The KP approximation under a weak Coriolis forcing

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Abstract

In this paper, we study the asymptotic behavior of weakly transverse water-waves under a weak Coriolis forcing in the long wave regime. We derive the Boussinesq-Coriolis equations in this setting and we provide a rigorous justification of this model. Then, from these equations, we derive two other asymptotic models. When the Coriolis forcing is weak, we fully justify the rotation-modified Kadomtsev-Petviashvili equation (also called Grimshaw-Melville equation). When the Coriolis forcing is very weak, we rigorously justify the Kadomtsev-Petviashvili equation. This work provides the first mathematical justification of the KP approximation under a Coriolis forcing.

1 Introduction

We consider the motion of an inviscid, incompressible fluid under the influence of the gravity $\boldsymbol{g} = -g\boldsymbol{e_z}$ and the rotation of the Earth with a rotation vector $\mathbf{f} = \frac{f}{2}\boldsymbol{e_z}$. We assume that the fluid has a constant density ρ and that no surface tension is involved. We assume that the surface is a graph above the still water level and that the seabed is flat. We denote by $X = (x, y) \in \mathbb{R}^2$ the horizontal variable and by $z \in \mathbb{R}$ the vertical variable. The fluid occupies the domain $\Omega_t := \{(X, z) \in \mathbb{R}^3 \,,\; -H < z < \zeta(t, X)\}$. We denote by $\mathbf{U} = (\mathbf{V}, \mathrm{w})^t$ the velocity in the fluid. Notice that $\mathbf{V}$ is the horizontal component of $\mathbf{U}$ and w its vertical component. Finally, we assume that the pressure $\mathcal{P}$ is constant at the surface of the fluid. The equations governing such a fluid are the free surface Euler-Coriolis equations[(1)]

$$\begin{cases} \partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_{X,z})\,\mathbf{U} + \mathbf{f} \times \mathbf{U} = -\dfrac{1}{\rho}\nabla_{X,z}\mathcal{P} - g\boldsymbol{e_z} \text{ in } \Omega_t, \\ \mathrm{div}\,\mathbf{U} = 0 \text{ in } \Omega_t, \end{cases} \tag{1}$$

with the boundary conditions

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[1]The centrifugal potential is assumed to be constant and included in the pressure term.

$$\begin{cases} \mathcal{P}_{|z=\zeta} = P_0, \\ \partial_t \zeta - \underline{\mathbf{U}} \cdot \mathbf{N} = 0, \\ \mathrm{w}_b = 0, \end{cases}$$

where P_0 is constant, $\mathbf{N} = \begin{pmatrix} -\nabla\zeta \\ 1 \end{pmatrix}$, $\underline{\mathbf{U}} = \begin{pmatrix} \underline{\mathbf{V}} \\ \underline{\mathrm{w}} \end{pmatrix} = \mathbf{U}_{|z=\zeta}$ and $\mathbf{U}_b = \begin{pmatrix} \mathbf{V}_b \\ \mathrm{w}_b \end{pmatrix} = \mathbf{U}_{|z=-H}$.

In this work, we do not directly work on the free surface Euler-Coriolis equations. We rather consider another formulation called the Castro-Lannes formulation (see [4]). This formulation generalizes the well-known Zakharov/Craig-Sulem formulation ([22, 6]) to a fluid with a rotational component. In [4], Castro and Lannes shown that we can express the free surface Euler equations thanks to the unknowns $\left(\zeta, \mathbf{U}_{/\!/}, \boldsymbol{\omega}\right)$[2] where $\boldsymbol{\omega} = \mathrm{Curl}\ \mathbf{U}$ is the vorticity of the fluid and

$$\mathbf{U}_{/\!/} = \underline{\mathbf{V}} + \underline{\mathrm{w}} \nabla\zeta.$$

Then, they provide a system of three equations on these unknowns. In [15], a similar work has been done to take into account the Coriolis forcing. It leads to the following system, called the Castro-Lannes system or the water waves equations with vorticity,

$$\begin{cases} \partial_t \zeta - \underline{\mathbf{U}} \cdot \mathbf{N} = 0, \\ \partial_t \mathbf{U}_{/\!/} + \nabla\zeta + \frac{1}{2}\nabla\left|\mathbf{U}_{/\!/}\right|^2 - \frac{1}{2}\nabla\left[\left(1 + |\nabla\zeta|^2\right)\underline{\mathrm{w}}^2\right] + \left(\nabla^\perp \cdot \mathbf{U}_{/\!/}\right)\underline{\mathbf{V}}^\perp + f\underline{\mathbf{V}}^\perp = 0, \\ \partial_t \boldsymbol{\omega} + \left(\mathbf{U} \cdot \nabla_{X,z}\right)\boldsymbol{\omega} = \left(\boldsymbol{\omega} \cdot \nabla_{X,z}\right)\mathbf{U} + f\partial_z \mathbf{U}, \end{cases} \tag{2}$$

where $\mathbf{U} = \begin{pmatrix} \mathbf{V} \\ \mathrm{w} \end{pmatrix} = \mathbf{U}[\zeta](\mathbf{U}_{/\!/}, \boldsymbol{\omega})$ is the unique solution in $H^1(\Omega_t)$ of the following Div-Curl equation

$$\begin{cases} \mathrm{curl}\ \mathbf{U} = \boldsymbol{\omega} \text{ in } \Omega_t, \\ \mathrm{div}\ \mathbf{U} = 0 \text{ in } \Omega_t, \\ \left(\underline{\mathbf{V}} + \underline{\mathrm{w}}\nabla\zeta\right)_{|z=\zeta} = \mathbf{U}_{/\!/}, \\ \mathrm{w}_b = 0. \end{cases}$$

The main goal of this paper is to study weakly transverse long waves. Therefore, we consider a nondimensionalization of the previous equations. Five physical parameters are involved in this work : the typical amplitude of the surface a, the typical longitudinal scale L_x, the typical transverse scale L_y, the characteristic water depth H and the typical Coriolis frequency f. We introduce four dimensionless parameters

$$\mu = \frac{H^2}{L_x^2},\ \varepsilon = \frac{a}{H},\ \mathrm{Ro} = \frac{a\sqrt{gH}}{HfL_x} \text{ and } \gamma = \frac{L_x}{L_y}.$$

[2]Notice that Castro and Lannes used the unknowns $\left(\zeta, \frac{\nabla}{\Delta} \cdot \mathbf{U}_{/\!/}, \boldsymbol{\omega}\right)$. However, as noticed in [16], the unknowns $\left(\zeta, \mathbf{U}_{/\!/}, \boldsymbol{\omega}\right)$ are better to derive shallow water asymptotic models.

The parameter μ is called the shallowness parameter. The parameter ε is called the nonlinearity parameter. The parameter Ro is the Rossby number and finally the parameter γ is called the transversality parameter. Then, we can nondimensionalize the Euler equations (1) and the Castro-Lannes equations (2) (see Part 1.2). In this work, we study the following asymptotic regime

$$\mathcal{A}_{\text{boussi}} = \left\{ (\mu, \varepsilon, \gamma, \text{Ro}) \,, 0 \leq \mu \leq \mu_0, \varepsilon = \mathcal{O}\left(\mu\right), \gamma \leq 1, \frac{\varepsilon}{\text{Ro}} = \mathcal{O}(\sqrt{\mu}) \right\},$$

This regime corresponds to a long wave regime ($\varepsilon = \mathcal{O}(\mu)$) under a weak Coriolis forcing $\frac{\varepsilon}{\text{Ro}} = \mathcal{O}(\sqrt{\mu})$. For an explanation of the first assumption, we refer to [12]. The second assumption is standard in oceanography. Rewriting $\frac{\varepsilon}{\text{Ro}} = \frac{f L_x}{\sqrt{gH}}$, this assumption means that the rotation period is assumed to be much smaller than the time scale of the waves. We refer to [9, 7] for more explanations about this assumption (see also [10, 17, 8, 14]).

We organize this paper in four parts. In Section 1.2, we explain how we nondimensionalize the equations and we provide a local wellposedness result. In Section 2, we derive and justify the Boussinesq-Coriolis equations in the asymptotic regime $\mathcal{A}_{\text{boussi}}$. The Boussinesq-Coriolis equations are a system of three equations on the surface ζ and the vertical average of the horizontal velocity denoted $\overline{\mathbf{V}}$ (defined in (9)). They correspond to a $\mathcal{O}(\mu^2)$ approximation of the water waves equations. These equations are

$$\begin{cases} \partial_t \zeta + \nabla^\gamma \cdot \left([1 + \varepsilon\zeta] \, \overline{\mathbf{V}} \right) = 0, \\ \left(1 - \frac{\mu}{3} \nabla^\gamma \nabla^\gamma \cdot \right) \partial_t \overline{\mathbf{V}} + \nabla^\gamma \zeta + \varepsilon \overline{\mathbf{V}} \cdot \nabla^\gamma \overline{\mathbf{V}} + \frac{\varepsilon}{\text{Ro}} \overline{\mathbf{V}}^\perp = 0. \end{cases} \tag{3}$$

Then, in Section 3, we study the KP approximation which corresponds to the asymptotic regime $\mathcal{A}_{\text{boussi}}$ with $\varepsilon = \mu$ and $\gamma = \sqrt{\mu}$. This second assumption corresponds to weakly transverse effects (see for instance [12]). In this regime, we derive two other asymptotic models. When the Coriolis forcing is weak $\left(\frac{\varepsilon}{\text{Ro}} = \sqrt{\mu}\right)$, we rigorously justify the modified-rotation Kadomtsev-Petviashvili equation (see Subsection 3.1), also called Grimshaw-Melville equation in the physics literature,

$$\partial_\xi \left(\partial_\tau k + \frac{3}{2} k \partial_\xi k + \frac{1}{6} \partial_\xi^3 k \right) + \frac{1}{2} \partial_{yy} k = \frac{1}{2} k.$$

Then, when the Coriolis forcing is very weak $\left(\frac{\varepsilon}{\text{Ro}} = \mu\right)$, we fully justify the KP equation (see Subsection 3.2)

$$\partial_\xi \left(\partial_\tau k + \frac{3}{2} k \partial_\xi k + \frac{1}{6} \partial_\xi^3 k \right) + \frac{1}{2} \partial_{yy} k = 0.$$

Finally, in Section 4, we compare the scalar asymptotic models we derive in Section 3 with the ones derived in [16] : the Ostrovsky equation and the KdV equation.

1.1 Notations/Definitions

- If $\mathbf{A} \in \mathbb{R}^3$, we denote by $\mathbf{A}_h$ its horizontal component.

- If $\mathbf{V} = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$, we define the orthogonal of $\mathbf{V}$ by $\mathbf{V}^\perp = \begin{pmatrix} -v \\ u \end{pmatrix}$.

- In this paper, $C(\cdot)$ is a nondecreasing and positive function whose exact value has no importance.

- Consider a vector field $\mathbf{A}$ or a function w defined on Ω. Then, we denote $\underline{\mathbf{A}} = \mathbf{A}_{|z=\varepsilon\zeta}$, $\underline{\mathrm{w}} = \mathrm{w}_{|z=\varepsilon\zeta}$ and $\mathbf{A}_b = \mathbf{A}_{|z=-1}$, $\mathrm{w}_b = \mathrm{w}_{|z=-1}$.

