A review of Functional Analytic tools for PDEs

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Introduction

The goal of this course is to provide you with the standard functional analysis material you will need when studying partial differential equations. Naturally, there is no way to cover all the relevant results and concepts, but we settled on a (fairly standard) choice that we hope will prove useful. Some of the lectures and notes are inspired by the lectures of D. Gontier (who was in charge of the course in 2024).

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CHAPTER 1

Lebesgue Spaces

In the first one or two lectures, we will be going over the classical theory of Lebesgue spaces L^p

1.1. Some reminders on measure theory and on Lebesgue measure

The notion of Lebesgue measure is a delicate one. Roughly speaking, there are two ways to construct the Lebesgue measure on $\mathbb{R}^d$:

(1) One starts from the construction of the one-dimensional Lebesgue measure and proceed by tensorisation.
(2) The second one proceeds from the Carathéodory extension theorem.

Although we refer to [7] for a comprehensive and efficient treatment of basic facts in measure theory, let us summarise the main points we will use:

(1) First, we define the Borel σ-algebra as the σ-algebra generated by open sets.
(2) Second, one needs to prove that there exists a (unique up to multiplication) "natural"[1] measure $|\cdot|$ on the Borel σ-algebra, which satisfies:
    (a) For any $(a_i; b_i) \in \mathbb{R}^2$, $a_i \leq b_i$, $|\prod_{i=1}^d (a_i; b_i)| = \prod_{i=1}^d |b_i - a_i|$,
    (b) For any Borel measurable set A, for any $x \in \mathbb{R}^d$, $|x + A| = |A|$.
    (c) For any measurable set A and any $\lambda \geq 0$, $|\lambda A| = \lambda^d |A|$.
(3) Third, one needs to define the notion of Lebesgue measurable sets; to this end, one must introduce the notion of zero Lebesgue measure set. Essentially, any Lebesgue measurable set is the union of a Borel measurable set and of a zero Lebesgue measure set. This fact relies on the inner and outer regularity of the Lebesgue measure. When a property is satisfied up to a set of zero measure, we say it is satisfied almost everywhere[2].

As always in measure theory, the key point is to keep in mind that things can behave weirdly, and that counter-examples are extremely important (there are Lebesgue measurable sets that are not Borel measurable, there are non-Lebesgue measurable sets, there are sets with empty interior that have positive measure etc). We also refer to [3] for a historical overview of the development of measure theory.

1.2. Integrable functions and L^p spaces

We work with functions $f : \mathbb{R}^d \to \mathbb{R}$. Recall that such a function is called **measurable** if, for any Lebesgue measurable set $A \subset \mathbb{R}$, $f^{-1}(A) \subset \mathbb{R}^d$ is Lebesgue

[1] in the sense of Haar measures.

[2] To be really precise, one should rather define it as Lebesgue almost everywhere; to keep things light, this will be implicit.

measurable. Among these, we single out simple functions, namely, functions that write

$$f = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$$

where $n \in \mathbb{N}^*$, $\{\alpha_i\}_{i=1,\dots,n} \in \mathbb{R}^d$ satisfies

$$\alpha_1 < \cdots < \alpha_n$$

and, for any $i\,,1 \leq i \leq n$, A_i is a Lebesgue-measurable set. We let $\mathcal{E}$ be the set of simple functions, and $\mathcal{E}_+$ be the set of non-negative simple functions.

1.2.1. Definition of integrable functions. For any $f \in \mathcal{E}_+$ writing $f = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i}$, we define

$$\int_{\mathbb{R}^d} f(x)dx = \sum_{i=1}^{n} \alpha_i |A_i|.$$

Now, for any non-negative measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$, we define

$$\int_{\mathbb{R}^d} f(x)dx = \sup \left\{ \int_{\mathbb{R}^d} h(x)dx : \, h \in \mathcal{E}_+\,, h \leq f \text{ almost everywhere} \right\}.$$

The key properties of this integral are the following:

(1) Linearity: for any $\lambda \geq 0$, for any $f\,, g$ measurable and non-negative,

$$\int_{\mathbb{R}^d} (\lambda f + g) = \lambda \int_{\mathbb{R}^d} f + \int_{\mathbb{R}^d} g.$$

(2) Monotonicity: for any $f\,, g$ measurable and non-negative, if $f \leq g$ almost everywhere, then

$$\int_{\mathbb{R}^d} f \leq \int_{\mathbb{R}^d} g.$$

(3) For any measurable, non-negative functions $f\,, g$, if $f = g$ a.e., then $\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} g$.

From this, we might define integrable functions as Lebesgue measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $|f|$ has a finite integral in the sense above. In this case, writing $f = f_+ - f_-$ with $f_\pm \geq 0$, we define

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} f_+ - \int_{\mathbb{R}^d} f_-.$$

We let $\mathcal{L}^1(\mathbb{R}^d)$ denote the set of integrable functions.

The three main theorems of the Lebesgue integral are the following:

Theorem 1.1 (Monotone Convergence Theorem). *If (f_j) is a sequence of measurable functions,* **increasing** *in the sense that $f_j \leq f_{j+1}$ a.e., then $f : x \mapsto f(x) := \lim f_j(x)$ is measurable, and*

$$\int_{\mathbb{R}^d} f(x)dx = \lim_{j \to \infty} \int_{\mathbb{R}^d} f_j(x)dx.$$

In other words, increasing sequence implies that we can commute limits $\int \lim = \lim \int$. The last value can be infinite, in which case f is not integrable.

REMARK 1.1 (Lebesgue is stable, Riemann is not). *Note the first result, stating that $f = \lim f_j(x)$ is measurable. This is strong statement, which fails in Riemann theory. Recall that the Riemann integral is defined on the set $C^0_{\rm pw}$ of piece-wise continuous functions (or to be more specific on its completion). However, this set is not closed when taking increasing limits. For instance, label the rationnal numbers by $\mathbb{Q} = \{q_1, q_2, \cdots\}$ (it is countable), and set $f_N(x) = \mathbb{1}(x \in \{q_1, \cdots, q_N\})$. Then f_N is piece-wise continuous (the points where f_N is discontinuous are isolated). We have $f_{N+1} \ge f_N$ and $\lim_{N\to\infty} f_N = \mathbb{1}_{\mathbb{Q}}$, which is not piece-wise continuous.*

Another related proposition is the following:

PROPOSITION 1.1. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of measurable functions. The set A of elements x such that $\lim_{k\to\infty} f_k(x)$ exists is measurable, and the function*

$$f_\infty : \begin{cases} A \ni x \mapsto \lim_{k\to\infty} f_k(x)\,, \\ A^c \ni x \mapsto 0 \end{cases}$$

is measurable.

We refer for instance to [7, Lemma 1.5].

THEOREM 1.2 (Fatou inequality). *Let (f_j) be a sequence of* **positive** *measurable functions. Then $f := \liminf_{j\to\infty} f_j$ is positive, measurable, and*

$$0 \le \int_{\mathbb{R}^d} f(x)dx \le \liminf_{j\to\infty} \int_{\mathbb{R}^d} f_j(x)dx.$$

In other words,

$$\int \liminf \le \liminf \int .$$

To remember the order of the inequality, just keep in mind the sequence

$$f_j : x \mapsto \mathbb{1}_{[j;+\infty]}(x).$$

Then for any $x \in \mathbb{R}$ we have $\liminf f_j(x) = 0$, although $\int_{\mathbb{R}^d} f_j = +\infty$.

Finally, we have the most important theorem, the dominated convergence theorem[3].

THEOREM 1.3 (Dominated convergence theorem). *Let (f_j) be a sequence of measurable functions which converges point-wise to a measurable function f a.e. Assume there is an integrable function G so that for any $j \in \mathbb{N}$ we have $|f_j| \le G$ a.e.. Then $|f| \le G$ a.e. and*

$$\int_{\mathbb{R}^d} f = \lim_{j\to\infty} \int_{\mathbb{R}^d} f_j\,, \int_{\mathbb{R}^d} |f_j - f| = 0.$$

In other words, domination implies that we can commute the limits:

$$\lim \int = \int \lim .$$

Here again, we refer to [7]. To understand why domination is important, the reader should keep in mind the following three **counterexamples**.

- **The mass goes to infinity**. Let $\psi \in C^\infty_0(\mathbb{R}^d, \mathbb{R}^+)$ and $e \in \mathbb{R}^d \setminus \{0\}$. Then $f_j(x) := \psi(x - je)$ converges point-wise to $f = 0$. However, $\int f_j = \int \psi > 0$, while $\int f = 0$.

[3] Although it could in several instances be replaced with the Egorov theorem; we refer to [12].

- **The mass spreads over**. Take $f_j(x) = j^d\psi(jx)$. Then f_j converge point-wise to f for all $x \neq 0$, so a.e. However, $\int f_j = \int \psi > 0$, while $\int f = 0$.
- **The mass concentrates**. Consider now $f_j(x) = j^{-d}\psi(x/j)$. Again, f_j converges point-wise to $f = 0$. However, $\int f_j = \int \psi > 0$, while $\int f = 0$.

Another fantastic result in measure theory is that, in some sense, the fundamental theorem of calculus is true almost everywhere; this is the Lebesgue differentiation theorem:

THEOREM 1.4. *Let $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then, for a.e. $x \in \mathbb{R}^d$,*

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{|\mathbb{B}(x;\varepsilon)|} \int_{\mathbb{B}(x;\varepsilon)} f.$$

This is one of the few theorems here that requires finer properties of the Lebesgue measure (through the Vitali covering lemma) and that can be reinterpreted through the lens of the Radon-Nykodym theorem. We refer the reader to [5].

1.2.2. L^p spaces. Analogously to $\mathcal{L}^1$, we define, for any $p \in [1;+\infty)$, the set $\mathcal{L}^p(\mathbb{R}^d)$ as

$$\mathcal{L}^p := \left\{f \text{ measurable}, |f|^p \in \mathcal{L}^1(\mathbb{R}^d)\right\}. \tag{1}$$

We introduce the notation

$$\|f\|^p_{L^p(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |f|^p. \tag{2}$$

For $p = \infty$, the set $\mathcal{L}^\infty$ is defined as the set of Lebesgue measurable functions that are bounded a.e. In that case, we introduce the notation

$$\|f\|_{L^\infty(\mathbb{R}^d)} := \inf\left\{\lambda \geq 0\,, |\{f > \lambda\}| = 0\right\}. \tag{3}$$

It is important to observe that the notations are consistence, in that, if $f \in L^\infty(\mathbb{R}^d)$ is compactly supported then

$$\|f\|_{L^\infty(\mathbb{R}^d)} = \lim_{p \to \infty} \|f\|_{L^p(\mathbb{R}^d)}.$$

It is easy to check that $\mathcal{L}^p\,, \mathcal{L}^\infty$ are vector spaces. However, the natural quantities $\|\cdot\|_{L^p(\mathbb{R}^d)}$ are not norms. Rather, they satisfy the following two first axioms of norms:

(1) Homogeneity: for any $\lambda \geq 0$, for any $p \in [1;+\infty]$, for any $f \in \mathcal{L}^p(\mathbb{R}^d)$,

$$\|\lambda f\|_{L^p(\mathbb{R}^d)} = |\lambda| \cdot \|f\|_{L^p(\mathbb{R}^d)}.$$

(2) Sub-additivity: for any $p \in [1;+\infty]$, for any $f\,, g \in L^p(\mathbb{R}^d)$, we have

$$\|f+g\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} + \|g\|_{L^p(\mathbb{R}^d)}. \tag{4}$$

This is the **Minkowski inequality**, which we prove below (Theorem 1.8).

The problem is the separation property: if $f = g$ a.e., but $f \not\equiv g$, then

$$\|f-g\|_{L^p(\mathbb{R}^d)} = 0.$$

This leads to defining the equivalence relation

$$f \sim g \text{ if, and only if, } f = g \text{ a.e.}$$

and to defining $L^p(\mathbb{R}^d)$ as

$$L^p(\mathbb{R}^d) := \mathcal{L}^p(\mathbb{R}^d)/\sim .$$

For any $f \in L^p(\mathbb{R}^d)$ (which should be thought of as an equivalence class), we can define

$$\|f\|_{L^p(\mathbb{R}^d)} = \|f_0\|_{L^p(\mathbb{R}^d)}$$

where f_0 is any representative of f (as an equivalence class).

The set $L^p(\mathbb{R}^d)$ is called the Lebesgue space with exponent p. Before studying the topological properties of $L^p(\mathbb{R}^d)$ we need to recall the main inequalities.

For any given measurable subset Ω of $\mathbb{R}^d$, we define in similar ways the spaces $\mathcal{L}^p(\Omega)$ and $L^p(\Omega)$.

1.3. The fundamental inequalities

1.3.1. The Jensen inequality. The key inequality is the Jensen inequality– note that one needs to prove it differently from the Riemann Jensen inequality.

THEOREM 1.5. *[Jensen inequality] Let $\varphi : \mathbb{R} \to \mathbb{R}$ be convex, $f : \mathbb{R}^d \to \mathbb{R}$ be measurable and $\mu : \mathbb{R}^d \to \mathbb{R}$ be measurable, with $\int_{\mathbb{R}^d} \mu = 1$. Then*

$$\varphi\left(\int_{\mathbb{R}^d} f\mu\right) \leq \int_{\mathbb{R}^d} \varphi(f)\mu.$$

In particular, for any measurable subset A of $\mathbb{R}^d$ and any function f such that $f\mathbb{1}_A \in L^1(\mathbb{R}^d)$ we have

$$\varphi\left(\frac{1}{|A|}\int_A f\right) \leq \frac{1}{|A|}\int_A \varphi(f).$$

REMARK 1.2. *It is awkward to take μ as a function rather than as a probability measure, but this is a consequence of focusing on the Lebesgue measure.*

PROOF OF THEOREM 1.5. Assume for the sake of simplicity that φ is $\mathscr{C}^1$ so that, for any $z\,, y \in \mathbb{R}$,

$$\varphi(y) \geq \varphi(z) + \varphi'(z)(y - z).$$

In particular, defining

$$y = f(x)\,, z = \int_{\mathbb{R}^d} f\mu$$

we obtain

$$\varphi(f(x)) \geq \varphi\left(\int_{\mathbb{R}^d} f\mu\right) + \varphi'\left(\int_{\mathbb{R}^d} f\mu\right)\left(f(x) - \int_{\mathbb{R}^d} f\mu\right).$$

Multiplying by μ and integrating yields the conclusion.

When φ is not $\mathscr{C}^1$, one simply needs to use the following fact (valid for any convex functions):

$$\forall \xi \in \mathbb{R}\,, \varphi(\xi) = \sup_{a\,,b\in\mathbb{R}\,,a\cdot+b\leq\varphi(\cdot)} (a\xi + b)$$

to conclude in the same way. $\square$

1.3.2. The Hölder inequality and inclusion of Lebesgue spaces. We now use the Jensen inequality to prove the following:

THEOREM 1.6. *Let $1 \leq p, q \leq \infty$ be such that*

$$\frac{1}{p} + \frac{1}{q} = 1$$

and let Ω be a measurable subset of $\mathbb{R}^d$. Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$, and

$$\int_\Omega |fg| \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

PROOF OF THEOREM 1.6. Without loss of generality, we may assume that f and g are non-negative. We introduce $G := g/\|g\|_{L^q}$, which satisfies $\|G\|_{L^q} = 1$. We then set $\mu(x) = G^q(x)$, $F(x) := f(x)/G^{q/p}(x)$ and $J(t) := t^p$, which is convex. We apply Jensen's inequality to (J, F, μ), which gives

$$\int_\Omega \left(\frac{f}{G^{q/p}}\right)^p G^q \geq \left(\int_\Omega \frac{f}{G^{q/p}} G^q\right)^p.$$

with (we use that $q(1 - \frac{1}{p}) = 1$ in the last equality)

$$\int_\Omega \left(\frac{f}{G^{q/p}}\right)^p G^q = \int_\Omega f^p, \quad \text{and} \quad \left(\int_\Omega \left(\frac{f}{G^{q/p}}\right) G^q\right)^p = \left(\int_\Omega f G^{q(1-\frac{1}{p})}\right)^p = \left(\int_\Omega fG\right)^p.$$

Consequently,

$$\left(\int_\Omega f \frac{g}{\|g\|_{L^q}}\right)^p = \left(\int_\Omega fG\right)^p \leq \int_\Omega f^p,$$

whence

$$\|fg\|_{L^1}^p = \left(\int_\Omega fg\right)^p \leq \|f\|_{L^p}^p \|g\|_{L^q}^p.$$

□

A crucial consequence of the Hölder inequality is the following:

COROLLARY 1.1. *If $|\Omega| < \infty$, then for any $1 \leq p \leq q \leq \infty$,*

$$L^q(\Omega) \subset L^p(\Omega).$$

REMARK 1.3. *This inclusion is obviously not true when $|\Omega| = \infty$. For instance, it suffices to take $\Omega = \mathbb{R}$, $f \equiv 1$, so that $f \in L^\infty(\mathbb{R})$ but f is not in any $L^p(\mathbb{R})$ for any $p < \infty$.*

You should also pay attention to the fact that not only is this not true for general measures, but that the reverse inequalities might actually holds. For instance, if we define for any $p \in [1; +\infty)$

$$\ell^p := \left\{\{x_k\}_{k\in\mathbb{N}}, \sum_{k=0}^\infty |x_k|^p < \infty\right\}$$

and

$$\ell^\infty := \{\{x_k\}_{k\in\mathbb{N}}, \{x_k\}_{k\in\mathbb{N}} \text{ is bounded }\}$$

then for any

$$1 \leq p \leq q \leq \infty$$

we have

$$\ell^p \subset \ell^q.$$

Finally, a good exercise is the following: fix any $p \in [1;+\infty]$. Find a function $f \in L^p(\mathbb{R}_+)$ such that, for any $q \neq p$ we have $f \notin L^q(\mathbb{R}_+)$.

Another corollary of the Hölder inequality is the following– the proof (which is easy) is left as an exercise:

THEOREM 1.7. *[General form of the Hölder inequality] Let $1 \leq p, q, r \leq \infty$ be such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^r(\Omega)$, and

$$\|fg\|_{L^r}(\Omega) \leq \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}.$$

1.3.3. The Minkowski inequality. Finally, we go back to the Minkowski inequality (4).

THEOREM 1.8. *For all $f, g \in L^p(\Omega)$ with $1 \leq p \leq \infty$, we have*

$$\|f+g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

In particular, the map $f \mapsto \|f\|_{L^p(\Omega)}$ is convex. If $1 < p < \infty$ this map is strictly convex.

PROOF OF THEOREM 1.8. We have

$$\int_\Omega |f+g|^p = \int_\Omega |f+g| \cdot |f+g|^{p-1} \leq \int_\Omega |f| \cdot |f+g|^{p-1} + \int_\Omega |g| \cdot |f+g|^{p-1}.$$

We use the Hölder inequality with $f \in L^p(\Omega)$ and $|f+g|^{p-1} \in L^q$ with $q = \frac{p}{p-1}$, and get

$$\int_\Omega |f| \cdot |f+g|^{p-1} \leq \left(\int_\Omega |f|^p\right)^{1/p} \left(\int_\Omega |f+g|^p\right)^{\frac{p-1}{p}}.$$

This gives $\|f+g\|^p_{L^p(\Omega)} \leq \left(\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}\right) \|f+g\|^{p-1}_{L^p(\Omega)}$, which is equivalent to the required inequality. □

1.4. L^p as vector spaces

In this section, we review several important topological facts about Lebesgue spaces.

1.4.1. Lebesgue spaces as Banach spaces. The first topological fact about Lebesgue spaces is that they are Banach spaces, in the sense that any Cauchy sequence admits a limit in that space. We summarise this in the following theorem:

THEOREM 1.9. *For any $1 \leq p \leq \infty$, the Lebesgue space $L^p(\mathbb{R}^d)$ is a Banach space.*

We will see, in the proof of this theorem, the following result, that we single out for easier referencing:

PROPOSITION 1.2. *Let $\{f_k\}_{k\in\mathbb{N}} \in L^p(\mathbb{R}^d)^{\mathbb{N}}$ and $f \in L^p(\mathbb{R}^d)$ be such that*

$$\|f_k - f\|_{L^p(\mathbb{R}^d)} \underset{k\to\infty}{\to} 0.$$

Then there exists a subsequence of $\{f_k\}_{k\in\mathbb{N}}$ that converges almost everywhere to f.

