\square$

In particular, for large diffusivities, any increasing optimisation problems behaves like an energy minimisation problem which, as we saw, enjoys many good geometric properties (*i.e.* convexity and symmetry if the domain is convex and symmetric, compatible with rearrangements in domains that allow for rearrangements etc). However, in the rest of this chapter, we want to argue that this common behaviour only holds for large diffusivities.

2.2.2. The low-diffusivity regime for general functions: the role of monotonicity in μ. In this section, which describes [**MNP22a**, **MRB21**], we want to showcase the fact that the geometric properties of the optimisers of (2.15) strongly depend on the exact non-linearity when considering the case of small diffusivities. This will be done through the study of two main examples:

(1) First, with the cost $j:x\mapsto x^3$, which amounts (see below for details) to minimising the energy of (2.13) and is thus amenable to rearrangement techniques.
(2) Second, with the total population size, that is with $j:x\mapsto x$, where we show that the perimeter of optimisers explodes as $\mu\to 0^+$.

2.2.2.1. *The case of $j_3 : x \mapsto x^3$.* We begin with the optimisation problem

$$\max_{m\in\mathcal{M}} \int_\Omega \theta_{m,\mu}^3 \text{ where } \theta_{m,\mu} \text{ solves (2.13).} \tag{2.20}$$

Introducing the energy functional

$$\mathscr{E}_{m,\mu} : W^{1,2}(\Omega) \ni u \mapsto \frac{\mu}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega mu^2 + \frac{1}{3}\int_\Omega |u|^3$$

it is readily seen that

$$\theta_{m,\mu} \text{ is the unique solution of } \min_{u\in W^{1,2}(\Omega)\,,u\geq 0} \mathscr{E}_{m,\mu}(u)$$

and that

$$\mathscr{E}_{m,\mu}(\theta_{m,\mu}) = -\frac{1}{6}\int_\Omega \theta_{m,\mu}^3.$$

Consequently, (2.20) is equivalent to the double minimisation problem

$$\min_{m\in\mathcal{M}} \min_{u\in W^{1,2}(\Omega)\,,u\geq 0} \mathscr{E}_{m,\mu}(u). \tag{2.21}$$

Now, let $m^* = \mathbb{1}_{E^*}$ solve (2.21). Then, using symmetrisation techniques (see Section 2.1.2) we can show the following facts:

(1) If we work in the one-dimensional case, the only two solutions to (2.21) are $\mathbb{1}_{[0;m_0)}$ and $\mathbb{1}_{(1-m_0;m_0)}$.
(2) If we work in the two-dimensional torus with periodic boundary conditions instead of Neumann boundary conditions, m^* is symmetric non-increasing in every direction (up to a translation).
(3) If Ω is a convex set, then E^* is convex.

In particular, for this functional, we expect a "nice" geometric behaviour.

2.2.2.2. *The case of $j_1 : x \mapsto x$.* We continue with the problem we started with, which we rewrite here for the sake of readability

$$\max_{m\in\mathcal{M}} \int_\Omega \theta_{m,\mu}. \tag{2.22}$$

Recall that as $\mu \to \infty$, the maximisers converge to solutions of (2.16) and in particular are expected to have nice perimeter bounds. However, as $\mu \to 0$, the situation is completely different. We summarise our main results below; they are taken from [**MNP22a**, **MRB21**].

Theorem 2.4. *Assume we work in the periodic case $\Omega = \mathbb{T}^d$. Let, for any $\mu > 0$, $m_\mu^* = \mathbb{1}_{E_\mu^*}$ be a solution of* (2.22)*. There exists a constant $C > 0$ such that for any $\mu > 0$ small enough*

$$\operatorname{Per}(E_\mu^*) \geq \frac{C}{\sqrt{\mu}}. \tag{2.23}$$

In particular, in the one-dimensional case, the number of connected components of E_μ^* explodes as $\mu \to 0$.

Some comments on this result

In the following section, we will give the main elements of the proof of Theorem 2.4, but we want to point out several facts: first, in [**MRB21**], in collaboration with Ruiz-Balet, we proved that $\operatorname{Per}(E_\mu^*) \underset{\mu\to 0^+}{\to} +\infty$ in the torus. Shortly thereafter, Heo & Kim [**131**] generalised this result to arbitrary domains with Neumann boundary conditions, and with Nadin & Privat [MNP22a] we were able, adapting the approach of Modica, to quantify the blow-up of the perimeter–we insist upon the fact that the $\sqrt{\mu}$ appearing in (2.25) is the same natural scaling appearing in [176]. Observe that we expect that

$$\mathrm{Per}(E^*_\mu) \underset{\mu\to 0^+}{\sim} C\sqrt{\mu}$$

but this estimate is for the time being out of reach. Finally, let us mention [186], which provides very precise fragmentation estimates for a discrete version of the problem; the techniques are very different, as well as [183], which derives precise perimeter bounds for a closely related problem.

The core element of the proof is the non-monotonicity in μ of $S : (m,\cdot) \mapsto S(m,\mu) = \int_\Omega \theta_{m,\mu}$. It was indeed proved by Lou [169] that, for any $m \geq 0$,

$$\lim_{\mu\to 0^+} S(m,\mu) = \lim_{\mu\to\infty} S(m,\mu) = \int_\Omega m = \inf_{\mu>0} S(m,\mu) \tag{2.24}$$
$$\text{and if } m \text{ is not constant, for any } \mu>0,\ S(m,\mu) > m_0.$$

The importance of this fact will be clear in the sketch of proof below, but let us also point out that this property is not true if m changes sign–we refer to the last paragraph of this section. At any rate, m being fixed, the finer properties of the map $\mu \mapsto S(m,\mu)$ are not completely clear and we refer for instance to [161] for more information.

Sketch of proof of Theorem 2.4. The proof goes in several steps but ultimately relies on a contradiction argument. We let, for any $\mu > 0$, m^*_μ be a solution of (2.22).

(1) As a first step, let us recall the result [169], namely, that for any $m \in \mathcal{M}$ there holds

$$\int_\Omega \theta_{m,\mu} \underset{\mu\to 0^+}{\to} m_0.$$

In [MRB21], we show that this convergence is uniform with respect to the BV norm of m or, in other words: setting, for any $M > 0$, $\mathcal{M}_M := \left\{m \in \mathcal{M}\,, \|m\|_{BV(\Omega)} \leq M\right\}$, there holds

$$\forall M>0\,,\forall\varepsilon>0\,,\exists\mu_\varepsilon>0\,,\forall\mu\leq\mu_\varepsilon\,,\forall m\in\mathcal{M}_M\,,\left|S(m,\mu)-\int_\Omega m\right|\leq\varepsilon.$$

In [MNP22a], we refined this estimate by adapting the arguments of Modica and by showing the existence of a constant C such that

$$\forall m\in\mathcal{M}\cap BV(\Omega)\,,\left|S(m,\mu)-\int_\Omega m\right|^3\leq C\sqrt{\mu}\|m\|_{BV(\Omega)}. \tag{2.25}$$

We think that this point is an important one, and deserves being mentioned. Nevertheless, as it can get quite technical, we refer to [MNP22a] for a detailed discussion of (2.25). Let us admit it for the time being.

(2) The second step of the proof consists in arguing by contradiction: assume that there exists a sequence $\{\mu_k\}_{k\in\mathbb{N}}$ converging to 0 and a sequence of maximisers $\{m^*_{\mu_k}\}_{k\in\mathbb{N}}$ such that

$$\liminf_{k\to\infty}\|m^*_{\mu_k}\|_{BV(\Omega)}<\infty.$$

If this is the case, if follows from the considerations of the first point that

$$\max_{m\in\mathcal{M}} S(m,\mu_k) - m_0 \underset{k\to\infty}{\to} 0,$$

and so the conclusion is reached provided we can construct a sequence $\{m_k\}_{k\in\mathbb{N}} \in \mathcal{M}^{\mathbb{N}}$ of competitors and a real number $\delta > 0$ such that

$$\forall k \in \mathbb{N}\,, S(m_k,\mu_k) - m_0 \geq \delta. \tag{2.26}$$

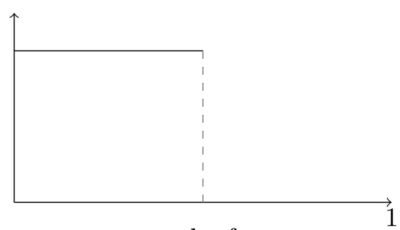
This sequence of competitors can be built as follows: fix any $m \not\equiv m_0$. By (2.24) we can fix any $\mu_0 > 0$ such that

$$S(m,\mu) > m_0 \text{ for any } \mu \in [\mu_0; 4\mu_0].$$

Now extend m by periodicity and set, for any $k \in \mathbb{N}$, $m_k(x) := m(kx)$. A simple change of variable shows that

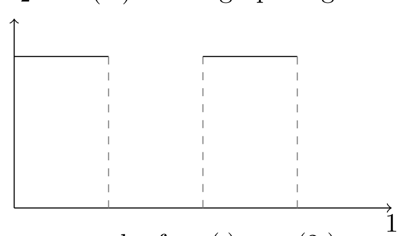
$$\forall k \in \mathbb{N}\,, S\left(m_k, \frac{\mu_0}{2^{2k}}\right) = S(m,\mu_0).$$

We illustrate this construction below in $(0;1)$: replacing the function m whose graph is given by



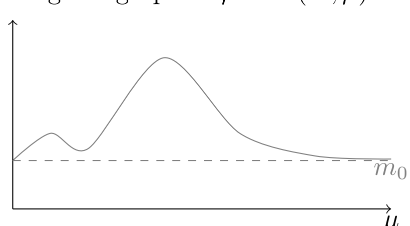
graph of m

with the function $m_2 = m(2\cdot)$ whose graph is given by



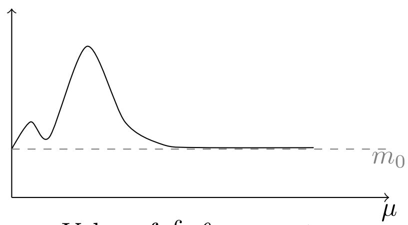
graph of $m_2(\cdot) = m(2\cdot)$

amounts to contracting the graph of $\mu \mapsto S(m,\mu)$



Value of $\int_\Omega \theta_{m,\mu}$ w.r.t μ

by a factor $1/4$, leading to



Value of $\int_\Omega \theta_{\overline{m},\mu}$ w.r.t μ

Using this construction gives

$$\forall \mu \leq \mu_0\,, \max_{m\in\mathcal{M}(\mathbb{T}^d)} S(m,\mu)-m_0 \geq \min_{\nu\in[\mu_0;4\mu_0]} S(m_k,\nu)-m_0 =: \delta>0.$$

(3) We can now conclude: letting m_μ^* be a solution of (2.22), it follows from (2.26)–(2.25) that there exists a positive constant C such that

$$\sqrt{\mu}\|m_\mu^*\|_{BV(\mathbb{T})} \geq C,$$

which yields the conclusion. Observe that we implicitly assume that m_μ^* is in BV, for there is otherwise nothing to prove.

$\square$

Remark 2.5 (On a result of Heo & Kim). Shortly after publishing our paper, Heo & Kim [**131**] extended the fragmentation result (*i.e* the fact that the BV-norms of maximiser diverges as $\mu\to 0$) to any smooth domain Ω with Neumann boundary conditions; the main difference is in finding a suitable sequence of competitors that would allow to prove (2.26).

Remark 2.6 (Is there a geometric interpretation to Theorem 2.4?). In the one-dimensional case, Theorem 2.4 immediately gives that, if we write for an optimiser m_μ^* of the total population size $m_\mu^* = \mathbb{1}_{E_\mu^*}$ then the number of connected components of E_μ^* explodes as $\mu\to 0$. In higher dimensions, the situation is unclear; although, from what we observe in our numerical simulations (see the next paragraph) we also expect that the number of connected components diverges. Nevertheless, it is unclear how one should approach this question.

Remark 2.7 (On the non-monotonicity of the size functional). As we explained in the proof, a key point is the non-monotonicity of the map $\mu\mapsto S(m,\mu)$. The question of the dependency of the total population size functional on the diffusivity μ is an intricate and interesting one. We refer on this matter to the works of Liang & Lou [**161**] and Liang & Zhang [**162**].

Some numerical simulations

The simulations of Figure 1 illustrate the results of Theorem 2.4; they are taken from [**MRB21**]. We use a thresholding scheme to simulate the optimal resources distributions m_μ^* (we refer to [**MRB21**] for details on the numerical procedure) in the two-dimensional torus. In each of the simulations below, the red zone corresponds to $\{m_\mu^*=1\}$, while the blue one corresponds to $\{m_\mu^*=0\}$. The average of the resources distribution $m_0=0.3$ is fixed, and we only vary the diffusivity μ.

The importance of the positivity of admissible profiles in the low diffusivity regime: heuristics

We would like to thank A. Zilio for raising the question that occupies the present paragraph, namely, does the fragmentation result given in Theorem 2.4 hold if we

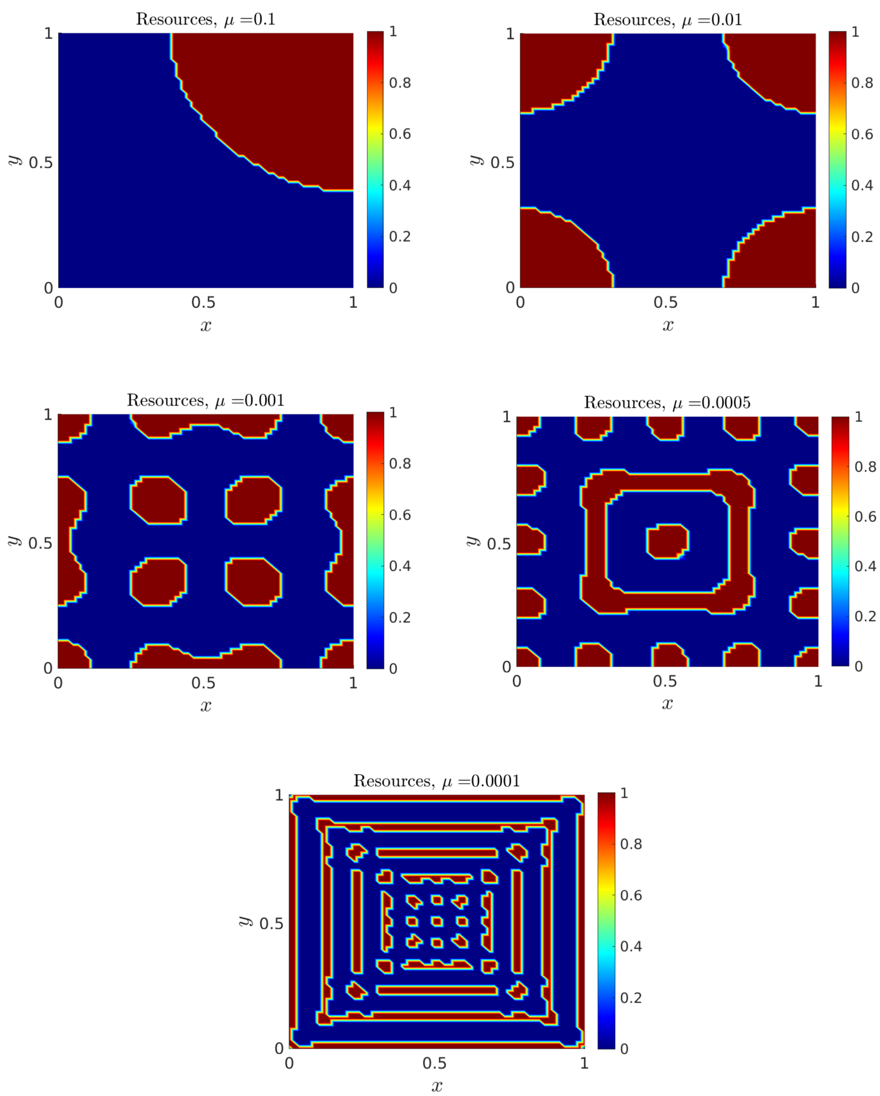


FIGURE 1. Illustration of the fragmentation phenomenon of Theorem 2.4

get rid of the positivity assumption $m \in [0;1]$ a.e.? Indeed, when m is constrained to lie in

$$\mathcal{M} := \left\{ m \in L^\infty(\Omega)\,, -\kappa \leq m \leq 1 \text{ a.e.}, \int_\Omega m = m_0 \right\}$$

where $\kappa\,, m_0 > 0$ then retaining the notation $S(m,\mu) := \int_\Omega \theta_{m,\mu}$, Lou showed in [169] that

$$S(m,\mu) \underset{\mu\to 0}{\to} \int_\Omega \max(0,m)\,, \quad S(m,\mu) \underset{\mu\to\infty}{\to} \int_\Omega m = m_0.$$

In that case, the fact that the total population size is minimised at $\mu=+\infty$ still holds, but it is no longer minimised at $\mu=0$, and thereby this bars the construction of a sequence of competitor such that (2.26) holds.

It seems in fact quite unlikely that Theorem 2.4 holds in this setting. Although we can not give a full proof here, let us present a heuristic argument in the one-dimensional case:

(1) First, Theorem 1.7 still holds, so that we can focus on the case of controls writing
$$m_E := -\kappa \mathbb{1}_{(0;1)\setminus E}+\mathbb{1}_E.$$
(2) Now, take $E=\sqcup_{i=1}^N (a_i;b_i)$ and take the limit $\kappa\to\infty$ (this is where things get tricky and might not be properly formalisable). The solution $\theta_{m_E,\mu}$ should behave as the solution to
$$\begin{cases}-\mu\Delta\Theta_{E,\mu}=\Theta_{E,\mu}(1-\Theta_{E,\mu}) & \text{in } E\,,\\ \Theta_{E,\mu}=0 & \text{on } \partial E.\end{cases}$$
(3) Finally, "gluing" the intervals together (see Figure 2 below) we can construct from $\Theta_{E,\mu}$ a sub-solution to the equation
$$-\mu\Delta\Theta=\Theta(1-\Theta) \text{ in } (0;m_0)\,,\Theta(0)=\Theta(m_0)=0\,,\Theta\geq,\not\equiv 0.$$

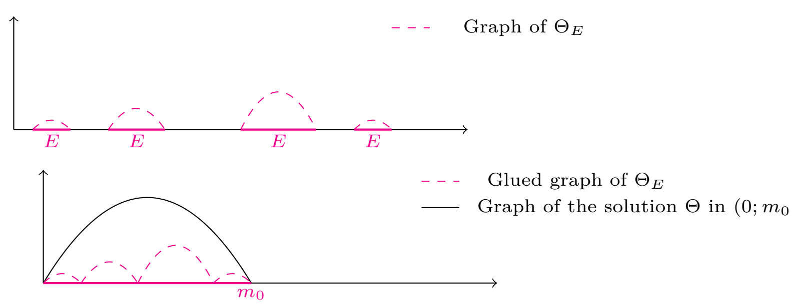


FIGURE 2. Illustration of the gluing procedure

Here again one needs to be careful with the constants but, provided m_0 is big enough, the existence of Θ_m is not a problem. In particular,
$$\int_E \Theta_{E,\mu}<\int_0^{m_0}\Theta,$$
so that fragmentation cannot occur.

We leave this question as an interesting open problem.

2.3. Symmetrisation of parabolic equations

2.3.1. Presentation of the problem and main results. The optimisation problem(s) In this section, we investigate some (non-)symmetry properties for optimisation problems in parabolic equations. In order to fix ideas, let us already mention that we will be working on the following problem: consider, for a fixed, time-space periodic function m (say, T-periodic in time, $\mathbb{T}^d$-periodic in x), henceforth referred to as *potential*, the principal eigenvalue $\lambda(m)$ associated with $\partial_t - \Delta - m$, that is, the unique real number $\lambda(m)$ such that the following problem

$$\begin{cases} \partial_t u_m - \Delta u_m = (m + \lambda(m))u_m & \text{in } [0;T] \times \mathbb{T}^d\,, \\ u_m(0,\cdot) = u_m(T,\cdot) & \text{in } \mathbb{T}^d\,, \\ u_m \geq 0\,, u_m \not\equiv 0 \end{cases} \tag{2.27}$$

has a solution.

The main problem under consideration is the following:

Minimise $\lambda(m)$ with respect to m.

Remark 2.8 (Biological interpretation). As we already discussed in Chapter 1 (see in particular Remark 1.3 for the elliptic case) there is a natural interpretation of this variational problem, as $\lambda(m)$ quantifies the chances of survival of a population evolving according to a logistic-diffusive equation and having access to a resources distribution m.

Remark 2.9 (Existence and uniqueness of a principal eigenvalue). Of course, as the operator $\partial_t - \Delta - m$ is not symmetric, the existence and uniqueness of a principal eigenvalue can not be obtained by minimisation of a Rayleigh quotient. Rather, these properties can be obtained using the Krein-Rutman theorem provided m is smooth enough or, for merely L^∞ potentials, by working on the Poincaré map defined as

$T : u_0 \mapsto u(T,\cdot)$

where u solves the initial value problem $\begin{cases} \partial_t u - \Delta u = mu & \text{in } (0;T) \times \mathbb{T}^d\,, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{T}^d. \end{cases}$

For this latter approach we refer to [**182**, Proof of Theorem 3.9]. In general, the monograph of Hess [**132**] is the reference work on periodic parabolic eigenvalue problems.

In terms of optimisation problems, we will be considering two types of controls:

(1) First, analogous to Problem (2.1), we consider the class

$$\mathcal{M}_T := \left\{ m \in L^\infty([0;T] \times \mathbb{T}^d) : 0 \leq m \leq 1 \text{ a.e.}, \iint_{(0;T)\times\mathbb{T}^d} m = m_0 \right\} \tag{2.28}$$

for some $m_0 \in (0; T|\mathbb{T}^d|)$. The associated optimisation problem is

$$\min_{m \in \mathcal{M}_T} \lambda(m). \tag{2.29}$$

(2) The second class of controls is different: we consider a fixed potential $V \in L^\infty(\mathbb{T})$ and the admissible controls are given by

$$m : (t,x) \mapsto c(t)V(x),$$

where the so-called *activation function* c belongs to the set

$$\mathcal{C}_T := \left\{ c \in L^\infty((0;T)) : 0 \leq c \leq 1 \text{ a.e.}, \int_0^T c = c_0 \right\} \tag{2.30}$$

for some $c_0 \in (0;T)$. We write $\lambda(c)$ for $\lambda(cV)$. The second optimisation problem under consideration is

$$\min_{c\in\mathcal{C}_T} \lambda(c). \tag{2.31}$$

REMARK 2.10. Actually, in [**BMFN25**] we consider more general classes of constraints by requesting that the admissible functions are dominated in the sense of rearrangements, but as the results are similar we stick to this simpler setting for the sake of readability.

What type of properties do we expect, and why? Recall that for the optimisation of potentials in the context of elliptic operators *i.e.* for (2.1) we obtain, in the torus, that any optimal potential V should be symmetric decreasing in space. As we emphasised, this property is intimately related to isoperimetric inequalities, which appear naturally when dealing with diffusion equations. Here, when dealing with parabolic problems, the main motivation behind our work [**BMFN25**] was to derive *symmetry and monotonicity properties* of the optimal control in *time*. This property is to be understood as "symmetry and monotonicity with respect to $\frac{T}{2}$". This might seem a surprising question at first, as the operator $\partial_t - \Delta - m$ only features a transport term in time, so it is not clear why one should expect this. In order to paint a complete picture, let us briefly review the main known results of λ:

(1) First of all, an easy concavity argument shows that any solution m^*, c^* of (2.29) and (2.31) respectively are bang-bang: $m^* = \mathbb{1}_E$ for some subset E of $[0;T] \times \mathbb{T}^d$ and $c^* = \mathbb{1}_I$ for some subset I of $[0;T]$. This begs the question of the geometry of E and I.
(2) Regarding the spatial properties of optimisers of (2.29), they are well-known in the one-dimensional case: Nadin, in [**182**], proved that, letting $m^\sharp$ denote the decreasing rearrangement in space of the function m, there holds
$$\lambda(m^\sharp) \leq \lambda(m).$$
(3) Regarding the monotonicity and symmetry in time (both of which have to be understood as "up to a translation in time") however, not much was known until [**BMFN25**], but there is one result that indicated that the geometry should be simple. Namely, these are results of Liu, Lou & Song [**167**] (which build on previous works of Liu, Lou & Peng [**166, 165**]): consider, for a given frequency $\tau > 0$ and a fixed $m \in L^\infty((0;T) \times \mathbb{T}^d)$, the principal eigenvalue $\lambda(\tau)$ of $\tau\partial_t - \Delta - m$. Then the function $\tau \mapsto \lambda(\tau)$ is non-decreasing (and is even increasing if m does not write $m(t,x) = h(x) + c(t)$). However, through a simple change of variables, this implies that for any function $m \in L^\infty((0;T) \times \mathbb{T}^d)$ that is not constant in time, if we define, for any $k \in \mathbb{N}$, $m_k : (t,x) \mapsto m(kt,x)$, the sequence $\{\lambda(m_k)\}_{k\in\mathbb{N}}$ is increasing. A possible interpretation of this result is that *oscillations in time are detrimental to spectral optimisation*, which hints that there might be an underlying simple geometric structure.

2.3.2. Main results. In collaboration with Bogosel & Nadin, we proposed in [**BMFN25**] an approach to the two problems (2.29)-(2.31); the results are not complete yet and pursuing these questions constitutes a major axis of our research in the short term. We state three theorems:

THEOREM 2.11. *Any solution* m^* *of* (2.29) *or* c^* *of* (2.31) *is symmetric in time: up to a translation, there holds*

$$m^*(T-t,\cdot)=m^*(t,\cdot)\,,c^*(T-t)=c^*(t).$$

The proof of this result, which we do not give here, relies on a saddle point formulation for non-symmetric eigenvalue problems which is due to Heinze [126]. Naturally, this does not give any information about the monotonicity in time of an optimal control. Nevertheless, we investigated this monotonicity in two distinct regimes in the case of activation functions. Recall that this means that we consider a fixed potential $V \in L^\infty(\mathbb{T}^d)$ and we want to solve (2.31). We let $c^\star$ be defined as

$$c^\star := \mathbb{1}_{\left(\frac{T-c_0}{2};\frac{T+c_0}{2}\right)},$$

so that the question of monotonicity and symmetry of optimisers boils down to: is it true that $c^\star$ is the unique solution to (2.31)? We were able to provide a positive answer in two different regimes.

First limit regime

The first limit regimes is the "large heat operator" regime. Namely, for any parameter μ and any function $c \in \mathcal{C}_T$, let $\lambda(c,\mu)$ be the principal eigenvalue associated with the system

$$\begin{cases}\mu\partial_t v_{c,\mu}-\mu\Delta v_{c,\mu}=cVv_{c,\mu}+\lambda(c,\mu)v_{c,\mu} & \text{in } (0;T)\times\mathbb{T}^d\,,\\ v_{c,\mu}(0,\cdot)=v_{c,\mu}(T,\cdot) & \text{in } \mathbb{T}^d\,,\\ v_{c,\mu}\geq 0\,,\neq 0\,,\\ \iint_{(0;T)\times\mathbb{T}^d} v_{c,\mu}^2=1.\end{cases} \tag{2.32}$$

We are interested in the problem

$$\min_{c\in\mathcal{C}_T}\lambda(c,\mu). \tag{2.33}$$

The first theorem is the following:

THEOREM 2.12. *Let, for any* $\mu>0$, c_μ^* *be a solution of* (2.33)*. Then, up to a translation,*

$$c_\mu^* \underset{\mu\to\infty}{\to} c^\star$$

strongly in any $L^p((0;T))$*,* $p\in[1;+\infty)$*.*

Second limit regime

The second limit regime is quite different and amounts to working in a "singular perturbation" setting. Namely, consider, fo any $\varepsilon>0$, the principal eigenfunction $\lambda_\varepsilon(c)$ associated with

$$\begin{cases}\varepsilon\partial_t u_{c,\varepsilon}-\varepsilon^2\Delta u_{c,\varepsilon}=cVu_{c,\varepsilon}+\lambda_\varepsilon(c)u_{c,\varepsilon} & \text{in } (0;T)\times\mathbb{T}^d\,,\\ u_{c,\varepsilon}(0,\cdot)=u_{c,\varepsilon}(T,\cdot) & \text{in } \mathbb{T}^d\,,\\ u_{c,\varepsilon}\geq 0\,,\neq 0\,,\\ \iint_{(0;T)\times\mathbb{T}^d} u_{c,\varepsilon}^2=1.\end{cases} \tag{2.34}$$

We consider

$$\min_{c\in\mathcal{C}}\lambda_\varepsilon(c). \tag{2.35}$$

We assume the following:
$(\mathbf{H}_V)$
$V \in \mathscr{C}^2(\mathbb{T}^d)$, $x^* = 0$ is the only local maximiser of V in $\mathbb{T}^d$ and $\nabla^2 V(0) \in S_d^{--}(\mathbb{R})$.