- If $N \in \mathbb{N}$ and f is a function on $\mathbb{R}^2$, $|f|_{H^N}$ is its H^N-norm, $|f|_2$ is its L^2-norm and $|f|_{L^\infty}$ its L^∞-norm. We denote by $(\ ,\ )_2$ the $L^2(\mathbb{R}^2)$ inner product.

- If f is a function defined on $\mathbb{R}^2$, We use the notation $\nabla^\gamma f = (\partial_x f, \gamma \partial_y f)^t$.

- If $u = u(X, z)$ is defined in Ω, we define

$$\overline{u}(X) = \frac{1}{1+\varepsilon\zeta} \int_{-1}^{\varepsilon\zeta(X)} u(X,z)dz \text{ and } u^* = u - \overline{u}.$$

- For $N \geq 0$, we define the Hilbert spaces $\partial_x H^N(\mathbb{R}^2)$

$$\partial_x H^N(\mathbb{R}^2) = \left\{ k \in H^{N-1}(\mathbb{R}^2),\, k = \partial_x \tilde{k} \text{ with } \tilde{k} \in H^N(\mathbb{R}^2) \right\}. \tag{4}$$

The function $\tilde{k}$ is denoted $\partial_x^{-1} k$ in the following.

- Similarly, for $N \geq 0$, we define the Hilbert spaces $\partial_x^2 H^N(\mathbb{R}^2)$.

- In the following definition, we recall the notion of consistence (see for instance [12]).

Definition 1.1. *We say that the Castro-Lannes equations* (7) *are consistent of order* $\mathcal{O}(\mu^2)$ *with a system of equations* S *for* ζ *and* $\overline{\mathbf{V}}$ *if for any smooth solutions* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)$ *of the Castro-Lannes equations* (7)*, the pair* $\left(\zeta, \overline{\mathbf{V}}[\varepsilon\zeta]\left(\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)\right)$ (defined in (9)) *solves* S *up to a residual of order* $\mathcal{O}(\mu^2)$*.*

1.2 Nondimensionalization

We recall the four dimensionless parameters

$$\mu = \frac{H^2}{L_x^2},\ \varepsilon = \frac{a}{H},\ \mathrm{Ro} = \frac{a\sqrt{gH}}{HfL_x} \text{ and } \gamma = \frac{L_x}{L_y}. \tag{5}$$

We nondimensionalize the variables and the unknowns. We introduce (see [12] or [15])

$$\begin{cases} x' = \dfrac{x}{L_x},\ y' = \dfrac{y}{L_y},\ z' = \dfrac{z}{H},\ \zeta' = \dfrac{\zeta}{a},\ t' = \dfrac{\sqrt{gH}}{L_x} t, \\ \mathbf{V}' = \sqrt{\dfrac{H}{g}} \dfrac{\mathbf{V}}{a},\ \mathrm{w}' = H \sqrt{\dfrac{H}{g}} \dfrac{\mathrm{w}}{a L_x} \text{ and } \mathcal{P}' = \dfrac{\mathcal{P}}{\rho g H}. \end{cases}$$

In the following, we use the following notations

$$\nabla^\gamma = \nabla^\gamma_{X'} = \begin{pmatrix} \partial_{x'} \\ \gamma \partial_{y'} \end{pmatrix},\ \nabla^{\mu,\gamma}_{X',z'} = \begin{pmatrix} \sqrt{\mu} \nabla^\gamma_{X'} \\ \partial_{z'} \end{pmatrix},\ \mathrm{curl}^{\mu,\gamma} = \nabla^{\mu,\gamma}_{X',z'} \times,\ \mathrm{div}^{\mu,\gamma} = \nabla^{\mu,\gamma}_{X',z'} \cdot .$$

We also define

$$\mathbf{U}^\mu = \begin{pmatrix} \sqrt{\mu} \mathbf{V}' \\ \mathrm{w}' \end{pmatrix},\ \boldsymbol{\omega}' = \frac{1}{\mu} \mathrm{curl}^{\mu,\gamma} \mathbf{U}^\mu, \tag{6}$$

and

$$\underline{\mathbf{U}}^\mu = \begin{pmatrix} \sqrt{\mu} \underline{\mathbf{V}}' \\ \underline{\mathrm{w}}' \end{pmatrix} = \mathbf{U}^\mu_{|z'=\varepsilon\zeta'},\ \mathbf{U}^\mu_b = \mathbf{U}^\mu_{|z'=-1},\ \mathbf{N}^{\mu,\gamma} = \begin{pmatrix} -\varepsilon \sqrt{\mu} \nabla^\gamma \zeta' \\ 1 \end{pmatrix}.$$

Remark 1.2. *Notice that the nondimensionalization of the vorticity presented in* (6) *corresponds to weakly sheared flows (see [3], [20], [18]).*

The nondimensionalized fluid domain is

$$\Omega'_{t'} := \{(X', z') \in \mathbb{R}^3\ ,\ -1 < z' < \varepsilon \zeta'(t', X')\}.$$

Finally, the Euler-Coriolis equations (1) become

$$\begin{cases} \partial_{t'} \mathbf{U}^\mu + \dfrac{\varepsilon}{\mu} \left(\mathbf{U}^\mu \cdot \nabla^{\mu,\gamma}_{X',z'} \right) \mathbf{U}^\mu + \dfrac{\varepsilon \sqrt{\mu}}{\mathrm{Ro}} \begin{pmatrix} \mathbf{V}'^\perp \\ 0 \end{pmatrix} = -\dfrac{1}{\varepsilon} \nabla^{\mu,\gamma}_{X',z'} \mathcal{P}' - \dfrac{1}{\varepsilon} \boldsymbol{e_z} \text{ in } \Omega'_t, \\ \mathrm{div}^{\mu,\gamma}_{X',z'}\ \mathbf{U}^\mu = 0 \text{ in } \Omega'_t, \end{cases}$$

with the boundary conditions

$$\begin{cases} \partial_{t'} \zeta' - \dfrac{1}{\mu} \underline{\mathbf{U}}^\mu \cdot \mathbf{N}^{\mu,\gamma} = 0, \\ \mathrm{w}'_b = 0. \end{cases}$$

In the following, we omit the primes. We can proceed similarly to nondimensionalize the Castro-Lannes formulation. We define the quantity

$$\mathbf{U}^{\mu,\gamma}_{/\!/} = \underline{\mathbf{V}} + \varepsilon \underline{\mathrm{w}} \nabla^\gamma \zeta.$$

Then, the Castro-Lannes formulation becomes (see [4] or [15] when $\gamma = 1$),

$$\begin{cases}\partial_t\zeta - \frac{1}{\mu}\underline{\mathbf{U}}^{\mu}\cdot\mathbf{N}^{\mu,\gamma} = 0,\\ \partial_t\mathbf{U}_{/\!/}^{\mu,\gamma}+\nabla^{\gamma}\zeta+\frac{\varepsilon}{2}\nabla^{\gamma}\left|\mathbf{U}_{/\!/}^{\mu,\gamma}\right|^2-\frac{\varepsilon}{2\mu}\nabla^{\gamma}\Big[\left(1+\varepsilon^2\mu\,|\nabla^{\gamma}\zeta|^2\right)\underline{\mathrm{w}}^2\Big]+\varepsilon\left(\nabla^{\perp}\cdot\mathbf{U}_{/\!/}^{\mu,\gamma}\right)\underline{\mathbf{V}}^{\perp}+\frac{\varepsilon}{\mathrm{Ro}}\underline{\mathbf{V}}^{\perp}=0,\\ \partial_t\boldsymbol{\omega}+\frac{\varepsilon}{\mu}\left(\mathbf{U}^{\mu}\cdot\nabla_{X,z}^{\mu,\gamma}\right)\boldsymbol{\omega}=\frac{\varepsilon}{\mu}\left(\boldsymbol{\omega}\cdot\nabla_{X,z}^{\mu,\gamma}\right)\mathbf{U}^{\mu}+\frac{\varepsilon}{\mu\mathrm{Ro}}\partial_z\mathbf{U}^{\mu},\end{cases} \tag{7}$$

where $\mathbf{U}^{\mu} = \begin{pmatrix}\sqrt{\mu}\mathbf{V}\\ \mathrm{w}\end{pmatrix} = \mathbf{U}^{\mu}[\varepsilon\zeta](\mathbf{U}_{/\!/}^{\mu,\gamma},\boldsymbol{\omega})$ is the unique solution in $H^1(\Omega_t)$ of

$$\begin{cases}\mathrm{curl}^{\mu,\gamma}\;\mathbf{U}^{\mu} = \mu\boldsymbol{\omega} \text{ in } \Omega_t,\\ \mathrm{div}^{\mu,\gamma}\;\mathbf{U}^{\mu} = 0 \text{ in } \Omega_t,\\ (\underline{\mathbf{V}}+\varepsilon\underline{\mathrm{w}}\nabla^{\gamma}\zeta)_{|z=\varepsilon\zeta} = \mathbf{U}_{/\!/}^{\mu,\gamma},\\ \mathrm{w}_b = 0.\end{cases}$$

In order to rigorously derive asymptotic models, we need an existence result for the Castro-Lannes formulation (7). We recall that the existence of solutions to the water waves equations is always obtained under the so-called Rayleigh-Taylor condition that assumes the positivity of the Rayleigh-Taylor coefficient $\mathfrak{a}$ (see Part 3.4.5 in [12] for the link between $\mathfrak{a}$ and the Rayleigh-Taylor condition or [15]) where

$$\mathfrak{a} := \mathfrak{a}[\varepsilon\zeta](\mathbf{U}_{/\!/}^{\mu,\gamma},\boldsymbol{\omega}) = 1+\varepsilon\left(\partial_t+\varepsilon\underline{\mathbf{V}}[\varepsilon\zeta](\mathbf{U}_{/\!/}^{\mu,\gamma},\boldsymbol{\omega})\cdot\nabla\right)\underline{\mathrm{w}}[\varepsilon\zeta](\mathbf{U}_{/\!/}^{\mu,\gamma},\boldsymbol{\omega}).$$

We explain in [15] how we can define the Rayleigh-Taylor coefficient $\mathfrak{a}$ at $t=0$. We also assume that the water depth is bounded from below by a positive constant

$$\exists\, h_{\min} > 0\;,\; 1+\varepsilon\zeta \geq h_{\min}.$$

The following theorem can be found in [15] and provide a local wellposedness result of the Castro-Lannes formulation (7) (see also Theorem 1.5 in [16]).