Proof of Theorem 1.9. We begin with the case $p < \infty$. We consider a Cauchy sequence (for the L^p topology) $\{f_k\}_{k\in\mathbb{N}}$. Of course, to obtain the result, it suffices to prove that there exists a subsequence of $\{f_k\}_{k\in\mathbb{N}}$ that admits a limit $f \in L^p$ (for the L^p topology), as the fact that we work with a Cauchy sequence will then entail the convergence of the entire sequence to f. Observe that, as the sequence is Cauchy, we can assume, up to extraction, that

$$\forall k \in \mathbb{N}\,, \|f_k - f_{k+1}\|_{L^p(\mathbb{R}^d)} \leq \frac{1}{2^k}. \tag{5}$$

Now, observe that the series

$$\sum_{i=1}^{k}(f_{i+1} - f_i)(x)$$

is absolutely converging for a.e. $x \in \mathbb{R}^d$. Indeed, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\left(\sum_{k=1}^{\infty}|f_{k+1} - f_k|\right)^p &= \lim_{N\to\infty}\int_{\mathbb{R}^d}\left(\sum_{k=1}^{N}|f_{k+1} - f_k|\right)^p \\ &\quad \text{by the monotone convergence theorem} \\ &= \lim_{N\to\infty}\left(\sum_{k=1}^{N}\|f_{k+1} - f_k\|\|_{L^p(\mathbb{R}^d)}\right)^p \\ &\quad \text{by the } L^p \text{ triangle inequality} \\ &< \infty \text{ by (5).}\end{aligned}$$

Consequently, for a.e. $x \in \mathbb{R}^d$, we have

$$\sum_{k=1}^{\infty}|f_{k+1} - f_k|(x) < \infty$$

which implies that, for a.e. $x \in \mathbb{R}^d$, the sequence $\{f_k(x)\}_{k\in\mathbb{N}}$ converges, say to $f(x)$. By Proposition 1.1, the function

$$f : x \mapsto f(x) \text{ if } \{f_k(x)\}_{k\in\mathbb{N}} \text{ converges, 0 otherwise}$$

is measurable. Furthermore,

$$f(x) = \lim_{N\to\infty} f_1(x) + \sum_{k=1}^{N}(f_{k+1}(x) - f_k(x))$$

and the Fatou lemma implies that $f \in L^p(\mathbb{R}^d)$. Finally,

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^d}|f - f_n|^p &= \lim_{N\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^d}|f_{N+1} - f_n|^p \\ &\leq \lim_{n\to\infty}\frac{1}{2^{np}} \\ &= 0.\end{aligned}$$

This concludes the proof in the case $p < \infty$. The case $p = \infty$ is straightforward. □

1.4.2. Separability of Lebesgue spaces. We now turn to another question, that of the separability of Lebesgue spaces. Recall that a Banach space is called **separable** if it admits a countable, dense subset. The main theorem is the following:

THEOREM 1.10.
- *For any $p \in [1;+\infty)$, the space $L^p(\Omega)$ is separable. The set of compactly supported, continuous functions is dense in $L^p(\mathbb{R}^d)$.*
- *$L^\infty(\Omega)$ is not separable.*

PROOF OF THEOREM 1.10. We begin with the case $p<\infty$. Of course, using the density of points with rational coordinates in $\mathbb{R}^d$, it suffices to show that the set of simple functions $f=\sum_{i=1}^N \alpha_i \mathbb{1}_{A_i}$, where the A_i's are measurable, is dense in L^p. Let $f \in L^p(\Omega)$. As we can split f into its positive and its negative part, we can assume that f is non-negative. We consider, for any $N \in \mathbb{N}$ and any $0 \leq j \leq N2^{N-1}$, the sets

$$A_N := \{f \geq N\}\,, B_{j,N} := \left\{\frac{j}{2^N} \leq f \leq \frac{j+1}{2^N}\right\}.$$

We finally define

$$f_N := N\mathbb{1}_{A_N} + \sum_{j=0}^{N2^{N-1}} \frac{j}{2^N}\mathbb{1}_{B_{j,N}}.$$

Then it is readily checked that this sequence is increasing, and converges pointwise to f. From the monotone convergence theorem, we deduce that

$$\|f_N - f\|_{L^p(\Omega)} \underset{N\to\infty}{\to} 0.$$

In particular, taking the α_i and the coordinates of the A_i's to be rational, we obtain the result.

Let us now show that $L^\infty(\Omega)$ is not separable. We claim that it suffices to show that $L^\infty(\mathbb{R}^d)$ is not separable. Indeed, should $L^\infty(\mathbb{R}^d)$ be separable, with a dense countable family $\{f_k\}_{k\in\mathbb{N}}$, then it is readily checked that the family $\{f_k \mathbb{1}_\Omega\}_{k\in\mathbb{N}}$ is a countable, dense family in $L^\infty(\Omega)$. Regarding the non-separability of $L^\infty(\mathbb{R}^d)$, simply consider the family $\{f_r\}_{r\in\mathbb{R}_+} := \{\mathbb{1}_{\mathbb{B}(0;r)}\}_{r\in\mathbb{R}_+}$. This family is uncountable and, for any $r \neq r'$, $\|f_r - f_{r'}\|_{L^\infty(\mathbb{R}^d)} = 1$. Consequently, $L^\infty(\mathbb{R}^d)$ can not admit a countable dense subset. □

1.4.3. Duality in Lebesgue spaces I: basic facts about duality. The last property of Lebesgue spaces as Banach spaces has to do with duality or, in other words, with linear forms on Lebesgue spaces. Recall that the dual of the space $L^p(\Omega)$ is the set of all **continuous** linear forms $T: L^p(\Omega) \to \mathbb{R}$. Before we state our main theorem (Theorem 1.13 below) we need to recall some basic facts about duality in general Banach spaces. As always, the main tool here is the Hahn-Banach theorem, which we give in analytic form:

THEOREM 1.11. *[Hahn-Banach, Analytic form] Let $(E, \|\cdot\|)$ be a Banach space, let F be a subspace of E and $T \in F'$. There exists $\tilde{T} \in E'$ such that*

$$\tilde{T}\Big|_F = T\,, \|\tilde{T}\|_{E'} \leq \|T\|_{F'}.$$

Observe that we do not require F to be closed, as any continuous linear form on F can be extended to the closure $\bar{F}$ (we leave this as an exercise). We do not prove this theorem here–in the finite dimensional setting, or in the case of separable

Banach spaces, this can be done completely by hand. However, we will use it in the non-separable case as well, and Theorem 1.11 then relies on the Zorn lemma.

A very useful consequence of the Hahn-Banach representation theorem is the following:

PROPOSITION 1.3. *Let $(E, \|\cdot\|)$ be a Banach space and E' be its dual, endowed with the norm. Then*

$$\forall x \in E\,, \|x\| = \sup_{T\in E'\,,\|T\|_{E'}\leq 1} T(x).$$

We also give the geometric version of the Hahn-Banach theorem:

THEOREM 1.12. *Let $(E, \|\cdot\|)$ be a Banach space, $A\,, B \subset E$ be two convex sets, with A compact, B closed and $A \cap B = \emptyset$. There exists $f \in E'$ such that*

$$\sup_A f < \inf_B f.$$

1.4.4. Duality in Lebesgue spaces II: the Riesz representation theorem. The main theorem in the study of duality is the Riesz representation theorem:

THEOREM 1.13. *[Riesz representation theorem]*

(1) *Let $p \in [1; +\infty)$ and define p', the Lebesgue conjugate exponent, as*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

There is an isometry between $L^p(\Omega)'$ and $L^{p'}(\Omega)$, which we often abbreviate as

$$L^p(\Omega)' = L^{p'}(\Omega).$$

(2) *For any $p \in (1; +\infty)$, $L^p(\Omega)$ is reflexive, meaning that the natural injection $L^p(\Omega) \hookrightarrow L^p(\Omega)''$ is a bijective isometry.*
(3) *The dual of $L^\infty(\Omega)$ is strictly larger than $L^1(\Omega)$.*

REMARK 1.4 (Regarding reflexivity). *The fact that for any $p \in (1; +\infty)$ the space $L^p(\Omega)$ is reflexive is a corollary of a much more general theorem, which states that any uniformly convex Banach space is reflexive. The fact that $L^2(\Omega)$ is reflexive is a consequence of the fact that any Hilbert space is reflexive.*

REMARK 1.5 (Regarding the dual of L^∞). *The dual of $L^\infty(\Omega)$ is, to put it mildly, a space you should stay away from, as it consists of a subset of finitely additive set-functions. We refer the brave reader to [2].*

To prove this theorem, we will rely on the Radon-Nikodym theorem:

THEOREM 1.14. *Assume μ and ν are two non-negative Borel measures and that μ is absolutely continuous with respect to ν in the sense that for any measurable set A we have*

$$\nu(A) = 0 \Rightarrow \mu(A) = 0.$$

Further assume that μ and ν are σ finite. Then there exists a ν-integrable function f such that for all measurable subset A we have

$$\mu(A) = \int_A f d\nu.$$

Proof of Theorem 1.14. We only treat the case where μ and ν have finite mass. We begin with the case $\mu \leq \nu$. The fact that $\nu(\mathbb{R}^d) < \infty$ implies that $L^2(\nu) \hookrightarrow L^1(\nu)$. Consider the map

$$\Phi : L^2(\nu) \ni f \mapsto \int f d\mu.$$

As Φ is continuous by the assumption $\mu \leq \nu$, we deduce that there exists $\psi \in L^2(\nu)$ such that

$$\forall f \in L^2(\nu)\,, \int f d\mu = \int \psi f d\nu.$$

In particular, for any measurable set A we have

$$\mu(A) = \int_A f d\nu.$$

We can then apply this to the general case: there exists a $\mu+\nu$ measurable function f such that for any measurable set A we have

$$\mu(A) = \int_A f d\mu + \int_A f d\nu.$$

This implies that $f \leq 1$ $\mu + \nu$ a.e., and that $\nu(\{f = 1\}) = 0$. Let $N := \{f = 1\}$. Then

$$\mu(A) = \mu(A \cap N^c) = \int_A \mathbb{1}_{N^c} \frac{f}{1-f} d\nu,$$

as claimed. ☐

We will not prove the fact that $L^p(\Omega)' = L^{p'}(\Omega)$ but we give some hints: consider a linear map $\Phi \in (L^p(\Omega))'$, and define

$$\mu : A \mapsto \Phi(\mathbb{1}_A).$$

Show that there exist two Radon measures $\mu_\pm$ such that $\mu = \mu_+ - \mu_-$, and such that $\mu_\mp$ is absolutely continuous with respect to the Lebesgue measure. Conclude.

It is important to keep in mind the proof that $L^\infty(\Omega)'$ is strictly larger than $L^1(\Omega)$.

Proof that $L^1(\Omega) \subsetneq L^\infty(\Omega)'$. Assume that Ω is open and bounded. The key is once again the Hahn-Banach theorem. The set $\mathscr{C}^0_c(\Omega)$ is closed in $L^\infty(\Omega)$ Define $T : \mathscr{C}^0_c(\Omega) \ni f \mapsto f(x_0)$ where x_0 is any given point in Ω. Then T can be extended to a continuous linear map on $L^\infty(\Omega)$, that vanishes on $\mathscr{C}^0_c(\Omega \backslash \{x_0\})$. Consequently, if there existed $\varphi \in L^1$ such that $\int_\Omega \varphi f = T(f)$, it would follows that $\psi = 0$, a contradiction.

Another, similar construction would be the following: choose a $f \in L^\infty$ such that $\mathrm{dist}(f, \mathscr{C}^0_b(\mathbb{R}^d)) > 0$ and define the linear map

$$T : \mathscr{C}^0_b(\mathbb{R}^d) \oplus \mathbb{R} f \ni g + tf \mapsto t$$

and show that T is continuous. Then, by the Hahn-Banach theorem, T extends to a non trivial linear form on $L^\infty(\mathbb{R}^d)$, which vanishes on $\mathscr{C}^0_b(\mathbb{R}^d)$. We can conclude as above. ☐

Henceforth, we will use the notation p' without redefining it every single time. A useful, related fact, is the following:

Proposition 1.4. *For all $1\leq p\leq \infty$, and all $f\in L^p(\Omega)$, we have*

$$\|f\|_{L^p}=\sup\left\{\int_\Omega fg,\quad g\in L^{p'}(\Omega),\quad \|g\|_{L^{p'}(\Omega)}\leq 1\right\}.$$

Proof of Proposition 1.4. Once again, we need to distinguish between $p\in(1;+\infty)$ and $p\in\{1,\infty\}$. Observe that by the Hölder inequality we have

$$\sup\left\{\int_\Omega fg,\quad g\in L^{p'}(\Omega),\quad \|g\|_{L^{p'}(\Omega)}\leq 1\right\}\leq \|f\|_{L^p(\Omega)}.$$

When $p\in(1;+\infty)$, we can take $g_0:=\frac{1}{\|f\|_{L^p}^{p-1}}|f|^{p-2}f$, so that $|g_0|=\frac{1}{\|f\|_{L^p}^{p-1}}|f|^{p-1}$. Since $f\in L^p(\Omega)$ and since $p'=p/(p-1)$, we have $g_0\in L^{p'}(\Omega)$ with

$$\|g_0\|_{L^{p'}}^{p'}=\int_\Omega |g_0|^{p'}=\int_\Omega |g_0|^{\frac{p}{p-1}}=\frac{1}{\|f\|_{L^p}^p}\int_\Omega |f|^p=1.$$

On the other hand, we have

$$\int_\Omega g_0 f=\frac{1}{\|f\|_{L^p}^{p-1}}\int_\Omega |f|^p=\|f\|_{L^p},$$

and the result follows.

When $p=1$, it suffices to take $g_0=\operatorname{sgn}(f)$. When $p=+\infty$, define, for any $\varepsilon>0$, $g_k:=\operatorname{sgn}(f)\frac{1}{|\{f\geq \|f\|_{L^\infty(\Omega)}-\varepsilon\}|}\mathbb{1}_{\{f\geq\|f\|_{L^\infty(\Omega)}-\varepsilon\}}$ and observe that

$$\int_\Omega g_k f\geq \|f\|_{L^\infty(\Omega)}-\varepsilon.$$

Passing to the limit $\varepsilon\to 0$ provides the conclusion. □

1.5. Convolution and density of smooth functions in Lebesgue spaces

In this section, we study some topological properties of Lebesgue spaces, with a particular emphasis on separability and density of particular families of functions. The main tool, which will also be used in the study of Sobolev spaces, is convolution.

1.5.1. Convolution in Lebesgue spaces. In this section, we take $\Omega=\mathbb{R}^d$. Let f,g be two measurable functions. We define the **convolution** $f*g$ by

$$(f*g)(x):=\int_{\mathbb{R}^d} f(y)g(x-y)dy.$$

It is readily checked that the convolution product is both commutative and associative. From the Hölder inequality (Theorem 1.6), the convolution product is well-defined for any $f\in L^p(\mathbb{R}^d)\,,g\in L^q(\mathbb{R}^d)$, with $\frac1p+\frac1q=1$ and, in that case,

$$\|f*g\|_{L^\infty(\mathbb{R}^d)}\leq \|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}.$$

The advantage of convolution is its regularising property, which we shall see in more details when studying elliptic equations and their fundamental solutions. In general, $f*g$ is at least as regular as f and as g. This is formalised in the following theorem:

Theorem 1.15. *Let $f\in L^p(\mathbb{R}^d)$ and let $g\in\mathscr{C}^k(\mathbb{R}^d)$ with compact support. Then $f*g\in\mathscr{C}^k(\mathbb{R}^d)$ and, for any $0\leq i\leq k$,*

$$\nabla^i(f*g)=f*(\nabla^i g).$$

Proof of Theorem 1.15. The proof of this theorem is essentially a combination of the dominated convergence theorem and of the fact that differentiability of a function follows from the continuity of the partial derivatives. Let e_i be the i-th canonical vector of $\mathbb{R}^d$. We have

$$\frac{1}{t}\left((f*g)(x+te_i)-(f*g)(x)\right)=\int_{\mathbb{R}^d} f(y)\left[\frac{1}{t}\left(g(x-y+te_i)-g(x-y)\right)\right]dy.$$

Since g is smooth, the term in bracket converges pointwise to $(\partial_i g)(x-y)$ as $t\to 0$. In addition, using the mean value theorem, there is c in the segment $[x-y,x-y+te_i]$ so that

$$\left|f(y)\left[\frac{1}{t}\left(j_\varepsilon(x+te_1-y)-g(x-y)\right)\right]\right|=|f(y)(\partial_i g)(c)|\le\|(\partial_i g)\|_{L^\infty}|f(y)|,$$

which is integrable in y, and independent of t. We can thus conclude by the dominated convergence theorem that $\partial_i(f*g)$ is well-defined. Applying the dominated convergence theorem once more, these partial derivatives are continuous, and we can conclude by induction that $f*g\in\mathscr{C}^k$. □

We now consider a compactly supported function $\psi\in\mathscr{C}^\infty(\mathbb{R}^d)$ that satisfies

$$0\le\psi\le 1\,,\mathrm{supp}(\psi)\subset\mathbb{B}(0;1)\,,\int_{\mathbb{R}^d}\psi=1.$$

The existence of such a function ψ is a standard exercise in analysis (think of $x\mapsto e^{-\frac{1}{x^2}}$...). We now introduce the sequence $\{\psi_k\}_{k\in\mathbb{N}}$, defined as follows:

$$\forall k\in\mathbb{N}\,,\psi_k:x\mapsto 2^{dk}\psi\left(2^k x\right).$$

Such a sequence is called a **mollifier** and serves to approximate any L^p function by a sequence of more regular functions.

Theorem 1.16. *Let $1\le p<\infty$. Let $f\in L^p(\mathbb{R}^d)$ and define, for any $k\in\mathbb{N}$, $f_k:=(f*\psi_k)$. Then:*

(1) *For any k, $f_k\in\mathscr{C}^\infty(\mathbb{R}^d)$.*
(2) *Furthermore,*

$$\|f_k-f\|_{L^p(\mathbb{R}^d)}\underset{k\to\infty}{\to}0. \tag{6}$$

In the proof of this theorem, we will be using the weak convergence of translation operators in L^p spaces.

Proposition 1.5. *Let $p\in[1;+\infty)$. Define, for any $\tau\in\mathbb{R}^d$ and any $f\in L^p(\mathbb{R}^d)$, $m_\tau(f):=f(\cdot-\tau)$. Then*

$$\forall f\in L^p(\mathbb{R}^d)\,,\|m_\tau(f)-f\|\underset{\tau\to 0}{\to}0.$$

Proof of Proposition 1.5. By the density of simple functions in $L^p(\mathbb{R}^d)$, it suffices to prove the result for f simple. However, this amounts to showing that, for any measurable orthotope $A=\prod_{i=1}^d(a_i;b_i)$ with finite measure,

$$\int_{\mathbb{R}^d}|\mathbb{1}_A-\mathbb{1}_{A+\tau}|^p\underset{\tau\to 0}{\to}0.$$

This is however a simple computation. Alternatively, one can also work with compactly supported continuous functions. □

Remark 1.6. *This proposition is obviously wrong when $p = +\infty$. Indeed, just take $f = \mathbb{1}_{[0;1]}$. Then, for any $\tau \neq 0$, $\|m_\tau(f) - f\|_{L^\infty(\mathbb{R}^d)} = 1$.*

Proof of Theorem 1.16. The first point is a consequence of Theorem 1.15. As for the second point, let us rewrite

$$(f_k - f)(x) = \int_{\mathbb{R}^d} f(y)\psi_k(x-y)dy - f(x)\int_{\mathbb{R}^d} \psi_k(x-y)dy = \int_{\mathbb{R}^d} \left(f(x-y) - f(x)\right)\psi_k(y)dy.$$

We deduce

$$|(f_k - f)(x)| \leq \int_{\mathbb{R}^d} |f - m_y(f)|(x) \cdot |\psi_k(y)|^{\frac{1}{p}} \cdot |\psi_k(y)|^{\frac{1}{p'}} dy$$

so that

$$\begin{aligned}
\int_{\mathbb{R}^d} |f_k - f|^p &\leq \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} |f - m_y(f)|(x) \cdot |\psi_k(y)|^{\frac{1}{p}} \cdot |\psi_k(y)|^{\frac{1}{p'}} dy \right|^p dx \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f - m_y(f)|^p(x) \cdot |\psi_k(y)| \cdot \|\psi_k\|_{L^1(\mathbb{R}^d)}^{\frac{p}{p'}} dydx \\
&\quad \text{by the Hölder inequality} \\
&\leq \int_{y \in \mathrm{supp}(\psi_k)} \sup_{z \in \mathrm{supp}(\psi_k)} \|m_z(f) - f\|_{L^p(\mathbb{R}^d)} \cdot |\psi_k(y)| dy \\
&\underset{k\to\infty}{\to} 0
\end{aligned}$$

where the last limit is obtained as a consequence of $\mathrm{supp}(\psi_k) \subset \mathbb{B}(0; 2^{-k})$ and from Proposition 1.5. □

Remark 1.7 (Uniform convergence for the strong L^p topology). *Can we make the previous result uniform, meaning, do we have*

$$\sup_{\|f\|_{L^p(\mathbb{R}^d)} \leq 1} \|f_k - f\|_{L^p(\mathbb{R}^d)} \underset{k\to\infty}{\to} 0? \tag{7}$$

This answer to this question is (naturally) no. To give an example, unfortunately, we have to jump ahead and to use a bit of (L^2) Fourier transform. Assuming that (7) held with $p = 2$, using the fact that the Fourier transform of f_k is

$$\hat{f}_k(\xi) = \hat{\psi}_k(\xi)\hat{f}(\xi)$$

and the fact that $\hat{\cdot}$ is an L^2 isometry, this would entail that

$$\sup_{\hat{f} \in L^2(\mathbb{R}^d), \|\hat{f}\|_{L^2(\mathbb{R}^d)} \leq 1} \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |1 - \hat{\psi}_k(\xi)|^2 d\xi \underset{k\to\infty}{\to} 0.$$

This in turn proves that, uniformly in ξ, k,

$$|1 - \hat{\psi}_k(\xi)|^2 \underset{k\to\infty}{\to} 0.$$

Nevertheless, take any ξ_0 such that $\hat{\psi}(\xi_0) \neq 0, \hat{\psi}(\xi_0) \neq 1$ (why does such a ξ_0 exist?), and observe that

$$\hat{\psi}_k(2^k\xi_0) = \hat{\psi}(\xi_0).$$

This concludes the proof.