Here, $S_d^{--}(\mathbb{R})$ denotes the class of symmetric negative definite matrix. The main theorem is the following:

THEOREM 2.13. *Let V be fixed and satisfy $(\mathbf{H}_V)$ and, for any $\varepsilon > 0$, $c^\varepsilon_{\min}$ be a solution of*

$$\min_{c \in \mathcal{C}} \lambda_\varepsilon(c). \tag{2.36}$$

Up to a translation,

$$c^\varepsilon_{\min} \underset{\varepsilon \to 0^*}{\to} c^\star$$

strongly in any $L^p((0;T))$, $p \in [1;+\infty)$.

Some preliminary comments Before we discuss some possible (but doomed to fail) strategies of proof, let us briefly comment on one aspect: although we can be satisfied with Theorems 2.12–2.13, it is not clear whether the method of proof we use is the "correct" one. Indeed, in both cases, we actually show how to boil down the limit problems to elliptic problems, which allows for the use of rearrangement inequalities. There is of course no hope of generalising such approaches, and it is likely that a completely new outlook is required. Let us also briefly mention some tempting possible approaches, and why they fail:

(1) As we already explained, the fact that parabolic equations only act as a transport in time prohibits using isoperimetric inequalities directly.
(2) However, it is possible to see a parabolic equation as a degenerate elliptic operators (see Eq. (2.11)) and one might thus be tempted to use the results of Hamel, Nadirashvili & Russ [**118**, **117**]. However, as already noted, such rearrangement inequalities also modify the transport term which, in the case of parabolic equations, amounts to breaking the parabolicity of the operator.
(3) A final possibility would be to use Talenti inequalities; however, we proved in [**Maz22a**] that there can be no "strong enough Talenti" for the time rearrangement of parabolic equations.

2.3.3. Sketches of proof. We briefly present the main elements of proofs, showing how the optimisation problems (2.33)–(2.35) reduce to elliptic problems.

SKETCH OF PROOF OF THEOREM 2.12. The basic idea is to use an asymptotic expansion, similar to Theorem 2.2. Namely, it is reasonable to expect that, as $\mu \to \infty$, the solution $v_{c,\mu}$ to (2.32) and the eigenvalue $\lambda(c,\mu)$ write

$$v_{c,\mu} \approx 1 + \frac{y_c}{\mu} + \frac{z_c}{\mu^2}\,, \lambda(c,\mu) \approx \lambda_0(c) + \frac{\lambda_1(c)}{\mu}.$$

Plugging these expansions in (2.32) we obtain, at a formal level

$$\begin{cases} \partial_t y_c - \Delta y_c = c(t)V(x) + \lambda_0(c) & \text{in } (0;T) \times \mathbb{T}^d\,, \\ y_c(T,\cdot) = y_C(0,\cdot) & \text{in } \mathbb{T}^d\,, \\ \iint_{(0;T)\times\mathbb{T}} y_c = 0. \end{cases} \tag{2.37}$$

Integrating this equation in $(0;T)\times\mathbb{T}^d$ gives

$$\lambda_0(c)=-c_0\int_{\mathbb{T}^d}V.$$

Going to the next order, we obtain

$$\begin{cases}\partial_t z_c-\Delta z_c=c(t)V(x)y_c+\lambda_0(c)y_c+\lambda_1(c) & \text{in } (0;T)\times\mathbb{T}^d\,,\\ z_c(T,\cdot)=z_c(0,\cdot) & \text{in } \mathbb{T}^d.\end{cases}\tag{2.38}$$

Integrating this equation in $(0;T)\times\mathbb{T}^d$ and using (2.37), we deduce

$$\lambda_1(c)=-\iint_{(0;T)\times\mathbb{T}^d}c(t)V(x)y_c(t,x)dtdx=-\iint_{(0;T)\times\mathbb{T}^d}|\nabla y_c|^2.\tag{2.39}$$

All this reasoning can be made rigorous, and we can deduce from this that, as $\mu\to\infty$, the solutions c^*_μ to (2.33) converge to a solution of the limit optimisation problem

$$\max_{c\in\mathcal{C}_T}\iint_{(0;T)\times\mathbb{T}}|\nabla y_c|^2\tag{2.40}$$

$$\text{subject to }\begin{cases}\partial_t y_c-\Delta y_c=c(t)V(x)-\iint_{(0;T)\times\mathbb{T}^d}cV & \text{in } (0;T)\times\mathbb{T}^d\,,\\ y_c(T,\cdot)=y_c(0,\cdot) & \text{in } \mathbb{T}^d\,,\\ \iint_{(0;T)\times\mathbb{T}^d}y_c=0.\end{cases}$$

We now claim

$$\forall c\in\mathcal{C}_T\,,c\neq c^\star(\text{up to a translation})\ ,\iint_{(0;T)\times\mathbb{T}}|\nabla y_c|^2<\iint_{(0;T)\times\mathbb{T}}|\nabla y_{c^\star}|^2.\tag{2.41}$$

To prove (2.41), decompose V as a Fourier series

$$V(x)=\sum_k a_k\phi_k(x)\Rightarrow y_c(t,x)=\sum_k\alpha_{k,c}(t)\phi_k(x)\text{ with }\begin{cases}\alpha_{k,c}'(t)+\lambda_k\alpha_{k,c}(t)=c(t)\,,\\ \alpha_{k,c}(0)=\alpha_{k,c}(T).\end{cases}$$

The question becomes:

$$\max_c\sum_k\lambda_k\int_0^T\alpha_{k,c}^2\text{ subject to }\begin{cases}\alpha_{k,c}'(t)+\lambda_k\alpha_{k,c}(t)=c(t)\,,\\ \alpha_{k,c}(0)=\alpha_{k,c}(T).\end{cases}$$

If we can show that

$$\forall k\in\mathbb{N}\,,\Lambda_k(c):=\int_0^T\alpha_{k,c}^2\leq\int_0^T\alpha_{k,c^\star}^2=\Lambda_k(c^\star)\tag{2.42}$$

then we are done. Let us show (2.42): introduce $\beta_{k,c}$ solution of

$$\begin{cases}-\beta_{k,c}'(t)+\lambda_k\beta_{k,c}(t)=\alpha_{k,c}(t)\,,\\ \beta_{k,c}(0)=\beta_{k,c}(T).\end{cases}$$

Then:

$$\int_0^T\alpha_{k,c}^2=\int_0^T\beta_{k,c}(t)c(t)dt.$$

Furthermore

$$-\beta_{k,c}''+\lambda_k^2\beta_{k,c}=\left(\frac{d}{dt}+\lambda_k\right)\left(-\frac{d}{dt}+\lambda_k\right)=c.$$

In other words, the Fourier coefficients of the solution satisfy an elliptic partial differential equation that has a nice variational formulation. Namely, introducing the energy functional

$$E_k(\cdot,c): W^{1,2}_{\mathrm{per}}((0;T))\ni\beta\mapsto\frac12\int_0^T|\beta'|^2+\lambda_k^2\int_0^T\beta^2-\int_0^T\beta c$$

where $W^{1,2}_{\mathrm{per}}((0;T))$ is the set of T-periodic Sobolev functions, the function $\beta_{k,c}$ is the unique minimiser of E_k in $W^{1,2}_{\mathrm{per}}((0;T))$. Furthermore, observe that for any $k\in\mathbb{N}$ there holds

$$E_k(\beta_{k,c},c)=-\int_0^T\beta_{k,c}c=-\Lambda_k(c).$$

Furthermore, from the Polyá-Szegö and Hardy-Littlewood inequalities (2.9)–(2.10), there holds

$$-\Lambda_k(c)=E_k\left(\beta_{k,c},c\right)\geq E_k\left(\beta^\sharp_{k,c},c^\sharp\right)\geq E_k\left(\beta_{k,c^\sharp},c^\sharp\right)=-\Lambda_k\left(c^\sharp\right).$$

Furthermore, using the equality cases in (2.9)–(2.10) we conclude that this inequality is strict unless, up to a translation, $c=c^\sharp$. This concludes the proof. ☐

Sketch of proof of Theorem 2.13. The idea is quite different. The bottom line is that, as $\varepsilon\to0$, we expect the function $u_{\varepsilon,c}$ to behave as a centred Gaussian around the maximiser $x^*=0$ of V. Up to a change of variable, we can further assume that $V(0)=0$. It is easy to show that

$$\lambda_\varepsilon(c)\underset{\varepsilon\to0}{\to}-V(0)\int_0^T c=-c_0V(0)=0;$$

Just as in the previous case, we need to go to the next order to obtain a non-trivial optimisation problem. We only present a formal derivation of the asymptotics. Consider the rescaled function $w_{\varepsilon,c}:(t,x)\mapsto u_{\varepsilon,c}(t,\sqrt{\varepsilon}x)$, where we choose the normalisation $u_{\varepsilon,c}(0,0)=1=\|u_{\varepsilon,c}\|_{L^\infty((0;T)\times\mathbb{T}^d)}$, which solves

$$\begin{cases}\partial_tw_{\varepsilon,c}-\Delta w_{\varepsilon,c}=\frac{\lambda_\varepsilon(c)w_{\varepsilon,c}}{\varepsilon}+\frac{cV(\sqrt{\varepsilon}\cdot)w_{\varepsilon,c}}{\varepsilon}\\ w_{\varepsilon,c}(0,\cdot)=w_{\varepsilon,c}(T,\cdot)\,,\\ w_{\varepsilon,c}(0,0)=1.\end{cases}\tag{2.43}$$

As $x^*=0$ is the unique maximiser of V and as $V(0)=0$ we have, locally uniformly in x,

$$V(\sqrt{\varepsilon}x)\underset{\varepsilon\to0}{\to}\frac12\langle\nabla^2V(0)x,x\rangle.$$

Defining $A:=-\nabla^2V(0)\in S_d^{++}(\mathbb{R})$ and using usual parabolic estimates (see also [**132**, Chapter 2]), we deduce that $(\lambda_\varepsilon(c),w_{\varepsilon,c})$ converges, locally uniformly, to a solution $(\overline{\lambda}(c),w_{0,c})$ of

$$\begin{cases}\partial_tw_{0,c}-\Delta w_{0,c}=\overline{\lambda}(c)w_{0,c}-\frac12c(t)\langle x,Ax\rangle w_{0,c}\quad\text{ in }[0,T]\times\mathbb{R}^d\,,\\ w_{0,c}(T,\cdot)=w_{0,c}(0,\cdot)\,,\\ w_{0,c}(0,0)=1\,,\\ 0\leq w_{0,c}\leq1.\end{cases}\tag{2.44}$$

As a consequence, we deduce that $\overline{\lambda}(c)$ is the principal eigenvalue for a problem set in all of $\mathbb{R}^d$ with a quadratic potential, and we similarly deduce that if, for any

$\varepsilon > 0$, c^*_ε is a solution of (2.36), then any weak $L^\infty - *$ closure point of $\{c^*_\varepsilon\}_{\varepsilon\to 0}$ is a solution of

$$\min_{c\in\mathcal{C}_T} \overline{\lambda}(c). \tag{2.45}$$

If we can show that $c^\star$ is the unique (up to a translation) solution of (2.45), we can conclude as before: as $c^\star$ is an extreme point of $\mathcal{C}_T$, it turns out that, up to a translation, $c^*_\varepsilon \underset{\varepsilon\to 0}{\to} c^\star$ strongly in any $L^p((0;T))$, $p \in [1;+\infty)$. Consequently, this means that it suffices to study (2.45) and to prove that for any $c \in \mathcal{C}_T$ there holds

$$\overline{\lambda}(c^\star) \leq \overline{\lambda}(c) \text{ with equality if, and only if, } c = c^\star. \tag{2.46}$$

To prove (2.46) assume, for the sake of simplicity, that $d = 1$ and that $A = \mathrm{Id}$. It turns out that (2.44) admits an explicit solution as

$$w_{0,c} : (t,x) \mapsto e^{\beta_c(t)-\gamma_c(t)\frac{x^2}{2}}$$

where (β_c, γ_c) solve

$$\begin{cases} \beta_c'(t) + \gamma_c(t) = \overline{\lambda}_c + c(t)\,, \\ \frac{\gamma_c'(t)}{2} + \gamma_c(t)^2 = \frac{c(t)}{2}, \\ \beta_c(T) = \beta_c(0)\,, \\ \gamma_c(T) = \gamma_c(0). \end{cases} \tag{2.47}$$

whence $\overline{\lambda}(c) = -c_0 + \frac{1}{T}\int_0^T \gamma_c$. Thus, (2.45) is equivalent to the minimisation problem

$$\min_{c\in\mathcal{C}_T} \int_0^T \gamma_c \text{ subject to } \begin{cases} \frac{\gamma_c'(t)}{2} + \gamma_c(t)^2 = \frac{c(t)}{2}, \\ \gamma_c(T) = \gamma_c(0). \end{cases} \tag{2.48}$$

The final step is to rewrite (2.48) as an elliptic spectral optimisation problem: introduce, for any $c \in \mathcal{C}_T$, the function $\varphi_c : t \mapsto e^{2\left(\int_0^t \gamma_c - \frac{t}{T}\int_0^T \gamma_c\right)} \in W^{1,2}_{\mathrm{per}}((0;T))$. Then it is readily checked that, with $\langle\gamma_c\rangle := \frac{1}{T}\int_0^T \gamma_c$, φ_c solves

$$\begin{cases} -\frac{d^2\varphi_c}{dt^2} - 4\langle\gamma_c\rangle\frac{d\varphi_c}{dt} = -4c\varphi_c + 4\langle\gamma_c\rangle^2\varphi_c \quad \text{ in } (0;T)\,, \\ \varphi_c(0) = \varphi_c(T)\,, \\ \varphi_c > 0. \end{cases} \tag{2.49}$$

Introduce the eigenvalue $\zeta(A,c)$ associated with the operator

$$-\frac{d^2}{dt^2} - 4A\frac{d}{dt} - 4A^2 + 4c,$$

endowed with periodic boundary conditions. As, with a bit more work, we can show that $\langle\gamma_c\rangle \geq 0$, $\langle\gamma_c\rangle$ is characterised as the (unique) positive root to

$$\zeta(\langle\gamma_c\rangle, c) = 0.$$

However, recall from (2.12) we know that

$$\zeta(\langle\gamma_c\rangle, c^\star) \leq \zeta(\langle\gamma_c\rangle, c) = 0 = \zeta(\langle\gamma_{c^\star}, c^\star),$$

and it is easy to see that $\mathbb{R}_+ \ni A \mapsto \zeta(A, c^\star)$ is decreasing, whence

$$\langle\gamma_{c^\star}\rangle \leq \langle\gamma_c\rangle.$$

Working a bit more to study the equality case provides the conclusion.

□

2.4. Research plan

We briefly describe here some research directions related to the problems of the chapter; as they are more prospective, the description is less detailed than in Chapter 1.

Further regularity in non-energetic problems

For this first research direction, we in fact refer to the conclusion of Chapter 1. Indeed, this first research line has to do with the geometric and regularity properties of optimal sets in general optimal control problems.

Other qualitative properties for spectral optimisation problems

A major mathematical challenge in optimal control problems for models coming from mathematical biology is the study of symmetry and monotonicity properties for general spectral optimisation problems. By "general", what we mean is that we do not simply investigate symmetric elliptic operators but, as was done in this chapter, either parabolic problems or problems with advection. In other words, we are interested in optimisation problems involving operators of the form

$$-\Delta + \langle X, \nabla \cdot \rangle - m$$

where X is a vector field that is not assumed to be divergence free. Observe that here we mostly describe the case of scalar eigenvalue problems (the case of system is of course very interesting, and we refer to Chapter 5). As we saw, monotonicity properties are of paramount importance when dealing with the underlying optimisation problems, and our research problem is thus the following:

(1) First, we plan on investigating the monotonicity properties and asymptotic behaviour of the principal eigenvalue of such operators with respect to its different parameters. In order to do so, the usual approach is to build appropriate sub and super-solutions [**135**, **149**, **166**, **165**, **167**]. It should be noted that the dynamical properties of the vector field X are expected to have a major impact on the behaviour of this eigenvalue–we refer in particular to the recent [**18**]. In a recent preprint [**MF25**] we proposed a different approach that only relies on the Hopf-Cole transform and only involves studying the invariant measure of the system. Some cases (typically, that of shear flows) are still elusive when considered from this perspective and we plan on developing this approach further.

(2) Second, we aim to study the minimisation of this eigenvalue with respect to the potential m, leaving the vector field X untouched. In this setting, although it is clear that the optimal potential m^* should, under appropriate constraints, be a bang-bang function $m^* = \mathbb{1}_{E^*}$, the emphasis will be placed on the geometric and regularity properties of the set E^*. Since we can not expect *a priori* symmetries of this optimal set E^* (as these should depend on the vector field X) we will be focusing on limit regimes (typically, the low volume asymptotics). The recent contributions of Ferreri, Mazzoleni, Verzini, Pellacci [**95**, **96**, **172**] in the symmetric case $X = 0$ will serve as our starting point.

CHAPTER 3

Quantitative estimates in optimal control problems

3.1. Introduction

In this chapter, we discuss the contributions [**Maz22b, MRB22a, Maz20**] which discuss quantitative inequalities in optimal control problems. More specifically, we will discuss the type of estimates we want to obtain and the general strategy of proof, before giving a list of selected examples and concluding the chapter with several applications of such inequalities to the convergence of numerical algorithms [**CMF24, CMFSP24**], to the optimal observability of partial differential equations [**MFPT25**] or to the turnpike property [**MRB22a**].

3.1.1. Main objective and initial discussion. Throughout, Ω refers to a bounded, smooth domain in $\mathbb{R}^d$. The expression "quantitative inequality", in the framework of optimal control, refers to the following: consider the admissible class

$$\mathcal{M} := \left\{ m \in L^\infty(\Omega) : 0 \leq m \leq 1 \,, \int_\Omega m = m_0 \right\}$$

for some volume constraint m_0 and an optimisation problem

$$\max_{m \in \mathcal{M}} J(m). \tag{3.1}$$

Here, the functional J can be the one discussed in the previous chapters. In order to fix ideas, let us use the following list of examples:

(1) $J(m) = -\lambda(m)$ where

$$\lambda(m) = \min_{u \in W_0^{1,2}(\Omega)\,, u \neq 0} \frac{\int_\Omega |\nabla u|^2 - \int_\Omega m u^2}{\int_\Omega u^2}. \tag{3.2}$$

Recall (see Remark 1.3) that this eigenvalue appears naturally in the context of mathematical biology.

(2)

$$J_{\text{ell}}(m) = \int_\Omega u_m^2 \text{ where } \begin{cases} -\Delta u_m = m & \text{in } \Omega\,, \\ u_m \in W_0^{1,2}(\Omega). \end{cases} \tag{3.3}$$

(3)

$$J_{\text{parab}}(m) = \iint_{(0;T)\times\Omega} y_m^2 \text{ where } \begin{cases} \partial_t y_m - \Delta y_m = m & \text{in } (0;T)\times\Omega\,, \\ y_m(t,\cdot) \in W_0^{1,2}(\Omega) & \text{in } (0;T)\,, \\ y_m = 0. \end{cases} \tag{3.4}$$

As will be detailed in Section 3.2.1, when $\Omega = \mathbb{B}(0;1)$, the last two examples are linked to quantitative Talenti inequalities.

In all of these examples, let $\mathcal{M}^*$ be the set of solutions to (3.1). Finding a quantitative inequality amounts to answering the following question: does there exist a norm $\|\cdot\|$, an exponent ν and a constant $C>0$ such that, letting m^* be any solution of (3.1),

$$\forall m \in \mathcal{M}\,, J(m) - J(m^*) \leq -C\mathrm{dist}_{\|\cdot\|}(m, \mathcal{M}^*)^\nu ? \tag{3.5}$$

In (3.5), the distance $\mathrm{dist}_{\|\cdot\|}$ is defined as

$$\mathrm{dist}_{\|\cdot\|}(f, \mathcal{M}^*) = \inf\{\|f - m^*\|\,, m^* \in \mathcal{M}^*\}.$$

There are multiple steps in proving such inequalities:

(1) First, determining the optimal norm and the optimal exponent ν.
(2) Second, even when these are correctly conjectured, finding a proof can be quite intricate.

At the very end of my PhD, I developed an approach for the minimisation of $m \mapsto \lambda(m)$ (see (3.2)), which we systematised with Ruiz-Balet [**MRB22a**] and we want to present in this chapter a general and flexible framework.

Why are such estimates important?

A legitimate question is: *why do we want such estimates?* There are several reasons:

(1) First, a purely mathematical one; it is indeed natural to try and obtain such inequalities, as they quantify the stability of solutions to optimal control problems.
(2) The second one is a more practical one: as we saw in Chapter 2, and in particular in Theorem 2.12, even when problems are *a priori* complicated (think of the total population size for instance), they can simplify in certain asymptotic regimes. For instance, a particular case of Theorem 2.12 is the following: as $\mu \to \infty$, the solutions of

$$\max_{m\in\mathcal{M}} J_\mu(m) := \int_\Omega \theta_{m,\mu} \text{ subject to } \begin{cases} -\mu\Delta\theta_{m,\mu} - \theta_{m,\mu}(m-\theta_{m,\mu}) = 0 & \text{in } \Omega\,, \\ \partial_\nu \theta_{m,\mu} = 0 & \text{on } \partial\Omega\,, \\ \theta_{m,\mu} \geq 0\,, \not\equiv 0 \end{cases} \tag{3.6}$$

converge to solutions of

$$\min_{m\in\mathcal{M}} J_\infty(m) := -\int_\Omega |\nabla\varphi_m|^2 \text{ subject to } \begin{cases} -\Delta\varphi_m = m_0(m-m_0) & \text{in } \Omega\,, \\ \partial_\nu\varphi_m = 0 & \text{on } \partial\Omega\,, \\ \int_\Omega \varphi_m = 0. \end{cases} \tag{3.7}$$

Now, Theorem 2.2 provides an expansion of the form

$$J_\mu(m) = m_0 - \frac{1}{m_0^2\mu} J_\infty(m) + \underset{\mu\to\infty}{O}\left(\frac{1}{\mu^2}\right). \tag{3.8}$$

Supposing we had an estimate of the form

$$J_\infty(m) - \min_{\mathcal{M}} J_\infty \geq C\mathrm{dist}_{\|\cdot\|}(m, \mathcal{M}^*)^\nu$$

where $\mathcal{M}^*$ is the set of minimisers of J_∞ and $C > 0$, this would actually give that, letting m^*_μ be any solution of (3.6), there holds

$$\mathrm{dist}_{\|\cdot\|}(m^*_\mu, \mathcal{M}^*) \leq \frac{C}{\mu^{\frac{1}{\nu}}}.$$

In other words, it allows to quantify the convergence rate to the limit problem. This will be, in particular, the type of reasoning used when dealing with optimal observability of partial differential equations, see Section 3.3.3.

(3) A final and crucial use of such inequalities has to do with the numerical simulations of such problems. Although we briefly touched upon that point in earlier chapters (see in particular the discussion of Chapter 1, Section 1.3.4), let us recall that numerical methods rely on a thresholding scheme. There are no convergence results for these methods, that are nevertheless used daily (apart from very specific cases where explicit computations are available, see [**141**] and the discussion of Section 3.3.2). Suffices to say, in this context, (sharp) quantitative inequalities play the role of stability estimates, and can be used to obtain convergence of the sequences generated by these algorithms.

Informal statement of our results

In this chapter, the main focus is going to be on the three problems (3.2)–(3.3)–(3.4), and we will be either showing, or identifying assumptions guaranteeing, quantitative inequalities of the following form: letting J denote the functional involved in (3.2)–(3.3)–(3.4), $\mathcal{M}^*$ denote the set of solutions of the underlying optimisation problem and m^* being any optimiser, we will prove

$$J(m) - J(m^*) \leq -C\mathrm{dist}_{L^1(\Omega)}(m, \mathcal{M}^*)^2. \tag{3.9}$$

We discuss this choice of norm and its sharpness in the next paragraph.

A foundational example: a linear control problem

As was the case in the previous chapter, this endeavour is at the crossroads of multiple old and recent developments in mathematical analysis–first and foremost, the study of quantitative geometric inequalities– that we will review in Section 3.1.2. Nevertheless, before we get into this, let us discuss a foundational example that will serve as justification for most of the conversation, particularly so for the choice of norm and of exponent. Namely, let $f \in L^\infty(\Omega)$ be a fixed reference function and consider the optimisation problem

$$\max_{m \in \mathcal{M}} T_f(m) := \int_\Omega fm. \tag{3.10}$$

This problem is very simple and has explicit solutions: namely, let $s \in \mathbb{R}$ be the unique real number such that

$$|\{f > s\}| \leq m_0\,, |\{f \geq s\}| \geq m_0.$$

Let $E_f := \{f > s\} \cup F$ where $F \subset \{f = s\}$ is such that $|E_f| = m_0$. Then $m_f := \mathbb{1}_{E_f}$ is a solution of (3.10). This result is known as either the bathtub principle, or the Hardy-Littlewood inequality. Of course, if we want to derive a quantitative inequality for this (linear) optimal control problem, we should at least require uniqueness of the optimiser, which in turn requires some stronger assumptions on f. For instance, if $f \equiv 1$, T_f is constant on $\mathcal{M}$. Although we could have sharper assumptions, let us assume

$$\forall s \in \mathbb{R}\,, |\{f = s\}| = 0. \tag{3.11}$$

Under this assumption, the map

$$m_f := \mathbb{1}_{\{f > s_0\}} \text{ where } |\{f > s_0\}| = m_0 \tag{3.12}$$

is the unique solution of (3.10), and we thus ask whether we have an estimate of the form

$$T_f(m) - T_f(m_f^*) \leq -C\|m - m_f\|^\nu \tag{3.13}$$

for some $C > 0$, some ν and some norm $\|\cdot\|$.

What we claim is the following: *the right norm is always the L^1 norm, but the exponent ν always depends on the regularity of f and on the values of its derivative on the set $\{f = s_0\}$*. To explain this, consider the one-dimensional case (this is very easily adaptable to the multi-dimensional setting) with $\Omega = (-1; 1)$. We will always be working under the assumption that f is decreasing, and with the volume constraint $m_0 = 1$, so that regardless of the specific choice of a decreasing f, the unique solution of (3.10) is always $m_f = m^* := \mathbb{1}_{(-1;0)}$. We refer to Fig. 1 below. We consider three cases, all represented on Fig. 1 below: one where f is discontinuous at 0, one where f is linearly decreasing at 0 and one where $f(x) - f(0) \underset{x \to 0}{\sim} -x^{2k+1}$ for $k \geq 1$ to illustrate the different optimal exponents we can expect.

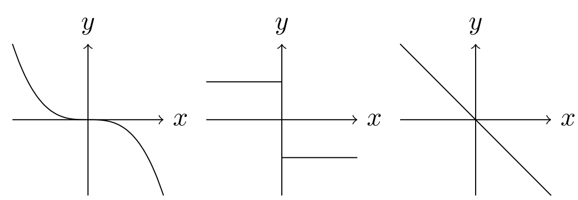


FIGURE 1. Illustration of the different behaviours of f at 0. Left: the case $f(x) \sim -x^{2k+1}$. Centre: the case of a discontinuous f. Right: the case $f'(0) \neq 0$.

The main approach to quantitative inequalities is the study of the small asymmetry regime: fix $\delta > 0$, and consider the penalised problem

$$\max_{m \in \mathcal{M}_\delta} \int_\Omega fm \text{ with } \mathcal{M}_\delta := \{m \in \mathcal{M} : \|m - m^*\|_{L^1} = \delta\}\,. \tag{3.14}$$

This problem also has an explicit solution m_δ^*, characterised as

$$m_\delta^* = \mathbb{1}_{\{f > s_\delta^+\}} + \mathbb{1}_{\{s_\delta^- < f < s_0\}}, \tag{3.15}$$

where $s_\delta^\pm$ are chosen so that

$$|\{s_0 < f < s_\delta^+\}| = |\{s_\delta^- < f < s_0\}| = \frac{\delta}{2}.$$

We refer to Fig. 2 below.

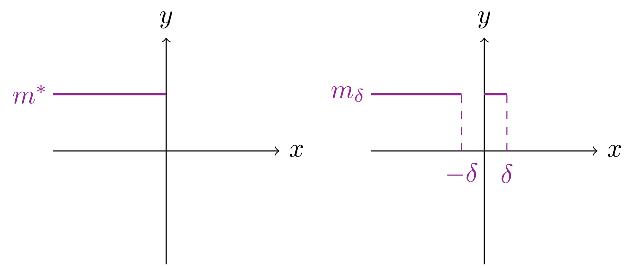


FIGURE 2. Left: graph of the optimiser m^*. Right: graph of the penalised optimiser m_δ.