Theorem 1.3. *Let* $A > 0$ and $\mathbf{N} \geq 5$. We suppose that $(\mu,\varepsilon,\gamma,\mathrm{Ro}) \in \mathcal{A}_{\mathrm{boussi}}$. *We assume that*

$$\left(\zeta_0,(\mathbf{U}_{/\!/}^{\mu,\gamma})_0,\boldsymbol{\omega}_0\right) \in H^{N+\frac{1}{2}}(\mathbb{R}^2)\times H^N(\mathbb{R}^2)\times H^{N-1}(\Omega_0),$$

with $\nabla^{\mu,\gamma}\cdot\boldsymbol{\omega}_0 = 0$ *and* $\nabla^{\gamma\perp}\cdot(\mathbf{U}_{/\!/}^{\mu,\gamma})_0 = \underline{\boldsymbol{\omega}_0}\cdot\begin{pmatrix}-\varepsilon\sqrt{\mu}\nabla^{\gamma}\zeta_0\\ 1\end{pmatrix}$. *Finally, we assume that*

$$\exists\, h_{\min},\, \mathfrak{a}_{\min} > 0\;,\; 1+\varepsilon\zeta_0 \geq h_{\min} \text{ and } \mathfrak{a}[\varepsilon\zeta_0]((\mathbf{U}_{/\!/}^{\mu,\gamma})_0,\boldsymbol{\omega}_0) \geq \mathfrak{a}_{\min},$$

and that

$$\left|\zeta_0\right|_{H^{N+\frac{1}{2}}} + \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}(\mathbf{U}_{/\!/}^{\mu,\gamma})_0\right|_{H^N} + ||\boldsymbol{\omega}_0||_{H^{N-1}} \leq A.$$

Then, there exists $T > 0$ and a unique classical solution $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)$ to the Castro-Lannes (7) on $[0,T]$ with initial data $\left(\zeta_0, (\mathbf{U}_{/\!/}^{\mu,\gamma})_0, \boldsymbol{\omega}_0\right)$. Moreover,

$$T = \frac{T_0}{\max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right)} \ , \ \frac{1}{T_0} = c^1,$$

$$\max_{[0,T]} \left(|\zeta(t,\cdot)|_{H^N} + \left| \frac{1}{\sqrt{1+\sqrt{\mu}|D|}} \mathbf{U}_{/\!/}^{\mu,\gamma}(t,\cdot) \right|_{H^{N-\frac{1}{2}}} + ||\boldsymbol{\omega}(t,\cdot)||_{H^{N-1}} \right) = c^2,$$

with $c^j = C\left(A, \mu_0, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}\right)$.

Remark 1.4. *Notice that thanks to Theorem 1.3 together with Part 5.5.1 in [4], the quantities* ζ, $\mathbf{U}_{/\!/}^{\mu,\gamma}$, $\boldsymbol{\omega}$, $\overline{\mathbf{V}}$, $\mathbf{U}$, $\partial_t\zeta$, $\partial_t\mathbf{U}_{/\!/}^{\mu,\gamma}$, $\partial_t\boldsymbol{\omega}$ and $\partial_t\mathbf{U}$ *remain bounded uniformly with respect to the small parameters during the time evolution of the flow.*

2 The Boussinesq-Coriolis equations

In this part, we derive and fully justify the Boussinesq-Coriolis equations (3) under a weak Coriolis forcing $\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}\left(\sqrt{\mu}\right)$. We recall the corresponding asymptotic regime

$$\mathcal{A}_{\mathrm{boussi}} = \left\{ (\mu, \varepsilon, \gamma, \mathrm{Ro}) \,, 0 \leq \mu \leq \mu_0, \varepsilon = \mathcal{O}(\mu), \gamma \leq 1, \frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}(\sqrt{\mu}) \right\}. \tag{8}$$

Notice that no assumption on γ is made in this part. The Boussinesq equations correspond to an order $\mathcal{O}(\mu^2)$ approximation of the water waves equations. Motivated by [16], we use the Castro-Lannes equations (7) to derive this asymptotic model. We introduce the water depth

$$h(t,X) = 1 + \varepsilon\zeta(t,X),$$

and the vertical average of the horizontal velocity

$$\overline{\mathbf{V}} = \overline{\mathbf{V}}[\varepsilon\zeta](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega})(t,X) = \frac{1}{h(t,X)} \int_{z=-1}^{\varepsilon\zeta(t,X)} \mathbf{V}[\varepsilon\zeta, \beta b](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega})(t,X,z)dz. \tag{9}$$

In the following we denote $\mathbf{V} = (u,v)^t$. More generally, if u is a function defined in Ω, we denote by $\overline{u}$ its vertical average and $u^* = u - \overline{u}$. We also have to introduce the "shear" velocity

$$\mathbf{V}_{\mathrm{sh}} = \mathbf{V}_{\mathrm{sh}}[\varepsilon\zeta](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega})(t,X) = \int_z^{\varepsilon\zeta} \boldsymbol{\omega}_h^{\perp}(t,X,z')dz'$$

and its vertical average

$$\mathbf{Q} = \overline{\mathbf{V}_{\mathrm{sh}}} = \frac{1}{h}\int_{-1}^{\varepsilon\zeta}\int_{z'}^{\varepsilon\zeta} \boldsymbol{\omega}_h^{\perp}.$$

As noticed in [4], these quantities appear when one wants to obtain an expansion with respect to μ of the velocity. We recall that

$$\mathbf{U}_{/\!/}^{\mu,\gamma} = \underline{\mathbf{V}} + \varepsilon \underline{\mathrm{w}} \nabla^{\gamma}\zeta.$$

2.1 Asymptotic expansions with respect to μ

In this part, we give an expansion of different quantities with respect to μ. These expansions will help us to derive the Boussinesq-Coriolis equations (3) in Section 2.2. The following proposition gives a link between $\overline{\mathbf{V}}$ and $\underline{\mathbf{U}}^{\mu} \cdot \mathbf{N}^{\mu,\gamma}$ (the proof is a small adaptation of Proposition 4.2 in [15]).

Proposition 2.1. *If* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (7)*, we have*

$$\underline{\mathbf{U}}^{\mu} \cdot \mathbf{N}^{\mu,\gamma} = -\mu \nabla^{\gamma} \cdot \left(h\overline{\mathbf{V}}\right).$$

Then we get the first equation of the Boussinesq-Coriolis system from the first equation of (7). We also need an expansion of $\mathbf{V}$ and w with respect to μ. We introduce the following operators

$$T\left[\varepsilon\zeta\right] f = \int_{z}^{\varepsilon\zeta} \nabla^{\gamma}\nabla^{\gamma} \cdot \int_{-1}^{z'} f \text{ and } T^{*}\left[\varepsilon\zeta\right] f = \left(T\left[\varepsilon\zeta\right] f\right)^{*},$$

In the following, we denote $T = T\left[\varepsilon\zeta\right]$ and $T^{*} = T^{*}\left[\varepsilon\zeta\right]$ when no confusion is possible.

Proposition 2.2. *In the Boussinesq regime $\mathcal{A}_{boussi}$, if* $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (7)*, we have*

$$\begin{aligned}
\mathbf{V} &= \overline{\mathbf{V}} + \sqrt{\mu}\mathbf{V}_{\mathrm{sh}}^{*} + \mu T^{*}\overline{\mathbf{V}} + \mu^{\frac{3}{2}} T^{*}\mathbf{V}_{\mathrm{sh}}^{*} + \mathcal{O}\left(\mu^{2}\right), \\
\underline{\mathbf{V}} &= \overline{\mathbf{V}} - \sqrt{\mu}\mathbf{Q} + \mu \underline{T^{*}\overline{\mathbf{V}}} - \mu^{\frac{3}{2}} \overline{T\mathbf{V}_{\mathrm{sh}}^{*}} + \mathcal{O}\left(\mu^{2}\right),
\end{aligned}$$

where

$$T^{*}\overline{\mathbf{V}} = \frac{1}{2}\left(\frac{h^{2}}{3} - [z+1]^{2}\right)\nabla^{\gamma}\nabla^{\gamma} \cdot \overline{\mathbf{V}} \text{ and } \underline{T^{*}\overline{\mathbf{V}}} = -\frac{h^{2}}{3}\nabla^{\gamma}\nabla^{\gamma} \cdot \overline{\mathbf{V}}.$$

We also have

$$\begin{aligned}
\mathrm{w} &= -\mu(z+1)\nabla^{\gamma}\overline{\mathbf{V}} + \mu^{\frac{3}{2}} \int_{-1}^{z} \nabla^{\gamma} \cdot \mathbf{V}_{\mathrm{sh}}^{*} + \mathcal{O}\left(\mu^{2}\right), \\
\underline{\mathrm{w}} &= -\mu h \nabla^{\gamma} \cdot \overline{\mathbf{V}} + \mathcal{O}\left(\mu^{2}\right),
\end{aligned}$$

Proof. This proof is an adaptation of part 2.2 in [3], Part 4.2 in [15] and Section 2.1 in [16]. First, using $\mathrm{curl}^{\mu,\gamma}\,\mathbf{U}^{\mu} = \mu\boldsymbol{\omega}$, we obtain that

$$\sqrt{\mu}\boldsymbol{\omega}_h = \partial_z \mathbf{V}^{\perp} - \nabla^{\gamma\perp}\mathrm{w}.$$

Then, we consider the ansatz $\mathbf{V} = \overline{\mathbf{V}} + \sqrt{\mu}\mathbf{V}_1$. By integrating the previous equation, we obtain

$$\sqrt{\mu}\partial_z \mathbf{V}_1 = -\sqrt{\mu}\boldsymbol{\omega}_h^{\perp} + \nabla^{\gamma\perp}\mathrm{w}.$$

Since $\overline{\mathbf{V}_1} = 0$, we get

$$\mathbf{V}_1 = \left(\int_z^{\varepsilon\zeta} \boldsymbol{\omega}_h^{\perp}\right)^* - \frac{1}{\sqrt{\mu}}\left(\int_z^{\varepsilon\zeta} \nabla^{\gamma}\mathrm{w}\right)^*.$$

Secondly, using Proposition 2.1 and the divergence-free assumption, we get

$$\mathrm{w} = -\mu\nabla^{\gamma}\cdot\left(\int_{-1}^{z}\mathbf{V}\right). \tag{10}$$

Then, gathering the previous two equality, we obtain

$$\mathbf{V} = \overline{\mathbf{V}} + \sqrt{\mu}\mathbf{V}_{\mathrm{sh}}^* + \mu T^*\mathbf{V}. \tag{11}$$

Finally, the expansion of $\mathbf{V}$ follows by applying the operator $Id + \mu T^*$ to the previous equality. For the second equality, we notice that $\underline{T^*\mathbf{V}_{\mathrm{sh}}^*} = -\overline{T\mathbf{V}_{\mathrm{sh}}^*}$. The third and fourth equalities follow from the fact that $\overline{\mathbf{V}}$ does not depend on z. The fifth equality are a consequence of Equalities (10) and (11). Finally, the sixth equality follows from the fact that $\overline{\mathbf{V}_{\mathrm{sh}}^*} = 0$ and that $\varepsilon = \mathcal{O}(\mu)$. □

We can also get an expansion of $\partial_t\mathbf{V}$ and $\partial_t\mathrm{w}$.