It is however possible to prove that if we have a uniform L^p bound on the gradient of f, then the answer is positive. In the case $p = 1$ it is an elementary but *good exercise to prove the following estimate: assume that $f \in \mathscr{C}^1_c(\mathbb{R}^d)$. Show that there exists a constant C that only depends on ψ such that*

$$\|f_k - f\|_{L^1(\mathbb{R}^d)} \le C 2^{-k} \|\nabla f\|_{L^1(\mathbb{R}^d)}.$$

1.6. Convergence in L^p spaces

The goal of this paragraph is to provide some information regarding the different types of convergences in L^p spaces.

1.6.1. A reminder on the Riesz theorem. We begin this section, devoted to the exploration of various aspects of compactness, with a reminder on the Riesz theorem:

THEOREM 1.17 (Riesz). *Let E be a Banach space. The following statements are equivalent:*

(1) *The unit ball is sequentially compact.*
(2) *E is finite dimensional.*

PROOF OF THEOREM 1.17. One direction is immediate. Now, assume that E is infinite dimensional. Fix a unit vector x_1. Then we claim that there exists $x_2 \in \S(0;1)$ such that $\|x_1 - x_2\| \ge \frac{1}{2}$. Fix $x_1' \in E \subset \mathbb{R}x_1$. As the vector space $V_1 := \mathbb{R}x_1$ is closed, $\delta := \mathrm{dist}(x_1', V_1) > 0$. In particular, there exists $y_1 \in V_1$ such that

$$\delta \le \mathrm{dist}(x_1', y_1) \le 2\delta.$$

We set

$$x_2 := \frac{x_1' - y_1}{\|x_1' - y_1\|}.$$

Then we deduce that

$$\begin{aligned} \|x_1 - x_2\| &= \left\| \frac{x_1' - z_1}{\|x_1' - y_1\|} \right\| \text{ for some } z_1 \in V_1 \\ &\ge \frac{\delta}{2\delta} \\ &\ge \frac{1}{2}. \end{aligned}$$

Iterating this construction, we deduce that there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that, for any $k\,, k' \in \mathbb{N}$,

$$\|x_k - x_{k'}\| \ge \frac{1}{2},$$

which implies that it can not have any converging subsequence, thereby concluding the proof. □

1.6.2. Different types of convergence: a crash course on weak topologies. In this first paragraph, we recall the basic definitions of weak and strong convergence in Banach spaces; the notations are the same as in paragraph 3.11. Let E be a Banach space. We can consider several topologies on E. The more natural one is the **strong topology**, defined by the following notion of convergence:

$$x_n \xrightarrow[n\to\infty]{} x, \quad \text{iff} \quad \|x_n - x\|_E \to 0. \qquad (\textbf{strong convergence}).$$

We can also define the **weak topology** of E. This one is defined by the following notion:

$$x_n \underset{n\to\infty}{\rightharpoonup} x, \quad \text{iff} \quad \forall T \in E', \quad T(x_n - x) \to 0. \qquad (\textbf{weak convergence}).$$

Finally, we sometime use the **weak-$*$ topology**. This only applies if $E = F'$ is already the dual space of another Banach space F. Then

$$x_n \underset{n\to\infty}{\rightharpoonup} x, \quad \text{iff} \quad \forall f \in F, \quad (x_n - x)(f) \to 0. \qquad (\textbf{weak-* convergence if } E = F').$$

REMARK 1.8 (A big word of caution). *Weak topologies have fewer open sets, and therefore have more compact sets, which is the main reason it was introduced and studied. The weak topology has some surprising properties (if, instead of defining it through convergence, we did so using open sets, then any open set would be unbounded whenever E is infinite dimensional; as we do not touch on topological vector spaces in these lectures we do not dwell on this aspect).*

In general, the dual of a Banach space is much better behaved than the Banach space itself when it comes to compactness and weak convergence. This is due to the Banach-Alaoglu theorem which we state below.

REMARK 1.9. *In the finite dimensional case, weak and strong convergence are equivalent.*

REMARK 1.10 (The basic example of weak convergence: Hilbert spaces). *Take any infinite dimensional separable Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with a Hilbert basis $\{e_k\}_{k\in\mathbb{N}}$. We know that by the Plancherel identity there holds, for any $x \in H$,*

$$\|x\|^2 = \sum_{k=0}^{\infty} |\langle x, e_k \rangle|^2$$

which implies that for any $x \in H$ we have $\langle x, e_k \rangle \underset{k\to\infty}{\to} 0$. From the Riesz representation theorem, we deduce that $e_k \underset{k\to\infty}{\rightharpoonup} 0$, although $\|e_k\| = 1$. This gives the example of a weakly converging sequence that is not strongly converging.

More generally, consider a sequence $\{x_k\}_{k\in\mathbb{N}} \in H^{\mathbb{N}}$ that writes

$$\forall k \in \mathbb{N}, x_k = \sum_{j=0}^{\infty} \alpha_{k,j} e_j.$$

Prove that

$$x_k \underset{k\to\infty}{\rightharpoonup} x = \sum_{j=0}^{\infty} \alpha_{\infty,j} e_j$$

if, and only if,

$$\forall j \in \mathbb{N}, \alpha_{k,j} \underset{k\to\infty}{\to} \alpha_{\infty,j}.$$

In other words, in the case of Hilbert spaces:

1. *For a bounded sequence, the weak convergence is equivalent to the pointwise convergence of the coefficients (is it true without the boundedness assumption?).*
2. *The strong convergence implies the uniform convergence of the coefficients (is the converse true?).*

The main theorem related to weak and strong convergence is the following:

THEOREM 1.18. (1) *If* $x_k \underset{k\to\infty}{\to} x$ *strongly in* E*, then* $x_k \underset{k\to\infty}{\rightharpoonup} x$ *weakly.*

(2) *If* E *is reflexive, then the weak-* convergence on* $(E')' = E$ *is equivalent to the weak convergence on* E*.*

(3) *If* $x_k \underset{k\to\infty}{\rightharpoonup} x$ *weakly in* E*, then* $\{x_k\}_{k\in\mathbb{N}}$ *is bounded in* E *and*

$$\|x\| \leq \liminf_{k\to\infty} \|x_k\|. \tag{8}$$

(4) *If* $x_k \underset{k\to\infty}{\to} x$ *strongly in* E*, and* $T_k \underset{k\to\infty}{\rightharpoonup} T$ *weakly in* E'*, then* $T_k(x_k) \underset{k\to\infty}{\to} T(x)$*.*

REMARK 1.11. *The last point of the theorem states that weak convergence+strong convergence implies convergence. In general, if* $x_k \underset{k\to\infty}{\rightharpoonup} x$ *weakly in* E *and if* $T_k \underset{k\to\infty}{\rightharpoonup} T$ *weakly in* $E*$*, there is no reason that*

$$T_k(x_k) \underset{k\to\infty}{\to} T(x).$$

A classical example is the following: take again a Hilbert space $(H, \langle \cdot, \cdot \rangle)$ *and a Hilbert basis* $\{e_k\}_{k\in\mathbb{N}}$*. We saw in Remark 1.10 that* $e_k \underset{k\to\infty}{\rightharpoonup} 0$*. However, defining* $T_k : x \mapsto \langle x, e_k \rangle$ *this gives*

$$\forall k \in \mathbb{N}, T_k(x_k) = 1.$$

PROOF OF THEOREM 1.18. The only non-immediate point is the boundedness of every weakly converging sequence. We do not prove it here, as it relies on the Banach-Steinhaus theorem. □

1.6.3. The Banach-Alaoglu theorem and reflexivity of Banach spaces. The most important theorem is the Banach-Alaoglu theorem:

THEOREM 1.19. *Let* $(E, \|\cdot\|)$ *be a* ***separable*** *Banach space. Then any bounded set* B *of* E' *is sequentially compact for the weak-* topology.*

PROOF OF THEOREM 1.19. As E is separable, let $\{u_k\}_{k\in\mathbb{N}}$ be a countable, dense subset of E. Let $\{T_j\}_{j\in\mathbb{N}} \in (E')^{\mathbb{N}}$ be a bounded family of linear forms on E. We argue by the Cantor diagonal process to show that there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N}, \{T_{n_j}(u_k)\}_{j\in\mathbb{N}} \text{ has a limit } T(x).$$

To this end: first find an extractor ϕ_0 such that

$$T_{\phi_0(j)}(u_0) \underset{j\to\infty}{\to} T(u_0),$$

then ϕ_1 such that

$$T_{\phi_0(\phi_1(j))}(u_1) \underset{j\to\infty}{\to} T(u_1)....$$

and finally set

$$T_{n_j} := T_{\phi_0(\phi_1(\dots(\phi_j(j))))}.$$

It is clear that for any k, k', $T(x_k + x_{k'}) = T(x_k) + T(x_{k'})$. Furthermore, since $\{T_j\}_{j\in\mathbb{N}}$ is bounded in E', so is T. This implies that T can be extended to a continuous linear form on E, and this completes the proof. □

For further reference, we reformulate this theorem in the case of Lebesgue spaces:

THEOREM 1.20. (1) *Let $1 < p < \infty$. If $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $L^p(\Omega)$, then there is a subsequence $\{\phi(k)\}_{k\in\mathbb{N}}$ and an element $f \in L^p(\Omega)$ such that $\{f_{\phi(k)}\}_{k\in\mathbb{N}}$ weakly (in the weak $L^p(\Omega)$ sense) or weakly-* (seeing the weak topology on $L^p(\Omega)$ as the weak-* topology on $L^{p'}(\Omega)'$)converges to f.*

(2) *If $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(\Omega)$, then there is a subsequence $\{\phi(k)\}_{k\in\mathbb{N}}$ and an element $f \in L^\infty(\Omega)$ such that $\{f_{\phi(k)}\}_{k\in\mathbb{N}}$ weakly-* converges to f.*

REMARK 1.12 (The case $p = 1$). *Theorem 1.20 is not true for $p = 1$. To see this, consider $d = 1$, $\Omega := (-1;1)$ and the sequence $f_k := 2^{k+1}\mathbb{1}_{[-2^{-k};2^k]}$. This sequence is bounded in L^1. However, it can not converge for the weak topology on L^1: for any $\phi \in \mathscr{C}^0_c(\Omega)$, $\int_\Omega f_k\phi \underset{k\to\infty}{\to} \phi(0)$. There is, however, no L^1 function f_∞ such that for any $\phi \in \mathscr{C}^0_c(\Omega)$ we have $\phi(0) = \int_\Omega f_\infty\phi$ (we leave this as an exercise). One would need to work in the space of Radon measures.*

We will actually show later on that $L^1(\Omega)$ is not the dual of any Banach space.

1.7. Strong convergence criteria in $L^p(\mathbb{R}^d)$

1.7.1. Which phenomena prevent strong convergence in Lebesgue spaces? There are essentially three phenomena that can prevent the strong convergence (up to a subsequence) of a bounded sequence in $L^p(\Omega)$.

(1) When $\Omega = \mathbb{R}^d$ (or is at any rate unbounded) the mass can go to infinity, think of $f_k := \mathbb{1}_{[k;k+1]}$. This is a bit of cheating as, up to a translation, the sequence converges strongly.
(2) When $\Omega = \mathbb{R}^d$, one can also have a vanishing type of phenomenon, where the mass spreads out. Think, for instance, of
$$\psi_k(x) := \frac{1}{\sqrt{k}}\psi\left(\frac{x}{k}\right)$$
where ψ is a fixed $\mathscr{C}^\infty_c(\mathbb{R}^d)$ function. Then $\int_{\mathbb{R}^d}\psi_k^2 = \int_{\mathbb{R}^d}\psi^2$, but $\{\psi_k\}_{k\in\mathbb{N}}$ converges to 0. More generally, in dimension d, we can prove that for any $u \in L^2(\mathbb{R}^d)$, the sequence defined by
$$u_k(x) := \frac{1}{k^{\frac{d}{2}}}u\left(\frac{x}{k}\right)$$
converges weakly to 0. Indeed, let $\varphi \in \mathscr{C}^\infty_c(\mathbb{R}^d)$ be supported in $\mathbb{B}(0;R)$. Then
$$\begin{aligned}\int_{\mathbb{R}^d} u_k\varphi &= k^{\frac{d}{2}}\int_{\mathbb{B}(0;\frac{R}{k})} u\varphi(k\cdot)\\ &\leq k^{\frac{d}{2}}\|u\|_{L^2(\mathbb{B}(0;\frac{R}{k}))}\|\varphi\|_{L^\infty(\mathbb{R}^d)}\left(\frac{R}{k}\right)^{\frac{d}{2}}\\ &\leq C\|u\|_{L^2(\mathbb{B}(0;\frac{R}{k}))}\\ &\underset{k\to\infty}{\to} 0.\end{aligned}$$
(3) Finally, oscillations. This is the main hurdle, especially in the case of bounded domain. The standard example is the following: consider, on

$(-1;1)$, a function $\psi \in L^p(\Omega)$ extended by periodicity and set

$$\psi_k(x) := \psi(kx).$$

Then

$$\psi_k \underset{k\to\infty}{\rightharpoonup} \fint_{-1}^{1} \psi.$$

This last convergence is weak if $p < \infty$ and weak-* if $p = \infty$. We leave the proof as an exercise.

1.7.2. The case of $L^1(\Omega)$: the Dunford-Pettis theorem. As we saw, the previous theorem does not apply to $L^1(\Omega)$, and the counter-example we gave was essentially that of a concentrating sequence of functions that converges to a measure (a dirac, in that case). This is actually the main thing that can go wrong, *per* the Dunford-Pettis theorem (which we do not show here):

THEOREM 1.21. *Assume Ω has finite measure and let $\{f_k\}_{k\in\mathbb{N}} \in L^1(\Omega)^{\mathbb{N}}$. Then the following properties are equivalent:*

(1) *$\{f_k\}_{k\in\mathbb{N}}$ has a L^1-weakly converging subsequence.*
(2) *$\{f_k\}_{k\in\mathbb{N}}$ is uniformly integrable in the following sense: for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $A \subset \Omega$,*
$$|A| \leq \delta \Rightarrow \int_A |f_k| < \varepsilon.$$
(3) *For any $\varepsilon > 0$, there exists M such that*
$$\forall k \in \mathbb{N}\,, \int_{\{f_k \geq M\}} |f_k| < \varepsilon.$$

1.7.3. The Riesz-Fréchet-Kolmogorov theorem. The main theorem providing some compactness result in Lebesgue spaces in the Riesz-Fréchet-Kolmogorov theorem (which we do not prove here), the Lebesgue analog of the Arzela-Ascoli theorem:

THEOREM 1.22. *[Riesz-Fréchet-Kolmogorov] Let $p \in [1;+\infty)$ and let C be a closed, bounded subset of $L^p(\mathbb{R}^d)$. Then C is (sequentially) compact if, and only if, it satisfies the two following properties:*

(1) *It is **equi-integrable**: there holds*
$$\lim_{R\to\infty} \sup_{f\in C} \|f \mathbb{1}_{\mathbb{R}^d \setminus \mathbb{B}(0;R)}\|_{L^p(\mathbb{R}^d)} = 0.$$
(2) *It is **equicontinuous**: with the notations of Proposition 1.5,*
$$\lim_{y\to 0} \sup_{f\in C} \|m_y(f) - f\|_{L^p(\mathbb{R}^d)} = 0.$$

REMARK 1.13 (Other aspects of the problem). *Other facets of weak vs. strong convergence in Lebesgue spaces or (for the next chapter) in Sobolev spaces are to be found in applications of the concentration-compactness principle [10]. We refer to the expository notes of Lewin [9].*

1.8. Related questions

1.8.1. Convexity, weak convergence and strong convergence.

1.8.1.1. *The Mazur lemma.* There are intimate ties between weak convergence and convexity. In this part, we offer an overview of some of these links, without dwelling on the proofs. We begin with the equivalence between weak and strong closedness:

THEOREM 1.23. *Let E be a Banach space and $C \subset E$ be a convex set. Then C is closed for the strong topology if, and only if, C is closed for the weak topology.*

PROOF OF THEOREM 1.23. One direction is obvious. Let C be a convex set that is closed for the strong topology, let $\{x_k\}_{k\in\mathbb{N}} \in C^{\mathbb{N}}\,, x_k \underset{k\to\infty}{\rightharpoonup} x_\infty$ and assume by contradiction that $x_\infty \notin C$. Then by the Hahn-Banach theorem, we can find a linear map f such that

$$\sup_{x\in C} f(x) < f(x_\infty),$$

in contradiction with the weak convergence. This concludes the proof. □

To some extent, this theorem means that weak convergence can "only" reach the convex hull of a set. As a consequence of this theorem, one deduces the Mazur lemma:

COROLLARY 1.2 (Mazur Lemma). *Let $\{x_k\}_{k\in\mathbb{N}} \in E$ converge weakly to $x_\infty \in E$. Then there exists a sequence of convex combinations of $\{x_k\}_{k\in\mathbb{N}}$ that converges strongly to x_∞.*

1.8.2. Weak convergence becoming strong.

1.8.2.1. *Weak convergence becoming strong.* We present two theorems that prove very useful in practice. The first theorem, which serves as an appetiser, is the following:

THEOREM 1.24. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Let $\{x_k\}_{k\in\mathbb{N}} \in H^{\mathbb{N}}$ converge weakly to x_∞, and assume that*

$$\|x_k\| \underset{k\to\infty}{\to} \|x\|.$$

Then the convergence is strong.

PROOF OF THEOREM 1.24. It suffices to write

$$\begin{aligned}\|x_k - x_\infty\|^2 &= \|x_k\|^2 + \|x_\infty\|^2 - 2\langle x_k, x^*\rangle \\ &\underset{k\to\infty}{\to} 0 \\ &\text{as } \langle x_k, x^*\rangle \underset{k\to\infty}{\to} 0.\end{aligned}$$

□

Theorem 1.24 does not seem to have much to do with convexity, when stated as such. The next theorem helps reframe it in a more general setting:

THEOREM 1.25. *Let $(E, \|\cdot\|)$ be a Banach space and assume that $\|\cdot\|$ is uniformly convex in the sense that*

$$\forall \varepsilon > 0\,, \exists \delta > 0\,, \forall x\,, y \in \mathbb{S}(0;1)\,, \|x-y\| \geq \varepsilon \Rightarrow \left\|\frac{x+y}{2}\right\| \leq 1-\delta.$$

Let $\{x_k\}_{k\in\mathbb{N}} \in E^{\mathbb{N}}$ converge weakly to $x_\infty \in E$ and assume that

$$\|x_k\| \underset{k\to\infty}{\to} \|x_\infty\|.$$

Then

$$x_k \underset{k\to\infty}{\to} x_\infty.$$

PROOF OF THEOREM 1.25. Up to considering the sequence $z_k := \frac{x_k}{\|x_k\|}$, which also converges weakly to x_∞, we may assume that $\|x_k\| = 1$ for any k. Take $T \in E'$ such that $T(x_\infty) = 1 = \|T\|$. By weak convergence, we deduce that

$$T(x_k) \underset{k\to\infty}{\to} 1.$$

Consequently,

$$\left\|T\left(\frac{x_\infty + x_k}{2}\right)\right\| \leq \|T\| \cdot \left\|\frac{x_k + x_\infty}{2}\right\| \leq 1.$$

In particular,

$$\left\|\frac{x_k + x_\infty}{2}\right\| \underset{k\to\infty}{\to} 1.$$

By uniform convexity, this implies $\|x_k - x_\infty\| \underset{k\to\infty}{\to} 0$. □

REMARK 1.14 (The Milman-Pettis theorem). *The Milman-Pettis theorem ensures that uniformly convex Banach spaces are reflexive, so that one does not need distinguish between the weak topology on E and the weak-* topology on E seen as the dual of E'.*

REMARK 1.15 (Some examples of (non-)uniformly convex spaces and failure of the theorem). *This theorem applies to L^p spaces for any $p \in (1; +\infty)$ (prove that they are uniformly convex). However, it is easy to see that L^∞ and L^1 are not uniformly convex. One might then ask if, in that case, the theorem might still be true, either with the weak of with the weak-* topology. This is not the case: for instance, consider, in $L^\infty(\Omega)$ endowed with the weak-* topology, the following sequence: $f_k := \mathbb{1}_E + g_k \mathbb{1}_{E^c}$ where g_k is a sequence of characteristic functions converging weakly to a constant. Then the weak limit f_∞ of f_k satisfies $\|f_\infty\|_{L^\infty(\Omega)} = 1 = \lim_{k\to\infty} \|f_k\|_{L^\infty(\Omega)}$, but the convergence is not strong. Similarly, consider $L^1(\Omega)$ endowed with the weak topology, and (any)weakly converging sequence of non-negative unctions $\{f_k\}_{k\in\mathbb{N}}$ satisfying*

$$\int_\Omega f_k = 1.$$

Then, f_∞ being the weak limit of that sequence, we have $\int_\Omega f_\infty = 1$. Thus, if the theorem held, it would imply that any weak limit is strong, which is obviously false (exercise: provide a counter-example).