This allows for an investigation of the optimal norms and exponents, by considering the quantity

$$\int_{-1}^{1} f(m_\delta^* - m^*).$$

As f is decreasing,

$$\{s_0 < f < s_\delta^+\} = \left(-\frac{\delta}{2}; m_0\right), \{s_\delta^- < f < s_0\} = \left(0; \frac{\delta}{2}\right)$$

and thus

$$\int_{-1}^{1} f(m_\delta^* - m^*) = -\int_{-\frac{\delta}{2}}^{0} f + \int_{0}^{\frac{\delta}{2}} f.$$

Consequently (see Figure 1):

(1) If f is discontinuous at 0, we deduce that

$$\int_{-1}^{1} f(m_\delta^* - m^*) \underset{\delta \to 0^+}{\sim} \delta\left(f(0^+) - f(0^-)\right)$$

and so we can expect that for any m

$$T_f(m) - T_f(m^*) \leq -C\|m - m^*\|_{L^1}.$$

(2) If f is $\mathscr{C}^{1,\alpha}$ at 0 and $f'(0) < 0$, we have

$$\begin{aligned}\int_{-1}^{1} f(m_\delta^* - m^*) &= -\int_{-\frac{\delta}{2}}^{0} f + \int_{0}^{\frac{\delta}{2}} f \\ &\approx -\int_{-\frac{\delta}{2}}^{0} (f(0) + f'(0)x)dx + \int_{0}^{\frac{\delta}{2}} (f(0) + f'(0)x)dx \\ &\approx \frac{f'(0)}{8}\delta^2\end{aligned}$$

and so we can expect that for any $m \in \mathcal{M}$,

$$T_f(m) - T_f(m^*) \leq -C\,|f'(0)|\,\|m - m^*\|_{L^1}^2. \tag{3.16}$$

(3) Finally, if we assume $f(x) \underset{x \to 0}{\sim} f(0) - \alpha x^{2k+1}$, we deduce from the same considerations that

$$\int_{-1}^{1} f(m_\delta^* - m^*) \underset{\delta \to 0^+}{\sim} -C\alpha\delta^{2(k+1)}$$

whence in that case the natural inequality is that, for any $m \in \mathcal{M}$ there holds

$$T_f(m) - T_f(m^*) \leq -C\|m - m^*\|_{L^1}^{2(k+1)}.$$

In the rest of this chapter we will show that, for the functionals under consideration, the estimate is analogous to (3.16), so that the natural and optimal quantitative inequality is with respect to the squared L^1 norm.

Plan of the chapter In the next paragraph, Section 3.1.2, we review parts of the vast literature devoted to quantitative inequalities in shape optimisation and in functional inequalities. In the following paragraph, Section 3.2 we will state several quantitative inequalities for the functionals we introduced at the beginning of the chapter. In particular, we will identify the structural assumptions necessary for the stability of (3.2) and for the aforementioned parabolic problem. We will then present a general strategy of proof, emphasising what we think the main points are. Finally, the last section of the chapter is devoted to applications of these quantitative inequalities to various problems–this is Section 3.3.

3.1.2. Related literature. Some quantitative geometric and functional estimates

The most important quantitative inequality is without a doubt the quantitative isoperimetric inequality, proved by Fusco, Maggi & Pratelli [**102**]. Following this core contribution, several geometric or functional inequalities (typically, the Faber-Krahn inequality or the Saint-Venant inequality [**36, 35**]) were given sharp quantitative versions. A typical way to quantify these inequalities is the use of the so-called Fraenkel asymmetry, defined, for a set Ω, as

$$A(\Omega) := \inf_{x \in \mathbb{R}^d} |\Omega \triangle \mathbb{B}(x; r)|$$

where $r > 0$ is such that $|\mathbb{B}(0; r)| = |\Omega|$ and $\triangle$ stands for the symmetric difference of sets. All the aforementioned inequality typically write: for any Ω such that $|\mathbb{B}(0; 1)| = |\Omega|$, there holds

$$F(\Omega) - F(\mathbb{B}(0; 1)) \geq CA(\Omega)^2$$

where $C > 0$, and both the exponent and the norm are sharp. The typical steps, in deriving these inequalities, are the following:

(1) First, establish the inequality for *smooth deformations* of the optimal domain Ω^*. In other words, establish it for domains Ω that write

$$\Omega = (\mathrm{Id} + \Phi)\Omega^*$$

for some smooth vector field Φ. This is done using shape derivatives and shape hessians. This step is quite technical and can not be bypassed. As a general reference for establishing such inequalities, we refer to the work of Dambrine & Lamboley [**87**], the methodology of which will also be foundational in the remainder of this chapter.

(2) Second, and this is usually a very intricate step, there needs to be a selection principle or a regularity step that aims at showing that one can reduce establishing the global quantitative inequality to the first step. A foundational example is the selection principle of Cicalese & Leonardi [**83**].

Some stability results in optimal control

The references of the previous paragraph deal with "true" shape optimisation problems (typically, the minimisation of the first Dirichlet eigenvalue), while the cases we have in mind come from optimal control problems; regarding these, there also exists a growing literature, and it is important to mention some of these contributions. The most important one to us is the work of Acerbi, Fusco & Morini [2], which essentially (we refer to [2] for the precise statement) deals with a problem of the form

$$\min_{m\in\mathcal{M}}\left(\|m\|_{BV(\Omega)}-\int_\Omega|\nabla u_m|^2\right)\text{ subject to }\begin{cases}-\Delta u_m=m-u_m\,,\\ \int_\Omega u_m=0.\end{cases}$$

What they prove is that the stability in the sense of shape perturbations (*i.e.* the positivity of the second-order shape derivative) implies a local quantitative inequality. The main difference with the case we have in mind here is the presence of the perimeter term (the BV norm of m); for technical reasons, in that case, the positivity of the shape hessian implies its coercivity in the right norm (see later for a discussion of this aspect), and it helps a lot when dealing with the selection principle as it provides a good amount of regularity for penalised problems. The situation under consideration in this chapter (*i.e.* with volume constraints) is quite different.

A second line of research that needs to be mentioned deals with stability in optimal control problems, but with different sets of constraints. To be more specific, the works of Brasco & Buttazzo [35] on the one hand and of Carlen, Frank & Lieb [68] on the other, deal with optimal control problems under L^p constraints. [35] investigates

$$\max_{V\in L^p(\Omega)\,,\int_\Omega|V|^p\leq 1}\left(E(V):=\min_{u\in W^{1,2}_0(\Omega)}\int_\Omega|\nabla u|^2+\int_\Omega Vu^2-2\int_\Omega fu\right),$$

f being a fixed source term, and they establish an inequality of the form

$$E(V)-E(V^*)\leq -C\|V-V^*\|^2_{L^p(\Omega)} \tag{3.17}$$

where V^* is the unique maximiser (here $p>2$). On the other hand, in [68], the problem is roughly speaking the following: letting $\Lambda(V)$ denote

$$\Lambda(V):=\inf_{\psi\in W^{1,2}(\mathbb{R}^d)\,,\int_{\mathbb{R}^d}\psi^2=1}\int_{\mathbb{R}^d}|\nabla\psi|^2+\int_{\mathbb{R}^d}V\psi^2$$

and letting $\mathcal{V}$ denote a solution of the problem

$$\max_{V\,,\|V_-\|_{L^p(\mathbb{R}^d)}=1}|\Lambda(V)|$$

then one has

$$\forall V\in L^p(\mathbb{R}^d)\,,|\Lambda(V)|\leq|\Lambda(\mathcal{V})|-C\inf_{a\in\mathbb{R}^d\,,b\in\mathbb{R}}\|V_--(b^2\mathcal{V}(b(\cdot-a))_-\|^2_{L^p(\mathbb{R}^d)}. \tag{3.18}$$

Likewise, this can be understood as a sort of weighted asymmetry, in the spirit of what was done for the stability of optimal Sobolev embeddings. At any rate, beyond their natural relationship with the problems at hand here, we want to emphasise that both (3.17) and (3.18) rely on quantitative Hölder inequalities, and that the aforementioned quantitative bathtub principle (3.16) can be interpreted as a quantitative L^1-L^∞ Hölder inequality (we also refer to quantitative versions of the Hardy-Littlewood inequality [82] that appear in a variety of guises in stability analysis of optimal control [70, 72, 71] but these will be discussed in due time).

In the conclusion of this chapter we will also review recent developments in quantitative Talenti inequalities, as they tie into our research plan.

3.2. Main results and general strategy of proof

In our opinion, the main contribution of [**CMF24**, **Maz22b**, **MRB22a**, **Maz20**] is the progressive development of a systematic approach to volume constrained optimal control problems; while we first detail specific examples, the core contribution is actually the section 3.2.2 which details the strategy of proof.

3.2.1. Main results: list of examples of quantitative inequalities. Optimisation of a Schrödinger eigenvalue in the ball

We begin by recalling a result obtained at the end of my PhD. We begin with a problem set in the ball $\Omega = \mathbb{B}(0;1)$, with $m_0 \in (0;|\mathbb{B}(0;1)|)$ and $\mathcal{M} = \{m \in L^\infty(\mathbb{B}) : 0 \leq m \leq 1\,, \fint_{\mathbb{B}} m = m_0\}$:

$$\min_{m\in\mathcal{M}} \left(\lambda(m) := \min_{u\in W^{1,2}_0(\mathbb{B})\,,\fint_{\mathbb{B}} u^2=1} \int_{\mathbb{B}} |\nabla u|^2 - \int_{\mathbb{B}} mu^2\right). \tag{3.19}$$

An easy rearrangement argument shows that the unique solution of this optimisation problem is

$$m^* := \mathbb{1}_{\mathbb{B}^*} \tag{3.20}$$

where $\mathbb{B}^*$ is a centred ball of volume m_0.

THEOREM 3.1. [**Maz20**] *In the two-dimensional case, there exists a constant $C > 0$ such that*

$$\forall m \in \mathcal{M}\,, \lambda(m) - \lambda(m^*) \geq C\|m - m^*\|^2_{L^1(\mathbb{B})}.$$

REMARK 3.2. The two-dimensional restriction is not needed and is technical: at the time [**Maz20**] was published, the way Theorem 3.1 was proved relied on an explicit diagonalisation of the shape Hessian; since then, in particular thanks to [**CMF24**], we understood how to bypass such computations and we claim Theorem 3.1 holds in higher dimensions.

A quantitative parabolic isoperimetric inequality in the ball

Shortly after Theorem 3.1 we investigated a parabolic problem. Namely, $\Omega = \mathbb{B}(0;1) = \mathbb{B}$ is still the unit ball, and we fix a time horizon $T > 0$. Now, for any $m \in L^\infty((0;T) \times \mathbb{B})$, let u_m be the unique solution of

$$\begin{cases} \partial_t u_m - \Delta u_m = m & \text{in } (0;T) \times \mathbb{B}\,, \\ u_m(t,\cdot) \in W^{1,2}_0(\mathbb{B}) & \text{in } (0;T)\,, \\ u_m(0,\cdot) = 0. \end{cases} \tag{3.21}$$

Finally, consider the two admissible classes

$$\mathcal{M} := \left\{m \in L^\infty(\mathbb{B}) : 0 \leq m \leq 1\,, \int_{\mathbb{B}} m = m_0\right\} \tag{3.22}$$

and

$$\mathcal{M}_T := \left\{m \in L^\infty((0;T) \times \mathbb{B}) : 0 \leq m \leq 1\,, \int_{\mathbb{B}} m(t,\cdot) = m_0 \text{ for a.e. } t\right\}. \tag{3.23}$$

REMARK 3.3. The main difference between $\mathcal{M}$ and $\mathcal{M}_T$ is that for $m \in \mathcal{M}_T$ the total mass of $m(t,\cdot)$ is fixed for every t, while for $m \in \mathcal{M}$ the L^1 norm of $m(t,\cdot)$ might vary in time.

Finally, consider the functional

$$J_{\text{parab}}(m) := \iint_{(0;T)\times\mathbb{B}} u_m^2$$

and the two optimisation problems

$$\max_{m\in\mathcal{M}} J_{\text{parab}}(m), \tag{3.24}$$

$$\max_{m\in\mathcal{M}_T} J_{\text{parab}}(m). \tag{3.25}$$

Let $m^* := \mathbb{1}_{\mathbb{B}^*}$ where $\mathbb{B}^*$ is a centred ball of volume m_0 and $m_T^* := \mathbb{1}_{\mathbb{B}^*}$ (we add a subscript to emphasise the fact we are dealing with possibly time-dependent controls). By standard parabolic Talenti inequalities [**218**], coupled with uniqueness arguments inspired by Rakotoson & Mossino [**179**] and Kesavan [**144**], we can show that m^*, resp. m_T^*, is the unique solution of (3.24), resp. (3.25). The main result is the following:

THEOREM 3.4. [**Maz22b**]

(1) There exists a constant $C_T > 0$ such that

$$\forall m \in \mathcal{M}\,, J_{\text{parab}}(m) - J_{\text{parab}}(m^*) \leq -C_T\|m-m^*\|^2_{L^1(\mathbb{B})}. \tag{3.26}$$

(2) There exists a continuous function $\omega : [0;T] \to [0;+\infty)$ such that

$$\forall s \in [0;T)\,, \omega(s) > 0\,, \omega(s) \underset{s\to T}{\to} 0$$

and

$$\forall m \in \mathcal{M}_T\,, J_{\text{parab}}(m) - J_{\text{parab}}(m_T^*) \leq -\int_0^T \omega(s)\|m(s,\cdot)-m_T^*(s,\cdot)\|^2_{L^1(\mathbb{B})}.$$

This theorem can be interpreted as a quantitative parabolic Talenti inequality. We refer to the conclusion of the chapter for a more detailed discussion of the research that still needs to be carried out on this theme.

General assumptions for stability, and some situations where they are fulfilled

While developing the two previous results, Ruiz-Balet and myself identified the structural assumption required to obtain stability in spectral optimisation problems of the form

$$\min_{m\in\mathcal{M}}\left(\lambda(m) = \min_{u\in W_0^{1,2}(\Omega)\,,\int_\Omega u^2=1}\int_\Omega|\nabla u|^2 - \int_\Omega u^2\right) \tag{3.27}$$

in general domains, not merely in the ball. These assumptions, the use of which will be clear in Section 3.2.2, are the following:

(1) First, letting $\mathcal{M}^*$ be the set of minimisers, we know that each $m^* \in \mathcal{M}^*$ writes $m^* = \mathbb{1}_{E^*}$ (such an E^* is called an optimal set), and we assume that

$$\text{Any such } E^* \text{ is } \mathscr{C}^{2,\alpha} \text{ for some large enough } \alpha.$$

(2) Second, we assume that there exists a constant $\delta > 0$ such that, letting u_m denote the eigenfunction associated with $\lambda(m)$, for any $m^* = \mathbb{1}_{E^*} \in \mathcal{M}^*$,
$$\inf_{\partial E^*} |\nabla u_{m^*}| \geq \delta > 0.$$
(3) Finally, we assume the following minimality in the sense of shapes: for any optimal set E^*, for any smooth, compactly supported vector field $\Phi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^d)$ such that
$$\int_{E^*} \nabla \cdot \Phi = 0$$
there holds
$$\frac{d^2}{dt^2}\bigg|_{t=0} \left(t \mapsto \lambda\left(\mathbb{1}_{(\mathrm{Id}+t\Phi)E^*}\right)\right) > 0.$$

We obtained the following result:

THEOREM 3.5. [**MRB22a**] *Under the assumptions above, there exists a constant $C > 0$ such that for any $m \in \mathcal{M}$,*
$$\lambda(m) - \min_{\mathcal{M}} \lambda \geq C \mathrm{dist}_{L^1}(m, \mathcal{M}^*)^2.$$

REMARK 3.6. It should be noted that the *a priori* regularity of optimal shapes is a delicate matter and that we do not know if it is in general true in dimension $d \geq 3$. We refer for more detail to the conclusion of Chapter 1.

In collaboration with Chambolle & Privat, we went back to such questions [**CMF24, CMFSP24**], and this allowed us to show quantitative inequalities for a variety of optimal control problems under large volume constraints. More specifically we considered the optimisation problem (3.27) as well as the following general convex optimisation problem:

$$\max_{m \in \mathcal{M}} \int_\Omega j(u_m) \text{ subject to } \begin{cases} -\Delta u_m = m & \text{in } \Omega\,, \\ u_m \in W_0^{1,2}(\Omega). \end{cases} \tag{3.28}$$

One of the results of [**CMF24, CMFSP24**] is the following:

THEOREM 3.7. *(1) For any $m_0 \in (0;|\Omega|)$ let $\mathcal{M}^*$ denote the set of optimisers of (3.27). There exists $\overline{m}_0 \in (0;|\Omega|)$ such that for any $m_0 \geq \overline{m}_0$, there exists $C > 0$ such that*
$$\forall m \in \mathcal{M}\,, \lambda(m) - \min_{m \in \mathcal{M}} \lambda \geq C \mathrm{dist}_{L^1}(m, \mathcal{M}^*)^2.$$
(2) Assume that j satisfies $j'' > 0$ in $(0;+\infty)$. For any $m_0 \in (0;|\Omega|)$ let $\mathcal{I}^*$ denote the set of optimisers of (3.28). There exists $\overline{m}_0 \in (0;|\Omega|)$ such that for any $m_0 \geq \overline{m}_0$, there exists $C > 0$ such that
$$\forall m \in \mathcal{M}\,, J(m) - \max_{m \in \mathcal{M}} J \leq -C \mathrm{dist}_{L^1}(m, \mathcal{I}^*)^2.$$

As we mentioned, the main objective of [**CMF24, CMFSP24**] was to establish the convergence of thresholding schemes, and this relies on such quantitative inequalities. We refer to Section 3.3.

In the remainder of this section, we present the general strategy of proof and we comment on specific problems when we feel it is necessary.

3.2.2. Global strategy of proof.

3.2.2.1. *Preliminary comments & strategy.* We now present the general strategy of proof.

The role of the bang-bang property

We begin with a preliminary comment: the common feature of the optimisation problems we consider, that is, (3.24)–(3.25)–(3.27)–(3.28) is that the bang-bang property is satisfied: any optimiser m^* to any of this problem–whether we can characterise it or not–is a characteristic function. This is a necessary condition to have a quantitative inequality; indeed, consider any functional $F:\mathcal{M}\to\mathbb{R}$ that is continuous for the weak $L^\infty-*$ topology, and which is minimised at some m^*. Assume that m^* is not bang-bang. Introduce the set $\omega^*:=\{\varepsilon<m^*<1-\varepsilon\}$ which, for some $\varepsilon>0$ small enough, has positive measure and consider any sequence $\{h_k\}_{k\in\mathbb{N}}\in L^\infty(\Omega)^{\mathbb{N}}$ such that $\|h_k\|_{L^\infty(\Omega)}\leq\varepsilon\,,\|h_k\|_{L^1(\Omega)}=\frac{\varepsilon}{2}$ and, for any $k\in\mathbb{N}$, h_k is supported in ω^* and $\int_\Omega h_k=0$. Finally, build the sequence in such a way that $h_k \underset{k\to\infty}{\rightharpoonup} 0$ in the weak $L^\infty-*$ sense, and define for any $k\in\mathbb{N}$ $m_k:=m^*+h_k$. Then

$$F(m_k)\underset{k\to\infty}{\to}F(m^*)\,,\|m_k-m^*\|_{L^1(\Omega)}=\frac{\varepsilon}{2}$$

and J can hence not admit a local quantitative inequality (neither in the L^1 nor, for that matters, in any L^p norm).

The second comment, which is in fact related but not equivalent to the first one, is that each of the functionals we consider is convex (in the case of maximisation problems) or concave (in the case of minimisation problems).

The general setting

We only discuss the case of elliptic problems; the case of parabolic problems is structurally similar, although one has to be delicate when constructing the weight ω. To fix ideas, we thus consider

$$\max_{m\in\mathcal{M}} J(m) \tag{3.29}$$

where J is defined as in (3.27) ($J=-\lambda$) or as in (3.28). We also assume for the sake of simplicity that J has a unique maximiser m^*:

$$\forall m\in\mathcal{M}\,,m\neq m^*\,,J(m)<J(m^*).$$

The role of the adjoint state–first-order optimality conditions

Furthermore, the notion of adjoint state, already encountered in Chapter 1, plays a crucial role. Let J denote any of the functionals involved in (3.24)–(3.25)–(3.27)–(3.28). It is fairly easy to show that the map $m\mapsto J(m)$ is Gateaux differentiable: for any $m\in\mathcal{M}$, for any perturbation h, one can define the quantity

$$\dot{J}(m)[h]:=\lim_{t\to0}\frac{J(m+th)-J(m)}{t},$$

and the linear form $\dot{J}(m)[\cdot]$ is continuous for some L^p topology (p might depend *a priori* depend on the problem); by the Riesz representation theorem, there exists $p_m\in L^{p'}(\Omega)$ such that

$$\forall h\,,\dot{J}(m)[h]=\int_\Omega p_m h.$$

The function p_m is called the *switch function*.

Examples 3.8 (Some examples of switch functions). We give some examples of switch functions:

(1) For the problem (3.2), there holds

$$\dot{\lambda}(m)[h] = -\int_\Omega h u_m^2 \text{ where } \begin{cases} -\Delta u_m = \lambda(m)u_m + m u_m & \text{in } \Omega\,, \\ u_m \in W_0^{1,2}(\Omega)\,, \\ u_m \geq 0\,, \int_\Omega u_m^2 = 1. \end{cases}$$

(2) For (3.28), there holds

$$\dot{J}(m)[h] = \int_\Omega j'(u_m)\dot{u}_m \text{ with } \begin{cases} -\Delta \dot{u}_m = h & \text{in } \Omega\,, \\ \dot{u}_m \in W_0^{1,2}(\Omega). \end{cases}$$

Thus, introducing the solution q_m to

$$\begin{cases} -\Delta q_m = j'(u_m) & \text{in } \Omega\,, \\ q_m \in W_0^{1,2}(\Omega) \end{cases}$$

we obtain

$$\dot{J}(m)[h] = \int_\Omega q_m h.$$

Consequently, the optimality conditions for the problem

$$\max_{m\in\mathcal{M}} J(m),$$

if the functional J has p_m for switch function, is that if m^* is an optimiser then for any $m \in \mathcal{M}(\Omega)$

$$\int_\Omega p_{m^*}(m - m^*) \leq 0.$$

Combined with the bang-bang property, this implies that any optimiser m^*can be written as a level-set of p_{m^*}: there exists c^* such that

$$m^* = \mathbb{1}_{\{p_{m^*} > c^*\}}. \tag{3.30}$$

Blueprints to obtain a quantitative inequality

To obtain a quantitative inequality, we always argue by contradiction, which allows to reduce to the case of small asymmetries. Using this argument, we can simply show that it suffices to obtain (in the elliptic case)

$$\lim_{\delta\to 0} \sup_{m\in\mathcal{M}\,, \|m-m^*\|_{L^1(\Omega)}=\delta} \frac{J(m) - J(m^*)}{\delta^2} < 0. \tag{3.31}$$

In other words, it suffices to have a local quantitative inequality. Now, we essentially consider two ways to be close, in the L^1 norm, to an optimiser $m^* = \mathbb{1}_{E^*}$:

(1) A first way is to have a change in topology: for instance, if $E^* = \mathbb{B}^* = \mathbb{B}(0; r_0)$, then we can have a competitor m defined as a

$$m = \mathbb{1}_{\mathbb{A}_\delta \cup \mathbb{B}_\delta}$$

where

$$\mathbb{B}_\delta = \mathbb{B}(0; r_\delta^+)\,, r_\delta^+ < r_0$$

and $\mathbb{A}_\delta$ is an annulus of the form

$$\mathbb{A}_\delta = \mathbb{B}(0; r_\delta^-) \setminus \mathbb{B}^*\,, r_\delta^- > r_0.$$

We call such competitors parametric perturbations of m^*.

(2) A second possibility is to have a smooth deformation of E^* or, in other words, to have a competitor of the form

$$m_\delta = \mathbb{1}_{(\mathrm{Id}+\Phi_\delta)E^*}$$

for some smooth, compactly supported vector field Φ_δ. We call such competitors shape competitors.

The message of our articles is that controlling these two (loosely defined at this stage) type of deformations allows to conclude.

The general strategy is the following one:

(1) Step 1: Reduce to the small asymmetry regime and argue by contradiction.
(2) Step 2: Consider a competitor m_δ such that $\|m_\delta - m^*\|_{L^1(\Omega)} = \delta$ and define

$$m'_\delta := \mathbb{1}_{\{p_{m_\delta} > c_\delta\}}$$

where p_{m_δ} is the adjoint state computed at $m = m_\delta$ and c_δ is chosen to ensure that $m'_\delta \in \mathcal{M}$. Alternatively, m'_δ can be defined as the unique solution to

$$\max_{m\in\mathcal{M}} \int_\Omega p_{m_\delta} m.$$

By convexity of the functional J, we have

$$J(m'_\delta) - J(m_\delta) \geq \dot{J}(m_\delta)[m'_\delta - m_\delta] = \int_\Omega p_{m_\delta}(m'_\delta - m_\delta).$$

This is where the quantitative bathtub principle comes into play: we can show that, under suitable regularity assumptions,

$$\forall m \in \mathcal{M}\,, \int_\Omega p_{m_\delta}(m'_\delta - m) \geq C\|m'_\delta - m\|^2_{L^1(\Omega)} \tag{3.32}$$

for some constant independent of $\delta > 0$ small enough. Note that we can write $m'_\delta = \mathbb{1}_{E_\delta}$ with $E_\delta = \{p_{m_\delta} > c_\delta\}$.
(3) Step 3: As $\|m_\delta - m^*\| \ll 1$, if $|\nabla p_{m^*}| > 0$ on $\partial E^* = \{p_{m^*} = c^*\}$, we can expect that $\partial E_\delta := \{p_{m_\delta} = c_\delta\}$ is a smooth deformation of ∂E^*, and to in fact have

$$E_\delta = \{p_{m_\delta} > c_\delta\} = (\mathrm{Id} + \Phi_\delta)\, E^*.$$

This is a consequence of elliptic regularity–it should be noted that our approach to this step was revisited in the recent [1]. Thus in this step, we need to show that for any smooth enough vector field Φ satisfying $\|\Phi\|_{W^{2,p}(\Omega)} \ll 1$ (the exact regularity needed is not discussed here) there holds

$$J(m^*) - J\left(\mathbb{1}_{(\mathrm{Id}+\Phi)(E^*)}\right) \geq C\left|E^* \triangle (\mathrm{Id}+\Phi)E^*\right|^2. \tag{3.33}$$

(4) Step 4: Finally, we thus deduce that

$$\begin{aligned}
J(m_\delta) - J(m^*) &\leq J(m'_\delta) - J(m^*) - C\|m'_\delta - m_\delta\|^2_{L^1(\Omega)} \text{ by (3.32)} \\
&\leq -C\left|E_\delta \triangle E^*\right|^2 - C\|m'_\delta - m_\delta\|^2_{L^1(\Omega)} && \text{by (3.33)} \\
&\leq -C'\left(\|m'_\delta - m_\delta\|_{L^1(\Omega)} + \|m'_\delta - m^*\|_{L^1(\Omega)}\right)^2 \\
&\leq -C\|m_\delta - m^*\|^2_{L^1(\Omega)}.
\end{aligned}$$

We represent these different steps in the following figures, taken from [MRB22a].

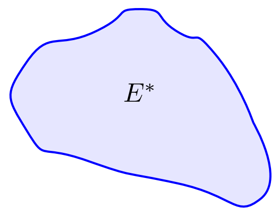


Figure 3. Depiction of the optimal set E^*.

For $\delta > 0$ small enough, we consider the solution m_δ of

$$\max_{m \in \mathcal{M}, \|m - m^*\|_{L^1(\Omega)} = \delta} J(m),$$

which we can expect to be the characteristic function of a set E_δ $(m_\delta = \mathbb{1}_{E_\delta})$, that will be very close to E^*:

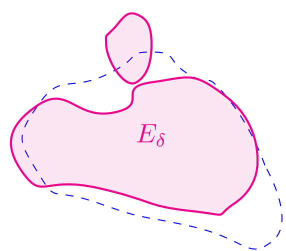


Figure 4. Depiction of the solution E_δ of the penalised problem; E^* is depicted in dashed blue.

We then use the quantitative bathtub principle: $c_\delta > 0$ be such that the level set of the eigenfunction p_{m_δ} associated with m_δ satisfies $|\{p_{m_\delta} > c\delta\}| = m_0$. We define $\tilde{E}_\delta := \{p_{m_\delta} > c_\delta\}$:

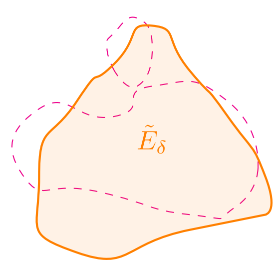


Figure 5. Level set $\tilde{E}_\delta$ of the eigenfunction p_{m_δ} associated with m_δ and satisfying the volume constraint; E_δ is depicted in dashed.

When $\delta > 0$ is small enough, $\tilde{E}_\delta$ should be a normal deformation of E^*:

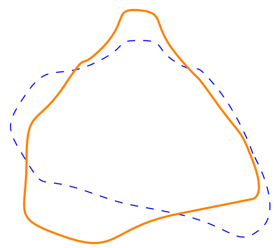


FIGURE 6. Comparison of $\tilde{E}_\delta$ (in orange) and of E^* (in dashed blue).