Proposition 2.3. *In the Boussinesq regime $\mathcal{A}_{boussi}$, if $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)$ satisfy the Castro-Lannes system* (7)*, we have*

$$\begin{aligned}
&\partial_t\left(\mathbf{V} - \overline{\mathbf{V}} - \sqrt{\mu}\mathbf{V}_{\mathrm{sh}}^* - \mu T^*\overline{\mathbf{V}} - \mu^{\frac{3}{2}}T^*\mathbf{V}_{\mathrm{sh}}^*\right) = \mathcal{O}\left(\mu^2\right),\\
&\partial_t\left(\underline{\mathbf{V}} - \overline{\mathbf{V}} + \sqrt{\mu}\mathbf{Q} - \mu\underline{T^*\overline{\mathbf{V}}} + \mu^{\frac{3}{2}}\overline{T\mathbf{V}_{\mathrm{sh}}^*}\right) = \mathcal{O}\left(\mu^2\right),\\
&\partial_t\left(\underline{\mathrm{w}} + \mu h\nabla^{\gamma}\overline{\mathbf{V}}\right) = \mathcal{O}\left(\mu^2\right).
\end{aligned}$$

Proof. The result follows from Proposition 2.1 and the equality

$$\mathbf{V} = (1 - \mu T^*)\left(\overline{\mathbf{V}} + \sqrt{\mu}\mathbf{V}_{\mathrm{sh}}^*\right) + \mu^2 T^*T^*\mathbf{V}.$$

□

Since we can not express Q and $\mathbf{V}_{\mathrm{sh}}^*$ with respect to ζ and $\overline{\mathbf{V}}$, we need an evolution equation at order $\mathcal{O}\left(\mu^{\frac{3}{2}}\right)$ of these quantities.

Proposition 2.4. *In the Boussinesq regime $\mathcal{A}_{boussi}$, if $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)$ satisfy the Castro-Lannes system (7), then Q satisfies the following equation*

$$\partial_t \mathbf{Q} + \varepsilon \overline{\mathbf{V}} \cdot \nabla^\gamma \mathbf{Q} + \varepsilon \mathbf{Q} \cdot \nabla^\gamma \overline{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\overline{\mathbf{V}} - \underline{\mathbf{V}} \right)^\perp = \mathcal{O}\left(\mu^{\frac{3}{2}}\right),$$

and $\boldsymbol{V}_{sh}^*$ satisfies the equation

$$\partial_t \mathbf{V}_{\mathrm{sh}}^* + \varepsilon \overline{\mathbf{V}} \cdot \nabla^\gamma \mathbf{V}_{\mathrm{sh}}^* + \varepsilon \mathbf{V}_{\mathrm{sh}}^* \cdot \nabla^\gamma \overline{\mathbf{V}} - \varepsilon \left[1 + z\right] \left(\nabla^\gamma \cdot \overline{\mathbf{V}} \right) \partial_z \mathbf{V}_{\mathrm{sh}}^* + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\mathbf{V} - \overline{\mathbf{V}} \right)^\perp = \mathcal{O}\left(\mu^{\frac{3}{2}}\right).$$

Proof. This proof is an adaptation of Part 2.3 in [3] and Part 2.2 in [16]. Thanks to the horizontal component of the vorticity equation of the Castro-Lannes formulation (7), we get

$$\partial_t \boldsymbol{\omega}_h + \varepsilon \mathbf{V} \cdot \nabla^\gamma \boldsymbol{\omega}_h + \frac{\varepsilon}{\mu} \mathrm{w} \partial_z \boldsymbol{\omega}_h = \varepsilon \boldsymbol{\omega}_h \cdot \nabla^\gamma \mathbf{V} + \frac{\varepsilon}{\sqrt{\mu}} \boldsymbol{\omega}_z \partial_z \mathbf{V} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \partial_z \mathbf{V}.$$

Furthermore, since $\mathrm{curl}^{\mu,\gamma}\, \mathbf{U}^\mu = \mu \boldsymbol{\omega}$, we have

$$\partial_z \mathbf{V} = -\sqrt{\mu} \boldsymbol{\omega}_h^\perp + \mathcal{O}(\mu) \text{ and } \boldsymbol{\omega}_z = \nabla^{\gamma\perp} \cdot \overline{\mathbf{V}} + \mathcal{O}\left(\sqrt{\mu}\right).$$

Then, using Proposition 2.2, we obtain

$$\partial_t \boldsymbol{\omega}_h + \varepsilon \overline{\mathbf{V}} \cdot \nabla^\gamma \boldsymbol{\omega}_h - \varepsilon \left[1 + z\right] \left(\nabla^\gamma \cdot \overline{\mathbf{V}} \right) \partial_z \boldsymbol{\omega}_h - \varepsilon \boldsymbol{\omega}_h \cdot \nabla^\gamma \overline{\mathbf{V}} - \varepsilon \left(\nabla^{\gamma\perp} \overline{\mathbf{V}} \right) \boldsymbol{\omega}_h^\perp - \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \partial_z \mathbf{V} = \mathcal{O}\left(\mu^{\frac{3}{2}}\right),$$

Then, integrating with respect to z, using the fact that $\partial_t \zeta + \nabla^\gamma \cdot \left(h \overline{\mathbf{V}} \right) = 0$, $\mathbf{V}_{\mathrm{sh}} = \int_z^{\varepsilon\zeta} \boldsymbol{\omega}_h^\perp$ and $\mathbf{Q}_x = \overline{\mathbf{V}_{\mathrm{sh}}^*}$ we get (see the computations in Part 2.3 in [3])

$$\partial_t \mathbf{V}_{\mathrm{sh}} + \varepsilon \overline{\mathbf{V}} \cdot \nabla^\gamma \mathbf{V}_{\mathrm{sh}} + \varepsilon \mathbf{V}_{\mathrm{sh}} \cdot \nabla^\gamma \overline{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\mathbf{V} - \underline{\mathbf{V}} \right)^\perp = \varepsilon \left[1 + z\right] \left(\nabla^\gamma \cdot \overline{\mathbf{V}} \right) \partial_z \mathbf{V}_{\mathrm{sh}} + \mathcal{O}\left(\mu^{\frac{3}{2}}\right).$$

and

$$\partial_t \mathbf{Q} + \varepsilon \overline{\mathbf{V}} \cdot \nabla^\gamma \mathbf{Q} + \varepsilon \mathbf{Q} \cdot \nabla^\gamma \overline{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\overline{\mathbf{V}} - \underline{\mathbf{V}} \right)^\perp = \mathcal{O}\left(\mu^{\frac{3}{2}}\right).$$

Finally, the second equation follows from the fact that $\mathbf{V}_{\mathrm{sh}}^* = \mathbf{V}_{\mathrm{sh}} - \mathbf{Q}$. □

2.2 Full justification of the Boussinesq-Coriolis equations

We can now establish the Boussinesq-Coriolis equations under a weak Coriolis forcing. The Boussinesq-Coriolis equations (3) are the following system

$$\begin{cases} \partial_t \zeta + \nabla^\gamma \cdot h \overline{\mathbf{V}} = 0, \\ \left(1 - \frac{\mu}{3} \nabla^\gamma \nabla^\gamma \cdot \right) \partial_t \overline{\mathbf{V}} + \nabla^\gamma \zeta + \varepsilon \overline{\mathbf{V}} \cdot \nabla^\gamma \overline{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}} \overline{\mathbf{V}}^\perp = 0. \end{cases}$$

First, we show a consistency result.

Proposition 2.5. *In the Boussinesq regime $\mathcal{A}_{Boussi}$, the Castro-Lannes equations* (7) *are consistent at order $\mathcal{O}(\mu^2)$ with the Boussinesq-Coriolis equations* (3) *in the sense of Definition 1.1.*

Proof. The first equation of the Boussinesq-Coriolis equations is always satisfied for a solution of the Castro-Lannes formulation by Proposition 2.1. We recall that the second equation of the Castro-Lannes formulation is

$$\partial_t \mathbf{U}_{/\!/}^{\mu,\gamma} + \nabla^\gamma \zeta + \frac{\varepsilon}{2} \nabla^\gamma \left| \mathbf{U}_{/\!/}^{\mu,\gamma} \right|^2 - \frac{\varepsilon}{2\mu} \nabla^\gamma \left[\left(1 + \varepsilon^2 \mu \left| \nabla^\gamma \zeta \right|^2 \right) \underline{\mathrm{w}}^2 \right] + \varepsilon \left(\nabla^\perp \cdot \mathbf{U}_{/\!/}^{\mu,\gamma} \right) \underline{\mathbf{V}}^\perp + \frac{\varepsilon}{\mathrm{Ro}} \underline{\mathbf{V}}^\perp = 0.$$

Thanks to Proposition 2.2, we know that ($\varepsilon = \mathcal{O}(\mu)$)

$$\mathbf{U}_{/\!/}^{\mu,\gamma} = \underline{\mathbf{V}} + \varepsilon \underline{\mathrm{w}} \nabla^\gamma \zeta = \underline{\mathbf{V}} + \mathcal{O}\left(\mu^2\right) = \overline{\mathbf{V}} - \sqrt{\mu} \mathbf{Q} + \mu \underline{T^* \overline{\mathbf{V}}} - \mu^{\frac{3}{2}} \overline{T \mathbf{V}_{\mathrm{sh}}^*} + \mathcal{O}\left(\mu^2\right),$$

and

$$\begin{aligned} \frac{\varepsilon}{2} \nabla^\gamma \left| \mathbf{U}_{/\!/}^{\mu,\gamma} \right|^2 =& \varepsilon \mathbf{U}_{/\!/}^{\mu,\gamma} \cdot \nabla^\gamma \mathbf{U}_{/\!/}^{\mu,\gamma} - \varepsilon \left(\nabla^{\gamma\perp} \cdot \mathbf{U}_{/\!/}^{\mu,\gamma} \right) \mathbf{U}_{/\!/}^{\mu,\gamma\perp} \\ =& \varepsilon \overline{\mathbf{V}} \cdot \nabla^\gamma \overline{\mathbf{V}} - \varepsilon \sqrt{\mu} \mathbf{Q} \cdot \nabla^\gamma \overline{\mathbf{V}} - \varepsilon \sqrt{\mu} \overline{\mathbf{V}} \cdot \nabla^\gamma \mathbf{Q} - \varepsilon \left(\nabla^{\gamma\perp} \cdot \mathbf{U}_{/\!/}^{\mu,\gamma} \right) \underline{\mathbf{V}}^\perp + \mathcal{O}\left(\mu^2\right). \end{aligned}$$

Furthermore, thanks to Proposition 2.4 and Proposition 2.2, we get $\left(\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}\left(\sqrt{\mu}\right) \right)$

$$\mu^{\frac{3}{2}} \partial_t \overline{T \mathbf{V}_{\mathrm{sh}}^*} = \mu^{\frac{3}{2}} \overline{T \partial_t \mathbf{V}_{\mathrm{sh}}^*} + \mathcal{O}\left(\mu^2\right) = -\mu^{\frac{3}{2}} \frac{\varepsilon}{\mathrm{Ro}} \overline{T \mathbf{V}_{\mathrm{sh}}^{*\perp}} + \mathcal{O}\left(\mu^2\right) = \mathcal{O}\left(\mu^2\right).$$