REMARK 1.16 (Are unit balls special?). *One might wonder whether the assumptions of this theorem can be weakened by asking: is it true that, if $K \subset E$ is a convex set, and if $\{x_k\}_{k\in\mathbb{N}} \in K^\mathbb{N}$ converges weakly to an extreme point x_∞ of K, then the convergence is strong? The answer is no: consider the Hilbert space $\ell^2(\mathbb{N})$, endowed with its canonical basis $\{e_k\}_{k\in\mathbb{N}}$. Consider the set $K = \mathrm{Conv}(\{e_k\}_{k\in\mathbb{N}})$ and the sequence $\{e_k\}_{k\in\mathbb{N}}$. By the Plancherel formula, $e_k \underset{k\to\infty}{\rightharpoonup} 0$. However, we can show that 0 is an extreme point of K: indeed, assuming that $0 = (1-t)x + ty$ for some $x, y \in K$, it follows that, for any $k \in \mathbb{N}$, $0 = (1-t)\langle x, e_k\rangle + t\langle y, e_k\rangle \geq 0$ whence $x = y = 0$. Since the convergence is not strong, the conclusion follows.*

1.8.2.2. *Some example of weak convergence becoming strong: the Visintin theorem.*

CHAPTER 2

Distributions and fundamental solutions

This is going to be a much shorter chapter, devoted to a very basic course in distribution theory. The main reference regarding distribution theory remains, to this day, the book of Schwartz. We also refer to [**6**, **11**].

Notations: multi index etc, ∂^α, ∂^α

2.1. Fundamental solutions and distributions

2.2. The basic idea(s) behind distributions

2.2.1. Basic definitions. Ω being an open set in $\mathbb{R}^d$, we let $\mathcal{D}(\Omega)$ denote the set of compactly supported, $\mathscr{C}^\infty$ functions.

Definition 2.1. *A **distribution** T is a linear map $T : \mathcal{D}(\Omega) \to \mathbb{R}$ that satisfies the following: for any compact set $K \subset \Omega$, there exists $N_K \in \mathbb{N}$ and c_K such that*

$$\forall \varphi \in \mathscr{C}_c^\infty(K)\,, |T(\varphi)| \leq c_K \sum_{|\alpha| \leq N_K} \|\partial^\alpha \varphi\|_{L^\infty(K)}.$$

The set of distributions is denoted $\mathcal{D}'(\Omega)$.

Remark 2.1 (Regarding the order of a distribution). *We list some comments:*

(1) *If $\Omega = \mathbb{R}^d$ and if the integer N_K can be chosen independent of K, say N, we say that T has finite order N.*
(2) *An example of distribution is $T : \varphi \mapsto \varphi(0)$, which is of order 0. On the other hand, it is easy to construct an example of a distribution that does not have finite order: define*
$$T : \varphi \mapsto \sum_{k=0}^\infty k \varphi^{(k)}(k).$$
Then T clearly defines a distribution, but it can not have finite order.
(3) *Schwartz quickly obtained a generalisation of the Riesz duality theorem (the set of Radon measures on a compact set K is the dual of $\mathscr{C}(K)$): on any compact set $K \subset \mathbb{R}^d$, there exist Radon measures μ_i's such that*
$$\forall \varphi \in \mathscr{C}_c^\infty(K)\,, T(\varphi) = \sum_{|\alpha| \leq N_k} \int_K \partial^\alpha \varphi, d\mu_\alpha.$$
We will revisit the last two points in the next paragraph.

The next thing we need to do is to define a topology on distributions; unsurprisingly, we are going to work with the analog of the weak-* topology (keep in mind that $\mathscr{C}_c^\infty(\mathbb{R}^d)$ can not be a Banach space for any reasonable topology).

Definition 2.2. *Let $\{T_k\}\in(\mathcal{D}'(\Omega))^{\mathbb{N}}$ be a sequence of distributions and $T_\infty\in\mathcal{D}'(\Omega)$. We say that the sequence converges to T_∞ in the sense of distributions and we write*

$$T_k \underset{k\to\infty}{\overset{\mathcal{D}'(\Omega)}{\to}} T_\infty$$

if

$$\forall\varphi\in\mathcal{D}(\Omega)\,, T_k(\varphi) \underset{k\to\infty}{\to} T_\infty(\varphi).$$

As a consequence of the Banach-Steinhaus theorem (for the interested reader, one would need to prove it in Fréchet spaces), we have the following marvellous property of distributions:

Proposition 2.1. *Let $\{T_k\}_{k\in\mathbb{N}}\in(\mathcal{D}'(\Omega))^{\mathbb{N}}$ and assume that there exists a linear function T_∞ such that*

$$\forall\varphi\in\mathcal{D}(\Omega)\,, T_k(\varphi) \underset{k\to\infty}{\to} T_\infty(\varphi).$$

Then $T_\infty\in\mathcal{D}'(\Omega)$ and

$$T_k \underset{k\to\infty}{\overset{\mathcal{D}'(\Omega)}{\to}} T_\infty.$$

Finally, we define the derivatives of distributions by analogy with the smooth case: indeed, recall that for any $f\in\mathscr{C}^1(\Omega)$, for any $\varphi\in\mathcal{D}(\Omega)$, as φ is compactly supported, it follows that for any $i\in\{1,\dots,d\}$,

$$\int_\Omega \frac{\partial f}{\partial x_i}\varphi = -\int_\Omega \frac{\partial \varphi}{\partial x_i} f.$$

Definition 2.3. *If $T\in\mathcal{D}'(\Omega)$, and α is a multi-index the ∂^α derivative of T is the distribution noted $\partial^\alpha T$ and defined by*

$$\partial^\alpha T(\varphi) := (-1)^{|\alpha|}T(\partial^\alpha\varphi).$$

Of course, this definition extends to any differential operator $L=\sum_\alpha a_\alpha(\cdot)\partial^\alpha$, for instance the Laplacian $L=-\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$, for instance by defining

$$-\Delta T : \mathcal{D}(\Omega)\ni\varphi\mapsto T(-\Delta\varphi).$$

The beautiful fact with distributions is that, almost by definition, we have the following stability result:

Proposition 2.2. *Assume that $T_k \underset{k\to\infty}{\overset{\mathcal{D}'(\Omega)}{\to}} T_\infty$. Then, for any multi-index α,*

$$\partial^\alpha T_k \underset{k\to\infty}{\overset{\mathcal{D}'(\Omega)}{\to}} \partial^\alpha T_\infty.$$

From the Schwartz theorem, all partial derivations commute for distributions.

2.2.2. Particular examples.

2.2.2.1. *Measure induced distributions.* As we already noted, for any Radon measure μ on Ω, the operator $T_\mu:\varphi\mapsto\int_\Omega\varphi d\mu$ defines a (zero order) distribution. Among these, the most important one is the dirac mass δ_0, whose associated distribution, still denoted δ_0, is given by

$$\delta_0:\varphi\mapsto\varphi(0).$$

Observe that the convergence in the sense of measures implies the convergence in the sense of distributions.

2.2.2.2. *Function induced distributions.* For $1 \leq p \leq \infty$, we set

$$L^p_{\text{loc}}(\Omega) := \{f \ \text{ measurable on } \Omega \text{ such that, for all compact } K \subset \Omega, \quad f \in L^p(K)\}\,. \tag{9}$$

The sets $L^p_{\text{loc}}(\Omega)$ **are not normed spaces** but it is an exercise to check that

$$L^p(\Omega) \subset L^p_{\text{loc}}(\Omega) \subset L^1_{\text{loc}}(\Omega),$$

so the space $L^1_{\text{loc}}(\Omega)$ contains all the $L^p(\Omega)$ spaces: if a result is true for all $f \in L^1_{\text{loc}}(\Omega)$, it is also true for $f \in L^p_{\text{loc}}(\Omega)$ and for $f \in L^p(\Omega)$.

The following theorem shows that one can distinguish L^1_{loc} functions (hence all L^p functions) among distributions.

Theorem 2.1 (L^1_{loc} functions are determined by distributions). *If $f \in L^1_{\text{loc}}(\Omega)$, then $T_f : \phi \mapsto \int_\Omega f\phi$ is a distribution. Furthermore, if $f, g \in L^1_{\text{loc}}(\Omega)$, then $T_f = T_g$ in $\mathcal{D}'(\Omega)$ if, and only if, $f = g$ a.e.*

The proof is elementary, and we leave it as an exercise.

An important example is that of the Heaviside function

$$H : \mathbb{R} \ni x \mapsto \mathbb{1}_{(-\infty:0]}(x).$$

Its importance comes from the fact that

$$H' = -\delta_0.$$

Indeed, it suffices to observe that, for any $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} H'\varphi &= -\int_{\mathbb{R}} H\varphi' \\ &= -\int_{-\infty}^0 \varphi' \\ &= -\varphi(0).\end{aligned}$$

2.2.2.3. *Convolution and distributions.* We can define several operations on distributions, but the one that will be most useful when discussing fundamental solutions of elliptic PDEs is the convolution with a smooth function. We only work in the case $\Omega = \mathbb{R}^d$, although one could make it work in an open set (but this would create irrelevant complications regarding the support of the function we take the convolution with). Observe that, for any (smooth enough) functions $f\,, g\,, \varphi$, we have

$$\int_{\mathbb{R}^d} (f{*}g)\phi := \iint_{(\mathbb{R}^d)^2} f(x)g(y{-}x)\phi(y)dydx = \iint_{(\mathbb{R}^d)^2} f(x)\phi(z)g(z{-}x)dxdz = \int_{\mathbb{R}^d} f(\widetilde{g}{*}\phi),$$

where we set $\widetilde{g}(x) := g(-x)$. Thus, if $T \in \mathcal{D}'(\mathbb{R}^d)$ and if g is a compactly supported L^1 function, we define the convolution product $T * g$ as

$$T * g : \mathcal{D}(\Omega) \ni \varphi \mapsto T\left(\widetilde{g} * \varphi\right).$$

The fact that $T * g$ is a distribution is immediate and left as an exercise. Just as was the case with L^p functions, convolution smoothes distributions out:

Theorem 2.2. *We retain the notations from Theorem 1.15. Let $\{\psi_k\}_{k\in\mathbb{N}}$ be a smooth approximation of unity. Then the sequence $\{T_k := T * \psi_k\}_{k\in\mathbb{N}}$ converges to T in $\mathcal{D}'(\mathbb{R}^d)$. In addition, for any $k \in \mathbb{N}$, T_k can be identified with a $\mathscr{C}^\infty(\mathbb{R}^d)$ function, and $\partial^\alpha T_k = T * (\partial^\alpha \psi_k)$.*

Then again, it suffices to work by duality and to invoke Theorem 1.15.

REMARK 2.2. *One can not define in a meaningful way the convolution of two distributions in general. It is however possible to define the convolution between a distribution and a compactly supported distribution (meaning a distribution T such that there exists a compact set K satisfying: for any $\varphi \in \mathscr{C}_c^\infty(\mathbb{R}^d \setminus K)$, $T(\varphi) = 0$).*

REMARK 2.3. *Observe that for any function $\varphi \in \mathcal{D}(\Omega)$ we have $\varphi * \delta_0 = \varphi$.*

We finally give the following:

PROPOSITION 2.3. *Let $T \in \mathcal{D}'(\Omega)$ be such that $\nabla T = 0$. Then there exists $c \in \mathbb{R}$ such that $T \equiv c$.*

PROOF OF PROPOSITION 2.3. Let $\{\psi_k\}_{k\in\mathbb{N}}$ be an approximation of unity. Then by the preceding theorem we obtain

$$\nabla(\rho_k * T) = 0$$

whence, for any $k \in \mathbb{N}$, $\rho_k * T$ is a constant, say c_k. The sequence $\{c_k\}_{k\in\mathbb{N}}$ converges, up to a subsequence, to some $c_\infty \in \mathbb{R}$, but also, in the sense of distribution, to T, whence $T = c_\infty$. □

2.3. The Fourier transform of tempered distributions

Another fundamental operation of analysis is the Fourier transform, and we would like to define the Fourier transform of distributions (this was actually one of the initial motivations of Schwartz). However, the situation is quite delicate.

2.3.1. The Fourier transform of functions. We begin by recalling some basic facts about the Fourier transform of functions. For a given function $f \in L^1(\mathbb{R}^d)$, we define its Fourier transform as

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} f(x)dx. \tag{10}$$

We summarise the following standard facts about the Fourier transform:

THEOREM 2.3.

(1) *For any $f \in L^1(\mathbb{R}^d)$, $\hat{f} \in \mathscr{C}^0(\mathbb{R}^d)$ and $\|\hat{f}\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)}$.*

(2) *For any $f \in \mathscr{C}^k(\mathbb{R}^d)$ such that, for any $|\alpha| \leq k$ $\nabla^\alpha f \in L^1(\mathbb{R}^d)$ we have*
$$\widehat{\nabla^\alpha f} = i^{|\alpha|}\xi^\alpha \hat{f}(\xi).$$

(3) *Conversely, if $k \in \mathbb{N}$, if $\int_{\mathbb{R}^d} \|x\|^k |f|(x)dx < \infty$ then $\hat{f} \in \mathscr{C}^k(\mathbb{R}^d)$ and, for any $|\alpha| \leq k$*
$$\nabla^\alpha \widehat{f}(\xi) = (-i)^{|\alpha|}\widehat{x^\alpha f}.$$

(4) *If $f, g \in L^1(\mathbb{R}^d)$ then*
$$\widehat{f * g} = (2\pi)^{\frac{d}{2}} \hat{f}\hat{g}.$$

(5) *If $f, \hat{f} \in L^1(\mathbb{R}^d)$ then*
$$\widehat{\hat{f}}(-\cdot) = f.$$

(6) *The Fourier transform can be defined as an isometry on $L^2(\mathbb{R}^d)$.*

These facts will not be proved here.

An important formula for Fourier analysis is the following theorem, which gives the Fourier transform of the Gaussian.

PROPOSITION 2.4. *Let, for any* $\alpha > 0$,

$$g_\alpha : x \mapsto e^{-\alpha\|x\|^2}.$$

Then

$$\hat{g}_\alpha(\xi) = \frac{1}{(2\alpha)^{\frac{d}{2}}} e^{-\frac{\|\xi\|^2}{4\alpha}}.$$

PROOF OF PROPOSITION 2.4. First of all, recall that

$$\hat{g}_\alpha(0) = \int_{\mathbb{R}^d} e^{-\alpha\|x\|^2} = \left(\frac{\pi}{\alpha}\right)^{\frac{d}{2}}.$$

We then claim that it suffices to compute it in the one-dimensional case (why?). To compute $\hat{g}_\alpha$ when $d = 1$,, it suffices to establish a differential equation, and to observe that, by derivation and integration by parts,

$$\begin{aligned}\hat{g}_\alpha'(\xi) &= -i \int_{\mathbb{R}} x e^{-\alpha x^2 - i\langle x, \xi\rangle} dx \\ &= -\frac{\xi}{2\alpha} \hat{g}_\alpha(\xi)\end{aligned}$$

and solving this ODE gives the desired result. □

2.3.2. The Fourier transform on $L^p(\mathbb{R}^d)$.

2.3.3. Tempered distributions and their Fourier transform. Of course, we would like to define the Fourier transform of a distribution T by duality, setting, for instance

$$\hat{T} : \mathcal{D}(\mathbb{R}^d) \ni \varphi \mapsto T(\hat{f}).$$

This is not possible, as one might very well have $\varphi \in \mathcal{D}(\Omega)$, although $\hat{\varphi} \notin \mathcal{D}(\mathbb{R}^d)$. Thus, we need to find a space that is bigger than $\mathcal{D}(\mathbb{R}^d)$, and that is stable by Fourier transform. Based on the properties we recalled above, the natural class to consider is the following:

DEFINITION 2.4. *The Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is defined as the set of functions $f \in \mathscr{C}^\infty(\mathbb{R}^d)$ such that*

$$\forall k \in \mathbb{N}, \sup_{x \in \mathbb{R}^d} \sup_{|\alpha| \leq k} (1 + |x|)^k |\nabla^\alpha f| < \infty.$$

This space is endowed with a Fréchet structure and has the following nice properties (which are left as exercises):

PROPOSITION 2.5. *$\mathcal{S}(\mathbb{R}^d)$ is stable by derivation and multiplication by a polynomial, and these operations are continuous for the Fréchet space structure on $\mathcal{S}(\mathbb{R}^d)$. Furthermore, $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$, and the Fourier transform is an isomorphism of $\mathcal{S}(\mathbb{R}^d)$.*

We can now define the class of distributions we can take the Fourier transform of–it is the dual of $\mathcal{S}(\mathbb{R}^d)$:

Definition 2.5 (Tempered distribution). *A distribution T is said to be tempered if there exist $k \in \mathbb{N}, C$ such that*

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^d), |T(\varphi)| \leq C \sup_{|\alpha| \leq k, x \in \mathbb{R}^d} (1+|x|)^k |\nabla^\alpha \varphi|(x).$$

In that case, we write $T \in \mathcal{S}'(\mathbb{R}^d)$.

Observe that by density of $\mathcal{D}(\mathbb{R}^d)$, this implies that a tempered distribution can be extended to an element of the dual of $\mathcal{S}(\mathbb{R}^d)$ and, conversely, any element in the dual of $\mathcal{S}(\mathbb{R}^d)$ defines a distribution. Thus, $\mathcal{S}'(\mathbb{R}^d) = (\mathcal{S}(\mathbb{R}^d))'$.

Observe that the class of tempered distributions is quite large: any function that grows at most polynomially defines a tempered distribution.

We can now define the Fourier transform oof a tempered distribution:

Definition 2.6 (Fourier transform of distributions). *Let $T \in \mathcal{S}'(\mathbb{R}^d)$. The Fourier transform of T is defined as*

$$\hat{T} : \mathcal{S}(\mathbb{R}^d) \ni \varphi \mapsto T(\hat{\varphi}).$$

Furthermore, it defines a tempered distribution.

2.4. Fundamental solution of partial differential equations

In this section we investigate a partial differential equation with (for the time being) constant coefficients

$$\sum_{|\alpha| \leq N} a_\alpha \partial^\alpha u = f \text{ in } \mathbb{R}^d \tag{11}$$

for a given function $f : \mathbb{R}^d \to \mathbb{R}$. The following theorem boils down the study of existence of a solution to (11) (we do not touch on the uniqueness, which does not hold without further information on the behaviour of the solution at infinity) to the solving of a distributional equation:

Theorem 2.4. *Suppose that there exists a distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ such that*

$$\sum_{|\alpha| \leq N} a_\alpha \partial^\alpha T = \delta_0.$$

Then, for any $f \in \mathscr{D}(\mathbb{R}^d)$, the $\mathscr{C}^\infty$ function $T * f$ is a solution of (11).

Proof of Theorem 2.4. The proof is immediate from Remark 2.3. □

2.4.1. Computation of certain fundamental solutions. In this part, we go over the basic partial differential equations and their fundamental solutions (as well as some consequences we can draw from it). We will be relying on the Fourier transform for tempered distributions.

2.4.1.1. *Green function for the Poisson problem.* We want to solve

$$-\Delta \Phi = \delta_0$$

in the sense of distributions.

To determine such a fundamental solution, we observe that the Laplacian is a rotation-invariant operator, as is δ_0 viewed as a distribution. This suggests searching for the fundamental solution in the form of a radial function, that is, a function of the form $\Phi = \varphi(|\cdot|)$. Then, for all $x \neq 0$, the hypothesis $\Delta\Phi = 0$ can be written in polar coordinates as:

$$\Delta\Phi(x) = \left(\partial_{rr}^2\varphi + \frac{n-1}{r}\partial_r\varphi\right)_{r=|x|} = 0.$$

We can solve this differential equation and obtain the following result:

DEFINITION 2.7. *Let $n \geq 2$. We define on $\mathbb{R}^d \setminus \{0\}$ the function Φ by*

$$\Phi(x) := -\frac{1}{2\pi}\ln(||x||_2) \quad \text{if } n = 2,$$

and

$$\Phi(x) := \frac{1}{(n-2)S_n|x|^{n-2}} \quad \text{if } n > 2,$$

where

$$S_n := \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

The function Φ is called the ***fundamental solution of the Laplacian****.*

LEMMA 2.1. *The function Φ defined above is indeed a fundamental solution of the Laplacian in the sense of distributions.*

PROOF. We only treat the case $n = 3$; the case $n = 2$ is simpler, and the case $n > 3$ is similar. Let us show that:

$$\Delta\left(\frac{1}{|x|}\right) = -4\pi\delta_0.$$

Already, for $|x| > 0$, we have $\Delta\left(\frac{1}{|x|}\right) = 0$ in the classical sense. Thus, in the sense of distributions:

$$\Delta\left(\frac{1}{|x|}\right) = \sum u_\alpha D^\alpha \delta_0,$$

where the u_α are real coefficients and the sum is finite[1].