This allows to conclude.

In the last sections of this paragraph, we briefly discuss the approaches and proof of Steps 2 and, more particularly, 3, as they are the most novel in the approaches we develop.

3.2.2.2. *The quantitative bathtub principle.* We first investigate the quantitative bathtub principle (3.32); in our previous works [**CMF24, Maz22b, MRB22a, Maz20**] we proved it using essentially the same proof as in the one-dimensional example given at the beginning of the chapter, see (3.16). The essential assumptions used in our approach are:

(1) The fact that ∂E^* has finite perimeter.
(2) The fact that $|\nabla p_{m^*}| \neq 0$ on ∂E^*.

These two assumptions allow us to reduce to the radially symmetric case $E^* = \mathbb{B}^*$, replacing p_{m^*} with its Schwarz rearrangement. We want to mention that there are different approaches that we were unaware of when first investigating this quantitative inequality:

(1) First, it is possible to see (3.32) as a quantitative Hardy-Littlewood inequality; for which results by Cianchi & Ferone [**82**] on the one hand, and by Lemou [**159**] on the other provide quantitative estimates. Although, upon first inspection of [**82**], their results do not encompass our case, we can adapt their approach (see in particular [**82**, Remark 1.4]) to provide the same result.
(2) We also want to mention that (3.32) appears in a completely different context in the work of Casas, Wachsmuth & Wachsmuth [**72**, Proposition 2.7], where it is shown to hold under the following assumption:

$$\lim_{\varepsilon\to 0}\frac{|\{c^*-\varepsilon<p_{m^*}<c^*+\varepsilon\}|}{\varepsilon}<\infty,$$

and where the quantitative estimate is obtained by completely different techniques.

3.2.2.3. *Diagonalisation and coercivity of shape hessians.* We now move to the discussion of (3.33). Relying on the general structure presented in the work of Dambrine & Lamboley [**87**], we can show that this so-called “shape stability” follows from the *coercivity of the second-order shape derivative*. To be more specific, we can define a shape functional as follows:

$$\mathscr{J} : E \mapsto J\left(\mathbb{1}_E\right).$$

Denoting

$$\mathscr{M} := \{E \subset \Omega\,, |E| = M_0\}$$

we deduce from the bang-bang property that

$$\mathscr{J}(E^*) = \min_{E\in\mathscr{M}} \mathscr{J}(E).$$

In order to discuss second-order optimality conditions, we need to introduce the Lagrangian associated with the volume constraint $|E| = m_0$. To properly introduce it, define the notion of shape derivative as follows: for any $\Phi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^d)$, define (provided it exists)

$$\mathscr{J}'(E)[\Phi] := \lim_{t\to 0} \frac{\mathscr{J}((\mathrm{Id}+t\Phi)E^*) - \mathscr{J}(E^*)}{t}.$$

The second-order derivative $\mathscr{J}''$ is defined analogously. The differentiability is usually guaranteed by some form of the implicit function theorem [**175**] but, most importantly, the first Hadamard structure theorem [**129**, Chapter 5] ensures that there exists a function p_E such that (here we assume some regularity on E)

$$\mathscr{J}'(E)[\Phi] = \int_{\partial E} p_E \langle \Phi, \nu\rangle$$

where ν is the outwards normal vector in ∂E^*. We choose the notation p_E or $p_{\mathbb{1}_E}$ indifferently as it turns out that this function p_m is exactly the adjoint state defined in the context of the previous paragraph. At any rate, the first-order optimality conditions for these variations is also

$$p_{E^*} \equiv c^* \text{ on } \partial E^*,$$

and this leads to defining the Lagrangian

$$\mathscr{L} : E \mapsto \mathscr{J}(E) + c^*|E|.$$

The derivative of the volume functional being

$$|\cdot|'(E)[\Phi] = \int_{\partial E} \langle \Phi, \nu\rangle,$$

the second order optimality condition reads

$$\forall \Phi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^d)\,, \int_{\partial E^*} \langle \Phi, \nu\rangle = 0 \Rightarrow \mathscr{L}''(E^*)[\Phi,\Phi] \leq 0. \tag{3.34}$$

The main hope is to prove that if

$$\forall \Phi \in \mathscr{C}_c^\infty(\Omega;\mathbb{R}^d)\,, \int_{\partial E^*} \langle \Phi, \nu\rangle = 0 \Rightarrow \mathscr{L}''(E^*)[\Phi,\Phi] < 0 \tag{3.35}$$

then, if Φ is small enough (say, in a $W^{2,p}$-norm), then

$$\mathscr{J}((\mathrm{Id}+\Phi)E^*) - \mathscr{J}(E) \leq -C\left|E^* \triangle (\mathrm{Id}+\Phi)E^*\right|^2.$$

The main difficulty here is the well-known two-norms discrepancy phenomenon, meaning that the norm required for the differentiability (say, $W^{2,p}$) is strongly stronger than the best coercivity norm for $\mathscr{L}''(E^*)$. Typically, in general PDE-constrained shape optimisation problems, consider the first Dirichlet eigenvalue $\mu(\Omega)$ of a domain Ω, which is minimised at the ball $\mathbb{B}$ (under a volume constraint). Then one has, $\mathscr{R}$ being the associated Lagrangian

$$\mathscr{R}''(\mathbb{B})[\Phi,\Phi] \geq C\|\langle \Phi, \nu\rangle\|^2_{W^{\frac12,2}(\partial\mathbb{B})}.$$

In fact, the main difficulty comes from *identifying the optimal coercivity for the second derivative of the Lagrangian* and one of core contributions [**Maz20**] is that the optimal coercivity norm is often the L^2 one: in several cases, there exist two constant $\alpha_0\,,\alpha_1 > 0$ such that, for the Lagrangian associated with the optimal control problem, E^* being an optimiser, for any Φ such that $\int_{\partial E^*}\langle \Phi,\nu\rangle = 0$,

$$\alpha_0\|\langle \Phi,\nu\rangle\|^2_{L^2(\partial E^*)} \leq \mathscr{L}''(E^*)[\Phi,\Phi] \leq \alpha_1\|\langle \Phi,\nu\rangle\|^2_{L^2(\partial E^*)}.$$

Now, before we proceed with explicit examples to make these considerations more grounded, let us observe that, traditionally, the coercivity of the shape Hessian (the second-order derivative) is obtained at specific geometries, typically the ball, where explicit diagonalisation basis are available. This makes several computations quite involved, and can heavily rely on fine properties of Bessel functions, see [**87**]. In our contributions, we proposed a new diagonalisation approach that lightens these computations.

The case of Schrödinger eigenvalues

When dealing with (3.27) the function is $J = -\lambda$ and the switch function is $-u_m^2$. Consequently, we deduce that the first-order optimality condition is

$$u_{m^*} \text{ is constant equal to some } c^* \text{ on } \partial E^*, \text{ with } m^* = \mathbb{1}_{E^*}.$$

Thus,

$$\mathscr{L}'(E)[\Phi] = \int_{\partial E}(u^2_{\mathbb{1}_E} - c^*)\langle \Phi,\nu\rangle.$$

After standard computations, we obtain, for the shape Hessian, the expression

$$\mathscr{L}''(E^*)[\Phi,\Phi] = 2\int_{\partial E^*} u_{m^*}u'_{m^*}\langle \Phi,\nu\rangle + 2\int_{\partial E^*} u_{m^*}\frac{\partial u_{m^*}}{\partial \nu}\langle \Phi,\nu\rangle^2.$$

In this expression, u'_{m^*} is the shape derivative of $E \mapsto u_{\mathbb{1}_E}$ at E^* in the direction Φ, which solves, in a weak form

$$\tag{3.36} \begin{cases} -\Delta u'_{m^*} = \lambda(m^*)u'_{m^*} + m^* u'_{m^*} & \text{in } \Omega\,, \\ [\![\partial_\nu u'_{m^*}]\!] = -u_{m^*}\langle \Phi,\nu\rangle & \text{on } \partial E^*\,, \\ u'_{m^*} \in W^{1,2}_0(\Omega)\,, \\ \int_\Omega u_{m^*}u'_{m^*} = 0, \end{cases}$$

where $[\![f]\!]$ denotes the jump of a function across a hypersurface. The weak form of (3.36) is the following:

$$\forall v \in W^{1,2}_0(\Omega)\,, \int_\Omega \langle \nabla u'_{m^*}, \nabla v\rangle = \lambda(m^*)\int_\Omega u'_{m^*}v + \int_{E^*} u'_{m^*}v + \int_{\partial E^*} u_{m^*}v\langle \Phi,\nu\rangle.$$

This equation comes from the differentiation of the weak formulation

$$\forall v \in W^{1,2}_0(\Omega)\,, \int_\Omega \langle \nabla u_{\mathbb{1}_E}, \nabla v\rangle = \lambda(\mathbb{1}_E)\int_\Omega u_{\mathbb{1}_E}v + \int_E u_{\mathbb{1}_E}v.$$

Say we want to obtain the best coercivity norm, that is, we want to obtain a bound of the form

$$\mathscr{L}''(E^*)[\Phi,\Phi] \leq -C\|\langle \Phi,\nu\rangle\|^2.$$

When E^* is a ball, the strategy adopted in [**Maz20**] is the standard one streamlined in [**87**]: decompose $\langle \Phi,\nu\rangle$ as a Fourier series, use the basis of spherical harmonics, and compute the associated coefficients of the quadratic form explicitly. This is also the approach used in [**Maz22b**]. We observed, in these two contributions, that the optimal norm was in fact the L^2 norm, and that it was optimal: namely, when $\Omega = \mathbb{B}(0;1)$ and $E^* = \mathbb{B}^*$, we were able to show that there exist two constants $a_1\,, a_2 > 0$ such that, for any Φ satisfying $\int_{\partial\mathbb{B}^*}\langle\Phi,\nu\rangle = 0$,

$$-a_1 \left\|\langle\Phi,\nu\rangle\right\|^2_{L^2(\partial\mathbb{B}^*)} \leq \mathscr{L}''(E^*)[\Phi,\Phi] \leq -a_2 \left\|\langle\Phi,\nu\rangle\right\|^2_{L^2(\partial\mathbb{B}^*)}\,,$$

and this was shown to be enough to obtain (3.33).

On the other hand, we proposed a much faster way to approach the problem in [**CMF24, CMFSP24**]: as u_{m^*} is constant on ∂E^*, we can simply focus on studying the quantity

$$\int_{\partial E^*} u'_{m^*}\langle\Phi,\nu\rangle + \int_{\partial E^*}\frac{\partial u_{m^*}}{\partial\nu}\langle\Phi,\nu\rangle^2.$$

Now, assume that $\inf_{\partial E^*}|\nabla u_{m^*}| > 0$ and introduce the eigen-pairs $\{\sigma_k,\psi_k\}_{k\in\mathbb{N}}$ of the (self-adjoint) problem

$$\begin{cases} -\Delta\psi_k = \lambda(m^*)\psi_k + m^*\psi_k & \text{in } \Omega\,,\\ [\![\partial_\nu\psi_k]\!] = -\frac{\sigma_k u_{m^*}}{|\nabla u_{m^*}|}\psi_k & \text{on } \partial E^*\,,\\ \psi_k \in W^{1,2}_0(\Omega)\,, \\ \int_\Omega u_{m^*}\psi_k = 0, \end{cases} \tag{3.37}$$

Then, introducing $\tilde\varphi := |\nabla u_{m^*}|\langle\Phi,\nu\rangle$ on ∂E^*, we have that, if

$$\tilde\varphi = \sum_{k=0}^\infty \alpha_k\psi_k$$

then

$$\mathscr{L}''(E^*)[\Phi,\Phi] = \sum_{k=0}^\infty\left(\frac{1}{\sigma_k} - 1\right)\alpha_k^2,$$

which has two immediate consequences:

(1) The negativity of the shape hessian is equivalent to the condition $\sigma_1 > 1$ (which, incidentally, is also a condition that appears in [**79, 177**]) and, in this case, there exist constants $a_1\,, a_2 > 0$ such that for any Φ satisfying $\int_{\partial E^*}\langle\Phi,\nu\rangle = 0$,

$$-a_1 \left\|\langle\Phi,\nu\rangle\right\|^2_{L^2(\partial\mathbb{B}^*)} \leq \mathscr{L}''(E^*)[\Phi,\Phi] \leq -a_2 \left\|\langle\Phi,\nu\rangle\right\|^2_{L^2(\partial\mathbb{B}^*)}\,.$$

(2) Thus, one only has to check one coefficient. In the case of the ball, this can be done by explicit computations. In [**CMF24, CMFSP24**], a key element is that, provided $m_0 > 0$ is large enough, $\sigma_1 > 1$ (which is proved by contradiction), which yields the required stability condition.

The case of general convex functionals

We now explain how to derive such coercivity for the general convex problem (3.28). Recall that in that case the adjoint state is the solution of

$$\begin{cases} -\Delta q_m = j'(u_m) & \text{in } \Omega\,,\\ q_m \in W^{1,2}_0(\Omega), \end{cases} \tag{3.38}$$

that the optimality condition reads: $\{q_{m^*} = c^*\} = \partial E^*$, and that the associated Lagrangian is

$$\mathscr{L} : E \mapsto \mathscr{J}(E) + c^*|E|.$$

The first-order shape derivative of the Lagrangian is

$$\mathscr{L}'(E)[\Phi] = \int_{\partial E} (q_{\mathbb{1}_E} - c^*)\langle \Phi, \nu\rangle,$$

and the second order-derivative is

$$\mathscr{L}''(E^*)[\Phi, \Phi] = \int_{\partial E} q'_{m^*}\langle \Phi, \nu\rangle + \int_{\partial E^*} \frac{\partial q_{m^*}}{\partial \nu}\langle \Phi, \nu\rangle^2.$$

In this expression, q'_{m^*} is the derivative of the map $\Phi \mapsto q_{\mathbb{1}_{(\mathrm{Id}+t\Phi)E^*}}$ at $t = 0$, which is given through the shape derivative u'_{m^*}; overall, standard computations show that u'_{m^*}, q'_{m^*} solve

$$\begin{cases} -\Delta u'_{m^*} = 0 & \text{in } \Omega\,, \\ [\![\partial_\nu u'_{m^*}]\!] = -\langle \Phi, \nu\rangle & \text{on } \partial E^*\,, \\ u'_{m^*} \in W_0^{1,2}(\Omega)\,, \end{cases} \quad \begin{cases} -\Delta q'_{m^*} = j''(u_{m^*})u'_{m^*} & \text{in } \Omega\,, \\ q'_{m^*} \in W_0^{1,2}(\Omega). \end{cases}$$

The equation on q'_{m^*} can be recast as

$$\begin{cases} \Delta\left(\frac{1}{j''(u_{m^*})}\Delta q'_{m^*}\right) = 0 & \text{in } \Omega \\ \left[\!\!\left[\frac{\partial \frac{1}{j''(u_m)}\Delta q'_{m^*}}{\partial \nu}\right]\!\!\right] = \langle \Phi, \nu\rangle & \text{on } \partial E^*\,, \\ q'_{m^*}\,, \Delta q'_{m^*} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.39}$$

In [**CMF24**, **CMFSP24**], we introduce the spectral basis $\{\sigma_k, \psi_k\}$ associated with the family of problems

$$\begin{cases} \Delta\left(\frac{1}{j''(u_{m^*})}\Delta \psi_k\right) = 0 & \text{in } \Omega \\ \left[\!\!\left[\frac{\partial \frac{1}{j''(u_m)}\Delta \psi_k}{\partial \nu}\right]\!\!\right] = \frac{\sigma_k}{|\nabla q_{m^*}|}\psi_k & \text{on } \partial E^*\,, \\ \psi_k\,, \Delta\psi_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.40}$$

once again provided that $\inf_{\partial E^*} |\nabla q_{m^*}| > 0$, and, by the very same reasoning, we obtain that, if

$$\langle \Phi, \nu\rangle = \frac{1}{|\nabla q_{m^*}|}\sum_{k=0}^{\infty} \alpha_k \psi_k$$

then

$$\mathscr{L}''(E^*)[\Phi, \Phi] = \sum_{k=0}^{\infty}\left(\frac{1}{\sigma_k} - 1\right)\alpha_k^2.$$

The collection follows as before, by showing that if m_0 is large enough then $\sigma_1 > 1$.

The main advantage of this diagonalisation procedure is that it avoids lengthy, geometry specifics computations, and immediately gives the right coercivity norm.

3.3. Applications of quantitative inequalities

We conclude this chapter with several applications of the quantitative inequalities we studied in this chapter.

3.3.1. An application to the turnpike property. We first give an application of the spectral quantitative inequality of Theorem 3.5 to the turnpike property; this is taken from [**MRB22a**]. Let us briefly recall what the turnpike property means in the context of optimal control (we also refer to Chapter 4 for a discussion of the phenomenon in the context of Mean-Field Games): essentially, given a parabolic optimal control problem, set in a very large time horizon, it is expected that, for a large portion of the time, the optimal parabolic control will resemble the optimal control of a related elliptic problem. This property has several nice consequences, typically by allowing to choose a reasonable initial condition in numerical simulations, and has known a great deal of interest in recent years [**151, 207, 217, 216, 222**]. Nevertheless, most of the references we are aware only deal with *linear* control problems, and it is unclear to us whether the formalism can in fact accommodate bilinear control problems. In [**MRB22a**], we proposed an approach to the following easy optimal control question:

$$\max_{m\in L^\infty((0;T)\times\Omega)\,,0\leq m\leq 1\,,\forall t\,,\int_\Omega m(t,\cdot)=m_0}\int_\Omega y_m(T,\cdot)^2 \tag{3.41}$$

$$\text{subject to }\begin{cases}\partial_t y_m-\Delta y_m=my_m & \text{in } (0;T)\times\Omega\,,\\ y_m(t,\cdot)\in W^{1,2}_0(\Omega) & \text{in } (0;T)\,,\\ y_m=y_0\,,y_0\neq 0\,,y_0\geq 0.\end{cases}$$

We let $\mathcal{M}^*$ denote the set of optimisers for the eigenvalue optimisation problem (3.27).

THEOREM 3.9. *Let, for any $T>0$, m_T^* be a solution of* (3.41). *Under the assumptions of Theorem 3.5, the quantity*

$$\int_0^T \mathrm{dist}_{L^1}(m_T^*(s,\cdot),\mathcal{M}^*)^2ds$$

is bounded uniformly in T.

Thus, on average and for large time horizons T, the optimisers of (3.41) behave like the minimisers of an elliptic spectral optimisation problem.

REMARK 3.10. In fact, this result can also be rewritten using the weak KAM theory for Hamilton-Jacobi equations; we think mentioning it here might be worthwhile, as it is a big part of the following chapter.

3.3.2. An application to the convergence of thresholding schemes. In [**CMF24, CMFSP24**] our objective was to investigate how quantitative estimates could imply convergence results for the convergence of the thresholding scheme, which aims at solving an optimal control problem using the adjoint state. As far as we are aware, apart from the specific one-dimensional case, where the linear convergence of a rearrangement algorithm was obtained by Kao, Mohammadi & Osting [**141**], there are no general proofs. To fix ideas, we consider the general convex optimisation problem (3.28):

$$\max_{m\in\mathcal{M}} J(m)=\int_\Omega j(u_m)\text{ subject to }\begin{cases}-\Delta u_m=m & \text{in } \Omega\,,\\ u_m\in W^{1,2}_0(\Omega).\end{cases}$$

The adjoint state $q_m \in W_0^{1,2}(\Omega)$ solves $-\Delta q_m = j'(u_m)$ in Ω and the optimality conditions read

$$m^* = \mathbb{1}_{\{q_{m^*} > c^*\}}.$$

The standard thresholding algorithm goes as follows:

(1) Fix an initialisation m_0.
(2) For any $k \in \mathbb{N}$, compute q_{m_k}, set $E_{k+1} := \{q_{m_k} > c_k\}$, where c_k is chosen so as to satisfy the volume constraint, and define

$$m_{k+1} = \mathbb{1}_{E_{k+1}}.$$

Until [**CMF24**], there were no convergence proof apart from the one-dimensional case [**141**], but we managed to obtain convergence of the sequence of iterates under large volume constraints using Theorem 3.7:

THEOREM 3.11. [**CMF24, CMFSP24**] *If $m_0 > 0$ is large enough, the sequence $\{m_k\}_{k\in\mathbb{N}}$ generated by the thresholding algorithm converges to a local maximiser of J strongly in $L^1(\Omega)$.*

The role of quantitative inequalities in this result is central as it allows to obtain an estimate of the form

$$\sum_{k=0}^{\infty} \|m_{k+1} - m_k\|^2_{L^1(\Omega)} < \infty,$$

which can be used to guarantee the uniqueness of closure points of the sequence.

3.3.3. An application to the optimal location of sensors. As a final application of quantitative inequalities, let us mention a recent work [**MFPT25**] in collaboration with Privat and Trélat devoted to the optimal observability constant– for the sake of simplicity, we merely state it for the heat equation, but it applies to a much broader range of systems. Namely, consider, for any $m \in \mathcal{M}$ and any time horizon $T > 0$ the *observability constant* associated with m

$$C_T(m) := \min_{y_0 \in \mathscr{C}_c^\infty(\Omega)\,, y_0 \neq 0} \left(\frac{\iint_{(0;T)\times\Omega} my^2}{\int_\Omega y^2(T,\cdot)} \text{ where } \begin{cases} \partial_t y - \Delta y = 0 & \text{in } (0;T)\times\Omega\,, \\ y(t,\cdot) \in W_0^{1,2}(\Omega) & \text{in } (0;T)\,, \\ y(0,\cdot) = y_0 \end{cases} \right).$$

This constant is of paramount importance in the controllability of partial differential equations, observability being the dual notion of controllability, as well as in the study of inverse problem: the larger the constant $C_T(m)$, the easier it is to reconstruct a signal from its observation, the observation being measured through m. When $m = \mathbb{1}_E$, this can be interpreted in a natural way as observing a signal on E. This leads to the optimisation problem

$$\delta_T := \max_{m \in \mathcal{M}} C_T(m). \tag{3.42}$$

This problem is notoriously difficult to handle and analyse, and has been successfully tackled only in suitably modified scenarii (typically, randomised situations [**197, 198, 199, 200, 201, 202, 203**]). In [**MFPT25**] we propose a detailed study of the case $T \to \infty$. To properly state the result, introduce the optimisation problem

$$\max_{m \in \mathcal{M}} \sigma(m) := \int_\Omega m\phi_\Omega \text{ where } \phi_\Omega \text{ is the first Dirichlet eigenfunction of } \Omega. \tag{3.43}$$

This problem has a unique solution m^* and, using among other things the quantitative bathtub principle, we were able to prove:

THEOREM 3.12. *Let, for any* $T > 0$, m_T^* *be a solution of* (3.42) *and* m^* *be the solution of* (3.43). *There exists a positive constant* $\omega > 0$ *and* C *such that*

$$\|m_T^* - m^*\|_{L^1(\Omega)} \leq Ce^{-\omega T}.$$

As far as we are aware, this is one of the first qualitative results about the "real"(*i.e* non-randomised) observability constant.

3.4. Research plan

We briefly sketch some of our current and future research interests.

3.4.1. Quantitative Talenti inequalities. There has recently been a surge in interest around quantitative version of the so-called Talenti inequalities, where one aims at comparing the solution of

$$\begin{cases} -\Delta u_f = f & \text{in } \Omega\,, \\ u_f \in W_0^{1,2}(\Omega) \end{cases}$$

with the solution of

$$\begin{cases} -\Delta v_{f^\sharp} = f^\sharp & \text{in } \Omega^\sharp\,, \\ v_{f^\sharp} \in W_0^{1,2}(\Omega^\sharp). \end{cases}$$

In this equations, $\Omega^\sharp$ and $f^\sharp$ stand for the Schwarz rearrangements of Ω and f respectively (see Section 2.1.2). More specifically, the Talenti inequality asserts that for any Ω, f, for any $p \in [1; +\infty]$, if $f \geq 0$

$$\int_\Omega u_f^p \leq \int_{\Omega^\sharp} v_{f^\sharp}^p. \tag{3.44}$$

Our first question is: can we guarantee a quantitative estimate of the form

$$\mathcal{A}(\Omega)^\nu + \mathcal{F}(f)^\theta \leq \int_{\Omega^\sharp} v_{f^\sharp}^p - \int_\Omega u_f^p$$

where $\mathcal{A}$ and $\mathcal{F}$ are measures of the asymmetry of Ω and f respectively? To some extent, Theorems 3.4–3.7 answer this question when $\Omega = \Omega^\sharp$ and f is assumed to be a characteristic function (in the case of Theorem 3.4, for $p = 2$ and, in the case of Theorem 3.7, for the case of large volume constraints).

Two very recent papers [**14, 146**] tackle this question in all generality, but derive what we believe are sub-optimal exponents. More recently still, Acampora & Lamboley [**1**] proved the sharp stability of the Talenti inequality when $\Omega = \Omega^\sharp$ is fixed and f is a characteristic functions. Their analysis completes and systematises the methods we initiated in the contributions presented in this chapter.

Our goal will be to derive sharp quantitative estimates for general domains and general source terms, first focusing on the case where $\Omega = \Omega^\sharp$ and where f is simply constrained to have a prescribed rearrangement. This is an ongoing work with P. Acampora and J. Lamboley.

3.4.2. Quantitative estimates in optimal control as consequences of quantitative geometric estimates. Another perspective that we would like to explore is how to obtain quantitative Talenti inequalities (or spectral inequalities) from the quantitative versions of more fundamental geometric inequalities. To illustrate what we have in mind, let us mention that, in the theory of rearrangement, the Riesz-Sobolev inequality

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x-y)h(x)g(y)dxdy \leq \iint_{\mathbb{R}^d\times\mathbb{R}^d} f^\sharp(x-y)h^\sharp(x)g^\sharp(y)dxdy \tag{3.45}$$

can be thought of as a “master inequality”, given that it implies both the Polyá-Szegö inequality (see [**163**, Lemma 7.17]) and the Talenti inequality [**12**]. However, (3.45) was recently proved in quantitative versions [**81, 101, 103**]. We would like to investigate how (3.45), in its quantitative form, might provide quantified estimates for (3.44) and, more generally, if one can deduce in a more systematic fashion quantitative estimates for PDE constrained problems from geometric estimates.

CHAPTER 4

Game theoretical models in the management of fisheries

In this chapter, we discuss the contributions [**KMFRB24a**, **KMFRB24b**, **MRB22b**], which all deal with various aspects of the following question: *what is the impact of competition of fishermen on the dynamics of fisheries*? This question is a natural one given the rising amount of pressure exerted on ecosystems by harvesting activities and, to tackle it from a mathematical perspective, we proposed three different approaches, which all aim at providing some examples for the *tragedy of the commons*: when harvesting from a common pool of resources, the competition between harvesters can lead to the extinction of resources, while the coordination between these harvesters might have allowed for a better pay-off for each harvester, and the survival of the resources.

4.1. Introduction

4.1.1. Scope and core concepts. We start by describing the goal of the chapter, and by informally introducing the main models we will be considering.

Scope of the chapter

In this series of works in collaboration with Kobeissi & Ruiz-Balet, which we are currently pursuing further with Lamonaca & Nadin, our goal was to contribute to the mathematical analysis of the collapse of aquatic ecosystems which, as part of the decline in biodiversity, is a major environmental, political and economical challenge of our present time [**20**, **85**, **88**, **119**, **192**, **193**]. There are several factors accounting for this collapse, including pollution or climate change, but we want to focus on a simple, paradigmatic model that only factors in fishing phenomena, with a special emphasis on the impact that the competition between fishermen has on the overall dynamics of the fish population. This is of course a major simplification that needs to be revisited (we also refer to the conclusion of the chapter), but for which the mathematical analysis proves multi-faceted and quite rich. In this way, our goal in this chapter is to present a gallery of simple models displaying a wealth of behaviours, that we hope sheds a new light on the complex interactions between human activities and ecosystems.

A point of terminology

Throughout this chapter, we will indistinctly refer to agents or to fishermen, while fishes will also sometimes be referred to as resources.

Competition between agents, Nash equilibria and the tragedy of the commons

Regardless of which model we consider, we are always working with two populations: a population of fishermen, and a population of fishes that is being harvested.

The fishermen can be in finite or infinite numbers, leading to two analytically different types of questions, but the main fact is that they are *competing* for the resources (*i.e* to harvest fish), each trying to maximise his own outcome. Through their harvesting, they in turn impact the fish population, which is thus subject to two phenomena: on the one hand, it has its own, intrinsic, dynamics which can be of either stable or unstable type, while on the other hand it is also being actively killed by fishermen.