Finally, using Proposition 2.2, Proposition 2.4, Proposition 2.3 and Remark 1.4, we obtain from the second equation of the Castro-Lannes formulation that

$$\left(1 - \frac{\mu}{3} \nabla^\gamma \nabla^\gamma \cdot \right) \partial_t \overline{\mathbf{V}} + \nabla^\gamma \zeta + \varepsilon \overline{\mathbf{V}} \cdot \nabla^\gamma \overline{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}} \overline{\mathbf{V}}^\perp = \mathcal{O}\left(\mu^2\right).$$

Notice that all the terms that involve $\mathbf{Q}$ disappear (this fact was pointed out in [3] and [15]). □

Remark 2.6. *In [16], the author points out the fact that under a strong Coriolis forcing $\left(\frac{\varepsilon}{Ro} \leq 1 \right)$, a new term appears in the Boussinesq-Coriolis equations. We would like to emphasize that this term is not present in this setting since we only study a weak Coriolis forcing $\left(\frac{\varepsilon}{Ro} = \mathcal{O}\left(\sqrt{\mu}\right) \right)$.*

The purpose of this part is to fully justify the Boussinesq-Coriolis equations (3). First, we give a local wellposedness result of the Boussinesq-Coriolis equations. We define the energy space

$$X_\mu^N(\mathbb{R}^2) = H^N(\mathbb{R}^2) \times H^N(\mathbb{R}^2) \times H^N(\mathbb{R}^2),$$

endowed with the norm

$$\left| (\zeta, \mathbf{V}) \right|_{X_\mu^N}^2 = \left| \zeta \right|_{H^N}^2 + \left| \mathbf{V} \right|_{H^N}^2 + \mu \left| \nabla^\gamma \cdot \mathbf{V} \right|_{H^N}^2.$$

Proposition 2.7. *Let $N \geq 3$ and $\left(\zeta_0, \overline{\mathbf{V}}_0\right) \in X_\mu^N(\mathbb{R}^2)$. Suppose that $(\mu, \varepsilon, \gamma, \mathrm{Ro}) \in \mathcal{A}_{\mathrm{boussi}}$. Assume that*

$$\exists\, h_{\min} > 0 \ , \ 1 + \varepsilon\zeta_0 \geq h_{\min}.$$

Then, there exists an existence time $T > 0$ and a unique solution $\left(\zeta, \overline{\mathbf{V}}\right)$ on $[0, T]$ to the Boussinesq-Coriolis equations (3) *with initial data $\left(\zeta_0, \overline{\mathbf{V}}_0\right)$. Moreover, $(\zeta, \overline{\mathbf{V}}) \in \mathcal{C}\left([0, T]; X_\mu^N(\mathbb{R}^2)\right)$ and*

$$T = \frac{T_0}{\mu} \ , \ \frac{1}{T_0} = c^1 \text{ and } \max_{[0,T]} \left|\left(\zeta, \overline{\mathbf{V}}\right)(t, \cdot)\right|_{X_\mu^N} = c^2,$$

with $c^j = C\left(\mu_0, \frac{1}{h_{\min}}, \left|\left(\zeta_0, \overline{\mathbf{V}}_0\right)\right|_{X_\mu^N}\right)$.

Proof. This proof is a small adaptation of the proof of Proposition 6.7 in [12]. We only give the energy estimates. We assume that $\left(\zeta, \overline{\mathbf{V}}\right)$ solves (3) on $\left[0, \frac{T_0}{\mu}\right]$ and that

$$1 + \varepsilon\zeta \geq \frac{h_{\min}}{2} \text{ on } \left[0, \frac{T_0}{\mu}\right].$$

We denote $U = \left(\zeta, \overline{\mathbf{V}}\right)^t$. We introduce the symmetric matrix operator

$$S(U) = \begin{pmatrix} 1 & 0 \\ 0 & hI_2 - \mu\frac{1}{3}\nabla^\gamma\left(h\nabla^\gamma\cdot\right) \end{pmatrix}$$

and the associated energy

$$\mathcal{E}^N(U) = \frac{1}{2}\sum_{|\alpha| \leq N} \left(S(U)\partial^\alpha U, \partial^\alpha U\right)_2.$$

Remark that there exists $c_1, c_2 = C\left(\frac{1}{h_{\min}}, |h|_{L^\infty}\right)$ such that

$$c_1 \left|\nabla^\gamma \cdot \mathbf{V}\right|_2^2 \leq \left(-\frac{1}{3}\nabla^\gamma\left(h\nabla^\gamma \cdot \overline{\mathbf{V}}\right), \overline{\mathbf{V}}\right)_2 \leq c_2 \left|\nabla^\gamma \cdot \mathbf{V}\right|_2^2.$$

We also notice that for $|\alpha| = N$,

$$\frac{d}{dt}\left(S(U)\partial^\alpha U\right) = \begin{pmatrix} \partial^\alpha\partial_t\zeta \\ h\left(1 - \frac{\mu}{3}\nabla^\gamma\nabla^\gamma\cdot\right)\partial^\alpha\partial_t\overline{\mathbf{V}} \end{pmatrix} - \begin{pmatrix} 0 \\ \varepsilon\frac{\mu}{3}\left(\nabla^\gamma\cdot\partial^\alpha\partial_t\overline{\mathbf{V}}\right)\nabla^\gamma\zeta \end{pmatrix} + \varepsilon\ l.o.t.$$

and that, denoting $\Delta^\gamma = \nabla^\gamma \cdot \nabla^\gamma$,

$$\mu\left|\nabla^\gamma\cdot\partial_t\overline{\mathbf{V}}\right|_{H^N} \leq \left|\left(1 - \frac{\mu}{3}\Delta^\gamma\right)^{-1}\mu\nabla^\gamma\cdot\left(\nabla^\gamma\zeta + \varepsilon\overline{\mathbf{V}}\cdot\nabla^\gamma\overline{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}}\overline{\mathbf{V}}^\perp\right)\right|_{H^N} \leq C\left(\mu_0, \mathcal{E}^N(U)\right).$$

Then, after some computations we obtain ($\varepsilon = \mathcal{O}(\mu)$)

$$\frac{d}{dt}\mathcal{E}^N(U) \leq \mu C\left(\mathcal{E}^N(U)\right)\mathcal{E}^N(U)$$

and the result follows from Grönwall's inequality. □

We also have a stability result for the Boussinesq-Coriolis equations (3).

Proposition 2.8. *Let the assumptions of Proposition 2.7 be satisfied. Suppose that there exists* $\left(\tilde{\zeta}, \tilde{\mathbf{V}}\right) \in \mathcal{C}\left(\left[0, \frac{T_0}{\mu}\right]; X_\mu^N(\mathbb{R}^2)\right)$ *satisfying*

$$\begin{cases} \partial_t \tilde{\zeta} + \nabla^\gamma \cdot \tilde{h}\tilde{\mathbf{V}} = R_1, \\ \left(1 - \frac{\mu}{3}\nabla^\gamma \nabla^\gamma \cdot\right)\partial_t \tilde{\mathbf{V}} + \nabla^\gamma \tilde{\zeta} + \varepsilon \tilde{\mathbf{V}} \cdot \nabla^\gamma \tilde{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}}\tilde{\mathbf{V}}^\perp = R_2. \end{cases}$$

where $\tilde{h} = 1 + \varepsilon\tilde{\zeta}$ *and with* $R = (R_1, R_2) \in L^\infty\left(\left[0, \frac{T_0}{\mu}\right]; H^{N-1}(\mathbb{R}^2)\right)$. *Then, if we denote* $\mathfrak{e} = \left(\zeta, \overline{\mathbf{V}}\right) - \left(\tilde{\zeta}, \tilde{\mathbf{V}}\right)$ *where* $U = (\zeta, \mathbf{V})$ *is the solution given in Proposition 2.7, we have*

$$|\mathfrak{e}(t)|_{X_\mu^{N-1}} \leq c_1\left(\left|\mathfrak{e}_{|t=0}\right|_{X_\mu^{N-1}} + t\,|R|_{L^\infty\left(\left[0, \frac{T_0}{\mu}\right]; H^{N-1}\right)}\right),$$

where

$$c_1 = C\left(\mu_0, \frac{1}{h_{\min}}, \left|(\zeta_0, \overline{\mathbf{V}}_0)\right|_{X_\mu^N}, \left|\left(\tilde{\zeta}, \tilde{\mathbf{V}}\right)\right|_{L^\infty\left(\left[0, \frac{T_0}{\mu}\right]; X_\mu^N\right)}, |R|_{L^\infty\left(\left[0, \frac{T_0}{\mu}\right]; H^{N-1}\right)}\right).$$

Proof. We proceed as in Proposition 2.7. We define the energy

$$\mathcal{F}^{N-1}(\mathfrak{e}) = \frac{1}{2}\sum_{|\alpha| \leq N-1} \left(S(U)\partial^\alpha \mathfrak{e}, \partial^\alpha \mathfrak{e}\right)_2.$$

After some computations, we get

$$\frac{d}{dt}\mathcal{F}^{N-1}(\mathfrak{e}) \leq \left(|R|_{H^{N-1}} + \mu C\left(\mu_0, \frac{1}{h_{\min}}, |U|_{X_\mu^N}, \left|\left(\tilde{\zeta}, \tilde{\mathbf{V}}\right)\right|_{X_\mu^N}, |R|_{H^{N-1}}\right)|\mathfrak{e}|_{X_\mu^{N-1}}\right)|\mathfrak{e}|_{X_\mu^{N-1}}.$$

Then the result follows from Gronwall's inequality. □

We can now rigorously justify the Boussinesq-Coriolis equations. We recall that the operator $\overline{\mathbf{V}}[\varepsilon\zeta](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega})$ is defined in (9).

Theorem 2.9. *Let* $\mathbf{N} \geq 12$ and $(\mu, \varepsilon, \gamma, \mathrm{Ro}) \in \mathcal{A}_{Boussi}$. *We assume that we are under the assumptions of Theorem 1.3. We define the following quantities*

$$\overline{\mathbf{V}}_0 = \overline{\mathbf{V}}[\varepsilon\zeta_0]((\mathbf{U}_{/\!/}^{\mu,\gamma})_0, \boldsymbol{\omega}_0)\ ,\ \overline{\mathbf{V}} = \overline{\mathbf{V}}[\varepsilon\zeta](\mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}).$$

Then, there exists a time $T > 0$ such that

(i) T has the form

$$T = \frac{T_0}{\max\left(\mu, \frac{\varepsilon}{Ro}\right)}, \text{ and } \frac{1}{T_0} = c^1.$$

(ii) There exists a unique classical solution $\left(\zeta_B, \overline{\mathbf{V}}_B\right)$ of (3) *with the initial data $\left(\zeta_0, \overline{\mathbf{V}}_0\right)$ on* $[0, T]$.

(iii) There exists a unique classical solution $\left(\zeta, \mathbf{U}_{/\!/}^{\mu,\gamma}, \boldsymbol{\omega}\right)$ of System (7) *with initial data $\left(\zeta_0, (\mathbf{U}_{/\!/}^{\mu,\gamma})_0, \boldsymbol{\omega}_0\right)$ on* $[0, T]$.