Let $f \in \mathcal{D}(\mathbb{R}^3)$. Then:

$$\begin{aligned}\langle\Delta\left(\frac{1}{|x|}\right), f\rangle &= \int_{\mathbb{R}^3}\frac{\Delta f(x)}{|x|}\,dx \\ &= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\Delta f(x)}{|x|}\,dx.\end{aligned}$$

Define, for $\varepsilon > 0$:

$$I_\varepsilon := \int_{|x|\geq\varepsilon}\frac{\Delta f(x)}{|x|}\,dx.$$

Let us compute this quantity using the Green formula; assume $\text{supp}(f) \subseteq \mathbb{B}(0,R)$. Then, Green's formula applied to $\mathbb{B}(0,R)\setminus\mathbb{B}(0,\varepsilon)$ gives:

$$I_\varepsilon = \int_{|x|=\varepsilon}\left\{\frac{1}{|x|}\cdot\frac{-y}{|y|}\cdot\nabla f(y) + f(y)\cdot\frac{y}{|y|}\cdot\nabla\frac{1}{|y|}\right\}dy \to -4\pi f(0) \quad \text{as } \varepsilon\to 0.$$

[1]This follows from the notion of homogeneity degree of a distribution.

Indeed,

$$\left|\int_{|x|=\varepsilon} \frac{y}{|y|^2} \cdot \nabla f(y)\, dy\right| \leq \frac{1}{\varepsilon}\|\nabla f\|_\infty \underbrace{\int_{|x|=\varepsilon} dy}_{=4\pi\varepsilon^2} \to 0 \quad \text{as } \varepsilon \to 0,$$

and

$$\begin{aligned}\int_{|x|=\varepsilon} -\frac{1}{|y|^2} f(x)\, dy &= -\int_{|x|=\varepsilon} \frac{1}{|y|^2}(f(y) - f(0) + f(0))\, dy \\ &= \underbrace{-\frac{1}{\varepsilon^2}\int_{|x|=\varepsilon} f(0)\, dy}_{=-4\pi f(0)} - \underbrace{\int_{|x|=\varepsilon} \frac{1}{\varepsilon^2}(f(y) - f(0))\, dy}_{\leq 4\pi \sup_{|x|=\varepsilon} |f(x)-f(0)| \to 0}.\end{aligned}$$

□

2.4.2. Green function for the heat equation. Let $g \in \mathcal{S}(\mathbb{R}^d)$. We aim to solve the **heat equation**:

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } [0;T] \times \mathbb{R}^d\,, \\ u(0,\cdot) = g \in \mathcal{S}(\mathbb{R}^d). \end{cases} \tag{12}$$

To do this, we will use Fourier theory, developed for this purpose.

Suppose we have a solution to (12) such that for all $t \geq 0$, $u(\cdot,t) \in \mathcal{S}(\mathbb{R}^d)$ and u is $\mathscr{C}^1$ in the time variable. Fix t, and take the spatial Fourier transform of the equation, using the convention:

$$\hat{u}(\xi,t) := \int_{\mathbb{R}^d} u(x,t) e^{-ix\cdot\xi}\, dx$$

so that the Fourier inversion formula reads:

$$u(x,t) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \hat{u}(\xi,t) e^{i\xi\cdot x}\, d\xi$$

With the assumptions above, the heat equation becomes:

$$\forall \xi \in \mathbb{R}^d, \forall t \geq 0, \quad \partial_t \hat{u}(\xi,t) = -|\xi|^2 \hat{u}(\xi,t)$$

This form is much nicer: whereas Δ is a differential operator and couples the spatial variables, the ξ variables are decoupled in the Fourier-transformed equation. Thus, in discretization, the resulting matrix will be diagonal, while the discretization of the Laplacian generally produces a non-diagonal matrix. In a way, the Fourier transform diagonalizes constant coefficient differential operators.

Solving this ordinary differential equation, we obtain:

$$\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{g}(\xi). \tag{13}$$

In other words, in Fourier space, the heat equation simply multiplies the initial data by a Gaussian. Furthermore, irreversibility appears in this expression: we cannot reverse time and allow $t < 0$, otherwise the inverse Fourier transform would become invalid.

To invert this transformation, we introduce the **heat kernel**:

$$\Phi(x,t) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\xi|^2 t + i\xi\cdot x}\, d\xi = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$$

The solution to 12 is then given by convolution:

$$(14) \qquad \forall x \in \mathbb{R}^d, \forall t > 0, \qquad u(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy$$

$$(15) \qquad \forall x \in \mathbb{R}^d, \qquad u(x,0) = g(x)$$

and this formula indeed satisfies the desired conditions.

Proposition 2.6. *If $g \in \mathscr{C}^0(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then u given by 14 satisfies:*

(1) $u \in \mathscr{C}^\infty(\mathbb{R}^d \times]0; +\infty[)$
(2) $\partial_t u(x,t) = \Delta u(x,t)$ *for all* $x \in \mathbb{R}^d$, $t > 0$
(3) *For all* $\overline{x} \in \mathbb{R}^d$, $u(x,t) \to g(\overline{x})$ *as* $x \to \overline{x}$ *and* $t \to 0^+$

Proof. (1) This follows from successive applications of general integration theorems.

(2) This second point is evident once we notice that $\partial_t \Phi = \Delta \Phi$.

(3) This point is a bit more delicate: set $y := x + \sqrt{t} z$. Then

$$u(x,t) = \frac{1}{(4\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{4}} g(x + z\sqrt{t})\, dz$$

and this formula is still valid for $t = 0$. We then conclude by dominated convergence:

$$|u(x,t) - g(\overline{x})| = \frac{1}{(4\pi)^{\frac{d}{2}}} \left| \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{4}} (g(x + z\sqrt{t}) - g(\overline{x}))\, dz \right| \to 0 \quad \text{as } x \to \overline{x}, t \to 0^+$$

□

Note that the assumptions made on g are by no means minimal: we could allow g to have polynomial growth, or merely require that g defines a tempered distribution.

A parabolic maximum principle appears: for example,

$$\sup_{x \in \mathbb{R}^d} \sup_{t \geq 0} u(x,t) \leq \sup g$$

and similarly for infimums. Moreover, $g \geq 0$ implies $u \geq 0$, and even $u(x,t) > 0$ for all x and $t > 0$, unless $g \equiv 0$: heat propagates instantaneously throughout the entire universe.

A natural question to ask is: is the solution u given by 14 the unique solution to this problem? Equivalently, if $g = 0$, is the zero solution the only solution? Actually, no. The issue is that the other solutions are not physically acceptable (e.g., they may grow like $e^{|x|^2}$ at infinity). Under slightly more restrictive assumptions, however, uniqueness can be guaranteed:

Proposition 2.7. *Let $u \in \mathscr{C}^0(\mathbb{R}^d \times [0;T]) \cap L^\infty(\mathbb{R}^d \times [0;T])$ such that*

$$\partial_t u = \Delta u \quad \text{in } \mathcal{D}'(\mathbb{R}^d \times]0;T[).$$

Then u is given by (14) with $g = u(\cdot, 0)$.

Proof. The proof is rather clever and uses the previous proposition. First, suppose that u is actually of class $\mathscr{C}^2$ on $\mathbb{R}^d \times [0;T]$ and that $x \mapsto \partial_x^\alpha \partial_t^\beta u(x,t)$ is bounded on $\mathbb{R}^d$, for $t \in]0;T[$ and $|\alpha| + |\beta| \leq 2$.

Let's move to the main idea: define, for $0 \leq s < t \leq T$, the function $v_{t,s}$[2] as

$$v(x,\tau) := \frac{1}{(4\pi(t-\tau))^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4(t-\tau)}} u(y,\tau)\, dy, \quad x \in \mathbb{R}^d, \tau \in]s;t[$$

So, omitting the spatial dependence:

$$v(\tau) = \Phi(t-\tau) * u(\tau), \quad \tau \in]s;t[$$

Differentiate both sides in τ. Differentiation under the integral on the right is justified by the hypotheses. Then:

$$\begin{aligned} \partial_\tau v(\tau) &= \partial_\tau \Phi(t-\tau) * u(\tau) + \Phi(t-\tau) * \partial_\tau u(\tau) \\ &= -\Delta\Phi(t-\tau) * u(\tau) + \Phi(t-\tau) * \Delta u(\tau) \\ &= 0 \end{aligned}$$

So v depends only on space. Then, taking the limits in t and s is straightforward. The limit $\tau \to s$ poses no problem, and the limit $\tau \to s$ is handled using Proposition 2.6. Note that:

$$u(t) = v(t) = v(s) = \Phi(t-s) * u(s)$$

Thus,

$$u(t) = \Phi(t-s) * u(s)$$

With $s = 0$, we obtain the result. If u does not satisfy the stronger assumptions made at the start of the proof, we regularize and proceed by approximation. □

How can we use this kind of reasoning to treat the heat equation with source terms? We now present a very general and useful method for studying certain evolution equations.

We are now interested in the following equation:

$$\text{(16)} \qquad \boxed{\begin{cases} \partial_t u - \Delta u &= f(x,t) \quad \text{for } t \geq 0 \\ u(x,0) &= g(x) \end{cases}}$$

Working with suitable regularity and taking the spatial Fourier transform, we obtain:

$$\forall \xi \in \mathbb{R}^d, \forall t \geq 0, \quad \partial_t \hat{u}(\xi,t) = -|\xi|^2 \hat{u}(\xi,t) + \hat{f}(\xi,t)$$

Using ODE theory and the method of variation of constants, we get, for fixed ξ:

$$\forall t \geq 0, \quad \hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}(\xi,0) + \int_0^t e^{-|\xi|^2(t-s)} \hat{f}(\xi,s)\, ds$$

Taking the inverse Fourier transform we finally obtain:

$$\forall x \in \mathbb{R}^d, \forall t \geq 0, \quad u(x,t) = \left(\Phi(\cdot,t) * u(\cdot,0)\right)(x) + \int_0^t \Phi(x,t-s) * f(s)\, ds$$

This is the **Duhamel formula**. We thus obtain a "fundamental solution of the heat equation" in $\mathbb{R}^{n+1}$: define $\tilde{\Phi} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ by:

$$\text{(17)} \qquad \tilde{\Phi}(x,t) := H(t) \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}}$$

[2] abbreviated as v to lighten notation

where H is the Heaviside function. Note that $\tilde{\Phi}$ is $\mathscr{C}^\infty$ on $\mathbb{R}^{n+1} \setminus \{0\}$. It is also locally integrable: indeed,

$$\int_{\mathbb{R}^d} \tilde{\Phi}(x,t)\,dx = H(t) \in L^1_{loc}(\mathbb{R}),$$

and the positivity of $\tilde{\Phi}$ allows the Fubini theorem to be applied. Thus, $\tilde{\Phi}$ defines a distribution.

2.4.3. Green function for the wave equation. In this section, we are interested in solving the following equation:

$$\partial_t^2 u(x,t) - \Delta u(x,t) = f(x,t), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R} \tag{18}$$

First, we assume that $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R})$. Note that, in the case of the heat equation, we only considered future times, whereas here we are interested in the behavior at all times. We can impose initial data. The fact that the equation is second order in time indicates that we must impose two initial conditions:

$$\begin{cases} u(x,0) &= u_0(x), \quad x \in \mathbb{R}^n \\ \partial_t u(x,0) &= v_0(x), \quad x \in \mathbb{R}^n \end{cases}$$

where u_0 and v_0 are assumed to belong to $\mathcal{S}(\mathbb{R}^n)$. Taking the spatial Fourier transform of (18), we obtain the differential equation:

$$\partial_t^2 \hat{u}(\xi,t) + |\xi|^2 \hat{u}(\xi,t) = \hat{f}(\xi,t), \quad \xi \in \mathbb{R}^n,\ t \in \mathbb{R}$$

We obtain an expression for the solution:

$$\hat{u}(\xi,t) = \cos(|\xi|t)\hat{u}_0(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\hat{v}_0(\xi) + \int_0^t \frac{\sin(|\xi|(t-s))}{|\xi|}\hat{f}(\xi,s)\,ds$$

Let us call this solution (A). We can also choose not to impose initial conditions and instead select the so-called **causal** or **retarded** solution, i.e., one that tends to zero as $t \to -\infty$:

$$\hat{u}(\xi,t) = \int_{-\infty}^t \frac{\sin(|\xi|(t-s))}{|\xi|}\hat{f}(\xi,s)\,ds$$

We call this solution (B). Solutions (A) and (B) correspond to two different physical problems. Let Φ be the tempered distribution on $\mathbb{R}^n \times \mathbb{R}$ defined by

$$\hat{\tilde{\Phi}}(\xi,t) = H(t)\frac{\sin(|\xi|t)}{|\xi|} \tag{19}$$

This is a fundamental solution of the wave equation; we recover a convolution product. In the study of the heat equation, Fourier decay provided regularity of the solution, but here, the decay is not fast enough, and we may encounter non-physical solutions. With this definition, (B) becomes

$$u = \tilde{\Phi} * f$$

where the convolution is taken in both space and time. Note that this is not the only fundamental solution; it is called the **causal** or **retarded fundamental solution**. In low dimensions, we have explicit expressions for $\tilde{\Phi}$:

PROPOSITION 2.8. *The retarded fundamental solution of the wave equation satisfies:*

(1) <u>*Case $n = 1$:*</u> $\tilde{\Phi}(x,t) = \frac{1}{2}H(t - |x|)$

(2) *Case $n=2$:* $\tilde{\Phi}(x,t) = \frac{1}{2\pi}\frac{H(t-|x|)}{\sqrt{t^2-|x|^2}}$
(3) *Case $n=3$:* $\tilde{\Phi}(x,t) = \frac{1}{4\pi t}\delta(t-|x|) = \frac{1}{4\pi|x|}\delta(t-|x|)$

Before moving to the proof, recall that $\delta(t-|x|)$ is defined, by duality, as the distribution on $\mathbb{R}^{n+1}$ given by:

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^{n+1}), \quad \langle \delta(t-|x|), \varphi \rangle = \int_0^{+\infty} \left\{ \int_{|x|=t} \varphi(x,t)\, dS(x) \right\} dt = \int_{\mathbb{R}^3} \varphi(x,|x|)\, dx$$

Proof. In this proof, we will take a shortcut: we start from the expressions of $\tilde{\Phi}$ and show that they satisfy (19).

(1) Case $n=1$: Compute the Fourier transform in space: if $\xi \in \mathbb{R},\ t \in \mathbb{R}_+$,

$$\hat{\tilde{\Phi}}(\xi,t) = \frac{1}{2}\int_{-t}^{t} e^{-i\xi x} dx = \frac{\sin(|\xi| t)}{|\xi|} H(t)$$

Thus the desired result is proved. Before moving on to other dimensions, let's make some comments: by direct calculation, the causal solution given by (B) becomes

$$u(x,t) = \frac{1}{2}\int_{\mathbb{R}\times\mathbb{R}} H(t-s-|x-y|) f(y,s)\, dy\, ds$$

The Heaviside function selects a specific domain in space-time:

$$u(x,t) = \frac{1}{2}\int_{C_-(x,t)} f(y,s)\, dy\, ds$$

where $C_-(x,t) := \{(y,s) : |x-y| \le t-s\}$ is called the **past cone of the point** (x,t).

$C_-(x,t)$ is sometimes called the **domain of dependence of the point** (x,t). Consider (y,s). Can it influence $u(x,t)$? Yes, but only if (x,t) lies in the **domain of influence of** (y,s), i.e., if $(x,t) \in C_+(y,s) := \{(x,t) : |x-y| \le t-s\}$. This is also called the **future cone of the point** (y,s). It is quite clear that

$$(y,s) \in C_-(x,t) \Leftrightarrow (x,t) \in C_+(y,s)$$

This reflects the finite speed of wave propagation, a phenomenon also known as the Huygens principle. Let us now turn to d'Alembert's formula, solution (A). It takes the form:

$$\forall t \ge 0,\ \forall x, \quad u(x,t) = \frac{1}{2}(u_0(x-t)+u_0(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\, dy + \frac{1}{2}\int_0^t \int_{x-\tau}^{x+\tau} f(y,t-\tau)\, dy\, d\tau$$

Again, we see the same idea: we can define domains of influence, dependence, etc.

(2) Case $d=2$: We cautiously skip this calculation for now, as it is more complex, and will return to it later.

(3) <u>Case $d = 3$:</u> Here, for fixed t, $\delta(t - |x|)$ is simply the surface measure on $S(0,t)$. Its Fourier transform in space is obtained by direct calculation using the Fourier transform of signed measures. If $\xi \in \mathbb{R}^3, t > 0$, then

$$\hat{\tilde{\Phi}}(\xi, t) = \frac{1}{4\pi t} \int_{S(0,t)} e^{-ix\cdot\xi} \, dS(x)$$

We're in trouble, but it's time to present a very instructive calculation specific to odd dimensions: define ψ by

$$\forall t > 0, \ \forall \xi \in \mathbb{R}^3, \quad \psi(\xi, t) := \int_{S(0,t)} e^{-i\xi\cdot x} \, dS(x)$$

Then

$$\psi(\xi, t) = 4\pi t \frac{\sin(|\xi| t)}{|\xi|}$$

Define the value at $\xi = 0$ by taking the limit. For $\xi = 0$, the formula is obvious. If $\xi \neq 0$, then observe that ψ is rotation invariant. Letting $e_3 := (0, 0, 1)$, we have, for all $t > 0$,

$$\psi(\xi, t) = \psi(|\xi| e_3, t)$$

Switch to spherical coordinates:

$$S(0,t) = \{t(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \mid 0 < \theta < \pi, \ 0 < \varphi < 2\pi\}$$

In this system, $dS(x) = t^2 \sin\theta \, d\theta \, d\varphi$, and so

$$\begin{aligned}
\psi(\xi, t) = \psi(|\xi| e_3, t) &= \int_0^{\pi} \left\{ \int_0^{2\pi} e^{-it\cos\theta|\xi|} t^2 \sin\theta \, d\varphi \right\} d\theta \\
&= 2\pi t^2 \int_0^{\pi} e^{-it\cos\theta|\xi|} \sin\theta \, d\theta \\
&= 4\pi t \frac{\sin(|\xi| t)}{|\xi|}
\end{aligned}$$

The miracle of odd dimension occurs in the last step where we arrive at a function that we can integrate simply. This concludes the proof.

Write formulas (A) and (B). The causal solution becomes, with convolution taken in space-time:

$$\begin{aligned}
u(x,t) &= (\tilde{\Phi} * f)(x,t) \\
&= \int_0^{+\infty} \left\{ \int_{|y|=s} \frac{1}{4\pi|y|} f(x-y, t-|y|) \, dS(y) \right\} ds \\
&= \int_{\mathbb{R}^3} \frac{1}{4\pi|y|} f(x-y, t-|y|) \, dy
\end{aligned}$$

We thus obtain the very elegant formula:

(20)
$$u(x,t) = \int_{\mathbb{R}^3} \frac{f(y, t-|y-x|)}{4\pi|x-y|} \, dy$$

This is one of the beautiful formulas of classical physics. Here, we integrate over $|x-y| = t-s$. What the source emits arrives with a delay $t - |x-y|$, and the received signal is attenuated, which is expressed by the factor $\frac{1}{|x-y|}$. In dimension 1, however, the signal travels either to the right or the left, and the received signal always has intensity equal to half the initial one. We also see that $u(x,t)$ depends only on the values of f on $\partial C_-(x,t)$: this is the strong Huygens principle.

Formula (A) now becomes the famous **Kirchhoff formula**:

(21)
$$u(x,t) = \frac{1}{4\pi t}\int_{|x-y|=t} v_0(y)\, dS(y) + \frac{d}{dt}\left(\frac{1}{4\pi t}\int_{|x-y|=t} u_0(y)\, dS(y)\right) + \frac{1}{4\pi}\int_{|x-y|\le t} \frac{f(y,t-|x-y|)}{4\pi|x-y|}\, dy$$

□

2.5. The Malgrange-Ehrenpreis theorem

We conclude this chapter with a discussion of the Malagrange-Ehrenpreis theorem, which guarantees the existence of fundamental solutions to partial differential equations with constant coefficients. We refer to subsequent chapters for a different proof for non-constant coefficients elliptic operators in domains using duality methods.

CHAPTER 3

Sobolev Spaces and weak formulation of elliptic equations

In this chapter, we study a particular class of distributions given by Sobolev space. The main reference is [8] (see also [1]).

3.1. The Dirichlet energy and the Poisson problem

One of the natural ways to motivate the introduction of Sobolev spaces is the resolution of the following boundary value problem, the so-called Poisson problem: for a given $f : \partial\Omega \to \mathbb{R}$, prove that there exists a unique solution u to the problem

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega\,, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{22}$$

Physically, this corresponds to

3.2. Definition of the Sobolev spaces and basic properties

3.2.1. Basics and topological properties of Sobolev spaces. Throughout, we fix an integer $k \in \mathbb{N}$ and an exponent $p \in [1;+\infty]$. We let Ω be an open subset of $\mathbb{R}^d$.