Each agent has to design his or her own fishing strategy, meaning here *where* they decide to fish (and, in some cases, if they decide to fish). In this type of situation, the agents or fishermen are looking for an *equilibrium* situation. When a finite number of players are involved, "equilibrium" refers to the central notion of Nash equilibrium; a bit of formalism is required. Assume N agents are involved, with the i-th chosing the strategy α_i. The pay-off of this agent is quantified by a function $j_i(\alpha_1, \dots, \alpha_N)$. A Nash equilibrium is a N-tuple of strategies $(\alpha_1^*, \dots, \alpha_N^*)$ such that

$$\forall i \in \{1, \dots, N\}, j_i(\alpha_1^*, \dots, \alpha_N^*) = \max_{\alpha_i \text{ admissible strategy}} j_i(\alpha_1^*, \dots, \alpha_{i-1}^*, \alpha_i, \alpha_{i+1}^*, \dots, \alpha_N^*). \tag{4.1}$$

The existence of Nash equilibrium, in a given competition situation, is in itself a central question in mathematical economics [**153**], and the results from [**MRB22b**] we choose to present mostly deal with this aspect. When an infinite number of players are involved, we will rely on the Mean Field Game (MFG) formalism of Lasry & Lions [**154, 155, 156**] and Caines, Huang & Malhamé [**136, 137**]. From the modelling perspective, in the MFG formalism, each agent has a negligible impact on the overall system, while in the case of a finite number of fishermen, each has a global influence on the system. Here again, we insist on the fact that we choose a simplistic set of modelling assumptions.

At any rate, our main qualitative questions are the following:

(1) Can we obtain the *existence* of Nash equilibria when a finite number of players are involved?
(2) Can we obtain *existence*, *uniqueness* and, possibly, the *stability* of optimal strategies when an infinite number of players are involved?
(3) Finally, can we give examples of the *tragedy of the commons*? This term, coined by, a term coined by W.F. Lloyd [**168**], is a general principle of economics [**120**] that states that the action of selfish players, each playing so as to maximise his outcome from a common resources, will eventually lead to the depletion of this resources. We will investigate this phenomenon in the different models under scrutiny.

Remark 4.1. One might remark at this stage that we ask questions of uniqueness and stability for the MFG setting, while we only investigate the existence of Nash equilibria in the case of a finite number of players; this is due to the particular nature of the equations. In the MFG setting, the existence of an equilibria is usually settled using fixed-point arguments. In contrast, the question of uniqueness and stability is much more involved.

4.1.2. Models under consideration & plan of the chapter. In this chapter, we focus on three different models.

Finite number of players and Nash equilibria

We first consider, in section 4.2, the case of a finite number N of players engaged in a competition for a population of fishes, denoted by its density θ, that follows a logistic-diffusive equation (se Chapter 1 and, in particular, Theorem 1.2 for modelling considerations): it has access to a resources distribution $K \in L^\infty(\Omega)$, dies according to a Malthusian death term and has an intrinsic diffusivity $\mu > 0$. Each player must choose a fishing strategy $\alpha_i \in L^\infty(\Omega)\,, \alpha_i \geq 0$, so that the fish population solves

$$\begin{cases} -\mu\Delta\theta - \theta\left(K - \sum_{i=1}^N \alpha_i - \theta\right) = 0 & \text{in } \Omega\,, \\ \partial_\nu \theta = 0 & \text{on } \partial\Omega\,, \\ \theta \geq 0\,, \theta \neq 0. \end{cases}$$

The i-th player is trying to solve the optimisation problem

$$\max_{\alpha_i} \int_\Omega \alpha_i \theta$$

and our emphasis will be on the existence of Nash equilibria, and on some of their qualitative (geometric) properties. We will also exemplify the tragedy of the commons in this case. This part of the chapter relies, at a technical level, on the tools and methods introduced in Chapter 1.

A monostable MFG model

In Section 4.3 we present the results of[**KMFRB24a**]: we also consider a fish population that evolves according to a logistic-diffusive equation, but we rely this time on the MFG formalism by assuming that an infinite number of players are competing in a time evolving system. Namely, in this setting, letting the fishes' density still be denoted by θ, it solves

$$\partial_t\theta - \mu\Delta\theta = \theta(K - m - \theta)$$

in the torus, where m denotes the macroscopic density of players. Each player is moving according to a controlled differential equation

$$\dot{x} = \alpha,$$

and tries to tune his or her strategy α so as to solve

$$\max_\alpha \int_0^T \left(\theta(t, x(t)) - \frac{\|\alpha(t)\|^2}{2}\right) dt,$$

resulting in a Mean Field Game systems of three coupled equations. In this section, the main emphasis will be on the *uniqueness* of solutions to the ensuing MFG system, as well as on its *long-time behaviour* and *convergence to the ergodic system*. We also give a telling instance of the tragedy of the commons.

A travelling waves' approach to the tragedy of the commons

In Section 4.4 we present the results of [**KMFRB24b**]; it is, once again, a MFG model, but this time the underlying fish population follows a *bistable* non-linearity. Bistable non-linearities, which will be defined in Section 4.4, account for the *Allee effect*, namely, for the fact that, when the population of fishes is below a certain threshold, it will eventually die out. The system will be set in all of $\mathbb{R}$ so that we can rely on the methodology of travelling waves. Assume that the non-linearity and initial conditions for the fish population are chosen in such way that, in the absence of fishermen, the solution would converge, in long time, to an invading travelling wave (meaning the fish population eventually invades the entire real line). Is it possible that, by acting selfishly, the fishermen actually drive the population to extinction? What can be said about the possible effects of coordination on the fish population? This section is quite distinct from the previous ones and the focus will mostly be qualitative.

4.2. A Nash equilibrium model

4.2.1. Main model under consideration. In this section, we cover some results of [MRB22b]. The model is the following: in a smooth, bounded domain Ω, we let $K \in L^\infty(\Omega)\,, K \geq 0\,, K \not\equiv 0\,,$ denote a resources distribution and we let $\mu > 0$ be a fixed diffusivity. We consider N players, competing to harvest from a fish population that can access resources K and that has an intrinsic diffusion rate $\sqrt{2\mu}$. Each of these players chooses a strategy $\alpha_i \in L^\infty(\Omega)$, that satisfies

$$0 \leq \alpha_i \leq 1 \text{ a.e.}, \int_\Omega \alpha_i = m_i.$$

The first constraint models the fact that the fishing intensity can not exceed a certain value, while the second one accounts for the globally limited fishing ability of the i-th player. Overall, considering an N-tuple $\vec{\alpha} := (\alpha_1, \dots, \alpha_N)$ of strategies, the k-th agent playing the strategy α_k, the fish population $\theta_{\vec{\alpha}}$ solves the logistic-diffusive equation

$$\begin{cases} -\mu\Delta\theta_{\vec{\alpha}} - \theta_{\vec{\alpha}}\left(K - \sum_{i=1}^N \alpha_i - \theta_{\vec{\alpha}}\right) = 0 & \text{in } \Omega\,, \\ \partial_\nu \theta_{\vec{\alpha}} = 0 & \text{on } \partial\Omega\,, \\ \theta_{\vec{\alpha}} \geq 0\,, \theta_{\vec{\alpha}} \neq 0. \end{cases} \tag{4.2}$$

In order for (4.2) to have a solution, it is sufficient to impose that

$$\sum_{i=1}^N m_i < \int_\Omega K; \tag{4.3}$$

we refer to Theorem 1.2. The admissible class for the i-th player is

$$\mathcal{M}_i := \left\{\alpha_i \in L^\infty(\Omega) :\ 0 \leq \alpha_i \leq 1 \text{ a.e.}, \int_\Omega \alpha_i \leq m_i\right\} \tag{4.4}$$

where the m_i ($i = 1, \dots, N$) are chosen to satisfy (4.3); we refer to the next paragraph for some crucial comments on this admissible class.

The optimisation problem that the i-th player wants to solve is the following: $\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_N$ being fixed, find a solution to

$$\max_{\alpha \in \mathcal{M}_i} \left(J_i(\alpha_1, \dots, \alpha_{i-1}, \alpha, \alpha_{i+1}, \dots, \alpha_N) := \int_\Omega \alpha\theta_{\alpha_1,\dots,\alpha_{i-1},\alpha,\alpha_{i+1},\dots,\alpha_N}\right). \tag{4.5}$$

As noted in the introduction, in this formulation, each agent has a macroscopic influence on the fish population. We define the notion of Nash equilibrium:

Definition 4.2. An N-tuple $\vec{\alpha} = (\alpha_1^*, \dots, \alpha_N^*)$ is a *Nash equilibrium* if

$$\forall i \in \{1, \dots, N\}\,, \forall \alpha \in \mathcal{M}_i\,, J_i(\alpha_1^*, \dots, \alpha_{i-1}^*, \alpha, \alpha_{i+1}^*, \dots, \alpha_N^*) \leq J_i(\alpha_1^*, \dots, \alpha_N^*).$$

Our main results deal with the existence of such equilibria but, before we get to that point, a few comments are in order.

Multiple agents or single fisher

Although this is not the main focus of the chapter, [MRB22b] also deals extensively with the single fisher problem (*i.e* $N = 1$) in which case the problem under study becomes

$$\max_{0\leq\alpha\leq 1 \text{ a.e.}\,,\int_\Omega \alpha\leq 1} \int_\Omega \alpha\theta_\alpha \text{ subject to } \begin{cases} -\mu\Delta\theta_\alpha - \theta_\alpha(K-\alpha-\theta_\alpha) = 0 & \text{in } \Omega\,, \\ \partial_\nu\theta_\alpha = 0 & \text{on } \partial\Omega\,, \\ \theta_\alpha \geq 0\,, \theta_\alpha \not\equiv 0, \end{cases}$$

which also plays a role in the analysis of Nash equilibria and serves to exemplify several phenomena (see for instance the next paragraph).

Some comments on the class of constraints: the overfishing phenomenon

The first point we want to address is the definition of (4.4), which, contrary to all the other examples encountered in this manuscript, features an L^1 inequality rather than an L^1 equality constraint. The main reason for that comes from modelling considerations: this allows for the i-the player to choose not only *where* he fishes, but also *whether or not he fishes* at maximum capacity. This would allow to illustrate overfishing phenomena: it is not always in the player's best interest to fish as much as he can, for he might harm the fish population so much it might end up being detrimental to his outcome. As a simple example of that phenomenon, consider the case of a constant resources distribution $K \equiv 1$ and the case of a single fisherman, trying to solve

$$\max_{0\leq\alpha\leq 1 \text{ a.e.}\,,\int_\Omega \alpha\leq 1} \int_\Omega \alpha\theta_\alpha \text{ subject to } \begin{cases} -\mu\Delta\theta_\alpha - \theta_\alpha(1-\alpha-\theta_\alpha) = 0 & \text{in } \Omega\,, \\ \partial_\nu\theta_\alpha = 0 & \text{on } \partial\Omega\,, \\ \theta_\alpha \geq 0\,, \theta_\alpha \not\equiv 0. \end{cases}$$

In that case,

$$\int_\Omega \alpha\theta_\alpha = \int_\Omega \theta_\alpha(1-\theta_\alpha).$$

As by the maximum principle $0 \leq \theta_\alpha \leq 1$ we deduce from studying the function $x \mapsto x(1-x)$ that

$$\int_\Omega \alpha\theta_\alpha \leq \frac{1}{4} \tag{4.6}$$

and that the optimal strategy is given by $\alpha \equiv \frac{1}{2}$. A possible reinterpretation of that is that there is also an optimal global fishing ability $m_{i,\text{opt}}$; this will play a role in the forthcoming analysis.

How does one usually prove the existence of Nash equilibria?

First, observe that in general situations there are no Nash equilibria–the typical example is the rock-paper-scissor game. Now, there are essentially ways to prove the existence of Nash equilibria:

(1) The first one is to exhibit a Nash equilibrium directly; although hopeless in the general case, it can provide good examples. A typical situation is once again the case of a constant resources distribution $K \equiv 1$ with N competing agents. Define, for $i \in \{1, \dots, N\}$,

$$\alpha_i^* \equiv \frac{1}{N+1}; \tag{4.7}$$

here, we take for instance $m_i = \frac{1}{2}$ so that the strategies are admissible. Then one can check that $(\alpha_1^*, \dots, \alpha_N^*)$ is a Nash equilibrium–this is essentially the same computation as in (4.6).

(2) As we said, it is in general a hopeless endeavour to exhibit an explicit equilibrium and, as far as we are aware, the main tool to prove the existence of equilibria actually dates back to the paper of Nash [188] itself–we also refer to [110]. The main tool is the Kakutani fixed point theorem: as the admissible set $\mathcal{M}_i$ is convex, if we can prove that, for any $i \in \{1, \dots, N\}$, for any $(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_N)$ the set of maximisers of $\alpha \mapsto J_i(\alpha_1, \dots, \alpha_{i-1}, \alpha, \alpha_{i+1}, \dots, \alpha_N)$ is convex, then the Kakutani fixed point theorem ensures the existence of a Nash equilibrium. This is guaranteed provided the map $\alpha \mapsto J_i(\alpha_1, \dots, \alpha_{i-1}, \alpha, \alpha_{i+1}, \dots, \alpha_N)$ is concave. This strategy, which will give some restrictions on the set of constraints, is the one we focus on. We will heavily rely on the tools of Chapter 1.

Related literature

Before we state our main results, let us review some related contributions. Nash equilibria problems in optimal control are an emerging and active area of research, as exemplified by the recent publications [69, 94] and the references therein. The focus of these papers is on tracking-type functionals, which behave in fundamentally different ways from the ones considered here; however, the main difference, from our point of view, with [94] is the fact that they consider linear control problems. As explained in Chapter 1, these behave extremely differently from bilinear control problems, especially from the concavity perspective (and this plays a central role in the analysis of the existence of Nash equilibria). [58] deals with Nash equilibria models for bilinearly controlled ODE models.

From the PDE perspective, the most related works are to us the research of Bressan, Coclite, Shen & Staicu [40, 41, 42], which investigate optimal harvesting problems either for single companies or in a competitive setting, but within the framework of elliptic equations with measure data, the measure under consideration representing the density of players; the measures are the optimisation variables. We note that we were not aware of [40, 42] at the time [MRB22b] was published. We would like to thank one of the anonymous reviewers of [KMFRB24a] for pointing these references out. Although the methods and models used differ, our work relates to theirs, both in their objectives and in the type of conclusion they reach for the existence of Nash equilibria results in [40] (namely, that they exist under a small fishing ability constraint, see Theorem 4.3 below), but with a different analytic and qualitative outlook. Furthermore, their analysis for the existence of Nash equilibria is done in the one-dimensional case, while ours holds in any dimension, for any type of $L^\infty - L^1$ constraints on the fishing strategies, and emphasises the link between the existence of equilibria and the overfishing phenomenon. Besides, we present several qualitative (geometric and pointwise) results, particularly so in the large diffusivity setting, and we carry out a number of numerical simulations illustrating the complexity of the problem.

4.2.2. Main results. We now move on to the description of the main results of [MRB22b].

Existence of Nash equilibria

We begin with some existence results for Nash equilibria:

THEOREM 4.3. *In the one-dimensional case $\Omega = (0;1)$, there exists $\delta > 0$ such that, if*

$$\sum_{i=1}^{N} m_i \leq \delta$$

there exists a Nash equilibrium for the fishing game.

In any dimension d, there exist $\delta > 0\,,\varepsilon > 0$ such that, if

$$\sum_{i=1}^{N} m_i \leq \delta\,, \left\| K - \fint_\Omega K \right\|_{L^\infty(\Omega)} \leq \varepsilon$$

there exists a Nash equilibrium for the fishing game.

In the one-dimensional case, the type of conditions we impose (small fishing capacity) is similar to the existence criterion given in [40] for measure-valued Nash equilibria (in particular, without the L^∞ constraints). In our analysis, this "small mass" condition appears as a natural consequence of the investigation of *monotonicity properties* of the harvesting functional for a single player game. Our proof relies on fine properties of the stationary logistic-diffusive equation. We refer to the sketches of proof for a detailed discussion of these aspects.

The tragedy of the commons

In [MRB22b] we also give a fairly elementary example of the tragedy of the commons. Namely, going back to the example (4.7), observe that the amount harvested by the i-th company is given by

$$J_i\left(\vec{\alpha^*}\right) = \frac{1}{(N+1)^2},$$

and the overall fish population satisfies

$$\theta_{\vec{\alpha^*}} = \frac{1}{N+1}.$$

Now, it is easy to create a "better" coordinated strategy: take, for any $i \in \{1, \dots, N\}$,

$$\alpha_i' := \frac{1}{2(N+1)}.$$

Then straightforward computations show that

$$J_i\left(\vec{\alpha'}\right) = \frac{1}{2(N+1)}\,, \theta_{\vec{\alpha'}} = \frac{1}{2}.$$

Consequently, had the fishermen coordinated, they would have been better off at an individual level, and the fish population would have survived. This simple example served as impetus for our further investigations of the tragedy of the commons.

Some precise geometric behaviour in the large diffusivity regime

We now study the large diffusivity case $\mu \to \infty$ in the case of two players. In particular, we show that the "small fishing ability" assumption of Theorem 4.3 is not sharp in general. This analysis is done under the assumption of a fixed fishing ability:

$$\alpha_i \in \mathcal{N}_i := \left\{\alpha_i \in L^\infty(\Omega) : 0 \leq \alpha_i \leq 1 \text{ a.e.}, \int_\Omega \alpha_i = m_i\right\} \tag{4.8}$$

for the sake of simplicity.

Owing to the same type of large diffusivity analysis that was carried out in Chapter 2, Theorem 2.12, we can show that, as $\mu \to \infty$, the limit Nash equilibrium problem is the following one: considering, for two strategies $(\alpha_1, \alpha_2) \in \mathcal{N}_1 \times \mathcal{N}_2$, the solution v_{α_1,α_2} of

$$\begin{cases} -\Delta v_{\alpha_1,\alpha_2} - M(K_0 - \alpha_1 - \alpha_2 - M) = 0 & \text{in } \Omega\,, \\ \frac{\partial v_{\alpha_1,\alpha_2}}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \\ \fint_\Omega v_{\alpha_1,\alpha_2} = \frac{1}{M^2} \fint_\Omega |\nabla v_{\alpha_1,\alpha_2}|^2, \end{cases}$$

where

$$M := \frac{\int_\Omega K - m_1 - m_2}{|\Omega|},$$

we need to find a Nash equilibrium for the functionals

$$J_{\infty,i} : \alpha_i \mapsto \int_\Omega \alpha_i v_{\alpha_1,\alpha_2}.$$

THEOREM 4.4. *Assume $\Omega = (0;1)$, that K is constant and that $m_1\,, m_2 > \frac{K}{4}$. Let*

$$\alpha_i^* = \mathbb{1}_{[0;m_i]} \ (i = 1, 2).$$

$(\alpha_1^*\,, \alpha_2^*)$ is a Nash equilibrium for the functionals $J_{\infty,i}$.

Two things should be remarked on this theorem:

(1) First, this proves that there is in general no uniqueness of the Nash equilibrium, as, if we take $K \equiv 1\,, m_1 = m_2 = \frac{1}{3}$, the strategies provided by (4.7) are still Nash equilibria.
(2) Second, this goes to prove that the "small fishing ability" assumption of Theorem 4.3 is not sharp.

Some numerical illustrations

We now present some numerical simulations of the optimal strategies for the case of two players to illustrate what we think are the main remaining challenges in the analysis of this problem, namely, the geometry of optimal fishing strategies: provided a Nash equilibrium (α_1^*, α_2^*) exists, what does it look like? Are the optimal strategies bang-bang (see Definition 1.4)? If so, should fishermen fish in the same location, or should they spread evenly? Can we observe fragmentation phenomena analogous to those of Chapter 2? We only present simulations in the one-dimensional case and refer to [MRB22b] for the two-dimensional case.

We observe, interestingly enough, a fragmentation phenomenon for small diffusivities; the analysis of such geometric properties remains an open and interesting problem.

4.2.3. Sketches of proof.

We now briefly comment on the proofs of the main theorems, highlighting what we think the main elements are.

SKETCH OF PROOF OF THEOREM 4.3. As we mentioned earlier, the existence of a Nash equilibrium rests upon the concavity of the functionals at hand, which can be analysed using a single player case. Namely: consider, for any $\alpha \in \mathcal{M}$, where

$$\mathcal{M} := \left\{\alpha \in L^\infty(\Omega) : \, 0 \leq \alpha \leq 1 \text{ a.e.}, \int_\Omega \alpha \leq m_0 \right\}$$

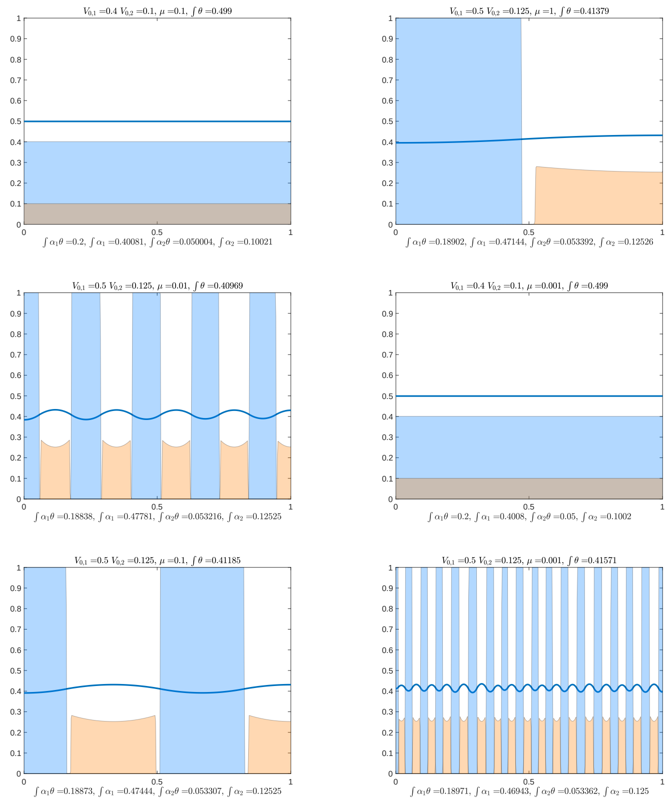


FIGURE 1. We use the notation $V_{0,i} = m_i$. The blue line represents the state $\theta_{\alpha_1,\alpha_2}$. The strategy of the first (and second) player, with higher (lower) capacity, has been depicted as a blue (orange) subgraph. The second player has a lower integral bound than the first player. In all these simulations, $K \equiv 1$ and the only varying parameter is the diffusivity μ.

the functional

$$\alpha \mapsto J(\alpha) := \int_\Omega \alpha\theta_\alpha \text{ subject to } \begin{cases} -\Delta\theta_\alpha - \alpha(K - \alpha - \theta_\alpha) = 0 & \text{in } \Omega\,, \\ \partial_\nu \theta_\alpha = 0 & \text{on } \partial\Omega\,, \\ \theta_\alpha \geq 0\,, \not\equiv 0. \end{cases}$$

We claim that for any $K \in L^\infty(\Omega)$ there exists $\delta > 0$ such that for any $m_0 \leq \delta$ the functional J is concave on $\mathcal{M}$. But this is akin to a bilinear optimal control problem of the type that we encountered in Chapter 1 and, owing to the same reasoning as for instance in Theorem 1.7, we claim that the concavity of J is related to its *monotonicity*. We do not detail the proof of this implication as it follows from the same line as in Chapter 1, and we rather focus on this monotonicity. Thus, we claim that the existence of Nash equilibria boils down to proving that, for $m_0 > 0$ small enough:

$$\forall \alpha\,, \alpha' \in \mathcal{M}\,, \alpha \leq \alpha' \text{ a.e.}, \;\Rightarrow J(\alpha) \leq J(\alpha').$$

This is where the overfishing phenomenon comes into play: the question of monotonicity can indeed be recast as "should we always fish as much as we can?". A key part of the proof of this theorem is thus showing the following fact:

(1) If $m_0 > 0$ is too large, then it can never be in our best interest to fish as much as we can. This is an extension of the considerations surrounding (4.6).
(2) If $m_0 > 0$ is small enough however, then it is always in our best interest to fish as much as we can.

□

Sketch of proof of Theorem 4.4. Similar to what was done when dealing with the total population size in the large diffusivity regime (see Theorem 2.12), the key-point is simply to formulate the limit optimisation problems as energetic minimisation problems and to use Talenti inequalities. We refer to Chapter 2 and to Theorem 2.12 for the manipulation of such tools. □

4.3. Mean Field Game models I: long-time behaviour for the optimal management of fisheries

4.3.1. Presentation of the model and main results. We now present our recent article [**KMFRB24a**], in which we deal with a similar question from a Mean-Field Game perspective (see the bibliographical paragraph, as well as the references in [**KMFRB24a**]). The setting is the following:

(1) We work in the d-dimensional torus $\mathbb{T}^d$.
(2) The fish population is still modelled through a diffusive-logistic equation and can access a resources distribution K.
(3) There are infinitely many players, the macrosopic density of which is represented by a Radon measure m.
(4) Each fisherman behaves according to the following rules:
    (a) He fishes in a small neighbourhood of its location (this is akin to having a fishing net); this is modelled through a smooth, compactly supported, convolution kernel ρ. The operation $\star$ denotes the convolution product. We assume that the impact of fishing on the population is quantified by a parameter $\varepsilon > 0$.
    (b) This fisherman's strategy is his trajectory: letting $x(t)$ be the trajectory of this agent, he can control it by acting on the differential equation

$$\frac{dx}{dt} = \alpha(t),$$

α being the control.

For a given time horizon $T > 0$, a given macroscopic density of fishermen m and a fixed initial condition $\theta_0 \geq 0\,, \not\equiv 0$, the fishes' density solves the evolution equation

$$\begin{cases} \partial_t \theta_m - \Delta \theta_m = \theta_m (K - \varepsilon \rho \star m - \theta_m) & \text{in } (0;T) \times \mathbb{T}^d\,, \\ \theta_m(0, \cdot) = \theta_0 & \text{in } \mathbb{T}^d. \end{cases} \tag{4.9}$$

Now, each player, at an individual level, is trying to solve the following optimisation problem: if he starts from the initial position x_0, solve
(4.10)

$$\max_{\alpha\in L^2((0;T);\mathbb{R}^d)}\int_0^T\left((\rho\star\theta_m)(t,x_\alpha(t))-\frac{\|\alpha\|^2(t)}{2}\right)dt \text{ subject to } \begin{cases}\frac{dx}{dt}=\alpha & \text{in } (0;T)\,,\\ x(0)=x_0.\end{cases}$$

REMARK 4.5. The presence of the convolution kernel makes sense from an applied perspective, as it models the use of fishing nets, but it is also crucial in our mathematical analysis. Also, observe that we consider first-order mean-field games; in [**KMFRB24a**] we also consider the case of second-order mean-field games, that is, adding a Brownian noise in the trajectories of agents, but we do not dwell on this in this presentation.

A core assumption in the MFG framework is the following one: *a single agent does not have an impact on the fish population θ_m. It is only the collective behaviour of all agents that influences the fishes' distribution.* This is in stark contrast with the situation of Section 4.2.

In that context, a Mean-Field equilibrium is a situation in which each individual players plays the best possible strategy given what the other players are doing, which is the analog of Nash equilibria for a finite number of players. In order to compute the optimal strategy for a given player, it is customary to introduce the value function

$$u(t,x)=\max_\alpha\int_t^T(\rho\star\theta)(s,X_s)ds-\int_t^T\frac{\|\alpha\|^2(s)}{2}ds \text{ subject to } \begin{cases}\frac{dx}{ds}=\alpha & \text{in (t;T)}\\ x(t)=x.\end{cases}$$

By a standard application of the Bellman dynamic programming principle, the function u is the unique viscosity solution of the Hamilton-Jacobi-Bellman equation

$$\begin{cases}-\partial_t u-\frac{\|\nabla_x u\|^2}{2}=\rho\star\theta_m \text{ in } (0,T)\times\mathbb{T}^d\,,\\ u(T,\cdot)\equiv 0\end{cases}\tag{4.11}$$

When, m being fixed, u is a solution of that equation, an optimal control solving (4.10) is given by

$$\alpha(t,x)=\nabla_x u(t,x).\tag{4.12}$$

Furthermore, when every agent follows a control in such a feedback form (*i.e* that only depends on t and x), the resulting density of fishermen m, initially distributed according to a probability distribution m_0, satisfies a Fokker-Planck-Kolmogorov (FPK) or continuity equation

$$\begin{cases}\partial_t m+\nabla\cdot(m\nabla_x u)=0 \text{ in } (0,T)\times\mathbb{T}^d\,,\\ m(0,\cdot)=m_0.\end{cases}\tag{4.13}$$

Overall, an MFG solution is a triplet (u^T,m^T,θ^T) of the system

$$
\begin{cases}
-\partial_t u^T - \frac{\|\nabla_x u^T\|^2}{2} = \rho \star \theta^T & \text{in } (0,T)\times\mathbb{T}^d\,,\\
u^T(T,\cdot)\equiv 0, & \\
\partial_t m^T + \nabla\cdot\left(m^T\nabla_x u^T\right) = 0 & \text{in } (0,T)\times\mathbb{T}^d\,,\\
m^T(0,\cdot) = m_0, & \\
\partial_t\theta^T - \mu\Delta\theta^T = \theta^T(K-\theta^T) - \varepsilon(\rho\star m)\theta^T & \text{in } (0,T)\times\mathbb{T}^d\,,\\
\theta^T(0,\cdot) = \theta_0. &
\end{cases}
\tag{4.14}
$$

As regards (4.14), the existence of a solution follows by a standard fixed point argument. Our main focus will be on the following more delicate qualitative queries:

(1) First, do we have *uniqueness* of the solution to (4.14)?
(2) Second, what can be said about the long-time behaviour of (4.14)?