(iv) The following error estimate holds, for $0 \leq t \leq T$,

$$\left|\left(\zeta, \overline{\mathbf{V}}\right) - \left(\zeta_B, \overline{\mathbf{V}}_B\right)\right|_{L^\infty([0,t]\times\mathbb{R}^2)} \leq \mu^2 t\, c^2,$$

with $c^j = C\left(A, \mu_0, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}\right)$.

Therefore, in the Boussinesq regime $\mathcal{A}_{Boussi}$ a solution of the water waves system (7) remains close to a solution of the Boussinesq-Coriolis equations (3) over a time $\mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$ with an accuracy of order $\mathcal{O}\left(\mu^{\frac{3}{2}}\right)$.

Remark 2.10. *Notice that if one considers a solution of a system and wants to show that this solution remains close to a solution of the waves equations over times $\mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$ with an accuracy of order $\mathcal{O}\left(\mu^{\frac{3}{2}}\right)$, it is sufficient to compare this solution with a solution of the Boussinesq-Coriolis equations* (3)*. We use this strategy in the following.*

3 The KP approximation

In this section, we consider the KP (Kadomtsev-Petviashvili) approximation under a weak Coriolis forcing. We assume that $\varepsilon = \mu$ (long waves) and $\gamma = \sqrt{\mu}$ (weakly transverse effects). We consider two different regimes. First, if $\frac{\varepsilon}{\text{Ro}} = \sqrt{\mu}$ (weak rotation), we derive the rotation-modified KP equation (12). Then, if $\frac{\varepsilon}{\text{Ro}} = \mu$ (very weak rotation), we derive the KP equation (19). We refer to [10] for more physical explanations about these two models (see also [11, 7, 9]).

3.1 Weak rotation, the rotation-modified KP equation

In the irrotational setting, the KP equation provides a good approximation of the water waves equation under the assumption that ε and μ have the same order and that γ and $\sqrt{\mu}$ have the same order (see [13] or Part 7.2 in [12]). When a Coriolis forcing is taken into account, Grimshaw and Melville ([10]) derived an equation for long waves, which is an adaptation of the KP equation

$$\partial_\xi \left(\partial_\tau k + \frac{3}{2} k \partial_\xi k + \frac{1}{6} \partial_\xi^3 k \right) + \frac{1}{2} \partial_{yy} k = \frac{1}{2} k. \tag{12}$$

This equation is called the rotation-modified KP equation or Grimshaw-Melville equation in the physics literature. Notice that this equation was originally derived for internal water waves ([10, 7]). In this part, we fully justify this equation. Inspired by [10, 7], we consider the asymptotic regime

$$\mathcal{A}_{\mathrm{RKP}} = \left\{ (\mu, \varepsilon, \gamma, \mathrm{Ro}), 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \gamma = \sqrt{\mu}, \frac{\varepsilon}{\mathrm{Ro}} = \sqrt{\mu} \right\}.$$

Then, the Boussinesq-Coriolis equations become ($\gamma = \sqrt{\mu}$)

$$\begin{cases} \partial_t \zeta + \nabla^\gamma \cdot \left([1 + \mu \zeta] \overline{\mathbf{V}} \right) = 0, \\ \left(1 - \frac{\mu}{3} \nabla^\gamma \nabla^\gamma \cdot \right) \partial_t \overline{\mathbf{V}} + \nabla^\gamma \zeta + \mu \overline{\mathbf{V}} \cdot \nabla^\gamma \overline{\mathbf{V}} + \sqrt{\mu} \overline{\mathbf{V}}^\perp = 0. \end{cases} \tag{13}$$

In the following, we denote $\mathbf{V} = (u, v)^t$. Our strategy is similar to the one used in [16] to fully justify the Ostrovsky equation. We consider an expansion of $(\zeta, \mathbf{V})$ with respect to μ. Inspired by [13] or Part 7.2 in [12], we seek an approximate solution $(\zeta_{app}, u_{app}, v_{app})$ of (13) in the form

$$\begin{aligned} \zeta_{app}(t, x, y) &= k(x - t, y, \mu t) + \mu \zeta_{(1)}(t, x, y, \mu t), \\ u_{app}(t, x, y) &= k(x - t, y, \mu t) + \mu u_{(1)}(t, x, y, \mu t), \\ v_{app}(t, x, y) &= \sqrt{\mu} v_{(1/2)}(t, x, y, \mu t) \end{aligned} \tag{14}$$

where $k = k(\xi, \tau)$ is a traveling water wave modulated by a slow time variable and the others terms are correctors. In the following, we denote by τ the variable associated to the slow time variable μt. Plugging the ansatz into Sytem (13), we obtain

$$\begin{cases} \partial_t \zeta_{app} + \nabla^\gamma \cdot \left([1 + \mu \zeta_{app}] \overline{\mathbf{V}}_{app} \right) = \mu R^1_{(1)} + \mu^2 R_1, \\ \left(1 - \frac{\mu}{3} \nabla^\gamma \nabla^\gamma \cdot \right) \partial_t \overline{\mathbf{V}}_{app} + \nabla^\gamma \zeta_{app} + \mu \overline{\mathbf{V}}_{app} \cdot \nabla^\gamma \overline{\mathbf{V}}_{app} + \sqrt{\mu} \overline{\mathbf{V}}^\perp_{app} = \sqrt{\mu} R^2_{(1/2)} + \mu R^2_{(1)} + \mu^{\frac{3}{2}} R_2. \end{cases} \tag{15}$$

where

$$R^1_{(1)} = \partial_t \zeta_{(1)} + \partial_x u_{(1)} + \partial_\tau k + 2k \partial_\xi k + \partial_y v_{(1/2)},$$
$$R^2_{(1/2)} = \begin{pmatrix} 0 \\ \partial_t v_{(1/2)} + \partial_y k + k \end{pmatrix} \text{ and } R^2_{(1)} = \begin{pmatrix} \partial_t u_{(1)} + \partial_x \zeta_{(1)} + \partial_\tau k + \frac{1}{3} \partial_\xi^3 k + k \partial_\xi k - v_{(1/2)} \\ 0 \end{pmatrix},$$

and

$$\begin{aligned} R_1 &= \partial_\tau \zeta_{(1)} + \partial_x \left(k u_{(1)} + k \zeta_{(1)} + \mu \zeta_{(1)} u_{(1)} \right) + \partial_y ((k + \mu \zeta_{(1)}) v_{(1/2)}), \\ R_2 &= (\sqrt{\mu} R_{2,1}, R_{2,2}) \end{aligned} \tag{16}$$

with

$$\begin{aligned}
R_{2,1} = {} & \partial_\tau u_{(1)} - \frac{1}{3}\partial_\xi^2\partial_\tau k - \frac{1}{3}\partial_x^2\partial_t u_{(1)} - \mu\frac{1}{3}\partial_x^2\partial_\tau u_{(1)} + \partial_x\left(ku_{(1)}\right) + \mu u_{(1)}\partial_x u_{(1)} \\
& - \frac{1}{3}\partial^3_{xyt}v_{(1/2)} - \frac{\mu}{3}\partial^3_{xy\tau}v_{(1/2)} + v_{(1/2)}\partial_y\left(k + \mu u_{(1)}\right), \\
R_{2,2} = {} & \partial_\tau v_{(1/2)} + \partial_y\zeta_{(1)} + k\partial_x v_{(1/2)} + u_{(1)} + \frac{1}{3}\partial_y\partial_\xi^2 k + \mu u_{(1)}\partial_x v_{(1/2)} + \mu v_{(1/2)}\partial_y v_{(1/2)} \\
& - \frac{\mu}{3}\left(\partial^3_{yx\tau}k + \partial^3_{yxt}u_{(1)} + \partial_y^2\partial_t v_{(1/2)} + \mu\partial_{yx\tau}u_{(1)} + \mu\partial_y^2\partial_\tau v_{(1/2)}\right).
\end{aligned}$$

Then, the strategy is to choose $(k, v_{(1/2)})$ such that, for all $(x,y) \in \mathbb{R}^2$, $t \in \left[0, \frac{T}{\mu}\right]$ and $\tau \in [0,T]$,

$$R^1_{(1)}(t,x,y,\tau) = 0 \text{ and } R^2_{(1/2)}(t,x,y,\tau) = R^2_{(1)}(t,x,y,\tau) = 0.$$

Remark 3.1. *As noticed in Part 7.2.2 in [12], we should a priori add $\sqrt{\mu}\zeta_{(1/2)}(t,x,y,\mu t)$, $\sqrt{\mu}u_{(1/2)}(t,x,y,\mu t)$, $v_{(0)}(t,x,y,\mu t)$, and $\mu v_{(1)}(t,x,y,\mu t)$ to the ansatz (14) for ζ_{app}, u_{app} and v_{app} respectively. But, it leads to $\zeta_{(1/2)} = u_{(1/2)} = v_{(0)} = v_{(1)} = 0$ if these quantities are initially zero.*

We focus first on the condition $R^2_{(1/2)}(t,x,y,\tau) = 0$. Assuming that $v_{(1/2)}$ and k vanish at $x = \infty$, this condition is equivalent to the equation

$$\partial_t\partial_x v_{(1/2)}(t,x,y,\tau) + \partial_\xi k(x-t,y,\tau) + \partial^2_{\xi y}k(x-t,y,\tau) = 0.$$

Then, using the fact that $\partial_t(k(x-t,y,\tau)) = -\partial_\xi k(x-t,y,\tau)$, we can integrate with respect to t and we get

$$\partial_x v_{(1/2)}(t,x,y,\tau) = \partial_x v^0_{(1/2)}(x,y) + k(x-t,y,\tau) + \partial_y k(x-t,y,\tau) - k^0(x,y) - \partial_y k^0(x-t,y,\tau),$$

where k^0 and $v^0_{(1/2)}$ are the initial data of k and $v_{(1/2)}$ respectively. Then, assuming that $k(\cdot,\tau) \in \partial_x H^N(\mathbb{R}^2)$ for all $\tau \in [0,T]$ (see (4)), we obtain

$$\begin{aligned}
v_{(1/2)}(t,x,y,\tau) = {} & v^0_{(1/2)}(x,y) + \partial_x^{-1}k(x-t,y,\tau) + \partial_x^{-1}\partial_y k(x-t,y,\tau) \\
& - \partial_x^{-1}k^0(x,y) - \partial_x^{-1}\partial_y k^0(x-t,y,\tau),
\end{aligned}$$

Secondly, we study the conditions $R^1_{(1)} = R^2_{(1)} = 0$. Denoting $w_\pm = \zeta_{(1)} \pm u_{(1)}$, we obtain

$$\begin{aligned}
&(\partial_t + \partial_x)\, w_+ + \left(2\partial_\tau k + 3k\partial_\xi k + \frac{1}{3}\partial_\xi^3 k + \partial_\xi^{-1}\partial_y^2 k - \partial_\xi^{-1}k\right)(x-t,\tau) + F^1_0 = 0, \\
&(\partial_t - \partial_x)\, w_- + \left(k\partial_\xi k - \frac{1}{3}\partial_\xi^3 k + \partial_\xi^{-1}\partial_y^2 k + 2\partial_\xi^{-1}\partial_y k + \partial_\xi^{-1}k\right)(x-t,\tau) + F^2_0 = 0,
\end{aligned} \tag{17}$$

where

$$F_0^1 = \partial_y v^0_{(1/2)} - v^0_{(1/2)} + \partial_\xi^{-1} k^0 - \partial_\xi^{-1}\partial_y^2 k^0,$$
$$F_0^2 = \partial_y v^0_{(1/2)} + v^0_{(1/2)} - \partial_\xi^{-1} k^0 - \partial_\xi^{-1}\partial_y^2 k^0 - 2\partial_\xi^{-1}\partial_y k^0.$$

The following Lemma (see Lemma 7.20 in [12] or Lemma 2 in [13]) gives us a Condition to control $\zeta_{(1)}$ and $u_{(1)}$.