DEFINITION 3.1. *The Sobolev space $W^{m,p}(\Omega)$ is defined as*

$$W^{m,p}(\Omega) := \left\{ f \in L^p(\Omega), \quad \forall \alpha \in \mathbb{N}^d,\ |\alpha| \leq m, \quad \partial^\alpha f \in L^p(\Omega) \right\},$$

where $\partial^\alpha f$ means that the distributional derivative $\partial^\alpha f$ can be identified with an L^p function. This space is endowed with the norm

$$\|f\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \|\partial^\alpha f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

REMARK 3.1. *We have by definition $W^{0,p}(\Omega) = L^p(\Omega)$.*

The case $p = 2$ plays a special role, and is sometimes abbreviated as $W^{m,2}(\Omega) = H^m(\Omega)$. Its specificity comes from the fact that it is particularly well-suited to handle elliptic equations, and that it is a Hilbert space for the scalar product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v.$$

As a consequence of similar theorems on Lebesgue spaces, we have the following topological properties of Sobolev spaces:

THEOREM 3.1. *Let $m \in \mathbb{N}\,, p \in [1;+\infty]$. Then:*

(1) *$W^{m,p}(\Omega)$ is a Banach space.*

(2) *If $p < \infty$, $W^{m,p}(\Omega)$ is separable.*
(3) *If $p \in (1; +\infty)$, $W^{m,p}(\Omega)$ is reflexive.*

Proof of Theorem 3.1. The fact that Sobolev spaces are Banach spaces is a simple consequence of the completeness of Lebesgue spaces: if $\{f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $W^{m,p}(\Omega)$, it follows that for any multi-index $\alpha\,, |\alpha| \leq k$ the sequence $\{\partial^\alpha f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$. In particular, for any α there exists g_α such that

$$\partial^\alpha f_k \underset{k\to\infty}{\rightharpoonup} g_\alpha.$$

To conclude, it remains to show that $\partial^\alpha g_0 = g_\alpha$. However, for any test function $\varphi \in \mathcal{D}(\Omega)$, in the sense of distribution, we have

$$\int_\Omega \varphi g_\alpha = \lim_{k\to\infty} \int_\Omega \partial^\alpha f_k \varphi = \lim_{k\to\infty} (-1)^{|\alpha|} \int_\Omega f_k \partial^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g_0 \partial^\alpha \varphi.$$

This allows to conclude.

Regarding the separability and reflexivity, recall that a closed subspace of a separable (resp. reflexive) Banach space is also separable (resp. reflexive) (exercise: prove it!). Let us then show that $W^{m,p}(\Omega)$ can be identified with a closed subspace of a reflexive, separable Banach space. To this end, consider the set $J := \{\alpha\,, |\alpha| \leq m\}$, and consider the embedding

$$\mathrm{inj} : W^{m,p}(\Omega) \ni f \mapsto (\partial^\alpha f)_{\alpha\in J} \in L^p(\Omega \times J),$$

where $L^p(\Omega \times J)$ is endowed with tensor product of the Lebesgue measure on Ω and of the counting measure on J. Then inj is injective, isometric and continuous and, by the same arguments that proved the completeness, its image is closed. We deduce that $W^{m,p}(\Omega)$ can be identified, as a Banach space, with a closed subspace of $L^p(\Omega \times J)$. As the latter is a reflexive, separable Banach space, the conclusion follows. □

Finally, to conclude the basic properties, let us provide the analog of Theorem 1.16:

Theorem 3.2. *Let $1 \leq p < \infty$ and $m \in \mathbb{N}$. Let $f \in W^{m,p}(\mathbb{R}^d)$ and define, for any $k \in \mathbb{N}$, $f_k := (f * \psi_k)$ (with the notations of Theorem 1.16). Then:*
(1) *For any k, $f_k \in \mathscr{C}^\infty(\mathbb{R}^d)$.*
(2) *Furthermore,*

$$\|f_k - f\|_{W^{m,p}(\mathbb{R}^d)} \underset{k\to\infty}{\to} 0. \tag{23}$$

This theorem is naturally also valid in any open set Ω, and leads to the following result:

Theorem 3.3. *Let $p \in [1; +\infty)$. Then $\mathscr{C}^\infty(\Omega) \cap W^{m,p}(\Omega)$ is dense in $W^{m,p}(\Omega)$.*

Remark 3.2. (1) *The set $\mathscr{C}^\infty(\Omega)$ does not coincide with $\mathscr{C}^\infty\left(\overline{\Omega}\right)$.*
(2) *The Sobolev space $W^{m,p}(\Omega)$ could have been defined as the topological completion of $\mathscr{C}^\infty(\Omega)$ with respect to the $\|\cdot\|_{W^{m,p}(\Omega)}$ norm.*
(3) *In the case where Ω has a smooth enough boundary we will see that $\mathscr{C}^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$.*

3.2.2. Some useful properties of Sobolev functions. We begin with something that always comes in handy:

PROPOSITION 3.1. *Let Ω be an open subset of $\mathbb{R}^d$, $m \in \mathbb{N}, m \geq 1$, $p \in [1; +\infty]$ and $u \in W^{m,p}(\Omega)$. If*

$$\nabla u = 0 \text{ a.e.}$$

then there exists a constant c such that

$$u = c \text{ a.e.}.$$

PROOF OF PROPOSITION 3.1. This is a consequence of Proposition 2.3. □

The other purely technical lemma is the following:

PROPOSITION 3.2. *Let Ω be an open set, $m \in \mathbb{N}, m \geq 1, p \in [1; +\infty]$ and $u \in W^{m,p}(\Omega)$. Assume that for a given $\lambda \geq 0$ we have*

$$|\{u = \lambda\}| > 0.$$

Then $\nabla u = 0$ a.e. in $\{u = \lambda\}$.

When dealing with PDEs, it is often quite convenient to compose Sobolev functions with other functions, for instance (see Section 3.9.2 for an application to the maximum principle) replacing a function u with $|u|$. The following proposition summarises the main information:

PROPOSITION 3.3. *Let Ω be an open set, $m \in \mathbb{N}, p \in [1; +\infty]$ and $u = u_+ - u_- \in W^{1,p}(\Omega)$. Then $|u| \in W^{1,p}(\Omega)$ and*

$$\nabla u = \mathbb{1}_{\{u \geq 0\}} \nabla u_+ - \mathbb{1}_{\{u \leq 0\}} \nabla u_-.$$

REMARK 3.3 (What happens in $W^{m,p}(\Omega)$ for $m > 1$?). *Proposition 3.3 is wrong for $u \in W^{m,p}(\Omega)$. For instance, in $W^{2,2}((-1; 1))$ consider $u = x$. Then $u \in W^{2,2}((-1; 1))$, but $v = |u| = |x|$ satisfies, in a distributional sense, $v' = \operatorname{sgn}(\cdot), v'' = \delta_0$ so $v \in W^{1,2}$, but $v \notin W^{2,2}$.*

To prove Proposition 3.3, we need the following Sobolev version of the chain rule:

PROPOSITION 3.4. *Let $\psi \in \mathscr{C}^1(\mathbb{R})$ be such that $\|\psi'\|_{L^\infty(\mathbb{R})} < +\infty$, let Ω be an open subset of $\mathbb{R}^d$, $p \in [1; +\infty]$ and $u \in W^{1,p}(\Omega)$. Then $\psi \circ u \in W^{1,p}(\Omega)$ and $\nabla(\psi \circ u) = \psi'(u) \nabla u$.*

REMARK 3.4. *One could adapt the proof to higher order Sobolev spaces, which would require more regularity of the function ψ (typically, $\mathscr{C}^m$ to obtain the stability of $\mathscr{C}^m$ by composition).*

PROOF OF PROPOSITION 3.4. We let $\{u_k\}_{k \in \mathbb{N}} \in \mathscr{C}^\infty(\Omega)$ be an approximation of u in $W^{1,p}(\Omega)$. Then, for any test function $\varphi \in \mathscr{C}_c^\infty(\Omega)$, we have

$$\begin{aligned} \int_\Omega v \partial_i \psi \circ u_k &= -\int_\Omega (\partial_i v) \psi \circ u_k \\ &= \int_\Omega v \psi'(u_k) \partial_i u_k \\ &\underset{k \to \infty}{\to} \int_\Omega v \psi'(u) \partial_i u \text{ by the dominated convergence theorem} \end{aligned}$$

and this concludes the proof. □

Proof of Proposition 3.3. We proceed by approximation, and we simply show that $u\mathbb{1}_{\{u\geq 0\}} \in W^{1,p}(\Omega)$. To this end, we let $\{\psi_\varepsilon\}_{\varepsilon\to 0}$ be a smooth approximation of $x \mapsto x^+$ satisfying $\|\psi'\|_{L^\infty} \leq 1$. Then it is clear from Proposition 3.4 that for any $\varepsilon > 0$, $\psi_\varepsilon \circ u \in W^{1,2}(\Omega)$, and that, in a L^p sense,

$$\nabla\psi_\varepsilon \circ u = \psi_\varepsilon'(u)\nabla u.$$

It is then an exercise tho check that $\psi_\varepsilon'(u) \to \mathbb{1}_{\{u\geq 0\}}$ a.e. in Ω, and it suffices to pass to the limit to obtain the conclusion.

□

3.3. The trace operator and $W_0^{m,p}(\Omega)$ spaces

Usually, when one considers a PDE set in a (possibly unbounded) domain, giving the equation itself is not enough, and it needs to be supplemented with some more conditions–this is of course even the case for differential equations. As an example, consider the PDE (in a distributional sense)

$$-\Delta u = f$$

set in $\mathbb{R}^d$. As, for any $b \in \mathbb{R}^d$, the map $v_b : x \mapsto \langle b, x\rangle$ is harmonic, if u solves the PDE, then so does $u + v_b$ for any b. On the other hand, if we impose, for instance, that u is bounded, then we have at most one solution (exercise: prove it using the Fourier transform of distributions).

Now, if we were working in a bounded domain, say $\Omega = \mathbb{B}(0; 1)$, a possibility would be to impose the value of u on the boundary of the domain, for instance $u = 0$ on $\partial\Omega$. Now, the question is: as we want to solve PDEs in Sobolev spaces, does this make any sense? Indeed, Sobolev functions are defined almost everywhere, and $\partial\Omega$ has zero Lebesgue measure. The goal of this paragraph is to explain why having some integrability of the gradient of u allows to define u on $\partial\Omega$ in a meaningful manner.

3.3.1. The trace operator. The main reference for this entire part is [8, Chapter 18]. To make sense, when $u \in W^{1,p}(\Omega)$, of $u : \partial\Omega \to \mathbb{R}$, we begin with the case $\Omega = \mathbb{R}^d_+ = \{x_d > 0\}$. In this case, we ca consider the map

$$\mathrm{Tr} : \mathscr{C}^\infty(\mathbb{R}^d_+) \ni u \mapsto u \in \mathscr{C}^\infty(\{x_d = 0\}).$$

We have the following theorem:

Theorem 3.4. *There exists a constant $C > 0$ such that, for any $u \in W^{1,p}(\mathbb{R}^d_+) \cap \mathscr{C}^\infty(\mathbb{R}^d_+)$ there holds*

$$\int_{\{x_d=0\}} |u|^p \leq C \int_{\mathbb{R}^d_+} |u|^p + |\nabla u|^p.$$

As a consequence, Tr *extends to a continuous operator* $\mathrm{Tr} : W^{1,p}(\mathbb{R}^d) \to L^p(\{x_d = 0\})$.

Proof of Theorem 3.4. Assume for the sake of simplicity that $u \in W^{1,p}(\mathbb{R}^d)$ (this step requires a extension operator, we refer to Section 3.5) and that it is compactly supported so that $u \in \mathscr{C}^\infty_c(\mathbb{R}^d)$. For all $\underline{x} \in \{x_d = 0\}$, we have

$$\left|u|_{\{x_d=0\}}\right|^p(\underline{x}) = |u|^p(\underline{x}, 0) \leq \int_{[0,\infty)} |\partial_{x_d}(|u|^p)|(\underline{x}, s)ds = p\int_{[0,\infty)} |u|^{p-1}|\partial_{x_d} u|(\underline{x}, s)ds.$$

Using the inequality $|a|^{p-1}|b| \leq |a|^p + |b|^p$ (consider the case $|a| \leq |b|$ and $|b| \leq |a|$), this gives the point-wise estimate

$$\left|u|_{\{x_d=0\}}\right|^p (\underline{x}) \leq p \int_{[0,\infty)} (|u|^p + |\nabla u|^p)(\underline{x}, s) ds.$$

Integrating in $\underline{x} \in \{x_d = 0\}$ proves that $\|u\|_{L^p(\{x_d=0\})} \leq p\|u\|_{W^{1,p}(\mathbb{R}^d)}$. By density of $C_0^\infty(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$, the result holds on the whole space $W^{1,p}(\mathbb{R}^d)$.

□

A natural question is whether any $L^p(\{x_d = 0\})$ function can be extended to a $W^{1,p}(\mathbb{R}^d)$ function. Unfortunately, the answer is no. In fact, the range of Tr is significantly smaller than $L^p(\{x_d = 0\})$. To understand it better, we would need to study fractional Sobolev spaces. We refer to Section 3.6. The moral of the story is that $\mathrm{Tr}(u)$ loses some regularity, but not too much, and an easy theorem is the following:

Theorem 3.5. *Let $m \geq 1$. There exists a constant $C > 0$ such that, for any $u \in W^{m,p}(\mathbb{R}^d_+) \cap \mathscr{C}^\infty(\mathbb{R}^d_+)$ there holds*

$$\|u\|_{W^{m-1,p}(\{x_d=0\})} \leq C\|u\|_{W^{m,p}(\mathbb{R}^d_+)}.$$

As a consequence, Tr *extends to a continuous operator* $\mathrm{Tr} : W^{m,p}(\mathbb{R}^d) \to W^{m-1,p}(\{x_d = 0\})$.

In the case of a bounded regular domain Ω, one can similarly define the trace operator, by flattening the boundary so that, locally, the situation is the same as in Theorem 3.4:

Theorem 3.6. *Let Ω be an open set with uniformly $\mathscr{C}^1$ boundary, $m \geq 1$ and $p \in [1; +\infty]$. There exists a constant $C_\Omega > 0$ such that, for any $u \in W^{m,p}(\Omega) \cap \mathscr{C}^\infty(\Omega)$ there holds*

$$\|u\|_{W^{m-1,p}(\partial\Omega)} \leq C\|u\|_{W^{m,p}(\Omega)}.$$

As a consequence, Tr *extends to a continuous operator* $\mathrm{Tr} : W^{m,p}(\Omega) \to W^{m-1,p}(\partial\Omega)$.

With these definitions at hand, we can now make sense of more general integration by parts formulas: for any $u \in W^{2,p}(\Omega)$ and any $v \in \mathscr{C}^\infty(\Omega)$, we have

$$\int_\Omega (-\Delta u)v = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v + \int_\Omega \langle \nabla u, \nabla v \rangle.$$

Finally, we can define the Sobolev space $W_0^{m,p}(\Omega)$:

Definition 3.2. *Let Ω be an open set with $\mathscr{C}^1$ boundary, $m \in, m \geq 1$ and $p \in [1; +\infty]$. The Sobolev space $W_0^{m,p}(\Omega)$ is defined as*

$$W_0^{m,p}(\Omega) := \ker(\mathrm{Tr})$$

where Tr *is the trace operator. When $u \in W_0^{m,p}(\Omega)$, we write $u = 0$ on $\partial\Omega$. When $p = 2$, this space is sometimes written $H_0^m(\Omega)$.*

3.3.2. An alternative definition of $W_0^{m,p}(\Omega)$. Another possibility is to define $W_0^{m,p}(\Omega)$ as the completion of of $\mathscr{C}_c^\infty(\Omega)$ with respect to the $W^{m,p}$ norm. This is not the most natural definition from the point of view of PDEs, but can also come in handy. The fact that when Ω is regular enough the two notions coincide is given by the following theorem:

THEOREM 3.7. *We let $\tilde W_0^{m,p}(\Omega)$ ($m \in \mathbb{N}, m \geq 1, p \in [1;+\infty]$ be the completion for the $W^{m,p}$ norm of $\mathcal{D}(\Omega)$. Then, if Ω has $\mathscr{C}^1$ boundary, it follows that*

$$W_0^{m,p}(\Omega) = \tilde W_0^{m,p}(\Omega).$$

3.4. Sobolev embeddings

In this paragraph, we review some of the main theorems of functional analysis in Sobolev spaces, the Sobolev embedding theorems. Essentially, these theorems mean that, if the gradient a function is extremely integrable, then the function has more regularity or, if a function has a high number of integrable derivatives, then the function is much more integrable than expected. The compact embedding theorems also provide a natural Sobolev extension of the Arzela-Ascoli theorem. Here, one should be very careful regarding the interplay between the degree of differentiability m, the degree of integrability p and the dimension d.

Caution: the proofs of all the following results are highly technical and should be omitted in your first read through. However, Remarks 3.8–3.7 are extremely important.

3.4.1. Sobolev embeddings when $p > d$. The main inequality is the Morrey embedding:

THEOREM 3.8. *[Morrey inequality] Assume that $\Omega = \mathbb{R}^d, \mathbb{R}^d_+$ or that Ω is a bounded open set with $\mathscr{C}^1$ boundary. Then, for any $p \in (d;+\infty)$, the space $W^{1,p}(\Omega)$ is continuously embedded in $\mathscr{C}^{0,1-\frac{d}{p}}(\Omega)$, meaning that for every $u \in W^{1,p}(\Omega)$ there exists $v \in \mathscr{C}^{0,1-\frac{d}{p}}(\Omega)$ such that $u = v$ a.e. and $\|v\|_{\mathscr{C}^{1-\frac{d}{p}}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$. This is abbreviated as: there exists a constant C such that*

$$\forall u \in W^{1,p}(\Omega), \|u\|_{\mathscr{C}^{0,1-\frac{d}{p}}(\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}.$$

As a corollary of the previous inequality and of some finer interpolation results we do not give here we deduce:

COROLLARY 3.1 (Morrey embedding theorem). *Let $\Omega = \mathbb{R}^d, \mathbb{R}^d_+$ or an open set with $\mathscr{C}^\infty$ boundary. Let $p \in (d;+\infty)$ and $m \in \mathbb{N}, m \geq 1$. If*

$$mp > d$$

then there exists a constant C such that

$$\forall u \in W^{m,p}(\Omega), \|u\|_{\mathscr{C}^{\ell,\gamma}(\overline{\Omega})} \leq \|u\|_{W^{m,p}(\Omega)}$$

where

$$\ell = \left\lfloor m - \frac{d}{p} \right\rfloor, \gamma = m - \ell - \frac{d}{p}$$

if $m - \frac{d}{p}$ is not an integer, and

$$\ell = m - \frac{d}{p}, \gamma \in [0;1)$$

if $m-\frac{\ell}{p}$ is an integer.

Assuming $\partial\Omega$ of class $\mathscr{C}^\infty$ is an overkill, one simply needs as many degrees of regularity as the functions defined in Ω.

REMARK 3.5. *The previous result does not hold when $p=d>1$. Indeed, should it be the case, it would follow that $W^{1,d}(\Omega)\subset L^\infty(\Omega)$, which is not true, as the following classical example shows: let $u:x\mapsto \psi(x)\ln\left(\ln\left(1+\frac{1}{|x|}\right)\right)$ where ψ is a smooth cut-off function supported in $\mathbb{B}(0;1)$. Then, focusing on the case $|x|\leq 1$ we obtain*

$$|\nabla u|(x)=\frac{1}{\frac{1}{1+|x|^{-1}}\ln\left(1+\frac{1}{|x|}\right)}.$$

By the Bertrand criterion and integration in polar coordinates, if $d>1$, we deduce that $\nabla u\in L^d(\mathbb{R}^d)$ and so $u\in W^{1,d}(\mathbb{R}^d)$. However, u is not bounded.