The two queries are related, and, to state things in an informal manner, both rely on the Lasry-Lions monotonicity condition, which is the following: letting, for any $m\in L^1(0,T;\mathcal{P}(\mathbb{T}^d))$, where $\mathcal{P}(\mathbb{T}^d)$ denotes the set of Radon measures on the d-dimensional torus $\theta_{T,m}$ denote the solution of

$$
\begin{cases}
\partial_t\theta_{T,m} - \Delta\theta_{T,m} - \theta_{T,m}\left(K - \varepsilon\rho\star m - \theta_{T,m}\right) = 0 & \text{in } (0;T)\times\mathbb{T}^d\,,\\
\theta_{T,m}(0,\cdot) = \theta_0 &
\end{cases}
\tag{4.15}
$$

the Lasry-Lions condition writes

$$
\forall m\,,m'\in L^1(0,T;\mathcal{P}(\mathbb{T}^d))\,,\ \iint_{(0;T)\times\mathbb{T}^d}\varepsilon(\theta_{T,m}-\theta_{T,m'})\rho\star(m-m')\leq 0.
\tag{4.16}
$$

Indeed, this monotonicity condition has been known, since [**154, 155**], to guarantee uniqueness of the solution, while Cardaliaguet [**66**] showed that a stronger version of the inequality provides the long-time behaviour of the system. To be more precise about this last point, we expect that, in large times, the solution (u^T,m^T,θ^T) behaves like

$$
u^T\approx\overline{\lambda}(T-t)+\overline{u}\,,m^T\approx\overline{m}\,,\theta^T\approx\overline{\theta}_{\overline{m}}
$$

where $(\overline{\lambda},\overline{u},\overline{m},\overline{\theta}_{\varepsilon,\overline{m}})$ solve the so-called *ergodic system*

$$
\begin{cases}
\overline{\lambda} - \frac{\|\nabla_x\overline{u}\|^2}{2} = \rho\star\overline{\theta}_{\varepsilon,\overline{m}} & \text{in } \mathbb{T}^d\,,\\
\nabla\cdot(\overline{m}\nabla_x\overline{u}) = 0 & \text{in } \mathbb{T}^d\,,\\
-\mu\Delta\overline{\theta}_{\varepsilon,\overline{m}} = \overline{\theta}_{\varepsilon,\overline{m}}(K-\overline{\theta}_{\varepsilon,\overline{m}}) - \varepsilon(\rho\star\overline{m})\overline{\theta}_{\varepsilon,\overline{m}} & \text{in } \mathbb{T}^d\,,\\
\overline{\theta}_{\varepsilon,m}\geq 0,\ \overline{\theta}_{\varepsilon,m}\not\equiv 0,\ \int_{\mathbb{T}^d}\overline{u}=0\,,\overline{m}\geq 0\,,\int_{\mathbb{T}^d}\overline{m}=1. &
\end{cases}
\tag{4.17}
$$

In order to state the main result in [**KMFRB24a**] we need to introduce the solution θ_K to

$$
\begin{cases}
-\mu\Delta\theta_K - \theta_K(K-\theta_K) = 0 & \text{in } \mathbb{T}^d\,,\\
\theta_K\geq 0\,,\not\equiv 0. &
\end{cases}
\tag{4.18}
$$

4.3.2. Main results. Uniqueness and long-time behaviour for the MFG system

The first main result of [**KMFRB24a**] is the following:

THEOREM 4.6. *There exists $\overline{\varepsilon} > 0$ and $\delta > 0$ such that: if*

$$K \in \mathscr{C}^2(\mathbb{T}^d)\,, \|\theta_0 - \theta_K\|_{\mathscr{C}^{2,\alpha}} \leq \delta$$

where θ_K is defined by (4.18) *then, for any $T > 0$, for any $\varepsilon \leq \overline{\varepsilon}$, there exists a unique solution (u^T, m^T, θ^T) to* (4.14)*, and there exists a unique solution $(\overline{\lambda}, \overline{u}, \overline{m}, \overline{\theta}_{\varepsilon,\overline{m}})$ to* (4.17)*. Furthermore, define the following rescaled functions: for $(s, x) \in [0,1] \times \mathbb{T}^d$,*

$$\Theta^T(s,x) := \theta^T(sT, x),\ M^T(s,x) := m(sT, x),\ W^T(s,x) := u^T(sT, x) \tag{4.19}$$

Then there holds:

(1) $\Theta^T \underset{T\to\infty}{\to} \overline{\theta}_{\varepsilon,\overline{m}}$ *strongly in $L^2(0,1;L^2(\mathbb{T}^d))$ and, in fact:*

$$\left\|\Theta^T - \overline{\theta}_{\varepsilon,\overline{m}}\right\|^2_{L^2((0;1)\times\mathbb{T}^d)} \leq \frac{C}{T}$$

for some constant C.

(2) $W^T/T \underset{T\to\infty}{\to} \overline{u}$ *strongly in $L^\infty(0,1;L^\infty(\mathbb{T}^d))$ and, in fact*

$$\left\|\frac{W^T}{T} - \overline{\lambda}(1-s)\right\|_{L^\infty((0;1)\times\mathbb{T}^d)} \leq \frac{C}{T^{\frac{1}{2}}}.$$

As we explained above, the proof of this theorem essentially rests on proving (4.16). Although we refer to the bibliographical paragraph for a comprehensive discussion of this property, let us underline some key aspects:

(1) First, there are traditionally two distinct sets of assumptions when proving uniqueness in MFG systems. The first one would correspond to taking, for a fixed T, a small enough $\varepsilon > 0$ (or, for a fixed $\varepsilon > 0$, a small enough T). The problem with this is that ε will be *a priori* exponentially small in T, and this is thus not a good idea to do this if we want to investigate the long-time behaviour of the system. The other possibility is, as we said, to prove the Lasry-Lions monotonicity condition, or related monotonicity criteria. As far as we are aware, this is the first contribution where these two regimes are blended (*i.e.* taking $\varepsilon > 0$ small enough to ensure the Lasry-Lions condition is satisfied).

(2) Second, once (4.16) is established, there is essentially nothing left to do but to adapt almost *verbatim* the work of Cardaliaguet [**66**] and of Cannarsa, Cheng, Mendico & Wang [**59**] in order to obtain the long-time behaviour of the system. For this reason, in our sketch of proof, we will be focusing on the proof of (4.16).

The tragedy of the commons

In [**KMFRB24a**] we also investigate another telling instance of the tragedy of the commons, working on the ergodic system (4.17). Our main result is the following:

THEOREM 4.7. *Let $d = 1$ and identify $\mathbb{T}$ with the open interval $(-0.5, 0.5)$ endowed with periodic boundary conditions. Let $K \in C^1(\mathbb{T})$ be even and such that $\partial_x K < 0$ on $(0, 0.5)$. Consider the system*

$$\begin{cases} \overline{\lambda} - \frac{(\overline{u}')^2}{2} = \overline{\theta}\,, \\ (\overline{m}u')' = 0\,, \\ -\mu\overline{\theta}'' = \overline{\theta}(K - \varepsilon\overline{m} - \overline{\theta})\,, \\ \overline{\theta} \geq 0,\ \overline{\theta} \not\equiv 0,\ \int_{\mathbb{T}} \overline{u} = 0\,, \overline{m} \geq 0\,, \int_{\mathbb{T}} \overline{m} = 1. \end{cases} \tag{4.20}$$

If $\varepsilon \in (0, \int_{\mathbb{T}} K)$*, there exists a unique solution* $(\overline{\lambda}, \overline{\theta}, \overline{u}, \overline{m})$ *of* (4.20) *with* $\overline{\theta} \neq 0$*. If* $\varepsilon > \int_{\mathbb{T}} K$*, any solution of* (4.20) *satisfies* $\overline{\theta} = 0$ *a.e. in* $\mathbb{T}$*.*

The reason we claim that this is an instance of the tragedy of the commons is that even for $\varepsilon > \int_{\mathbb{T}^d} K$ (and up to a certain threshold) it is naturally possible to find a distribution of players $\overline{m}_1$ such that the equation

$$-\mu\overline{\theta}'' = \overline{\theta}(K - \varepsilon\overline{m}_1 - \overline{\theta})$$

has a non-zero solution. Thus, because of the competition between players, the only equilibrium situation is a situation where all fishes are dead and where no player gets a positive reward, whereas, had they coordinated their action, they could have found a way to ensure survival of the fishes and a positive outcome.

4.3.3. Related literature. The literature devoted to Mean Field Games is extensive and it is hopeless to provide an exhaustive bibliography; to signal the references most relevant to us, let us mention the foundational works of Lasry & Lions on the one hand [**154, 155, 156**] and of Caines, Huang & Malhamé [**137, 136**] on the other. A large part of the theory was developed in the lectures of Lions at Collège de France, for which we refer to the notes of Cardaliaguet [**65**] and to those of Cardaliaguet & Porretta [**67**]. The most salient aspects of the theory, keeping in mind our problems, are on the one hand uniqueness properties, which will be discussed in details in due time, and the long-time behaviour of the MFG system. In this chapter, we heavily rely on the framework developed by Cardaliaguet [**66**], using the weak KAM theory, to understand the long-time behaviour of first order MFG systems and their convergence to the so-called ergodic system. We also refer to the work of Cannarsa, Cheng Mendico & Wang [**59**]. Let us insist on the fact that in [**59, 66**], the structural assumptions made to guarantee convergence to the ergodic system essentially boil down to a strengthened Lasry-Lions monotonicity condition. We leave as an interesting question for future research the possible emergence of multiple solutions for such MFG systems for the optimal management of fisheries–we refer to the conclusion of the chapter.

4.3.4. Sketches of proof.

Sketch of proof of Theorem 4.6. As we explained, most of the proof boils down to proving (4.16) and this, in turn hinges on the methodology introduced in Chapter 1. To be more specific, we introduce the functional

$$G : (m, m') \mapsto \int_{(0;T)\times\mathbb{T}^d} (\theta_{T,\varepsilon,m} - \theta_{T,\varepsilon,m'})\rho \star (m - m')$$

where $\theta_{T,\varepsilon,m}$ solves the Cauchy problem (4.15) (we add the subscript $\varepsilon > 0$ to emphasise the dependence in this variable). Our strategy of proof is thus to study the following optimisation problem: for a fixed $m \in L^1(0, T; \mathcal{P}(\mathbb{T}^d))$, solve

$$\max_{m' \in L^1(0,T);\mathcal{P}(\mathbb{T}^d)} G(m, m'). \tag{4.21}$$

Of course, the monotonicity condition is guaranteed if we can prove that m solves (4.21). Introduce, for notational simplicity,

$$g : m' \mapsto G(m, m').$$

It is easy to check that $m' = m$ is a critical point of g. The difficult part is showing that, if $\varepsilon > 0$ is small enough then, for any $T > 0$, g is (strictly) concave in m'. This is exactly what we prove and, without getting into details, let us underline that this is done in the same way that we proved Theorem 4.3, namely, by exploiting the ideas of Chapter 1: morally, as long as the functional is monotone increasing with respect to m', it is concave with respect to m'.

□

SKETCH OF PROOF OF THEOREM 4.7. We recall that the objective is to build solutions to

$$\begin{cases}\overline{\lambda}-\frac{(\overline{u}')^2}{2}=\overline{\theta}\,,\\ (\overline{m}u')'=0\,,\\ -\mu\overline{\theta}''=\overline{\theta}(K-\varepsilon\overline{m}-\overline{\theta})\,,\\ \overline{\theta}\geq 0,\ \overline{\theta}\not\equiv 0,\ \int_{\mathbb{T}}\overline{u}=0\,,\overline{m}\geq 0\,,\int_{\mathbb{T}}\overline{m}=1.\end{cases}$$

The first step is to observe that for any solution of this system there holds

$$\operatorname{supp}(m)\subset\{\overline{\theta}=\|\overline{\theta}\|_{L^\infty}\}.$$

This is simply a rewriting of the weak KAM formula. The proof then follows from the analysis the system of ODE, and boils down to proving the following facts:

(1) First, $\overline{m}$ has not atoms.
(2) Second, $\operatorname{supp}(m)$ is connected. This can easily be seen, as otherwise we can find $a,b\in\operatorname{supp}(m)$, $(a;b)\notin\operatorname{supp}(m)$. Then $\overline{\theta}'(a)=\theta'(b)=0\,,\overline{\theta}(a)=\overline{\theta}(b)=\|\overline{\theta}\|_{L^\infty}$. This is, however, in contradiction with the condition $\partial_x K<0$, which implies that $\partial_x\overline{\theta}\leq 0$ in $(a;b)$.
(3) By the same type of reasoning, the monotonicity of K implies that $0\in\operatorname{supp}(m)$, and that $\operatorname{supp}(m)=(-a;a)$.

The final step in the proof is showing that these conditions fully determine $\overline{m}$.

Now, in order to prove that if $\varepsilon>\int K$ then any solution is trivial, observe that, if a non-trivial solution exists, then there exists $c\neq 0$ such that

$$\begin{cases}\overline{\lambda}-\frac{(\overline{u}')^2}{2}=\overline{\theta}\,,\\ (\overline{m}u')'=0\,,\\ -\mu\overline{\theta}''=\overline{\theta}(K-\varepsilon\overline{m}-\overline{\theta})\,,\\ \overline{\theta}\geq 0,\ \overline{\theta}\not\equiv 0,\ \int_{\mathbb{T}}\overline{u}=0\,,\overline{m}\geq 0\,,\int_{\mathbb{T}}\overline{m}=1.\end{cases}\qquad\begin{cases}\operatorname{supp}(m)\subset\{\overline{\theta}=\|\overline{\theta}\|_{L^\infty}=c\}\\ \operatorname{supp}(m)=(-a;a).\end{cases}$$

Consequently, from $\theta\equiv c>0$ in $\operatorname{supp}(m)$ we get

$$K-\varepsilon\overline{m}\equiv c\Rightarrow\int K>\varepsilon.$$

Thus, if $\varepsilon>\int K$, $\overline{\theta}=0$.

□

4.4. Mean Field Game models II: a travelling waves' approach to the tragedy of the commons

4.4.1. Presentation of the model and main results. We finally present the results of [**KMFRB24b**] where the focus is on a travelling waves' approach to the tragedy of the commons. More specifically, we consider a bistable non-linearity f, which we assume writes

$$f:[0;1]\ni x\mapsto x(1-x)(x-\eta)$$

for some $\eta \in (0;1)$. This non-linearity has two stable roots, 0 and 1, and one unstable root η. We refer to Figure 2 below.

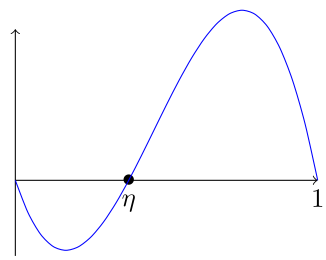


FIGURE 2. Graph of a bistable non-linearity

Such non-linearities are ubiquitous in the field in population dynamics and model the so-called Allee effect: considering a differential equation model (or, equivalently, a spatially homogeneous case)

$$\begin{cases} \theta'(t) = f(\theta)\,, \\ \theta_0 \in (0;1)\,, \end{cases}$$

then, if $\theta_0 < \eta$, then the population will die out, while if $\theta_0 > \eta$, it will survive in long time. We refer to [**97, 98**] for further modelling considerations. We further assume

$$\int_0^1 f(\xi)d\xi > 0. \tag{4.22}$$

In the spatially homogeneous case, that is, when we consider the equation (set on the real line)

$$\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} = f(\theta) \text{ in } (0;+\infty)\times\mathbb{R} \tag{4.23}$$

Assumption (4.22) allows to give a clear meaning to the wording "invasive species". Indeed, one can prove the following result [**97, 98**]:

(1) Under Assumption (4.22), there exists a unique speed $c^* > 0$ and a unique (up to a translation) profile $\overline{\theta}$ such that

$$\begin{cases} \theta : (t,x) \mapsto \overline{\theta}(x - c^* t) \text{ solves } (4.23), \\ \overline{\theta}(-\infty) = 1\,, \\ \overline{\theta}(+\infty) = 0. \end{cases} \tag{4.24}$$

This solution is called an "invading travelling wave".

(2) Furthermore, this travelling wave is dynamically attractive: for any initial condition θ_0 such that

$$\lim_{s\to-\infty} \theta_0 \in [0;\eta)\,,\ \lim_{s\to+\infty} \theta_0 \in (\eta;1],$$

there exists $x_0 \in \mathbb{R}\,, w > 0$ such that the solution of

$$\begin{cases} \frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} = f(\theta) & \text{in } (0;+\infty)\times\mathbb{R}\,, \\ \theta(0,\cdot) = \theta_0 & \text{in } \mathbb{R} \end{cases} \tag{4.25}$$

satisfies

$$\sup_{x\in\mathbb{R}} |\theta(t,x) - \overline{\theta}(x - c^*t - x_0)| = \underset{t\to\infty}{O}(e^{-wt}). \tag{4.26}$$

We refer to the monograph of Fife & McLeod [98].

Now, the question we seek to answer in [KMFRB24b]: is it possible that the competition between fishermen acting in their self-interest lead to a *reversal* of this invading travelling waves? Can we also, in this context, prove a "tragedy of the commons" phenomenon?

The framework we adopt is once again that of Mean Field Games. From the same modelling assumptions that were considered in Section 4.3, we consider a density m of players, each seeking to maximise their own gain by controlling their trajectory through a control α. For a given density of players m, we let θ_m be solution to the Cauchy problem

$$\begin{cases} \partial_t\theta_m - \frac{\partial^2\theta_m}{\partial x^2} = f(\theta_m) - m\theta_m & \text{in } (0;T)\times\mathbb{R}\,, \\ \theta_m(0,\cdot) = u_0 & \text{in } \mathbb{R}. \end{cases} \tag{4.27}$$

An agent is trying to maximise his outcome, this time in an infinite time horizon, albeit with a discount factor quantified by a parameter $\lambda > 0$ (the greater λ, the more short term minded the agent is). Although in the previous paragraph we took a quadratic penalisation of the control cost, here we allow for a general Lagrangian L that satisfies usual assumptions (in particular, strong convexity, smoothness and superlinearity), and a player starting from the initial position x_0 solves the optimisation problem

$$\sup_{\alpha\in L^\infty(\mathbb{R})} J(x_0,\alpha,\theta) := \int_0^\infty e^{-\lambda t}\left(\theta_m(t,x_\alpha(t)) - L(\alpha)\right)dt \tag{4.28}$$

$$\text{subject to } \begin{cases} x_\alpha'(t) = \alpha(t)\,, \\ x_\alpha(0) = x_0 \end{cases}.$$

We underline that the penalisation of the control does play an important role here. Similar to the previous paragraph, we introduce the value function

$$u : (t_0,x_0) \mapsto \sup_{\alpha\in L^\infty((t_0;\infty))} \left(\int_{t_0}^\infty e^{-\lambda(t-t_0)}\left(\theta_m(t,x_\alpha(t)) - L(\alpha)\right)dt\right) \tag{4.29}$$

$$\text{subject to } \begin{cases} x_\alpha'(t) = \alpha(t)\,, \\ x_\alpha(t_0) = x_0. \end{cases}$$

If we introduce the associated Hamiltonian

$$H : p \mapsto \sup_{\alpha\in\mathbb{R}} \left(p\alpha - L(\alpha)\right)$$

then an application of the Bellman dynamic programming principle yields, similar to (4.14), the following MFG system:

$$\begin{cases} \lambda u - \partial_t u - H(\partial_x u) = \theta & \text{in } (0;T)\times\mathbb{R}\,, \\ \partial_t m + \partial_x\left(H'(\partial_x u)m\right) = 0 & \text{in } (0;T)\times\mathbb{R}\,, \\ m(0,\cdot) = m_0, \\ \partial_t\theta - \partial_{xx}^2\theta = f(\theta) - m\theta & \text{in } (0;T)\times\mathbb{R}\,, \\ \theta(0,\cdot) = \theta_0. \end{cases} \tag{4.30}$$

In contrast to Section 4.3, the focus here is not on the uniqueness of solutions or on their long-time behaviour but rather on exhibiting particular solutions–the long-time behaviour of generic solutions seems out of reach at the moment. To be more specific, the focus of [**KMFRB24b**] is on the existence of *reversed travelling waves*: is it possible to find a triplet (U, M, Θ) and a *negative* speed $c < 0$ such that:

(1) $\Theta(-\infty) = 1$,
(2) $\Theta(+\infty) = 0$,
(3) For any $x_0 \in \mathrm{supp}(M)$, the constant control $\overline{\alpha} \equiv c$ is optimal in

$$u : (t_0, x_0) \mapsto \sup_{\alpha \in L^\infty((t_0;\infty))} \left(\int_{t_0}^{\infty} e^{-\lambda(t-t_0)} \left(\Theta(x_\alpha(t) - ct) - L(\alpha)\right) dt \right) \quad \text{subject to } \begin{cases} x'_\alpha(t) = \alpha(t)\,, \\ x_\alpha(t_0) = x_0. \end{cases} \tag{4.31}$$

In particular, the triplet $(t, x) \mapsto (U(x - ct), M(x - ct), \Theta(x - ct))$ solves (4.30).

The term "reversed" travelling wave comes from the fact that, in the absence of fishermen, with such an initial condition Θ, the result of Fife & McLeod entails that in long time the fish population would survive.

This question pertains to a growing literature on the existence (and stability) of travelling waves in MFG systems, which we give an overview of in the bibliographical paragraph.

Our results are twofold, first dealing with the existence of such reversed travelling waves, second with the tragedy of the commons.

THEOREM 4.8. *[Existence of reversed travelling waves,* [**KMFRB24b**]*] For any $\lambda > 0$, there exists $c_0(\lambda) > 0$ such that, for any $c \in (0, c_0(\lambda))$ there exists a unique monotonous (in the sense that $\Theta' > 0$) reversed MFG travelling wave with velocity c.*

In particular, in such a situation, the competition between fishermen drives the fish population to extinction.

REMARK 4.9. In [**KMFRB24b**] we also construct periodic reversed travelling waves (discarding the conditions $\Theta(+\infty) = 0\,, \Theta(-\infty) = 1$) and we provide finer estimates on $c_0(\lambda)$ but for the sake of clarity we do not present these results here.

REMARK 4.10. Observe that we do not deal with the attractivity of these reversed travelling waves in the MFG setting; this seems to be an out of reach question and we refer to the conclusion of the chapter for more comments on this aspect.

Our next result shows another instance of the tragedy of the commons, by exhibiting, starting from the initial condition $\theta_0 = \Theta$ (where Θ is the reversed travelling profile given by Theorem 4.8), a coordinated strategy that both allows the fishes population to survive and gives each fisherman a higher harvest. This is where we will need some freedom on the choice of the Lagrangian L.

THEOREM 4.11 (The tragedy of the commons revisited, [**KMFRB24b**]). *There exist a Lagrangian L, a discount factor λ and $c > 0$ such that:*

(1) There exists a monotonous reversed MFG travelling wave $(c, \lambda, M, \Theta, U)$,
(2) There exists an explicit strategy $\alpha_{\text{co}} \in L^\infty(\mathbb{R})$ that satisfies:

$$U(x_0) < J(x_0, \alpha_{\text{co}}, \theta_{\alpha_{\text{co}}}) \quad \textit{where} \quad \begin{cases} \partial_t \theta_{\alpha_{\text{co}}} - \partial^2_{xx} \theta_{\alpha_{\text{co}}} = f(\theta_{\alpha_{\text{co}}}) - m_{\alpha_{\text{co}}} \theta_{\alpha_{\text{co}}}, \\ \theta_{\alpha_{\text{co}}}(0) = \Theta, \\ \partial_t m_{\alpha_{\text{co}}} + \partial_x (\alpha_{\text{co}} m_{\alpha_{\text{co}}}) = 0, \\ m_{\alpha_{\text{co}}}(0) = M. \end{cases}$$

We will present sketches of proof of these two theorems after reviewing some related references.

4.4.2. Related literature. There are two major research lines that bear a strong link to the endeavours of this paragraph. The first one has to do with optimal control involving travelling waves, while the second ones deals more specifically with the emergence of travelling waves in MFG systems.

Optimal control problems involving travelling waves

As travelling waves are a robust paradigm to describe and analyse the invasion of species, it has seen a growing interest in recent years when discussing the prevention of invasions, in particular of disease-carrying species, the typical question being: given an invasive species, where and how should one act in order to block its propagation and eventual invasion? We would like to refer in particular to [**7, 8**] and the references therein. Belonging to this line of work and of particular relevance to [**KMFRB24b**] is the work of Bressan, Chiri & Salehi [**38**] (as well as their subsequent contributions [**30, 37, 39**]) which aims at blocking an invasive species in the most efficient way possible, using measures as controls. Although the objective (for them, killing an invasive species in the most effective way) is different from ours, our analysis uses, like theirs, a phase plane analysis of the system.

Travelling waves in MFG systems

Travelling waves are central in the analysis of nonlinear equations as they can often accurately predict the long-time behaviour of reaction-diffusion systems. In recent years, there has been a surge of interest in travelling waves solutions for a MFG system related to knowledge growth models, starting with the work of Lucas & Moll [**171**]. We refer to the contributions of Porretta & Rossi [**196**], Burger, Kanzler, Lorz & Wolfram [**53, 54**] and of Berestycki, Novikov, Roquejoffre & Ryzhik [**26**].

4.4.3. Sketches of proof.

Sketch of proof of Theorem 4.8. The proof of this theorem goes in two steps:

(1) We first construct a travelling wave solution (with negative speed) of the MFG system–observe that as the MFG system is an optimality condition, this does not give the result. This step relies heavily on the phase plane analysis of bistable equations.
(2) We then check that the strategy given by this MFG system is an optimal strategy. This is a delicate endeavour. Although [**KMFRB24b**] contains several results encompassing a wide class of Lagragians, for the sake of readability, we only present a proof in the case of quadratic Lagrangians.

Construction of a travelling wave solution of the MFG system

We begin by observing that if a reversed travelling wave solution (U, M, Θ) with speed $c < 0$ exists, then it must satisfy the MFG system

$$\begin{cases} \lambda V + cV' - H(V') = \Theta & \text{in } \mathbb{R} \\ H'(V') = c & \text{in } \operatorname{supp}(M)\,, \\ -\Theta'' - c\Theta' = f(\Theta) - M\Theta & \text{in } \mathbb{R}. \end{cases} \tag{4.32}$$

Now, the idea is quite simple (we skip many technical details here): we consider the phase-portrait for the bistable equation

$$-\frac{d^2\Theta}{dt^2} - c\frac{d\Theta}{dt} = f(\Theta)$$

in the absence of fishermen which rewrites as

$$\frac{d}{dt}\begin{pmatrix}\Theta' \\ \Theta\end{pmatrix} = \begin{pmatrix}c\Theta' + f(\Theta) \\ \Theta'\end{pmatrix}.$$

This equation has three particular equilibria, $(0,1), (0,0)$ and $(0,\eta)$. There are essentially two different phase portraits associated with this system depending on the value of c which has an impact on the nature of the equilibrium $(0,\eta)$ but suffices to say that, for $0 < c < 2\sqrt{f'(\eta)}$ this equilibrium is a sink: the linearised system has two complex conjugate eigenvalues with negative real parts. We represent it in Fig. 3.

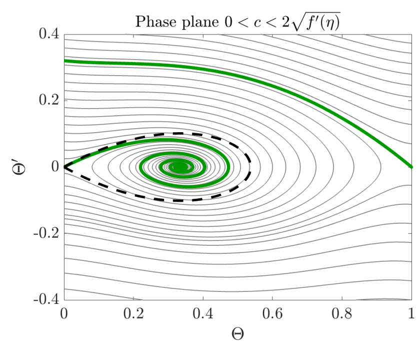


FIGURE 3. Phase portrait for the uncontrolled dynamics $(M = 0)$

There are two curves if interest (in green in Fig. 3): the first one is the unstable manifold associated with $(0,0)$, that links $(0,0)$ to $(0,\eta)$, and the second one is the stable manifold associated with $(0,1)$. The idea is to build a "candidate" density of players M to link these two manifolds and that does so in such a way that (4.32) is satisfied. Since we already know that in the support of M we must have $H'(V')$ constant, this implies that V' is constant and, in turn, that Θ must be linear, which also gives Θ' constant in the support of M. This suggests the following construction:

(1) First, follow the unstable manifold starting at $(0,0)$.
(2) At some point, use the control M to jump off this unstable manifold to move along the line $\Theta' =$ constant until the stable manifold associated with $(0,1)$ is reached.
(3) Stop acting after this point.