Lemma 3.2. *Let $c_1 \neq c_2$. Let $k_1, k_2 \in H^2(\mathbb{R}^2)$ with $k_2 = \partial_x K_2$ and $K_2 \in H^3(\mathbb{R}^2)$. We consider the unique solution k of*

$$\begin{cases} (\partial_t + c_1\partial_x)k = k_1(x - c_1 t, y) + k_2(x - c_2 t, y), \\ k_{|t=0} = 0. \end{cases}$$

Then, $\lim_{t\to\infty} \left|\frac{1}{t}k(t,\cdot)\right|_{H^2} = 0$ if and only if $k_1 \equiv 0$ and in that case

$$|k(t,\cdot)|_{H^2} \leq C\frac{t}{1+t}|K_2|_{H^3}.$$

Hence, since we want to avoid a linear growth of the solution of (17), we must impose

$$\partial_\tau k + \frac{3}{2}k\partial_\xi k + \frac{1}{6}\partial_\xi^3 k + \frac{1}{2}\partial_\xi^{-1}\partial_y^2 k - \frac{1}{2}\partial_\xi^{-1} k = 0 \tag{18}$$

which is the the rotation-modified KP equation defined in (12). In the following, we provide a local existence result for this equation. This proposition generalizes Theorem 1.1 in [5].

Proposition 3.3. *Let $N \geq 4$ and $k_0 \in \partial_x H^N(\mathbb{R}^2)$. Then, there exists a time $T > 0$ and a unique solution $k \in \mathcal{C}\left([0,T]; \partial_x H^N(\mathbb{R}^2)\right)$ to the rotation-modified KP equation (18) and one has*

$$\left|\partial_\xi^{-1} k(t,\cdot)\right|_{H^N} \leq C\left(T, \left|\partial_\xi^{-1} k_0\right|_{H^N}\right).$$

Furthermore, if $k_0, \partial_y^2 k_0 \in \partial_x^2 H^{N-2}(\mathbb{R}^2)$, then $k \in \mathcal{C}\left([0,T]; \partial_x^2 H^{N-2}(\mathbb{R}^2)\right)$ and one has

$$\left|\partial_\xi^{-2} k(t,\cdot)\right|_{H^{N-2}} \leq C\left(T, \left|\partial_\xi^{-2} k_0\right|_{H^{N-2}}, \left|\partial_\xi^{-2}\partial_y^2 k_0\right|_{H^{N-2}}, |\partial_\xi k_0|_{H^N}\right).$$

Finally, if $N \geq 6$ and $k_0, \partial_y^2 k_0 \in \partial_x^2 H^{N-4}(\mathbb{R}^2)$, then $\partial_y^2 k \in \mathcal{C}\left([0,T]; \partial_x^2 H^{N-4}(\mathbb{R}^2)\right)$ and one has

$$\left|\partial_\xi^{-2} k(t,\cdot)\right|_{H^{N-4}} \leq C\left(T, \left|\partial_\xi^{-2} k_0\right|_{H^{N-4}}, \left|\partial_\xi^{-2}\partial_y^2 k_0\right|_{H^{N-4}}, \left|\partial_\xi^{-1} k_0\right|_{H^N}\right).$$

Proof. The first point follows from Theorem 1.1 in [5]. We only have to prove the second and the third points. This proof is similar to the proof of Lemma 7.22 in [12] for the KP equation and the proof of Proposition 3.8 in [16] for the Ostrovsky equation. In the following, we denote by $S(t)$ the semi-group of the linearized rotation-modified KP equation

$$\partial_\tau k + \frac{1}{6}\partial_\xi^3 k + \frac{1}{2}\partial_\xi^{-1}\partial_y^2 k - \frac{1}{2}\partial_\xi^{-1} k = 0.$$

One can check that this semi-group acts unitary on $H^N(\mathbb{R}^2)$. We also define $\tilde{k} = \partial_\tau k$. We can check that

$$\partial_\tau \tilde{k} + \frac{1}{6}\partial_\xi^3 \tilde{k} + \frac{1}{2}\partial_\xi^{-1}\partial_y^2 \tilde{k} - \frac{1}{2}\partial_\xi^{-1} \tilde{k} + \frac{3}{2}\partial_\xi\left(\tilde{k}k\right) = 0.$$

Using the Duhamel's formula we obtain

$$\partial_\xi^{-1}\tilde{k}(\tau) = S(t)\partial_\xi^{-1}\tilde{k}_0 - \frac{3}{2}\int_0^\tau S(t-s)\left(k(s)\tilde{k}(s)\right) ds.$$

We can see by product estimates that $\partial_\xi^{-1}\tilde{k}_0,\ k(s)\tilde{k}(s) \in H^{N-4}(\mathbb{R}^2)$ and then that $\tilde{k} \in \mathcal{C}\left([0,T]; \partial_x H^{N-3}(\mathbb{R}^2)\right)$. Then, we consider the following equality

$$\frac{1}{2}\left(1 - \partial_y^2\right)\partial_\xi^{-2} k = \partial_\xi^{-1}\tilde{k} + \frac{3}{4}k^2 + \frac{1}{6}\partial_\xi^2 k,$$

For the second point, we get that $(1 - \partial_\xi^2 - \partial_y^2)\partial_\xi^{-2} k \in H^{N-4}(\mathbb{R}^2)$ and the result follows easily. For the third point, we obtain from the second point that $\partial_y^2\partial_\xi^{-2} k \in H^{N-4}(\mathbb{R}^2)$. □

We can now rigorously justify the rotation-modified KP equation. The following theorem is the main theorem of this part.

Theorem 3.4. *Let $k^0 \in \partial_x^2 H^{12}(\mathbb{R}^2)$ such that $1 + \varepsilon k^0 \geq h_{\min} > 0$ and $v^0 \in \partial_x H^8(\mathbb{R}^2)$. Suppose that $(\mu, \varepsilon, \gamma, \mathrm{Ro}) \in \mathcal{A}_{\mathrm{RKP}}$. Then, there exists a time $T_0 > 0$, such that we have*

(i) a unique classical solution (ζ_B, u_B, v_B) of (13) with initial data $\left(k^0, k^0, \sqrt{\mu}v^0\right)$ on $\left[0, \frac{T_0}{\mu}\right]$.

(ii) a unique classical solution k of (18) with initial data k^0 on $[0, T_0]$.

(iii) If we define $(\zeta_{RKP}, u_{RKP})(t,x,y) = (k(x-t,y,\mu t), k(x-t,y,\mu t))$ we have the following error estimate for all $0 \leq t \leq \frac{T_0}{\mu}$,

$$|(\zeta_B, u_B) - (\zeta_{RKP}, u_{RKP})|_{L^\infty([0,t]\times\mathbb{R}^2)} \leq C\frac{\mu t}{1+t}(1 + \sqrt{\mu}t)$$

where $C = C\left(\frac{1}{h_{\min}}, \mu_0, \left|\partial_x^{-2}k^0\right|_{H^{12}}, \left|\partial_x^{-1}v^0\right|_{H^8}\right)$.

Proof. In order to simplify the technicality of this proof, C is a constant of the form

$$C = C\left(\frac{1}{h_{\min}}, \mu_0, \left|\partial_x^{-2}k^0\right|_{H^{12}}, \left|\partial_x^{-1}v^0\right|_{H^9}\right)$$

The first and second point follow from Proposition 2.7 and Proposition 3.3. Then, from System (17) and Lemma 3.2, we obtain

$$\left|\zeta_{(1)}\right|_{H^2} + \left|u_{(1)}\right|_{H^2} \leq C\frac{t}{1+t}.$$

We also notice that we can control all the derivatives with respect to x, y or τ of u and v be differentiating (17). Hence, we get a control for the remainders R_1 and R_2 and we obtain, for $0 \leq t \leq \frac{T}{\mu}$,

$$|R_1(t)|_{H^3} + |R_2(t)|_{H^3} \leq C.$$

Then, using Proposition 2.8, on can have

$$|(\zeta_B, u_B, v_B) - (\zeta_{app}, u_{app}, v_{app})|_{L^\infty([0,t]\times\mathbb{R}^2)} \leq C\mu^{\frac{3}{2}}t.$$

Finally, from the ansatz (14) and Lemma 3.2, we have

$$|(\zeta_{app}, u_{app}) - (\zeta_{RKP}, v_{RKP})|_{L^\infty([0,t]\times\mathbb{R}^2)} \leq \mu\frac{t}{1+t},$$

and the result follows easily.

□

This theorem provides the first mathematical justification of the rotation-modified KP equation. Notice that the condition $k^0 \in \partial_x^2 H^{10}(\mathbb{R}^2)$ is quite restrictive. As noted in [12] Part 7.2.1 and in [16] for the Ostrovsky equation, using the strategy developed in [1], we can hope to weaken the assumption on k^0 into $k^0 \in \partial_x H^9(\mathbb{R}^2)$.