REMARK 3.6. *In the case $d=1$, the situation is much simpler, as shown by the following theorem (exercise: prove it!): For any $p\in[1;+\infty]$, $W^{1,p}(\mathbb{R})$ is continuously embedded into $L^\infty(\mathbb{R})\cap\mathscr{C}^0(\mathbb{R})$.*

REMARK 3.7 (Regarding the optimality of the regularity exponent). *We can use a scaling argument to show that the Morrey exponent is optimal. Indeed, assume that $W^{1,p}(\Omega)\hookrightarrow\mathscr{C}^{0,\gamma}(\overline{\Omega})$, so that for some constant C we have*

$$\forall x\,,y\,,|u(x)-u(y)|\leq C\|x-y\|^\alpha\|u\|_{W^{1,p}(\Omega)}.$$

Now define, for any $\varepsilon>0$, $u_\varepsilon:=u\left(\frac{\cdot}{\varepsilon}\right)$. Then direct computations show that

$$\|u_\varepsilon\|_{L^p}=\varepsilon^{\frac{d}{p}}\|u\|_{L^p}$$

as well as

$$\|\nabla u_\varepsilon\|_{L^p}=\varepsilon^{\frac{d-p}{p}}\|\nabla u\|_{L^p}$$

so that, for $\varepsilon>0$ small enough we have

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)}\sim\varepsilon^{\frac{d-p}{p}}\|\nabla u\|_{L^p(\Omega)}.$$

Consequently,

$$|u(x)-u(y)|=|u_\varepsilon(\varepsilon x)-u_\varepsilon(\varepsilon y)|\leq C\varepsilon^\alpha\varepsilon^{\frac{p-d}{p}}\|\nabla u\|_{L^p(\Omega)}$$

and so we must have

$$\alpha+\frac{d-p}{p}\leq 0.$$

3.4.2. Sobolev embedding for $p<d$. When $p<d$, we do not gain regularity, but we do obtain more integrability. This is the Gagliardo-Nirenberg-Sobolev inequality:

THEOREM 3.9. *[Gagliardo-Nirenberg-Sobolev inequality] Assume that $\Omega=\mathbb{R}^d\,,\mathbb{R}^d_+$ or that Ω is a bounded open set with $\mathscr{C}^1$ boundary. Then:*

1. *$W^{1,1}(\Omega)$ is continuously embedded into $L^{\frac{d}{d-1}}(\Omega)$.*
2. *For any $p\in(1;d)$, $W^{1,p}(\Omega)$ is continuously embedded in $L^{\frac{dp}{d-p}}(\Omega)$.*

In the general case, we have the following:

Theorem 3.10. *Assume that* $\Omega=\mathbb{R}^d\,,\mathbb{R}^d_+$ *or that* Ω *is a bounded open set with* $\mathscr{C}^\infty$ *boundary. Then, for any* $m\in\mathbb{N}\,,m\geq 1\,,p\in(1;+\infty)$ *such that*

$$mp<d$$

the space $W^{m,p}(\Omega)$ *is continuously embedded in* $L^q(\Omega)$ *with*

$$\frac{1}{q}=\frac{1}{p}-\frac{m}{d}.$$

When $m=d$ *and* $p=1$*, the space* $W^{d,1}(\Omega)$ *is continuously embedded into* $L^\infty(\Omega)$*.*

A final remarkable theorem related to Sobolev functions with high integrability is the Rademacher theorem:

Theorem 3.11. *Assume that* $\Omega=\mathbb{R}^d\,,\mathbb{R}^d_+$ *or that* Ω *is an open set with* $\mathscr{C}^1$ *boundary. Assume that for some* $p\in(d;+\infty]$ $u\in W^{1,p}(\Omega)$*. Then* u *is differentiable almost-everywhere, and its weak derivative coincides with is (classically defined) derivative.*

Remark 3.8 (Regarding the optimality of the integrability exponent). *The Lebesgue exponents are optimal in the previous theorem. Indeed, assume that for some constant* C *we have*

$$\forall u\in W^{1,p}(\Omega)\,,\|u\|_{L^q(\Omega)}\leq C\|u\|_{W^{1,p}(\Omega)}.$$

Let $u\in W^{1,p}(\Omega)$ *and set*

$$u_\varepsilon:x\mapsto u\left(\frac{x}{\varepsilon}\right).$$

Then, for any $\varepsilon>0$*, we have*

$$\|u_\varepsilon\|_{L^q}=\varepsilon^{\frac{d}{q}}\|u\|_{L^q}$$

as well as

$$\|\nabla u_\varepsilon\|_{L^p}=\varepsilon^{\frac{d-p}{p}}\|\nabla u\|_{L^p}.$$

In particular, we deduce that

$$\varepsilon^{\frac{d}{q}}\leq C\varepsilon^{\frac{d-p}{p}}$$

whence

$$\frac{d}{q}-\frac{d}{p}\geq 1.$$

This gives

$$\frac{1}{q}\geq\frac{d-p}{pd}.$$

3.4.3. Sobolev embedding when $p=d$**.** The case $p=d$ is very delicate, and we merely mention that the natural embedding space is that of functions of mean bounded oscillations. We refer to [8] and merely state the following result:

Theorem 3.12. *Assume that* $\Omega=\mathbb{R}^d\,,\mathbb{R}^d_+$ *or that* Ω *is an open set with* $\mathscr{C}^1$ *boundary. Then for any* $q\in[1;+\infty)$ *we have* $W^{1,d}(\Omega)\hookrightarrow L^q(\Omega)$*, but* $W^{1,d}(\Omega)$ *is not included in* $L^\infty(\Omega)$*.*

3.4.4. Compact embeddings. To conclude regarding Sobolev embeddings, we go back to the introduction of this chapter, Section 3.1: we would like to obtain some compactness properties. This is provided by the following theorem, due to Rellich and Kondrachov, which is set in the framework of bounded open sets Ω. To simplify things a bit, we say that the embedding of a Banach space X into Y is compact if any X-bounded sequence admits a strongly Y-converging subsequence.

THEOREM 3.13. *[Rellich-Kondrachov] Let Ω be a bounded open subset with $\partial\Omega$ of class $\mathscr{C}^1$. Then:*

1. *For any $p \in (d;+\infty)$, for any $\alpha \in \left(0;1-\frac{d}{p}\right)$, the embedding $W^{1,p}(\Omega) \hookrightarrow \mathscr{C}^{0,\beta}\left(\overline{\Omega}\right)$ is compact.*
2. *For ay $p \in [1;d)$, for any $q \in \left[1;p^*=\frac{pd}{d-p}\right)$, the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.*

The following corollary will be useful, we single it out for further reference:

COROLLARY 3.2. *If Ω is a bounded open subset with a $\mathscr{C}^1$ boundary and $p \in [1;+\infty]$, the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.*
In general, assume $p < d$. Letting p^* be defined as $p^* = \frac{pd}{d-p}$, $W^{1,p}(\Omega)$ is embedded in $L^q(\Omega)$ for any $q \in [1;p^*]$ and the embedding is compact for any $q < p^*$.

REMARK 3.9 (The critical case of the embedding). *We can wonder whether the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ could be compact. Here again, to show that it is not the case, we re-use the scaling properties already encountered in Remarks 3.8–3.7. Indeed, consider a smooth, radially symmetric and compactly supported function $u : \mathbb{B}(0;1) \to \mathbb{R}$, as well as, for any $\varepsilon > 0$,*

$$u_\varepsilon : x \mapsto u\left(\frac{x}{\varepsilon}\right).$$

Then, for any $\varepsilon > 0$, we have

$$\|u_\varepsilon\|_{L^q} = \varepsilon^{\frac{d}{q}}\|u\|_{L^q}$$

as well as

$$\|\nabla u_\varepsilon\|_{L^p} = \varepsilon^{\frac{d-p}{p}}\|\nabla u\|_{L^p},$$

whence if $q = p^*$ the two quantities scale the same. In particular,

$$v_\varepsilon := \frac{u_\varepsilon}{\varepsilon^{\frac{d-p}{p}}\|\nabla u\|_{L^p}}$$

is bounded in $W^{1,p}$ and in L^{q^*}. As it converges a.e. to 0, it can not converge strongly in L^q.

As always, we have the following more general version (the previous one is often sufficient in practice):

THEOREM 3.14. *Let Ω be an open bounded subset of $\mathbb{R}^d$ with a $\mathscr{C}^\infty$ boundary. For any m,ℓ,p,q with $(m,\ell) \in \mathbb{N}^2$, $(p,q) \in [1;+\infty]$ such that*

$$m-\frac{d}{p} > \ell-\frac{d}{q}$$

the embedding

$$W^{m,p}(\Omega) \hookrightarrow W^{\ell,q}(\Omega)$$

is compact.

Remark 3.10 (What happens when Ω is not bounded?). *When the domain Ω is not bounded, the situations is much more delicate but can be handled using the concentration-compactness principle of Lions; we refer once more to [9].*

3.5. Complement*: Sobolev extensions

3.6. Complement*: fractional Sobolev spaces

In this class, the main interest of fractional Sobolev spaces is to understand the range of the Trace operator and, more specifically, to obtain the following result:

Corollary 3.3. *Let Ω be an open bounded subset with $\mathscr{C}^1$ boundary. Then the trace operator* $\mathrm{Tr} : W^{1,2}(\Omega) \to L^2(\partial\Omega)$ *is compact.*

3.7. Complement*: dual of Sobolev spaces

3.8. Poincaré inequalities

We now prove the main functional inequality of Sobolev spaces, the Poincaré inequality, which will be revisited when discussing the spectral theory of compact operators:

Theorem 3.15. *Let Ω be a bounded open set with $\mathscr{C}^1$ boundary and $p \in [1;+\infty)$. There exist two constant $C_1\,, C_2$ such that the following hold:*

(1) *Poincaré inequality in $W^{1,p}_0(\Omega)$:*

$$\forall u \in W^{1,p}_0(\Omega)\,, \int_\Omega |u|^p \leq C_1 \int_\Omega |\nabla u|^p. \tag{24}$$

(2) *Poincaré inequality with normalised mean: if in addition Ω is connected*

$$\forall u \in W^{1,p}(\Omega)\,, \int_\Omega \left|u - \fint_\Omega u\right|^p \leq C_2 \int_\Omega |\nabla u|^p. \tag{25}$$

Proof of Theorem 3.15. We argue by contradiction and assume that (24) does not hold. In particular, and up to normalisation, there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \in W^{1,p}_0(\Omega)$ such that, for any $k \in \mathbb{N}$,

$$\int_\Omega |\nabla u_k|^p \leq \frac{1}{k}\,, \int_\Omega |u_k|^p = 1.$$

Since the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, up to a subsequence, there exists $u_\infty \in W^{1,p}(\Omega)$ such that $\{u_k\}_{k\in\mathbb{N}}$ converges L^p strongly, $W^{1,p}(\Omega)$ weakly (by reflexivity of $W^{1,p}(\Omega)$) to u_∞. By compactness of the trace operator, $u_\infty = 0$ on $\partial\Omega$ and

$$\int_\Omega |u_\infty|^p = 1\,, \int_\Omega |\nabla u_\infty|^p \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^p = 0$$

so that u_∞ is constant. Since $u_\infty = 0$ on $\partial\Omega$, we have $u_\infty \equiv 0$.

$\square$

Regarding (24), taking $u = 0$ on $\partial\Omega$ is an overkill, and one can easily show with the exact same method of proof the following:

THEOREM 3.16. *Let Ω be a bounded open set with $\mathscr{C}^1$ boundary and $p \in [1;+\infty]$. Let $\Gamma \subset \partial\Omega$ be a regular subset of $\partial\Omega$ with positive Hausdorff measure, and let $W^{1,p}(\Omega,\Gamma)$ denote the set of functions in $W^{1,p}(\Omega)$ such that $u = 0$ on Γ. Then, there exists C_3 such that*

$$\forall u \in W^{1,p}(\Omega,\Gamma)\,, \int_\Omega |u|^p \leq C_1 \int_\Omega |\nabla u|^p. \tag{26}$$

3.9. Existence of solutions to elliptic PDEs: minimisation procedures

In this section, we put the general theory developed up util that point to the study of existence of solutions to the Poisson problem in a bounded, smooth domain, by minimising the Dirichlet energy introduced in Section 3.1.

3.9.1. The general method for the Poisson problem. We want to solve, in a bounded domain Ω with $\mathscr{C}^1$ boundary, the PDE

$$\begin{cases} -\Delta u = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

Which sense should we give to this equation? A natural one is the notion of weak solutions, which requires the use of less derivatives of the solution u. Namely, assuming every integration by parts goes well, we have, if $u \in W^{1,2}_0(\Omega)$ solves this PDE, that

$$\forall v \in W^{1,2}_0(\Omega)\,, \int_\Omega \langle \nabla u, \nabla v\rangle = \int_\Omega fv.$$

This appears, as we already observed, as the criticality condition for the functional

$$\mathscr{E} : u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu.$$

Observe that we now have a natural space on which $\mathscr{E}$ is defined, the Sobolev space $W^{1,2}_0(\Omega)$. This suggests looking for minimisers of $\mathscr{E}$ in $W^{1,2}_0(\Omega)$, so that we should consider a minimising sequence $\{u_k\}_{k\in\mathbb{N}}$. There will be two steps:

(1) The first one, akin to the case of finite-dimensional optimisation, is to show that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded. This *coercivity property* is linked to the Poincaré inequality.
(2) Once boundedness is established, we need to be able to pass to the limit to deduce that the limit of a minimising sequence is a minimiser. This is where things can become tricky, due to the different topologies at play, but we stress that here, the main tool is the Rellich-Kondrachov compact embedding theorem.

We will prove the following theorem, and use it to obtain easy generalisations:

THEOREM 3.17. *For any $f \in L^2(\Omega)$, the functional*

$$\mathscr{E} : W^{1,2}_0(\Omega) \ni u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu$$

has a unique critical point u_f which is also a global minimiser of $\mathscr{E}$. u_f satisfies

$$\forall v \in W^{1,2}_0(\Omega)\,, \int_\Omega \langle \nabla u, \nabla v\rangle = \int_\Omega fv.$$

Finally, we have the regularity estimate

$$\|u\|_{W^{1,2}(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

for a constant C that only depends on Ω.

Proof of Theorem 3.17. We consider a minimising sequence $\{u_k\}_{k\in\mathbb{N}}$. First of all, by the Poincaré inequality,

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}$$

for some constant C. Consequently, the Cauchy-Schwarz inequality implies that

$$\int_\Omega |\nabla u_k|^2 \lesssim 1 + \|\nabla u_k\|_{L^2(\Omega)},$$

so that the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,2}_0(\Omega)$. By reflexivity of $W^{1,2}_0(\Omega)$ and by the Rellich-Kondrachov theorem, there exists $u_\infty \in W^{1,2}_0(\Omega)$ such that

$$u_k \underset{k\to\infty}{\to} u_\infty \text{ weakly in } W^{1,2}_0(\Omega), \text{ strongly in } L^2(\Omega).$$

We now need to check that u_∞ is a minimiser of $\mathscr{E}$. On the one hand, by strong L^2 convergence we have

$$\int_\Omega f u_\infty = \lim_{k\to\infty} \int_\Omega f u_k.$$

On the other hand, by the semi-continuity of the norm we also have

$$\int_\Omega |\nabla u_\infty|^2 \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2$$

whence

$$\mathscr{E}(u_\infty) \leq \liminf_{k\to\infty} \mathscr{E}(u_k) = \min \mathscr{E}.$$

Thus, u_∞ is a minimiser and the Euler-Lagrange equation for the minimisation of $\mathscr{E}$ reads

$$\forall v \in W^{1,2}_0(\Omega)\,, \int_\Omega \langle \nabla u, \nabla v\rangle = \int_\Omega f v.$$

As for the uniqueness, it follows from the strong convexity of $\mathscr{E}$ (on $W^{1,2}_0(\Omega)$; be careful, it is not strictly convex on $W^{1,2}(\Omega)$!!). □

Remark 3.11 (The case of inhomogeneous Dirichlet boundary conditions). *It is easy to adapt the previous theorem to the case of inhomogeneous boundary conditions in the following sense: let $g \in \text{range}(\text{Tr})$, $f \in L^2(\Omega)$. Consider the partial differential equation*

$$\begin{cases} -\Delta u = f & \text{in } \Omega\,, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Introduce the space

$$X_g := \{u \in W^{1,2}(\Omega) :\ u = g \text{ on } \partial\Omega\}.$$

Then, by the same arguments as before, show that

$$\mathscr{E} : X_g \ni u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f u$$

has a unique minimum on X_g, which is a solution of that PDE.

Remark 3.12 (The L^2 setting). *The fact that the Poisson equation with L^2 right hand side has a variational formulation is tied to two crucial facts: first, $W^{1,2}(\Omega)$ is a Hilbert space. Second, the operator $-\Delta$ is symmetric. A natural question is then: does a similar result hold when the right-hand side f only satisfies $f \in L^p(\Omega)$ and when the operator is no longer symmetric? The answer is yes, and we will discuss this in Section 3.11.*

3.9.2. Application to the maximum principle. An interesting consequence of the variational formulation of Theorem 3.17 is the following weak version of the maximum principle:

Theorem 3.18 (Maximum principle, weak form). *Let $f \in L^2(\Omega)\,, f \geq 0$. Let u be the solution of*

$$\begin{cases} -\Delta u = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \geq 0$ in Ω.

Proof of Theorem 3.18. For any $u \in W_0^{1,2}(\Omega)$ we also have $|u| \in W_0^{1,2}(\Omega)$ and, from Proposition 3.3

$$\int_\Omega |\nabla |u||^2 = \int_\Omega |\nabla u|^2.$$

As $f \geq 0$,

$$\int_\Omega fu \leq \int_\Omega f|u|$$

whence

$$\mathscr{E}(u) \geq \mathscr{E}(|u|).$$

The conclusion follows by uniqueness of the minimiser of $\mathscr{E}$. □

3.9.3. Generalisation: the Lax-Milgram theorem. A generalisation of the minimisation procedure introduced in the previous paragraph in terms of bilinear forms is the Lax-Milgram theorem, which holds in great generality (whose main applications are to be found in parabolic equations and numerical analysis, in particular for finite element methods), and overcomes the symmetry assumption:

Theorem 3.19. *[Lax-Milgram theorem] Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert spaces. Let $a : H \times H \to \mathbb{R}$ be a bilinear form which is continuous and coercive, and let $b : H \to \mathbb{R}$ be a continuous linear map. Then there is unique $u \in H$ so that*

$$\forall f \in H, \quad a(u, f) = b(f).$$

In addition, if a is symmetric then u is the unique minimiser of $J : H \to \mathbb{R}$ defined by

$$J(v) := \frac{1}{2} a(v, v) - b(v).$$

Proof of Theorem 3.19. We do not prove the properties related to symmetric forms a, as the proof is exactly similar to that of Theorem 3.17 (by studying the underlying minimisation problem). The proof proceeds in two steps: first, the case of a finite dimensional space, and second an approximation procedure. If H is finite dimensional, let A be the matrix of the bilinear form a and z be such that $b(f) = \langle z, f \rangle$. We are thus looking for u such that

$$\langle Au, f \rangle = \langle z, f \rangle.$$

As A is coercive, A is invertible and it suffices to set $u = A^{-1}z$. In the case of an infinite dimensional space H, let $\{e_k\}_{k\in\mathbb{N}}$ be a Hilbert basis of H and set, for any $N \in \mathbb{N}$, $E_N := \text{Vect}(e_1, \dots, e_N)$. Let, for any $N \in \mathbb{N}$, u_N denote the solution (with transparent notations) of $Au_N = z$. By continuity and coercivity of a, we obtain

$$\|u_N\|^2 \lesssim \|z\|^2$$

where $\lesssim$ means "$\leq$ up to a multiplicative constant". The sequence $\{u_N\}_{N\in\mathbb{N}}$ is thus bounded. Let u_∞ be a weak limit. Passing to the limit, we deduce that

$$\forall N \in \mathbb{N}\,, \forall f \in E_N, a(u_\infty, f) = b(f).$$

As $\{e_k\}_{k\in\mathbb{N}}$ is a Hilbert basis, the conclusion follows.

□

3.9.4. What happens with lower-order terms?

3.9.5. What happens with other boundary conditions? We conclude this brief presentation of variational methods with a presentation of the Neumann problem, where the homogeneous Dirichlet boundary condition is replaced with

$$\partial_\nu u = 0 \text{ on } \partial\Omega.$$

In other words, we consider

$$\begin{cases} -\Delta u = f & \text{in } \Omega\,, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \tag{28}$$

First of all, observe that by the Green formula, if (28) has a solution, then

$$\int_\Omega f = \int_\Omega (-\Delta u) = -\int_{\partial\Omega} \partial_\nu u = 0.$$

Second, observe that, if u is a solution, so is $u + c$ for any constant c. So this problem seems rather ill-posed. We can solve the last problem by imposing, for instance, $\int_\Omega u = 0$, in order to fix such a constant. Furthermore, it is not clear how we should look for a solution. Let us look at the weak formulation of this PDE: it should read

$$\forall v \in W^{1,2}(\Omega)\,, \int_\Omega \langle \nabla u\,, \nabla v\rangle = \int_\Omega fv.$$

This suggests looking at the functional

$$J : W^{1,2}(\Omega) \ni u \mapsto \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fv.$$

Let us check whether critical points of this functional can be interpreted as solutions of (28) (discarding any discussion of the existence of critical points). Assume furthermore that the critical point u is of class $W^{2,2}(\Omega)$, so that $\partial_\nu u$ can be interpreted in the sense of traces. Then we would have, for any $v \in \mathscr{C}_c^\infty(\Omega)$,

$$\int_\Omega (-\Delta u)v = \int_\Omega fv.$$

In particular, $-\Delta u = f$ a.e. Thus, the criticality condition implies

$$\forall v \in W_0^{1,2}(\Omega)\,, \int_{\partial\Omega} v\partial_\nu u = 0,$$

which in turn gives $\partial_\nu u = 0$. Thus, this seems to be the right-functional (although we would need to study the *a priori* regularity of minimisers).

This does not provide a way around the difficulties tied to the existence of minimisers. In fact, show that J has a (non-necessarily unique) minimiser if, and only if, $\int_\Omega f = 0$.