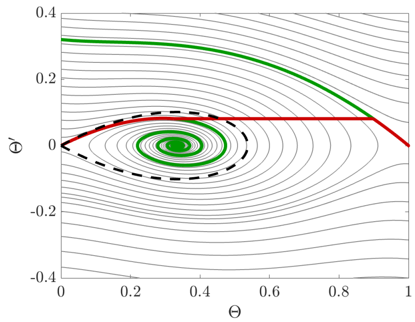


FIGURE 4. Construction of the control M, acting on the part bridging the unstable manifold associated with $(0,0)$ and the stable manifold associated with $(0,1)$.

This construction is illustrated in Fig. 4.

The difficulty here is to check that if M is constructed in such a way then it is non-negative and integrable; this can be done by tuning the moment where one jumps from the unstable manifold.

Checking the optimality of the constant speed strategy

The construction of the first step provides a solution to (4.32), but does not show that the constant speed strategy $\alpha \equiv c$ is optimal (which would be required for the triplet (U, M, Θ) to be a reversed travelling wave). In general, this is a delicate endeavour, but it is particularly simple in a specific case, that of quadratic Lagrangian. Namely, we consider, as Lagrangians,

$$L_\varepsilon(\alpha) := \frac{\alpha^2}{2\delta},$$

and we now show that, for $\delta > 0$ small enough, the triplet constructed in the first step is indeed a reversed travelling wave. To this end, recall that we solve the optimisation problem: for any $x_0 \in \operatorname{supp}(M)$,

$$\max_{\alpha \in L^\infty((t_0;\infty))} \left(J(\alpha) := \int_{t_0}^{\infty} e^{-\lambda(t-t_0)} \left(\Theta(x_\alpha(t) - ct) - L_\varepsilon(\alpha)\right) dt \right).$$

The fact that (U, M, Θ) solves (4.32) proves that $\alpha \equiv c$ is a critical point for this functional. Now, it is easy to observe that whenever $\delta > 0$ is small enough the functional $\alpha \mapsto J(\alpha)$ is strictly concave, which allows to conclude.

□

SKETCH OF PROOF OF THEOREM 4.11. To prove this theorem, we actually relies on the stability of invasion travelling waves 4.26. Here again, we skip the main technical details, and we focus on the essential idea: we construct a "spreading" population of fishermen. Namely, letting M be the density of fishermen provided by the reversed travelling wave, we introduce the coordinated strategy

$$\alpha_{\mathrm{co}} : (t, x) \mapsto \frac{x}{s_{\mathrm{ref}} + t}$$

where s_{ref} is a parameter provided in the construction of the reversed travelling wave, so that the density of players following this strategy is given by

$$m:(t,x)=\frac{2s_{\text{ref}}}{2s_{\text{ref}}+t}M\left(\frac{2s_{\text{ref}}+t}{2s_{\text{ref}}}x\right).$$

In particular, whenever t is large enough, $\|m\|_{L^\infty(\mathbb{R})}\ll 1$, and we can thus apply (4.26) to conclude that in this case the fish population is still invading in large times. Of course, the delicate part regards showing that this does better, for each fisherman, than the reversed travelling wave's strategy. This is where one needs to tune the family of Lagrangians. This step is quite technical and we refer to [**KMFRB24b**].

□

4.5. Research plan

In the conclusion of this chapter, we review what we believe some of the next important problems are in the management of fisheries. We focus on the qualitative interpretation of these questions.

Non-uniqueness for monostable MFG models

This pertains to the analysis of Section 4.3 dealing with monostable MFG models. As we explained, one of the main features of [**KMFRB24a**] was an investigation of the uniqueness for (4.14), which was achieved by showing the monotonicity property (4.16). With G. Lamonaca & G. Nadin, we are investigating situations where this monotonicity might be broken, which would then allow to apply the bifurcation framework developed by Cirant [**84**]. We also refer to [**19**] for a global discussion of non-uniqueness of solutions to MFG systems.

Optimal control in the management of fisheries

A second future research line is that of collaborative models, in which a central planner gives every agent its strategy. In this category of MFG, the main question we would like to investigate is the question of optimal resources managements: namely, considering a situation in a which a fishery is commonly exploited by agents, how should these agents feed the fish in order to maximise their eventual harvest? This question also ties to the considerations of Chapters 1–2.

More complex models

As we stated at the beginning of the chapter, the models we investigated thus far are simplistic in their assumptions. There are several further phenomena that have yet to be adequately modelled:

(1) The first one has to do with the interactions between fishermen; namely, at this stage, we have assumed that all fishermen were indistinguishable and that they all carried the same weight in the game. It would be interesting to incorporate "follow the leader" types of dynamics or to propose models scaling the relative importances of the different agents or fishing companies.

(2) The second one deals with the fact that, in the MFG models under consideration, no choice is given to the player to fish or not: they simply choose where they should be fishing. Incorporating a fishing activation parameter that allows the agent to decide whether or not he wants to fish seems like a natural next step. Another modelling possibility would be to

consider a situation where fishermen, as long as they are out to sea, fish, but can decide when to exit the game [29].

(3) We conclude with the most exploratory of these aspects, namely, improving the relevance of the model. There are several routes to this: a possibility would be incorporating more structure in the fish population itself. Although this is not yet clear what the good compromise between mathematical tractability and ecological relevance might be, let us observe that recent contributions by theoretical ecologists (see in particular [220] and the references therein) might serve as a starting point.

CHAPTER 5

Some vectorial shape optimisation problems

This last chapter is split into two distinct, independent parts, each corresponding to a currently developing line of our research. The common theme is the study of optimisation problems involving systems of PDEs:

(1) In Section 5.1 we study optimisation problems for systems of reaction-diffusion equations with a particular emphasis on cooperative models. The relevant article is [**GM21**], in collaboration with L. Girardin.

(2) In Section 5.2 we focus on a shape optimisation problem coming from the study of the Stokes operator, with a particular attention to the Faber-Krahn inequality. This corresponds to the article [**HMFP24**].

As both these contributions correspond to recent developing questions, this chapter is shorter than the previous ones and we note that the interesting open problems are listed at the end of each section, rather than at the end of the chapter.

5.1. Optimal control of systems arising from population dynamics

5.1.1. First considerations. In the first paragraph of this chapter we discuss the contribution [**GM21**] which deal with cooperative systems of reaction-diffusion equations. Broadly speaking, a cooperative system of N species is a system that writes

$$\forall i \in \{1, \dots, N\}\,, \begin{cases} \partial_t u_i - \mu_i \Delta u_i = f_i(u_1, \dots, u_N)\,, \\ u_i(0, \cdot) = u_i^{(0)} \geq 0\,, \not\equiv 0 \end{cases} \tag{5.1}$$

endowed with Neumann or Dirichlet boundary conditions–here and throughout, the equation is set in a smooth, bounded domain $\Omega \subset \mathbb{R}^d$. The wording *cooperative* refers, in this chapter, to the condition

$$\forall i \neq j\,, \frac{\partial f_i}{\partial u_j} \geq 0. \tag{5.2}$$

This condition can be recast as: any sub-species interacting with another one is in fact helping it grow. Mathematically, such models are extremely well-behaved, as (5.1) essentially amounts to having a (sometimes strong) maximum principle. Because of that, most of these systems share striking similarities with scalar reaction-diffusion equations and, broadly speaking (as one would need to distinguish between irreducible systems etc), the results of Theorem 1.2 holds: the long-time behaviour of (5.1) is governed by a spectral quantity, the principal eigenvalue of the system.

Type of optimisation problems under consideration In this section, we discuss a small part of the memoir [**GM21**], in which we undertake a systematic study of eigenvalue problems for cooperative systems. Namely, the two classes of results we present are, first, the optimisation of mutation strategies in cooperative system, for which we obtain a bang-bang like property (see Theorem 5.2) and, second, the optimisation of cooperative eigenvalues with respect to the potentials (see Theorem 5.3). This requires discussing the notion of principal eigenvalues for such systems.

Principal eigenvalue for periodic cooperative systems

For the sake of simplicity, and although [**GM21**] works with general space-time periodic coefficients, we assume that we are working in the torus $\mathbb{T}^d$ with a fixed time horizon $T>0$. We use the following notations:

(1) $\mathbf{u}$ denotes the vector valued function $\mathbf{u}:(t,x)\mapsto(u_1(t,x),\dots,u_N(t,x))$.
(2) The notation $\mathbf{u}\geq 0$, resp. $\mathbf{u}\gg 0$, stands for:
$$\forall i\in\{1,\dots,N\}\,,\forall(t,x)\in(0;T)\times\mathbb{T}^d\,,u_i(t,x)\geq 0,$$
resp.
$$\forall i\in\{1,\dots,N\}\,,\forall(t,x)\in[0;T]\times\mathbb{T}^d\,,u_i(t,x)>0.$$
(3) The notation $\operatorname{Diag}(\mu_1,\dots,\mu_N)$ stands for the diagonal matrix $\begin{pmatrix}\mu_1 & 0 & \dots & 0\\ 0 & \mu_2 & \dots & 0\\ \vdots & & \ddots & \vdots\\ 0 & \dots & 0 & \mu_N\end{pmatrix}$.

We consider the eigenvalue problem

$$\begin{cases}\partial_t\mathbf{u}-\operatorname{Diag}(\mu_1,\dots,\mu_N)\Delta\mathbf{u}=L\mathbf{u}+\lambda(L)\mathbf{u} \quad \text{ in } (0;T)\times\mathbb{T}\,,\\ \mathbf{u}(T,\cdot)=\mathbf{u}(0,\cdot)\,,\\ \mathbf{u}\geq 0\,,\mathbf{u}\not\equiv 0,\end{cases} \tag{5.3}$$

where:

(1) For any $i\in\{1,\dots,N\}$, $\mu_i>0$,
(2) L is a matrix valued function $L=(\ell_{i,j})_{i,j=1,\dots,N}$ and, for any $i,j\in\{1,\dots,N\}$, $\ell_{i,j}\in L^\infty((0;T)\times\mathbb{T})$.

Of course, without any assumption on L, there is no guarantee that there exists a unique $(\lambda(L),\mathbf{u})$ satisfying (5.3). Let us discuss how to define a principal eigenpair $(\lambda(L),\mathbf{u})$ in a meaningful way, first under a simple assumption, and then in a more general setting.

A simple set of assumptions ensuring the existence and uniqueness of an eigenpair Assume the following:

(1) The system is cooperative:
$$\forall i\neq j\,,\ell_{i,j}\geq 0 \text{ in } (0;T)\times\mathbb{T}.$$
(2) The system is irreducible: the matrix
$$\overline{L}:=\left(\|\ell_{i,j}\|_{L^\infty((0;T)\times\mathbb{T}^d)}\right)_{i,j=1,\dots,N}$$
is irreducible: it does not have any invariant subspace of the form $(e_{i_1},\dots,e_{i_K})$ where $(e_i)_{i=1,\dots,N}$ is the canonical basis for $K<N$.

Under these properties, which ensure that the parabolic operator has a strong maximum principle, (5.3) has a unique solution $(\lambda(L),\mathbf{u})$, and that we furthermore have $\mathbf{u}\gg 0$. This can be proved using the Perron-Frobenius approach to principal eigenvalues. The link with the evolution equation (5.1) is the same as in Chapters 1–2 (see in particular Remark 1.3): (5.3) corresponds to a linearisation of (5.1) around the trivial equilibrium $(0,\dots,0)$, and the positivity or negativity of $\lambda(L)$ entails survival or extinction for the system (5.1) under assumption (5.2).

REMARK 5.1. There are situations–typically when studying age-structured equations–where the non-linear system is not cooperative but where the linearisation around the trivial equilibrium nevertheless is. This leads to complicated dynamics that have been the subject of an intense research activity. We do not dwell on this aspect here and we refer for instance to [**3**, **34**, **43**, **63**, **64**, **106**, **105**, **107**, **108**, **109**, **111**, **112**, **125**, **178**].

How to proceed when the matrix L is not irreducible?

In some of our optimisation results, since we will be optimising L among large classes of matrices, and since irreducibility is not closed under weak or strong convergence (as one can see by taking $L_\varepsilon := \begin{pmatrix}1-\varepsilon & \varepsilon\\ \varepsilon & 1-\varepsilon\end{pmatrix}$ and letting $\varepsilon\to 0$), we need to briefly explain how to define the correct notion of principal eigenvalue in the general case. Throughout, we still assume that L is essentially non-negative in the sense that

$$\forall i\neq j\in\{1,\dots,N\}\,,\ \text{for a.e. } (t,x)\in(0;T)\times\mathbb{T}^d\,, \ell_{i,j}(t,x)\geq 0. \tag{5.4}$$

In that case, the notion of principal eigenvalue has to be defined in the fashion of Berestycki, Nirenberg & Varadhan [**25**] as

$$\lambda_1(L):=\sup\left\{\lambda\in\mathbb{R}:\ \exists \mathbf{u}\in\mathscr{C}^{1,2}_{\text{per}}((0;T)\times\mathbb{T})\,,\partial_t\mathbf{u}-\operatorname{Diag}(\mu_1,\dots,\mu_N)\Delta\mathbf{u}\geq L\mathbf{u}+\lambda\mathbf{u}\right\}.$$

where $\mathscr{C}^{1,2}_{\text{per}}((0;T)\times\mathbb{T})$ is the set of $\mathscr{C}^1$ in time, $\mathscr{C}^2$ in space, T-periodic vector fields. A core point of [**GM21**] is proving that this quantity $\lambda_1(L)$ is indeed an eigenvalue. Observe that in this case $\lambda_1(L)$ is not simple in general, and its multiplicity is a delicate question.

While we refer to [**GM21**] for a systematic study of these eigenvalues, an important case in the next sections is when L can be decomposed in a certain basis as

$$L=\begin{pmatrix}L_1^{\triangle} & 0 & \dots & 0\\ 0 & L_2^{\triangle} & \dots & 0\\ \vdots & & \ddots & \vdots\\ 0 & \dots & 0 & L_N^{\triangle}\end{pmatrix} \tag{5.5}$$

where each of the $L_i^{\triangle}$ $(i=1,\dots,N)$ is essentially non-negative and irreducible. In this setting, letting $\lambda_1(L_i^{\triangle})$ be the principal eigenvalue of the relevant subsystem, we can prove that

$$\lambda_1(L)=\min_{i=1,\dots,N}\lambda_1(L_i^{\triangle}),$$

meaning that the question fully decouples between the different irreducible subsystems. We refer to [**GM21**, Theorem 1.5] for a proof of this fact. This will be useful when dealing with optimal mutation strategies.

Scope of the section

The main question under study in this section is the following:

What is the matrix L that minimises $\lambda(L)$?

Here, unlike the scalar case, we need to distinguish between the type of interactions we allow. We consider essentially two situations:

(1) The first one corresponds to the case where L can be decomposed as
$$L = (S - \mathrm{Id}) - \mathrm{Diag}(m_1, \dots, m_N)$$
where $m_i \in L^\infty((0;T) \times \mathbb{T}^d)$ and where S is a bistochastic matrix. This corresponds to *mutations*, the mutation strategy being the matrix S, and this is the first question we seek to answer: what is the best mutation strategy?
(2) The second one is more finely tied with the scalar case, and we simply allow ourselves to optimise every entry $\ell_{i,j}$ of the matrix L under L^∞ and L^1 constraints. There, the main results deal with Talenti inequalities.

5.1.2. Optimisation of mutation strategies. Our first result deals with the case of mutations. To set the stage, we let
$$\mathcal{S} := \left\{S \in L^\infty((0;T) \times \mathbb{T}^d; M_N(\mathbb{R})) : S \text{ is bistochastic for a.e. } (t,x)\right\}$$
where we recall that a matrix $S = (s_{i,j})_{i,j=1,\dots,N}$ is bistochastic if for any $i\,,j \in \{1, \dots, N\}$,
$$\sum_{k=1}^N s_{i,k} = \sum_{k=1}^N s_{k,i} = 1.$$
Among bistochastic matrices, permutation matrices play a crucial role: we define
$$\mathcal{S}^\dagger := \left\{S \in L^\infty((0;T) \times \mathbb{T}^d; \mathcal{S}(\mathbb{R})) : \forall i\,,j\,, s_{i,j}(t,x) \in \{0,1\} \text{ for a.e. } (t,x)\right\}.$$
The Birkhoff-Von Neumann theorem states that $S^\dagger$ is the set of extreme points of $\mathcal{S}$.

We let, the N-tuple $(m_1, \dots, m_N)$ being fixed, $\lambda_1(S)$ denote the principal (*i.e.* Perron-Frobenius) eigenvalue of the operator
$$\partial_t - \mathrm{Diag}(\mu_1, \dots, \mu_N)\Delta - (S - \mathrm{Id}) - \mathrm{Diag}(m_1, \dots, m_N)$$
and we investigate the two optimisation problems
$$\min_{S \in \mathcal{S}} \lambda(S)\,, \quad \max_{S \in \mathcal{S}} \lambda(S).$$
More specifically, we investigate whether the solutions to these problems can be chosen as permutations; as, as already noted, permutations are extreme points of the set of bistochastic matrices, this amounts to investigating the validity of the bang-bang property for such problems, in the fashion of Chapter 1. Our main result is the following:

Theorem 5.2. [**GM21**] *There holds*
$$\min_{S \in \mathcal{S}} \lambda(S) = \min_{S \in \mathcal{S}^\dagger} \lambda(S)\,, \quad \max_{S \in \mathcal{S}} \lambda(S) = \max_{S \in \mathcal{S}^\dagger} \lambda(S)$$

Observe that one of the main differences with the results of Chapter 1 is that we investigate both the maximisation and the minimisation of λ_1 and that, in both cases, we can pick extremisers among extreme points. This is in sharp contrast with the study of the lowest eigenvalue of $-\Delta - m$ under L^∞ and L^1 constraints on m where, under Neumann boundary conditions, as is easily seen by a simple concavity argument. Surprisingly, the proof of Theorem 5.2 does not rely on a concavity or convexity argument. As a final comment, observe that it is not clear whether *any* extremiser is in fact a permutation. This theorem is an extension of a result of Newmann & Sze [**189**] in the case of matrices.

5.1.3. Symmetry results for cooperative systems. As we mentioned, the second optimisation result of [**GM21**] deals with the spatial rearrangement of coefficients. To fix ideas, recall that the notion of decreasing rearrangement was introduced in Chapter 2 and amounts to the following in the one-dimensional case: for any given function $f \in L^\infty(\mathbb{T})$ there exists a unique function $f^\sharp$ that is symmetric with respect to 0 and non-increasing (here we identify $\mathbb{T}$ with $(-\pi;\pi)$), and that has the same distribution function as f: for any t,

$$|\{f \geq t\}| = |\{f^\sharp \geq t\}|.$$

As we saw in Chapter 2, letting (with a slight conflict of notations), $\lambda_1(-\Delta - m)$ denote the lowest eigenvalue of $-\Delta - m$ in the torus, there always holds

$$\lambda_1(-\Delta - m^\sharp) \leq \lambda_1(-\Delta - m),$$

and we saw that this was easily proved using either variational formulations or Talenti inequalities. We refer to Section 2.1.1 for details. Using this last approach which, as was noted, can also accommodate non-symmetric operators, we can obtain the following result:

THEOREM 5.3. *Assume the spatial dimension is* 1*.* $L \in L^\infty((0;T)\times\mathbb{T}; M_N(\mathbb{R}))$ *being a field of matrices such that*

$$\forall i\,,j\,,\ell_{i,j} \geq 0 \text{ a.e.},$$

let $L^\sharp$ *denote the entry-wise space decreasing rearrangement of* L*. Then*

$$\lambda_1(L^\sharp) \leq \lambda_1(L).$$

It should be noted that the assumption that L just has non-negative entries (including the diagonal ones) is not problematic in practice, as we can replace L with $L+\alpha\mathrm{Id}$ for some large constant α. We will not sketch the proof of this Theorem here as it is mostly a generalisation of Talenti inequalities to the cooperative case, which is an easy consequence of the cooperative nature of the system. We refer to [**GM21**] for details. As a final comment, we mention the recent paper [**75**] of Celentano, Nitsch & Trombetti that investigates symmetry problems (in the guise of over-determined boundary problems) for cooperative systems.

5.1.4. Sketch of proof of Theorem 5.2.

SKETCH OF PROOF OF THEOREM 5.2. As we mentioned, the proof hinges on an adaptation of the approach of Neumann & Sze [**189**]. Say we consider the optimisation problem

$$\min_{S\in\mathcal{S}} \lambda(S), \tag{5.6}$$

as the proof for the maximisation problem follows from exactly the same lines. The first step is to establish the existence of an optimal $S^* \in \mathcal{S}$, which in itself is not immediate and relies on the particular structure of bistochastic matrices. Nevertheless, let us admit that there exists one. The second step is to show that we can, without loss of generality, assume that the matrix S^* is irreducible. This uses the Frobenius normal form for bistochastic matrices, which all write under the form (5.5) (here again, this is very specific to bistochastic matrices). We thus assume that S^* is irreducible. Letting $M := -\mathrm{Id} + \mathrm{Diag}(m_1,\dots,m_N)$ the underlying eigenvalue equation rewrites

$$\partial_t \mathbf{u} - \Delta\mathbf{u} = S^*\mathbf{u} + M\mathbf{u} + \lambda(S^*)\mathbf{u}\,,\mathbf{u} \gg 0$$

and, by irreducibility, the eigenvalue $\lambda(S^*)$ is simple, which gives differentiability. Now, argue by contradiction and assume that S^* is not permutation a.e., and let

$$E_0 := \{(t,x) \in (0;T) \times \mathbb{T}\,, S^* \notin \mathcal{S}^\dagger\}.$$

To ease notations and the description of the set of admissible perturbations, let us assume that $s^*_{i,j} > 0$ in E_0 a.e. for any i, j. Discarding this assumption would essentially mean picking adequate subsystems, but this can quickly become notationally heavy. At any rate, we know consider a small perturbation H of S^*, located in E_0. In terms of admissibility, the fact that for $|t|$ small enough $S^* + tH$ should still be admissible rewrites as follows: H should write

$$H(t,x) = \eta(t,x) \otimes \xi(t,x)$$

for two vector fields $\eta\,,\xi$ satisfying

$$\sum_{k=1}^N \eta_k = \sum_{k=1}^N \xi_k = 0 \text{ a.e.} \tag{5.7}$$

Letting $\dot\lambda(S^*)[H]$ denote the Gateaux derivative of λ at S^* in the direction H, we deduce, through the same computations as in Chapter 1, that

$$0 = \dot\lambda(S^*)[H] = -\iint_{(0;T)\times\mathbb{T}^d} \langle H\mathbf{u}, \mathbf{v}\rangle \tag{5.8}$$

where $\mathbf{v}$ solves the adjoint equation

(5.9)

$$\begin{cases} -\partial_t \mathbf{v} - \mathrm{Diag}(\mu_1, \dots, \mu_N)\Delta\mathbf{v} = (S^*)^T\mathbf{v} + M^T\mathbf{v} + \lambda(S^*)\mathbf{v} & \text{in } (0;T)\times\mathbb{T}^d\,,\\ \mathbf{v}(0,\cdot) = \mathbf{v}(T,\cdot) & \text{in } \mathbb{T}^d\,,\\ \mathbf{v} \gg 0. \end{cases}$$

Because $H = \eta \otimes \xi$, (5.8) rewrites

$$0 = \iint_{(0;T)\times\mathbb{T}^d} \langle \mathbf{u}, \xi\rangle \cdot \langle \mathbf{v}, \eta\rangle. \tag{5.10}$$

From (5.7), (5.10) implies that almost everywhere in E_0

$$(\forall i\,, j \quad \mathbf{u}_i = \mathbf{u}_j) \text{ or } (\forall i\,, j \quad \mathbf{v}_i = \mathbf{v}_j)\,. \tag{5.11}$$

Assume, for the sake of readability, that we have $\mathbf{u}_i = \mathbf{u}_j$ for any $i\,, j$ almost everywhere in E_0. Then this means that, on E_0, we can modify the matrix S^* to be any permutation T^*. The key property here is that

$$\forall t \in (0;1), \lambda_1(\mathbb{1}_{E_0^c}S^* + (1-t)\mathbb{1}_{E_0}S^* + t\mathbb{1}_{E_0}T^*) = \lambda_1(S^*)$$

as the property that $\mathbf{u}_i = \mathbf{u}_j$ on E_0 implies that $\mathbf{u}$ is still a positive eigenfunction for $\mathbb{1}_{E_0^c}S^* + (1-t)\mathbb{1}_{E_0}S^* + t\mathbb{1}_{E_0}T^*$ for any $t \in (0;1)$. This gives another mutation matrix that does as good, and that is furthermore a permutation a.e.

□

5.1.5. Ongoing research. In this section, we briefly sketch some ongoing and future research lines that we seek to tackle, that all have to do with the optimal control of systems of reaction-diffusion equations.

Bulk-surface systems

The first problem we want to mention has to do with cooperative systems, with the species interacting at the boundary. To be more specific, we let Ω be a smooth, compact subset of $\mathbb{R}^d$ and $\Sigma = \partial\Omega$, and we consider a bulk-surface system of the form

$$\begin{cases} -\Delta\theta_\Omega - \theta_\Omega\,(f-\theta_\Omega) = 0 & \text{in } \Omega\,, \\ \frac{\partial\theta_\Omega}{\partial\nu} + \theta_\Omega = \theta_\Sigma & \text{on } \Sigma\,, \\ -\Delta_\Sigma\theta_\Sigma - \theta_\Sigma(g-\theta_\Sigma) = \theta_\Omega - \theta_\Sigma\,, & \text{on } \Sigma\,, \\ \theta_\Omega\,, \theta_\Sigma > 0, & \text{in } \Omega\,, \end{cases} \tag{5.12}$$

where $f\,, g \geq 0$. This system (and related ones) have become the subject of an increasing research activity [**27, 28**], particularly so in the context of road field models. Ω can be thought of as *bulk*, while Σ is interpreted as a *surface*. Motivated by the recent contribution of Bogosel, Giletti & Tellini [**33**], A. Gentile, R. Prunier and myself are currently investigating several shape optimisation and optimal control problems related to this equation, with a particular emphasis on the underlying spectral problem (*i.e.* minimising the lowest eigenvalue of this cooperative system with respect to $f\,, g$ and Ω) and on Talenti inequalities for such bulk-surface coupling.

Cooperative systems and optimal control problems

Another type of question that we will tackle in the future and for which we already have partial results is the following: we consider any cooperative system of the form

$$\partial_t \mathbf{u} - \mathrm{Diag}(\mu_1, \dots, \mu_N)\Delta\mathbf{u} = \mathbf{f}(\mathbf{u}) + \mathrm{Diag}(m_1, \dots, m_N)\mathbf{u},$$

where $\mathbf{f}$ satisfies (5.1). Letting, for any $i \in \{1, \dots, N\}$,

$$\mathcal{M}_i := \left\{ m \in L^\infty((0;T)\times\Omega)\,:\, 0 \leq m \leq 1 \text{ a.e.}, \iint_{(0;T)\times\mathbb{T}} m_i = \bar{m}_i \right\}$$

we want to investigate the optimal control problem

$$\max_{m_i \in \mathcal{M}_i\,, i \in \{1,\dots,N\}} \iint_{(0;T)\times\mathbb{T}} j_i(\mathbf{u}_i)$$

under natural monotonicity assumptions. Here, the focus will be the same as in Chapters 1–2, with a strong emphasis on the bang-bang property; we expect that the behaviour will be similar to the scalar case, but this is a first step that needs to be finalised before moving on to the next problem, which is more relevant, but also much more delicate.

An important research line: predator-prey systems

As we underlined several times during this section, the cooperative nature of the systems under consideration makes them bear significant resemblance to their scalar counterparts. What we believe one of the important queries for future research in this field is the study of predator-prey systems, where one tries to optimise the population of a predator by acting on its prey. A prototypical system is the following:

$$\begin{cases}\partial_t u \mu_1 \Delta u = u(m-u) - \alpha v\,, \\ \partial_t v - \mu_2 \Delta v = v(K(x)-v) + \gamma u\end{cases} \tag{5.13}$$

where $\alpha\,,\gamma > 0$, and where a typical problem would be

$$\max_{0\leq m\leq 1 \text{ a.e.},\ \int_{(0;T)\times\Omega} m = m_0} \int_\Omega v.$$

We explained in Chapter 1 the importance of monotonicity regarding, for instance, the bang-bang property of optimisers. Here, this monotonicity is not clear at all: indeed, it is not always clear that we should feed the prey u as much as possible, as can be seen from the emergence of limit cycles: if we feed u too much early on, it might lead to a transitory growth of the population of predators, in turn leading to a sharp decline in preys etc. We expect the main difficulties in the study of such optimisation problems to come from the underlying dynamics.

5.2. Shape optimisation for vectorial problems arising from physics

5.2.1. General introduction. The final problem we want to discuss in this memoir is the topic of a collaboration with A. Henrot & Y. Privat [**HMFP24**] and deals with a seemingly innocuous question: is the Faber-Krahn inequality true for the Stokes operator? This is a vectorial shape optimisation problem coming from the study of electromagnetics or fluid dynamics.