3.10. Generalisations to other lower semi-continuous functionals

If we inspect the previous section, it appears that a fundamental tool is the minimisations of functionals defined on general Banach spaces. What did we use on the functional $\mathscr{E}$ used to derive existence of a solution for the Poisson problem? That it was coercive, and that it behaved well under weak convergence. Regarding the coercivity, not much can be done: we need to ensure that a minimising sequence remains bounded, or that we can choose at least one bounded minimising sequence. Now, regarding the behaviour under weak convergence, the functional $\mathscr{E}$ was not continuous, merely lower-semi-continuous (l.s.c.) for the weak topology. Likewise in general, it is not necessary to have continuity of the functional for the weak topology (there are way too few weakly continuous maps), but just to have l.s.c in the following sense:

Definition 3.3. *Let $(E, \|\cdot\|)$ be a Banach space and $J : E \to \mathbb{R}$. We say that J is weakly (resp. strongly) lower semi-continuous if, for any sequence $\{x_k\}_{k\in\mathbb{N}}$ weakly (resp. strongly) converging to x_∞ there holds*

$$J(x_\infty) \leq \liminf_{k\to\infty} J(x_k).$$

Of course, a weakly l.s.c. functional is strongly l.s.c. The following result, which should be taken as yet another instance of the interplay between convexity and weak topologies, shows that the converse is also true:

Theorem 3.20. *If $J : E \to \mathbb{R}$ is convex, then J is strongly l.s.c. if, and only if, J is weakly l.s.c.*

Proof of Theorem 3.20. Assume that J is convex and, for the sake of simplicity, that J is differentiable. In particular, for any $x\,, y \in E$,

$$J(y) \geq J(x) + dJ(x)[y - x].$$

Now, let $x_k \underset{k\to\infty}{\rightharpoonup} x_\infty$. It follow that

$$J(x_k) \geq J(x_\infty) + dJ(x_\infty)[x_k - x_\infty]$$

whence

$$\liminf_{k\to\infty} J(x_k) \geq J(x_\infty)$$

thereby establishing the proof of the weak lower semi-continuity. The proof in the case J l.s.c, with respect to the strong topology follows along the same lines, once the existence of a supporting hyperplane is established. □

On the other hand, the following result, taken from the monograph of Evans [4], shows that most natural functionals are to some extent weakly continuous if and only if they are convex. To be more specific, when dealing with PDEs, most functionals to optimise write

$$E : u \mapsto \int_\Omega j(\nabla u) + \int_\Omega j_1(u).$$

Usually, as one gets compactness in a stronger Sobolev norm, the dominated convergence theorem suffices to entail the continuity of. $\int_\Omega j_1(\cdot)$ with respect to the weak topology. Thus, the difficult part is to understand whether

$$J : W^{1,p}(\Omega) \ni u \mapsto \int_\Omega j(\nabla u)$$

is weakly l.s.c. The answer is the following

THEOREM 3.21. *The functional*

$$J : W^{1,p}(\Omega) \ni u \mapsto \int_\Omega j(\nabla u)$$

is weakly l.s.c. if, and only if, j is convex.

We refer to [4, Chapter 2].

3.11. Duality methods

3.11.1. Beyond the L^2 framework.

3.11.2. Existence of fundamental solutions.

3.12. Beyond symmetric operators

CHAPTER 4

Spectral methods, solvability of elliptic PDEs and basic regularity theory

In this chapter, we go back to the Sobolev space $W_0^{1,2}(\Omega)$ to study the eigenvalues and eigenfunctions of elliptic operators. We will begin with a general framework.

4.1. The spectral theorem

4.1.1. Main goal: spectral decomposition of the Laplacian. The main goal of this section is to obtain the following spectral decomposition theorem for the Laplacian:

THEOREM 4.1. *Let Ω be a bounded domain in $\mathbb{R}^d$ with boundary $\partial\Omega$ of class C^1. Then, there is a Hilbert basis $\{e_k\}_{k\in\mathbb{N}}$ of $L^2(\Omega)$ as well as a non-decreasing sequence of eigenvalues*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

such that

$$\forall k \in \mathbb{N}^*, e_k \text{ solves } \begin{cases} -\Delta e_k = \lambda_k e_k & \text{in } \Omega\,, \\ e_k = 0 & \text{on } \partial\Omega \end{cases}.$$

In addition, every eigenvalue has finite multiplicity. Furthermore, up to further normalisation, $\{e_k/\|e_k\|_{W^{1,2}(\Omega)}\}$ is a Hilbert basis of $W_0^{1,2}(\Omega)$.

There many ways to prove this theorem by hand but we will rather deduce it from the general spectral decomposition theorem.

4.1.2. The general framework: spectral decomposition of compact, symmetric operators. One needs to be careful when handling the spectrum of operators in the infinite dimensional case, especially in the case of general Banach spaces. Fortunately, the situation is quite simple when dealing with Hilbert spaces (and this is for the time being the framework we limit ourselves to).

4.1.2.1. *Compact, self-adjoint operators.* We let $(H, \langle\cdot,\rangle)$ be a Hilbert space and we let $A \in \mathcal{L}(H)$ be a bounded operator. We say that A is **self-adjoint** if

$$\forall x\,, y \in H\,, \langle Ax, y\rangle = \langle Ay, x\rangle.$$

We say that A is **compact** if for any bounded sequence $\{x_k\}_{k\in\mathbb{N}} \in H^{\mathbb{N}}$ the sequence $\{Ax_k\}_{k\in\mathbb{N}}$ admits a strongly converging subsequence.

4.1.2.2. *The spectral decomposition of self-adjoint operators.* For the sake of time, we first give a version of the spectral theorem in the case of injective operators A; we postpone the general statement to the next paragraph.

THEOREM 4.2. *Let $A \in \mathcal{L}(H)$ be a compact self-adjoint operator. Assume it is injective. Then:*

(1) *A has countably many eigenvalues $\{\sigma_k\}_{k\in\mathbb{N}}$ whose absolute values $\{|\sigma_k|\}_{k\in\mathbb{N}}$ can be ordered non-increasingly as, for any $\sigma > 0$, there are finitely many eigenvalues of A in $(-\sigma;\sigma)^c$. The associated eigenvectors are orthogonal and the eigenvalues can be variationally characterised as follows:*

$$|\sigma_1| = \sup_{\|x\|^2=1} |\langle Ax, x\rangle|\,,$$

$$\forall k > 1\,, |\sigma_k| = \sup_{\|x\|^2=1\,,\langle x,e_1\rangle=\cdots=\langle x,e_{k-1}\rangle=0} |\langle Ax, x\rangle|.$$

(2) *0 is the only accumulation point of the sequence of eigenvalues.*
(3) *Each non-zero eigenvalue has finite multiplicity.*
(4) *There holds*

$$\forall x \in H,\ Ax = \sum_{k=0}^{\infty} \sigma_k \langle x, e_k\rangle e_k.$$

(5) *The sequence $\{\sigma_k\}_{k\in\mathbb{N}}$ thus produced contains all possible non-zero eigenvalues of A.*
(6) *Letting, for any $k \in \mathbb{N}$, $E(\sigma_k) = \ker(A - \sigma_k \mathrm{Id})$, the closure of $\oplus_{k=0}^{\infty} E_k$ is H so that*

$$\forall x \in H,\ x = \sum_{k=0}^{\infty} \langle x, e_k\rangle e_k.$$

In other words, $\{e_k\}_{k\in\mathbb{N}}$ is a Hilbert basis of H.

Remark 4.1. *There are some (purely notational) difficulties when considering non-injective operators.*

To prove this theorem, we will rely on the following lemma:

Lemma 4.1. *For any compact self-adjoint operator A, we have*

$$\|A\| = \sup_{\|x\|^2=1} |\langle Ax, x\rangle|\,.$$

Proof of Lemma 4.1. Define

$$|\sigma_1| := \sup_{\|x\|^2=1} |\langle Ax, x\rangle|\,.$$

By definition of the operator norm and the Cauchy-Schwarz inequality, we have

$$|\sigma_1| \leq \|A\|.$$

Now, recall (exercise: prove it) that

$$\|A\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle|\,.$$

By symmetry of A,

$$\begin{aligned}
\langle Ax, y\rangle &= \frac{1}{2}\left(\langle Ax, y\rangle + \langle Ay, x\rangle\right)\\
&= \frac{1}{4}\left(\langle A(x+y), x+y\rangle - \langle A(y-x), y-x\rangle\right)\\
&\leq \frac{|\sigma_1|}{4}\left(\|x+y\|^2 + \|x-y\|^2\right)\\
&= |\sigma_1|,
\end{aligned}$$

which concludes the proof. □

We will also be using the following lemma:

Lemma 4.2. *Assume T is compact and $x_k \underset{k\to\infty}{\rightharpoonup} x_\infty$. Then $Tx_k \underset{k\to\infty}{\to} Tx_\infty$.*

Proof of Lemma 4.2. By compactness of T, there exists y_∞ such that $Tx_k \underset{k\to\infty}{\to} y_\infty$. Let $f \in E'$. Then $f\circ T \in E'$, and thus $(f\circ T)(x_k) \underset{k\to\infty}{\to} (f\circ T)(x_\infty)$. On the other hand, $f(Tx_k) \underset{k\to\infty}{\to} f(y_\infty)$. Thus, for any $f \in E'$, $f(Tx_\infty) = f(y_\infty)$, which implies that $Tx_\infty = y_\infty$. □

Proof of Theorem 4.2. We begin by observing that any eigenspace $E(\lambda) = \ker(A-\lambda \mathrm{Id})$ is finite dimensional if $\lambda \neq 0$. Indeed, assume that it is infinite dimensional, so that we can find a infinite orthonormal family $\{y_k\}_{k\in\mathbb{N}}$ of $E(\lambda)$. Then, as $y_k = \frac{1}{\lambda}Ty_k$ and as $\{y_k\}_{k\in\mathbb{N}}$ is bounded, $y_k \underset{k\to\infty}{\to} y_\infty$ for some $y_\infty \in E(\lambda)$ with norm 1. On the other hand, by the Plancherel formula, $y_k \underset{k\to\infty}{\rightharpoonup} 0$, a contradiction. The same proof shows that there can be only finitely many eigenvalues of A in $(-\sigma;\sigma)^c$ for any $\sigma > 0$ so that we have at most a countable set of eigenvalues.

We then proceed inductively. First of all, we define

$$|\sigma_1| := \max_{\|x\|=1} \langle Ax, x\rangle = \|A\| \tag{29}$$

and we observe that this variational problem has a solution e_1. Indeed, let $\{x_k\}_{k\in\mathbb{N}}$ be a maximising sequence for the problem:

$$\langle Ax_k, x_k\rangle \underset{k\to\infty}{\to} \sigma_1\,, \sigma_1 = \sup_{\|x\|=1} \langle Ax, x\rangle.$$

Then, up to a subsequence, $\{x_k\}_{k\in\mathbb{N}}$ converges weakly in H to some x_∞. As A is compact, we deduce that $Ax_k \underset{k\to\infty}{\to} Ax_\infty$ (strongly). We can not however guarantee that $\|x_\infty\| = 1$. However, observe that

$$\|Ax_k - \sigma_1 x_k\|^2 = -2\sigma_1\langle Ax_k, x_k\rangle + \sigma_1^2 + \|Ax_k\|^2 \leq \sigma_1^2 + \|A\|^2 - 2\sigma_1\langle Ax_k, x_k\rangle.$$

From Lemma 4.1, the right-hand side converges to zero so that $\{Ax_k - \sigma_1 x_k\}_{k\in\mathbb{N}}$ is strongly converging. As $\{Ax_k\}_{k\in\mathbb{N}}$ is also strongly converging, we deduce that $\{x_k\}_{k\in\mathbb{N}}$ converges strongly to x_∞, that $Ax_\infty = \sigma_1 x_\infty$, and that x_∞ solves (29). We let $e_1 := x_\infty$.

We can then create an iterative sequence by the variational procedure introduced above, by working on $\langle e_1\rangle^\perp$. It suffices to observe that A restricted to this orthogonal subspace is still compact and self-adjoint (Exercise: check that $\langle x_1\rangle^\perp$ is stable by A). The sequence thus obtained is non-increasing in norm, and, as already noted, the only possible accumulation point of the sequence is 0.

Let us now prove

$$\forall x \in H,\ Ax = \sum_{k=0}^{\infty} \sigma_k \langle x, e_k\rangle e_k.$$

To this end, let $x \in H$ and set

$$y_k := x - \sum_{i=1}^{k} \langle x, e_i\rangle e_i.$$

Then y_k is orthogonal to the first k eigenfunctions *i.e.* $y_k \in F_k^\perp$ with $F_k = \oplus_{1\leq i\leq k}\mathrm{Vect}(e_i)$, and, by the Plancherel theorem,

$$\|y_k\| \leq \|x\|.$$

This implies that

$$\|Ax - \sum_{i=1}^k \sigma_k\langle x, e_k\rangle e_k\| = \|Ay_k\| \leq \|y_k\|\cdot\|A\|_{\mathcal{L}(F_k^\perp)} = |\sigma_{k+1}|\cdot\|x\|.$$

Thus,

$$\|Ax - \sum_{i=1}^k \sigma_k\langle x, e_k\rangle e_k\| \underset{k\to\infty}{\to} 0.$$

Now, let λ be a non-zero eigenvalue of A and $x \in H\setminus\{0\}$ such that $Ax = \lambda x$. In particular,

$$\lambda x = \sum_{i=1}^\infty \lambda_k\langle x, e_k\rangle e_k = 0$$

as two eigenvectors associated with different eigenvalues are orthogonal. However, $\lambda x \neq 0$, a contradiction.

Finally, let us prove that $\{e_k\}_{k\in\mathbb{N}}$ is a Hilbert basis of H when A is injective. To this end, consider $F_\infty := \oplus_{i=1}^\infty \mathrm{Vect}(E_i)$, and let us show that $F_\infty^\perp = \{0\}$. Assume $F_\infty^\perp$ is not 0. Then $A \in \mathcal{L}(F_\infty)$, A is self-adjoint and compact, and so we can reiterate the previous construction to obtain yet another eigenvalue. □

4.1.3. Application to the Laplacian. We can now discuss Theorem 4.1. Of course, the main difficulty is that the Laplacian is certainly not compact. Much rather, we are going to work on the inverse of the Laplacian, that is, on the operator

$$(-\Delta)^{-1} : L^2(\Omega) \ni f \mapsto u_f \text{ solution of } \begin{cases} -\Delta u_f = f & \text{in } \Omega\,, \\ u_f \in W_0^{1,2}(\Omega). \end{cases}$$

Let us check that this operator satisfies all the assumptions of the spectral theorem:

(1) First of all, we need to check that it is symmetric. However, for any $f\,, g \in L^2(\Omega)$, we have

$$\begin{aligned}\int_\Omega u_f g &= \int_\Omega u_f(-\Delta u_g) \\ &= \int_\Omega u_g(-\Delta u_f) \\ &= \int_\Omega u_g f.\end{aligned}$$

(2) Second, we need to study its compactness. Observe that by the standard elliptic estimates we obtained before, we have, up to a multiplicative constant C,

$$\|u_f\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}.$$

By compactness of the Sobolev embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$, we deduce that the operator is compact.

Consequently, there exists a sequence $\{\sigma_k\}_{k\in\mathbb{N}}$ of eigenvalues converging to 0, and a sequence of normalised eigenfunctions $\{\varphi_k\}_{k\in\mathbb{N}}$ such that, for any $k\in\mathbb{N}$,

$$\varphi_k \in W^{1,2}_0(\Omega)\,,(-\Delta)^{-1}\varphi_k=\lambda_k\varphi_k.$$

Letting $\lambda_k := \frac{1}{\sigma_k}$, this gives an increasing sequence of eigenvalues such that

$$\forall k\in\mathbb{N}\,,\begin{cases}-\Delta\varphi_k=\sigma_k\varphi_k & \text{in }\Omega\,,\\ \varphi_k\in W^{1,2}_0(\Omega).\end{cases}$$

4.2. Spectral characterisation of $W^{k,2}(\mathbb{T}^d)$

4.3. The Fredholm alternative and solvability of second order elliptic equations

Another fundamental aspect of the theory of compact operators to elliptic PDEs is the Fredholm alternative. Un the finite dimensional case, the non-invertibility does not imply both non-injectivity and non-surjectivity. Some typical examples are the Bernoulli shift operators:

$$T_1:\ell^2(\mathbb{N})\ni u\mapsto (u_1,\dots,)\,,T_2:\ell^2(\mathbb{N})\ni u\mapsto(0,u_1,\dots).$$

The situation is different when the operators are assumed to be compact perturbations of the identity. This is the content of the Fredholm alternative, which reads as follows:

Theorem 4.3. *Let $(E,\|\cdot\|)$ be a reflexive Banach space and $T\in\mathcal{L}(E)$ be a compact operator. Then, for any $\mu\neq 0$, $\mu\mathrm{Id}-T$ is injective if, and only if, it is onto.*

This theorem does not require reflexivity of E but it simplifies one part of the proof.

Proof of Theorem 4.3. The idea of the proof is to argue by contradiction. Assume first that $\mu\mathrm{Id}-T=K_\mu$ is injective but not onto so that $\mathrm{Range}(K_\mu)$ is a proper subspace of E. Observe that this subspace is necessarily closed. Indeed, an easy contradiction argument shows that up to a multiplicative constant we have

$$\forall x\in E\,,\|K_\mu x\|\gtrsim\|x\|. \tag{30}$$

In particular, consider a sequence $\{K_\mu x_k\}_{k\in\mathbb{N}}$ that converges to some y_∞. Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ converges weakly to some x_∞. Writing $\mu x_k=K_\mu x_k-Tx_k$ and using the fact that $Tx_k\underset{k\to\infty}{\to}Tx_\infty$ we deduce that $x_k\underset{k\to\infty}{\to}x_\infty$, which concludes the proof. Thus, $\mathrm{Range}(K_\mu)$ is a closed proper subspace of E and K_μ is an isomorphism of E onto $\mathrm{Range}(K_\mu)$. Define $V_m:=\mathrm{Range}(K_\mu^{m+1})$, which is a strictly decreasing (as K_μ is an isomorphism from E to its range) sequence of closed subspaces. In particular, we can use the Riesz Lemma again (see Theorem 1.17) to find, for any $k\in\mathbb{N}$, some unit vector $y_k=K_\mu x_k\in V_k$ such that $\mathrm{dis}(y_k,V_k)\geq\frac{1}{2}$. Note that by (30) the sequence $\{x_k\}_{k\in\mathbb{N}}$ is bounded. Furthermore, for any $k\in\mathbb{N}$ and any $j\geq k+1$, we have $y_j\in V_{k+1}$ so that

$$Tx_k-Tx_j=K_\mu(x_k-x_j)-\mu x_j+\mu x_k\in\mu x_k+V_{k+1}$$

whence

$$\|Tx_k-Tx_j\|\geq\frac{\mu}{2}.$$

This contradicts the compactness of T. □

Let us see how to apply this result to the solvability of elliptic equations. We consider a differential operator $\mathcal{L}$ of the form

$$\mathcal{L} = -\nabla \cdot (A\nabla \cdot) + \langle b, \nabla \cdot \rangle + c\cdot$$

and we are interested in the solvability of the PDE

$$\begin{cases} \mathcal{L}u = f \in L^2(\Omega)\,, \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{31}$$

We assume that the matrix A is uniformly elliptic in the sense that there exists $\underline{a} > 0$ such that

$$\forall \xi \in \mathbb{R}^d\,, \underline{a}\|\xi\|^2 \leq \langle Ax, x\rangle$$

and, for the sake of simplicity, assume that $b\,, c \in \mathscr{C}^1(\overline{\Omega})$. The Fredholm alternative provides us with the following nice result: the (unique) solvability of (31) boils down to the study of the homogeneous equation

$$\mathcal{L}u = 0.$$

Although we present it for Dirichlet boundary conditions, we will see in the next paragraph how to extend it to other types of boundary conditions.

Theorem 4.4. *The following statements are equivalent:*

1. *For any $f \in L^2(\Omega)$, there exists a unique solution u to* (31)*.*
2. *The unique solution to $\mathcal{L}u = 0\,, u \in W_0^{1,2}(\Omega)$ is $u \equiv 0$.*

Proof of Theorem 4.4. Let us assume condition (2). Then, let us show that, for M large enough, the operator $\mathcal{L}_M = \mathcal{L} + M$ is invertible and has a compact resolvent. This is a simple consequence of the Lax-Milgram theorem: observe that, for any $u \in W_0^{1,2}(\Omega)$, we have, for any $\delta > 0$,

$$\begin{aligned} \langle \mathcal{L}_M u, u\rangle &= \int_\Omega \langle A\nabla u, \nabla u\rangle + \int_\Omega \langle b\nabla u, u\rangle + \int_\Omega cu^2 + M\int_\Omega u^2 \\ &\geq \underline{a}\int_\Omega |\nabla u|^2 - \frac{\|b\|_{L^\infty(\Omega)}\delta}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2\delta}\int_\Omega u^2 - \|c\|_{L^\infty(\Omega)}\int_\Omega u^2 + M\int_\Omega u^2 \\ &\gtrsim \int_\Omega |\nabla u|^2 + \int_\Omega u^2 \end{aligned}$$

where in the last line we first fixed $\delta > 0$ small enough and then M large enough. In particular, from similar arguments and the Lax-Milgram theorem, we deduce that for any $f \in L^2$ there exists $u_{M,f}$ such that $\mathcal{L}_M u_{M,f} = f$, and $\mathcal{L}_M$ has a compact inverse.

Now, how does this apply to the solvability of (31)? Observe that

$$\mathcal{L}u = f$$

if, and only if,

$$\mathcal{L}_M u = Mu + f$$

or, put differently,

$$(M\mathcal{L}_M^{-1} - \mathrm{Id})u = -\mathcal{L}_M^{-1} f.$$

Observe that the operator

$$\mathrm{Id} - M\mathcal{L}_M^{-1}$$

has a kernel reduced to $\{0\}$ as the homogeneous problem only has the trivial solution. By the Fredholm alternative, this operator is thus onto so that, for any g,

$$(M\mathcal{L}_M^{-1} - \mathrm{Id})u = g$$

has a solution u in $L^2(\Omega)$. This concludes the proof. □