The classical Faber-Krahn inequality

Before we discuss our aims and results, let us briefly recall the classical Faber-Krahn inequality: for a given, smooth, bounded set $\Omega \subset \mathbb{R}^d$, let $\mu_1(\Omega)$ denote its first Dirichlet eigenvalue, defined as

$$\mu_1(\Omega) = \min_{u\in W^{1,2}_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2}.$$

This first eigenvalue is *simple* and the celebrated Faber-Krahn inequality asserts that, if $\mathbb{B}$ is a ball with the same volume as Ω, then

$$\mu_1(\Omega) \geq \mu_1(\mathbb{B}). \tag{5.14}$$

We will review later on some strategies of proof, but we underline that this is the foundational problem in spectral optimisation [**127, 129**] and that it is by now almost perfectly understood. Typically, as was mentioned in Chapter 3, we have a quantitative version of (5.14), due to Brasco, De Philippis & Velichkov [**36**] : with the same notations as before, $\mathbb{B}$ being a ball with the same volume as Ω, there exists $C > 0$ that only depends on the volume of Ω such that

$$\mu_1(\Omega) - \mu_1(\mathbb{B}) \geq C \inf_{x\in\mathbb{R}^d} |\Omega \triangle (x+\mathbb{B})|^2.$$

The Stokes operator, its first eigenvalue and the optimisation problem

We now introduce the Stokes operator: let $W^{1,2}_0(\Omega)^d$ be the set of vector fields $\mathbf{u} = (u_1, \dots, u_d)$ such that for any $i \in \{1, \dots, d\}$, $u_i \in W^{1,2}_0(\Omega)$. We define the first Dirichlet-Stokes eigenvalue of the domain Ω as the quantity

$$\lambda_1(\Omega) := \min_{\mathbf{u}\in W^{1,2}_0(\Omega)^d\setminus\{0\}\,,\nabla\cdot\mathbf{u}=0} \frac{\int_\Omega |\nabla \mathbf{u}|^2}{\int_\Omega |\mathbf{u}|^2}, \tag{5.15}$$

where

$$|\nabla \mathbf{u}|^2 = \sum_{i,j=1}^{d} \left(\frac{\partial u_i}{\partial x_j}\right)^2 .$$

This Rayleigh quotient is associated with the equation

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \lambda_1(\Omega)\mathbf{u} & \text{in } \Omega\,, \\ \mathbf{u} = 0 & \text{on } \partial\Omega\,, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega\,, \\ \int_\Omega |\mathbf{u}|^2 = 1. \end{cases} \tag{5.16}$$

The pressure ∇p is a Lagrange multiplier associated with the incompressibility constraint $\nabla \cdot \mathbf{u} = 0$. This first eigenvalue problem appears naturally in a variety of contexts–as we already mentioned, this is the case in electromagnetics, but also in fluid mechanics, as the eigenvalues of the (linear) Stokes operator also play an important role in studying the long-time behaviour of the (nonlinear) Stokes equation [**21, 100**]. Our main optimisation problem is

$$\min_{\Omega \text{ quasi-open, } |\Omega| \leq V_0} \lambda_1(\Omega), \tag{5.17}$$

where V_0 is a volume constraint. It should be noted that in (5.17), "quasi-open" means that the admissible sets can be less regular than open in the following sense:

Definition 5.4. A subset Ω of $\mathbb{R}^d$ is called **quasi-open** if there exists a non-increasing sequence of open sets $\{\omega_k\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \operatorname{cap}(\omega_k) = 0 \quad \text{and} \quad \forall k \in \mathbb{N}, \quad \Omega \cup \omega_k \text{ is open},$$

where cap denotes the capacity of an open set. Recall that it is defined as

$$\operatorname{cap}(\omega) := \sup_{\substack{K \text{compact} \\ K \subset \omega}} \inf_{\substack{v \in \mathscr{C}_c^\infty(\mathbb{R}^d) \\ v \geq 1 \text{ on } K}} \int_{\mathbb{R}^d} |\nabla v|^2 .$$

This is the natural setting for the study of existence of optimal shapes in general shape optimisation [**219**], as we will briefly mention later on.

The main questions under study are the following:

(1) Does there exist an optimal quasi-open set Ω^* for (5.17)? We will state an existence result, Theorem 5.5.
(2) Do we have the analog of (5.14) for the Stokes operator, *i.e.* is the ball a solution of (5.17)? Here, we will see the strong influence that the dimension has on the problem; these are Theorems 5.6–5.8.

We state our main results in the next subsection. Before we do, let us contextualise our work by discussing a few things in the next paragraphs:

(1) First, we will briefly go over the main strategies of proof for the classical Faber-Krahn inequality (5.14) and explain why they are doomed to fail when dealing with the Stokes operator.
(2) Second, we will discuss some aspects of the Stokes operator in dimensions 2 and 3, which will allow to explore closely related works.

How is the Faber-Krahn inequality (5.14) **usually proved?** There are a variety of ways to prove (5.14), the two main ones being–as far as we are aware–the following:

(1) First, a rearrangement approach that relies on the Polyá-Szegö inequality (see Eq. (2.9)). Here, such an approach does not seem doable, as, as far we know, there is no possible symmetrisation procedure that would preserve incompressibility constraint. We refer to [127].
(2) The second one also relies on symmetrisation, but, unlike the Schwarz rearrangement, does not immediately rearrange the set into a ball. Rather, it goes as follows:
    (a) First, establish the existence of an optimal quasi-open set–this is doable for the Stokes problem.
    (b) Second, prove that the set is open and connected–as we shall see, this is unknown in the case of the Stokes operator and seems highly challenging.
    (c) Third, prove that the set is a ball through successive symmetrisations with respect to hyperplanes. Here again, this is doomed to fail when taking into account the incompressibility condition $\nabla \cdot \mathbf{u} = 0$.

For the second approach, we refer to the seminar of Bucur & Freitas [48]. Now, there is another question that might be raised: can we prove that, if a very smooth (say, $\mathscr{C}^3$) domain Ω^* is optimal, then it has to be the ball? In the case of the classical Faber-Krahn inequality, it is possible using the Serrin theorem; namely, the first order optimality condition for the minimisation of μ_1 under volume constraints read:

$$\frac{\partial \varphi_{\Omega^*}}{\partial \nu} = c \text{ on } \partial\Omega^*$$

for some constant c, where φ_{Ω^*} is the first Dirichlet eigenfunction of Ω^*. Consequently, φ_{Ω^*} solves

$$\begin{cases} -\Delta \varphi_{\Omega^*} = \lambda_1(\Omega^*)\varphi_{\Omega^*} & \text{in } \Omega^*\,, \\ \varphi_{\Omega^*}\,, \partial_\nu \varphi_{\Omega^*} \text{ are constant} & \text{on } \partial\Omega^*, \\ \varphi_{\Omega^*} > 0 & \text{in } \Omega^*. \end{cases}$$

Then the Serrin theorem [209] (we refer to the survey [190] of Nitsch & Trombetti for various approaches to this theorem) states that Ω^* has to be a ball. In the case of the Stokes operator, and while first order optimality conditions will be instrumental in proving our results, the overdetermined boundary value problem is quite enigmatic at first sight and, at any rate, all the proofs of the Serrin theorem rely in some capacity on the comparison principle, which is meaningless for such vectorial operators.

The Stokes operator in dimension 2

Now, let us discuss, in the following two paragraphs, what we expect the optimiser Ω^* should be. The first thing to note is that in $\mathbb{R}^2$ the Stokes operator can be reduced to a (scalar) fourth order operator: assume that the domain Ω is simply connected, so that the first Stokes eigenfunction $\mathbf{u}$ can be written as

$$\mathbf{u} = \begin{pmatrix} -\partial_y \psi \\ \partial_x \psi \end{pmatrix}$$

for some function ψ. Plugging this form in (5.16), we obtain

$$-\Delta^2 \psi = \lambda_1(\Omega)\Delta\psi.$$

Furthermore, as $\begin{pmatrix}-\partial_y\psi\\ \partial_x\psi\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$ on $\partial\Omega$, ψ is constant on $\partial\Omega$ and $\partial_\nu\psi=0$. Overall, ψ thus solves the problem

$$\begin{cases}-\Delta^2\psi=\lambda_1(\Omega)\Delta\psi & \text{in } \Omega\,,\\ \psi=\partial_\nu\psi=0 & \text{on } \partial\Omega.\end{cases}$$

In fact, with a bit more work, for simply connected domains, one can see that $\lambda_1(\Omega)$ coincides with the eigenvalue

$$\Lambda(\Omega):=\min_{v\in W^{2,2}(\Omega)\cap W^{1,2}_0(\Omega)}\frac{\int_\Omega(\Delta v)^2}{\int_\Omega|\nabla v|^2},$$

and so, if we restrict our search of optimal domains to the class of simply connected domains, the minimisation of λ_1 amounts to the *buckling eigenvalue problem*:

$$\inf_{\Omega\subset\mathbb{R}^2\,,\Omega\text{ quasi-open}\,,|\Omega|\leq V_0}\Lambda(\Omega). \tag{5.18}$$

We refer to [**127**, Chapter 11] for more details, but let us mention two things: first, the existence of an optimal shape for (5.18) remained open for a long-time until the contribution of Ashbaugh & Bucur [**16**] and, later, of Stollenwerk [**213, 214**]. Second, a long-standing conjecture by Pólyà & Szegö asserts that the ball is a solution of this variational problem: for any set Ω such that $|\Omega|=|\mathbb{B}|$, $\Lambda(\Omega)\geq\Lambda(\mathbb{B})$. This means that, in dimension 2, it is both reasonable to expect the ball to minimise the first Stokes eigenvalue, and to be humble in our expectations. Thus, we will focus on the local optimality of the ball.

The Stokes operator in dimension 3

In the three dimensional case, the situation is completely different, and, rather than reducing the Stokes operator to a fourth-order scalar operator, the natural way to see it is rather as the square of the **curl** operator: as $\nabla\cdot\mathbf{u}=0$, we have

$$\Delta\mathbf{u}=\mathbf{curl}(\mathbf{curl}(\mathbf{u})).$$

Consider the **curl**-eigenbasis: it is possible to prove (see [**204**])that there exist sequences $(\eta_{k,\pm}\,,\mathbf{u}_{k,\pm})_{k\in\mathbb{N}}$ such that

$$-\infty\underset{k\to\infty}{\leftarrow}\eta_{k,-}\leq\dots\leq\eta_{1,-}<0<\eta_{1,+}\leq\dots\leq\eta_{k,+}\underset{k\to\infty}{\rightarrow}+\infty$$

with

$$\begin{cases}\mathbf{curl}(\mathbf{u}_{k,\pm})=\eta_{k,\pm}\mathbf{u}_{k,\pm} & \text{in } \Omega\,,\\ \langle\mathbf{u}_{k,\pm},\nu\rangle=0 & \text{on } \partial\Omega\end{cases}$$

and that $\{\mathbf{u}_{k,\pm}\}_{k\in\mathbb{N}}$ is a Hilbert basis of

$$H:=\{\mathbf{u}\in W^{1,2}_0(\Omega)\,,\langle\mathbf{u},\nu\rangle=0\,,\nabla\cdot\mathbf{u}=0\}.$$

Decomposing the first Stokes eigenfunction $\mathbf{u}$ in this basis yields the estimate

$$\lambda_1(\Omega)\geq\min(\eta_{1,+}(\Omega)^2\,,\eta_{1,-}(\Omega)^2)\,,\lambda_1(\Omega)\in\{\eta^2_{k,\pm}(\Omega)\}_{k\in\mathbb{N}},$$

and there can sometimes be equality. The reason we mention this link is that an old shape optimisation problems in electromagnetics is the curl isoperimetric problem which, as far as we are aware, was first investigated by Chandrasekhar [**76**]. To the best of our knowledge, there are two main lines of works on spectral optimisation of the curl eigenvalues:

(1) The first one rather comes from the theoretical physics community, in particular with the works of Canatrella, DeTurck, Gluck and Teytel [**60, 61**] : using explicit computations, they observe the optimality of the ball among concentric spherical shells with a given volume constraint. More importantly for us, though, they conjecture that the minimiser of the **curl**-isoperimetric problem is a "spheromak", that is, a sphere where the south and north poles are glued together.

(2) The second one comes from the works of Enciso, Gerner & Peraltas-Sala (although they also consider additional orthogonality conditions on the vector fields, but we do not get into details here to focus on our main takeaway from their works) [**92, 93, 104**]. In [**93**], Enciso & Peraltas-Sala prove that a regular solution of the **curl**-isoperimetric problem can not have axial symmetry, and can not be convex; this is proved by investigating the first-order optimality conditions of the problem and by using the Hopf-Poincaré theorem. Gerner, in [**104**], went further and showed that a necessary condition for a domain Ω^* to be optimal is that the lowest **curl** eigenvalue be simple. We mention this as such characterisation will be at the core of our analysis.

In particular, in this setting, it is probably unreasonable to expect that the ball is a minimiser of the first Stokes eigenvalue.

5.2.2. Our main results. We are now in a position to state the three main results of [**HMFP24**]. We begin with an existence result:

THEOREM 5.5. [**HMFP24**] *There exists an optimal quasi-open set Ω^* for* (5.17).

This result, for which we gratefully acknowledge the help of D. Bucur, is proved using the concentration-compactness theorem of Lions [**164**] in the way it was adapted by Bucur to handle shape optimisation problems [**44, 45, 49, 48**]. We will not give a proof here, and we simply mention two things:

(1) As in general shape optimisation problems, the main difficulty is proving that we can restrict our search to a class of uniformly bounded domains (concentration). Once this is done, the proof of existence follows from an application of the Buttazzo-DalMaso theorem [**56**] (see also Chapter 1, Section 1.3.4).

(2) The framework that allows to reduce to a bounded design region is well-understood for scalar operators (we refer in particular to [**49**]), but the main difficulty here is the incompressibility condition, which requires care when being handled.

We continue with the optimality of the ball in dimension 2.

THEOREM 5.6. [**HMFP24**] *In dimension 2, the centred ball is locally optimal in the sense of Hadamard calculus: for any vector field $\Phi \in \mathscr{C}^\infty_c(\mathbb{R}^d;\mathbb{R}^d)$ such that*

$$\int_{\partial\mathbb{B}} \langle \Phi, \nu\rangle = 0$$

if we define

$$f_\Phi : t \mapsto \lambda_1\left((\mathrm{Id}+t\Phi)\mathbb{B}\right)$$

then

$$f'_\Phi(0) = 0\,,\, f''_\Phi(0) \geq c\|\langle \Phi, \nu\rangle\|^2_{W^{\frac{1}{2},2}(\partial\mathbb{B})}$$

for some $c > 0$, where ν is the unit normal vector on $\partial\mathbb{B}$.

Likewise we will not sketch the proof of this theorem, which follows from very technical computations. To give the blueprints: the idea is to use the connection between the Stokes problem and the buckled plate problem to then diagonalise the shape hessian. We note that this result in particular provides the diagonalisation of the shape hessian the ball for the buckling problem, which fits into the growing literature devoted to sensitivity analysis for polyharmonic operators. Regarding shape derivatives, we refer to [**113**] and, as concerns shape derivatives for polyharmonic operators, we point to the numerous works of Buoso & Lamberti [**50, 51, 52**].

We now move to our final result; to give it more meaning, we first recall the following result (that follows from explicit computations [**143, 157, 208**]):

PROPOSITION 5.7. *Let $\mathbb{B}_3$ be the unit ball in $\mathbb{R}^3$. $\lambda_1(\mathbb{B}_3)$ is triple.*

Our next theorem establishes, in particular, that the ball can not be a solution of (5.17).

THEOREM 5.8. [**HMFP24**] *Let Ω^* be a $\mathscr{C}^{2,\alpha}$ solution of* (5.17). *Then:*
(1) $\lambda_1(\Omega^*)$ is simple.
(2) Any connected component of $\partial\Omega^*$ is homeomorphic to a torus.

The rest of this section will be devoted to a sketch of proof of Theorem 5.8.

5.2.3. Sketch of proof.

SKETCH OF PROOF OF THEOREM 5.8. This proof relies on first-order optimality conditions: letting Ω^* be a $\mathscr{C}^{2,\alpha}$ solution of (5.17), we consider a vector field $\Phi \in \mathscr{C}^\infty_c(\mathbb{R}^d; \mathbb{R}^d)$. Now, if we knew that $\lambda_1(\Omega^*)$ were simple, the very same computations as in the case of the scalar Dirichlet Laplacian would imply that the map

$$f_\Phi : t \mapsto \lambda_1\left((\mathrm{Id} + t\Phi)\Omega^*\right)$$

is differentiable at $t = 0$, and that, $\mathbf{u}$ being an L^2 normalised eigenfunction,

$$f'_\Phi(0) = -\int_{\partial\Omega^*} |(\nabla\mathbf{u})\nu|^2 \langle \Phi, \nu\rangle. \tag{5.19}$$

There is a problem, however, which is that we do not know that $\lambda_1(\Omega^*)$ is simple. Nevertheless, and although the map $t \mapsto \lambda_1\left((\mathrm{Id} + t\Phi)\Omega^*\right)$ is not differentiable at 0, we can always compute the semi-derivative of λ_1 at Ω^* in the direction Φ; this quantity, which we denote by $\lambda_1'(\Omega^*)[\Phi]$, is defined as

$$\lambda_1'(\Omega^*)[\Phi] := \lim_{t\to 0\,, t>0} \frac{\lambda_1\left((\mathrm{Id} + t\Phi)\Omega^*\right) - \lambda_1(\Omega^*)}{t}.$$

The fact that this quantity is well-defined is actually a consequence of standard considerations in nonsmooth analysis, and we refer for instance to the works of Cox [**86**], Chenais & Rousselet [**206**] and Chatelain & Choulli [**80**]. Let us also indicate the recent paper of Caubet, Dambrine & Mahadevan [**73**], which developed this paradigm within the framework of linear elasticity. At any rate, the point is the following: if we let E_1 be the eigenspace associated with $\lambda_1(\Omega^*)$, the formula for $\lambda_1'(\Omega^*)[\Phi]$ is very similar to (5.19) and we have

$$\lambda_1'(\Omega^*)[\Phi] = \min_{\mathbf{u}\in E_1}\left(-\int_{\partial\Omega^*}|(\nabla\mathbf{u})\nu|^2\,\langle\Phi,\nu\rangle\right). \tag{5.20}$$

Fix $(\mathbf{u}_1,\dots,\mathbf{u}_N)$ an L^2 orthonormal basis of E_1. The first order optimality condition reads that there exists a Lagrange multiplier c such that

$$\forall\Phi\in\mathscr{C}_c^\infty(\mathbb{R}^d;\mathbb{R}^d)\,,\lambda_1'(\Omega^*)[\Phi]+c\int_{\partial\Omega^*}\langle\Phi,\nu\rangle\geq 0$$

rewrites

$$\forall\Phi\in\mathscr{C}_c^\infty(\mathbb{R}^d;\mathbb{R}^d)\,, \tag{5.21}$$

$$\min_{\sum_{i=1}^N\alpha_i^2=1}\left(-\sum_{i,j=1}^N\alpha_i\alpha_j\int_{\partial\Omega^*}\langle(\nabla\mathbf{u}_i)\nu,(\nabla\mathbf{u}_j)\nu\rangle\langle\Phi,\nu\rangle+c\sum_{i=1}^N\alpha_i^2\int_{\partial\Omega^*}\langle\Phi,\nu\rangle\right)\geq 0.$$

In particular, if we introduce the matrix M_Φ as

$$M_\Phi:=\left(-\int_{\partial\Omega^*}\langle(\nabla\mathbf{u}_i)\nu,(\nabla\mathbf{u}_j)\nu\rangle\langle\Phi,\nu\rangle\right)_{i,j=1,\dots,N}+c\mathrm{Id}$$

and its spectrum $\mathrm{Spec}(M_\Phi)$, (5.21) becomes

$$\forall\Phi\in\mathscr{C}_c^\infty(\mathbb{R}^d;\mathbb{R}^d)\,,\mathrm{Spec}(M_\Phi)\subset[0;+\infty). \tag{5.22}$$

As $M_{-\Phi}=-M_\Phi$, we finally deduce that if Ω^* is optimal then

$$\forall\Phi\in\mathscr{C}_c^\infty(\mathbb{R}^d;\mathbb{R}^d)\,,M_\Phi=0. \tag{5.23}$$

Inspecting the diagonal and off-diagonal terms of M_Φ and computing c explicitly, this implies:

(1) First, that there exists a constant $c\neq 0$ such that, for any $i\in\{1,\dots,N\}$,

$$|(\nabla\mathbf{u})\nu|^2=c\text{ on }\partial\Omega^*. \tag{5.24}$$

(2) Second, that, for any $i\neq j$,

$$\langle(\nabla\mathbf{u}_i)\nu,(\nabla\mathbf{u}_j)\nu\rangle=0\text{ on }\partial\Omega^*. \tag{5.25}$$

The conditions (5.24)–(5.25) already imply that $N\leq 3$. Furthermore, it is easy to see, from the homogeneous Dirichlet boundary conditions and the incompressibility of the vector fields that, for any $i\in\{1,\dots,N\}$,

$$\langle(\nabla\mathbf{u})\nu,\nu\rangle=0,$$

which gives two informations:

(1) First, $N\leq 2$–this already rules out the ball as a possible solution of (5.17).
(2) Second, from the Hopf-Poincaré theorem, the existence of a tangential vector field that does not cancel implies that any connected component of $\partial\Omega^*$ is homeomorphic to a torus.

So it remains to prove that $N=1$ at an optimiser. Here, we adapt some of the ideas developed by Gerner in [**104**]; it should be noted that in the context of [**104**], optimal domains for the **curl** isoperimetric problems have a nice property as the **curl** eigenfunctions at an optimal shape have a much clearer geometric meaning in terms of Beltrami fields. This is not the case for the Stokes operator, and the construction is more involved, although the core idea remains the same. Namely, argue by contradiction and assume that $\lambda_1(\Omega^*)$ has multiplicity 2. For $i=1,2$, let $\mathbf{v}_i$ be the solution of

$$\begin{cases} -\Delta \mathbf{v}_i + \nabla p_i = \lambda_1(\Omega^*)\mathbf{v}_i + \omega \mathbf{u}_i & \text{in } \Omega^*\,, \\ \nabla \cdot \mathbf{v}_i = 0 & \text{in } \Omega^*\,, \\ \mathbf{v}_i \in E_1^\perp\,, & \\ \mathbf{v}_i = -(\nabla \mathbf{u}_i)\nu & \text{on } \partial\Omega^*, \end{cases} \tag{5.26}$$

where $E_1^\perp$ is the L^2-orthogonal subspace to E_1 and the constant ω is defined by

$$\omega = -\int_{\partial\Omega^*} \|(\nabla \mathbf{u}_i)\nu\|^2. \tag{5.27}$$

The fact that, for $i=1,2$, $\mathbf{v}_i$ is uniquely defined is a consequence of the optimality of Ω^*. The key point, which we sweep under the rug here for the sake of simplicity, is that

$$0 = \langle \mathbf{v}_1, (\nabla \mathbf{v}_2)\mathbf{v}_1\rangle = \langle \mathbf{v}_2, (\nabla \mathbf{v}_1)\mathbf{v}_2\rangle = -\langle \mathbf{v}_2, (\nabla \mathbf{v}_2)\mathbf{v}_1\rangle \text{ on } \partial\Omega^*.$$

This is shown through tedious computations

Consequently, $(\nabla \mathbf{v}_2)\mathbf{v}_1$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, so that there exists a function $f : \partial\Omega^* \to \mathbb{R}$ satisfying

$$(\nabla \mathbf{v}_2(x))\mathbf{v}_1(x) = f(x)\nu(x) \text{ on } \partial\Omega^*.$$

For the same reason,

$$(\nabla \mathbf{v}_1(x))\mathbf{v}_2(x) = g(x)\nu(x) \text{ on } \partial\Omega^*$$

for some function g. Letting $[X,Y]$ denote the Lie bracket between vector fields,

$$[\mathbf{v}_1, \mathbf{v}_2] = (\nabla \mathbf{v}_1)\mathbf{v}_2 - (\nabla \mathbf{v}_2)\mathbf{v}_1 = (f-g)\nu \qquad \text{on } \partial\Omega.$$

However, as $\mathbf{v}_1$ and $\mathbf{v}_2$ are both tangential, so is their Lie bracket, whence $[\mathbf{v}_1, \mathbf{v}_2] = 0$. In particular, $\partial\Omega^*$ admits a set of two pointwise orthogonal, commuting vector fields. This yields that $\partial\Omega^*$ has zero Gauß curvature [**158**, Theorem 7.3 and its proof], which is impossible as $\partial\Omega^*$ is the boundary of a compact domain. This concludes the proof.

□

5.2.4. Ongoing research. There are several questions related to vectorial optimisation, and we merely list a few research directions below.

Other vectorial operators coming from physics

Shortly after our paper [**HMFP24**], Henrot, Lemenant & Privat [**128**] investigated a similar question for the linear elasticity system; there, the incompressibility constraint is replaced with a penalisation of the divergence, which creates several differences with the case under consideration here (typically, it makes the question of regularity slightly easier, as they can go beyond quasi-open sets, but at the moment there is no proof that at an optimiser the lowest Lamé eigenvalue should be simple).

With A. Henrot, Y. Privat & D. Stantejsky, we are currently investigating similar spectral minimisation problems for eigenvalues arising in electromagnetics, in particular Maxwell eigenvalues.

Another interesting research direction has to do with non-linear models and, in particular, with the Navier-Stokes equation, where we would like to answer the following question: what is the optimal shape of a pipe, where optimal refers to "least energy dissipation"? Henrot & Privat studied [130] the question from the point of view of optimality conditions, showing the non-optimality of certain designs. A challenging goal, seeing the Stokes operator as the limit of Navier-Stokes operators in the high viscosity regime, would be to investigate the stability of optimal shapes when going from Stokes to Navier-Stokes. Here, we would start with a two dimensional stability analysis.

Regularity for vectorial shape optimisation problems

A completely open (but that we deem quite important) problem regards the regularity of optimal shapes in vectorial optimisation under incompressibility constraint. Observe that it is not clear whether we should expect regularity, but, at any rate, the conclusion of Theorem 5.5 is not fully satisfactory, as we do not know whether the optimal set Ω^* is even open–that is due to the fact that most techniques used to go from quasi-open to open are not fit to incorporate incompressibility constraints.

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RÉSUMÉ

Nous présentons dans ce mémoire une synthèse de nos différentes activités de recherche dans le domaine du contrôle optimal, de l'optimisation de formes, et de leurs applications à la dynamique des populations. De manière plus spécifique,
nous nous intéressons dans ce manuscrit à plusieurs aspects de ces questions:
• Dans un premier temps, nous nous penchons sur les propriétés ponctuelles des contrôles optimaux. Cette question est motivée par de nombreux facteurs, en particulier par les algorithmes d'approximation numérique mis en œuvre dans la résolution de ces problèmes.
• Nous nous penchons ensuite sur les propriétés géométriques des contrôles optimaux, en insistant sur les cas où l'on observe l'émergence de structures complexes.
• Une autre section de ce mémoire est dédiée à la question de la stabilité quantitative des contrôles optimaux; la motivation est ici double: en plus de fournir des renseignements fins sur les problèmes sous-jacents, de telles estimations ont des applications nombreuses, y compris pour l'approximation numérique des problèmes de contrôle optimal.
• Nous présentons ensuite plusieurs modèles de type "Théorie des jeux" pour la gestion optimale des pêcheries, en distinguant les cas où les pêcheurs présents sont en nombre fini (ce qui mène à des questions de type "équilibres de Nash") ou infini (ce qui mène à des modèles de type "jeux à champ moyen"). Une grande importance est accordée à l'étude de phénomènes tels que la tragédie des communs.
• Enfin, dans un dernier chapitre, nous présentons deux axes de recherche plus récents consacrés à des problèmes d'optimisation vectoriels motivés ou par la physique ou par la biologie.

ABSTRACT

In this manuscript, we present an overview of our research activity within the fields of shape optimisation and optimal control and their applications to population dynamics. More specifically, we focus on the following aspects of these questions:
• First, we analyse pointwise properties of optimal controls. This question has several motivations, stemming in particular from the study of the numerical methods put forth to solve such problems.
• We then focus on the study of geometric properties of the optimal controls, with a special emphasis on cases where complex structures emerge.
• Another section of this memoir is devoted to the quantitative stability of optimal controls. There are at least two reasons for such a study: beyond shedding a new light on the fine properties of the underlying optimal control problems, such estimates prove very useful in a variety of domains, including in the numerical approximation of optimal controls.
• We then present several game theoretical models for the optimal management of fisheries, distinguishing between cases where only a finite number of fishermen are present (leading to ``Nash equilibria" type questions), or where there are infinitely many of them (leading to ``Mean Field Games" models). We devote a large part of our study to the understanding of phenomena such as the tragedy of the commons.
• Finally, we present two more recent research directions, devoted to the understanding of vectorial shape optimisation problems arising either in physics or in biology.