3.2 Very weak rotation, the KP equation

In this part we study the situation of a very weak Coriolis forcing. We derive and fully justify the KP equation. We show that if $\frac{\varepsilon}{\mathrm{Ro}}$ is small enough, we can derive the KP equation

$$\partial_\xi\left(\partial_\tau k + \frac{3}{2}k\partial_\xi k + \frac{1}{6}\partial_\xi^3 k\right) + \frac{1}{2}\partial_{yy}k = 0. \tag{19}$$

Inspired by [10], we consider the following asymptotic regime

$$\mathcal{A}_{\mathrm{KP}} = \left\{(\mu, \varepsilon, \gamma, \mathrm{Ro}), 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \gamma = \sqrt{\mu}, \frac{\varepsilon}{\mathrm{Ro}} = \mu\right\}.$$

The Boussinesq-Coriolis equations become ($\gamma = \sqrt{\mu}$)

$$\begin{cases} \partial_t\zeta + \nabla^\gamma \cdot \left([1+\mu\zeta]\overline{\mathbf{V}}\right) = 0, \\ \left(1 - \frac{\mu}{3}\nabla^\gamma\nabla^\gamma\cdot\right)\partial_t\overline{\mathbf{V}} + \nabla^\gamma\zeta + \mu\overline{\mathbf{V}}\cdot\nabla^\gamma\overline{\mathbf{V}} + \mu\overline{\mathbf{V}}^\perp = 0. \end{cases} \tag{20}$$

Proceeding as in the previous part, we denote $\mathbf{V} = (u, v)^t$ and we seek an approximate solution $(\zeta_{app}, u_{app}, v_{app})$ of (20) in the form

$$\begin{aligned} \zeta_{app}(t,x) &= k(x-t,y,\mu t) + \mu\zeta_{(1)}(t,x,y,\mu t), \\ u_{app}(t,x) &= k(x-t,y,\mu t) + \mu u_{(1)}(t,x,y,\mu t), \\ v_{app}(t,x) &= \sqrt{\mu}v_{(1/2)}(t,x,y,\mu t) + \mu v_{(1)}(t,x,y,\mu t). \end{aligned} \tag{21}$$

Then, we plug the ansatz into Sytem (20) and we get

$$\begin{cases} \partial_t\zeta_{app} + \nabla^\gamma\cdot\left([1+\mu\zeta_{app}]\overline{\mathbf{V}}_{app}\right) = \mu R^1_{(1)} + \mu^{\frac{3}{2}}\partial_y v_{(1)} + \mu^2 R_1, \\ \left(1-\frac{\mu}{3}\nabla^\gamma\nabla^\gamma\cdot\right)\partial_t\overline{\mathbf{V}}_{app} + \nabla^\gamma\zeta_{app} + \mu\overline{\mathbf{V}}_{app}\cdot\nabla^\gamma\overline{\mathbf{V}}_{app} + \mu\overline{\mathbf{V}}^\perp_{app} = \sqrt{\mu}R^2_{(1/2)} + \mu R^2_{(1)} + \mu^{\frac{3}{2}}R_2. \end{cases}$$

where

$$\begin{aligned} &R^1_{(1)} = \partial_t\zeta_{(1)} + \partial_x u_{(1)} + \partial_\tau k + 2k\partial_\xi k + \partial_y v_{(1/2)}, \\ &R^2_{(1/2)} = \begin{pmatrix} 0 \\ \partial_t v_{(1/2)} + \partial_y k \end{pmatrix} \text{ and } R^2_{(1)} = \begin{pmatrix} \partial_t u_{(1)} + \partial_x\zeta_{(1)} + \partial_\tau k + \frac{1}{3}\partial^3_\xi k + k\partial_\xi k \\ \partial_t v_{(1)} + k \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} R_1 &= \partial_\tau\zeta_{(1)} + \partial_x\left(ku_{(1)} + k\zeta_{(1)} + \mu\zeta_{(1)}u_{(1)}\right) + \partial_y((k+\mu\zeta_{(1)})(v_{(1/2)} + \mu v_{(1)})), \\ R_2 &= \left(-(v_{(1/2)} + \sqrt{\mu}v_{(1)}) + \sqrt{\mu}\tilde{R}_{2,1}, R_{2,2}\right) \end{aligned}$$

with

$$\begin{aligned} \tilde{R}_{2,1} &= \partial_\tau u_{(1)} - \frac{1}{3}\partial^2_\xi\partial_\tau k - \frac{1}{3}\partial^2_x\partial_t u_{(1)} - \mu\frac{1}{3}\partial^2_x\partial_\tau u_{(1)} + \partial_x\left(ku_{(1)}\right) + \mu u_{(1)}\partial_x u_{(1)} \\ &\quad - \frac{1}{3}\partial^3_{xyt}(v_{(1/2)} + \sqrt{\mu}v_{(1)}) - \frac{\mu}{3}\partial^3_{xy\tau}(v_{(1/2)} + \sqrt{\mu}v_{(1)}) + (v_{(1/2)} + \sqrt{\mu}v_{(1)})\partial_y\left(k + \mu u_{(1)}\right), \\ R_{2,2} &= \partial_\tau v_{(1/2)} + \partial_y\zeta_{(1)} + k\partial_x(v_{(1/2)} + \sqrt{\mu}v_{(1)}) + u_{(1)} + \frac{1}{3}\partial_y\partial^2_\xi k + \mu u_{(1)}\partial_x(v_{(1/2)} + \sqrt{\mu}v_{(1)}) \\ &\quad - \frac{\mu}{3}\left(\partial^3_{yx\tau}k + \partial^3_{yxt}u_{(1)} + \partial^2_y\partial_t(v_{(1/2)} + \sqrt{\mu}v_{(1)}) + \mu\partial_{yx\tau}u_{(1)} + \mu\partial^2_y\partial_\tau(v_{(1/2)} + \sqrt{\mu}v_{(1)})\right) \\ &\quad + \mu(v_{(1/2)} + \sqrt{\mu}v_{(1)})\partial_y(v_{(1/2)} + \sqrt{\mu}v_{(1)}). \end{aligned}$$

Then, we choose $(k, v_{(1/2)}, v_{(1)})$ such that, for all $(x,y)\in\mathbb{R}^2$, $t\in\left[0,\frac{T}{\mu}\right]$ and $\tau\in[0,T]$,

$$R^1_{(1)}(t,x,y,\tau) = 0 \text{ and } R^2_{(1/2)}(t,x,y,\tau) = R^2_{(1)}(t,x,y,\tau) = 0.$$

First, we obtain that

$$
\begin{aligned}
v_{(1/2)} &= \partial_x^{-1}\partial_y k + v^0_{(1/2)} - \partial_x^{-1}\partial_y k^0,\\
v_{(1)} &= \partial_x^{-1} k + v^0_{(1)} - \partial_x^{-1} k^0.
\end{aligned}
$$

Then, denoting $w_\pm = \zeta_{(1)} \pm u_{(1)}$, we get

$$
\begin{aligned}
&(\partial_t + \partial_x)\, w_+ + \left(2\partial_\tau k + 3k\partial_\xi k + \frac{1}{3}\partial_\xi^3 k + \partial_\xi^{-1}\partial_y^2 k\right)(x-t,\tau) + F_0 = 0,\\
&(\partial_t - \partial_x)\, w_- + \left(k\partial_\xi k - \frac{1}{3}\partial_\xi^3 k + \partial_\xi^{-1}\partial_y^2 k\right)(x-t,\tau) + F_0 = 0,
\end{aligned}
$$

where

$$
F_0 = \partial_y v^0_{(1/2)} - \partial_\xi^{-1}\partial_y^2 k^0.
$$

Therefore, in order to avoid a linear growth (see Lemma 3.2), k must satisfies the KP equation (19). The following Lemma is a local wellposedness result for the KP equation (see Lemma 7.22 in [12] or [19, 2, 21]).

Proposition 3.5. *Let $N \geq 5$ and $k_0 \in \partial_x H^N(\mathbb{R}^2)$. Then, there exists a time $T > 0$ and a unique solution $k \in \mathcal{C}\left([0,T]; \partial_x H^N(\mathbb{R}^2)\right)$ to the KP equation* (19) *and one has*

$$
\left|\partial_\xi^{-1} k(t,\cdot)\right|_{H^N} \leq C\left(T, \left|\partial_\xi^{-1} k_0\right|_{H^N}\right).
$$

Furthermore, if $N \geq 6$ and $\partial_y^2 k_0 \in \partial_x^2 H^{N-4}(\mathbb{R}^2)$, then $\partial_y^2 k \in \mathcal{C}\left([0,T]; \partial_x^2 H^{N-4}(\mathbb{R}^2)\right)$ and one has

$$
\left|\partial_y^2 k_0 \partial_\xi^{-2} k(t,\cdot)\right|_{H^{N-4}} \leq C\left(T, \left|\partial_\xi^{-2}\partial_y^2 k_0\right|_{H^{N-4}}, \left|\partial_\xi^{-1} k_0\right|_{H^N}\right).
$$

We can now establish a rigorous justification of the KP equation.

Theorem 3.6. *Let $k^0 \in \partial_x^2 H^{12}(\mathbb{R}^2)$ such that $1+\varepsilon k^0 \geq h_{\min} > 0$ and $v^0_{(1/2)} \in \partial_x H^8(\mathbb{R}^2)$, $v^0_{(1)} \in H^7(\mathbb{R}^2)$. Suppose that $(\mu, \varepsilon, \gamma, \mathrm{Ro}) \in \mathcal{A}_{\mathrm{KP}}$. Denote $v^0 = \sqrt{\mu} v^0_{(1/2)} + \mu v^0_{(1)}$. Then, there exists a time $T_0 > 0$, such that we have*

(i) a unique classical solution (ζ_B, u_B, v_B) of (13) *with initial data $\left(k^0, k^0, v^0\right)$ on $\left[0, \frac{T_0}{\mu}\right]$.*

(ii) a unique classical solution k of (19) *with initial data k^0 on $[0, T_0]$.*

(iii) If we define $\left(\zeta_{KP}, u_{KP}\right)(t,x) = \left(k(x-t,y,\mu t), k(x-t,y,\mu t)\right)$ we have the following error estimate for all $0 \leq t \leq \frac{T_0}{\mu}$,

$$
\left|(\zeta_B, u_B) - (\zeta_{KP}, u_{KP})\right|_{L^\infty([0,t]\times\mathbb{R}^2)} \leq C\frac{\mu t}{1+t}(1+\sqrt{\mu}t)
$$

where $C = C\left(\frac{1}{h_{\min}}, \mu_0, \left|\partial_x^{-2} k^0\right|_{H^{12}}, \left|\partial_x^{-1} v^0_{(1/2)}\right|_{H^8}, \left|v^0_{(1)}\right|_{H^7}\right)$.

Proof. The proof is very similar to the proof of Theorem 3.4. □

Remark 3.7. *Contrary to the justification of the KP equation in the irrotational setting (see Part 7.2 in [12] or [13]), the transverse part of the horizontal velocity v must contain an order $\mathcal{O}(\mu)$ contribution. Notice that if one considers a weaker Coriolis forcing, for instance $\frac{\varepsilon}{Ro} = \mu^{\frac{3}{2}}$, this assumption is no more necessary.*

4 Which equation for which asymptotic regime ?

4.1 The Ostrovsky and KdV equations

In Section 3, we derived two asymptotic models in the long wave regime ($\varepsilon = \mu$). First, if $\gamma = \sqrt{\mu}$ and $\frac{\varepsilon}{\mathrm{Ro}} = \sqrt{\mu}$, we derived the rotation-modified KP equation

$$\partial_\xi \left(\partial_\tau k + \frac{3}{2} k \partial_\xi k + \frac{1}{6} \partial_\xi^3 k \right) + \frac{1}{2} \partial_{yy} k = \frac{1}{2} k.$$

Then, if $\gamma = \sqrt{\mu}$ and $\frac{\varepsilon}{\mathrm{Ro}} = \mu$, we obtained the KP equation

$$\partial_\xi \left(\partial_\tau k + \frac{3}{2} k \partial_\xi k + \frac{1}{6} \partial_\xi^3 k \right) + \frac{1}{2} \partial_{yy} k = 0.$$

In [16], we performed a similar derivation in the long wave regime under the assumption that $\gamma = \mathcal{O}(\mu^2)$. When $\frac{\varepsilon}{\mathrm{Ro}} = \sqrt{\mu}$, we derived the Ostrovsky equation

$$\partial_\xi \left(\partial_\tau k + \frac{3}{2} k \partial_\xi k + \frac{1}{6} \partial_\xi^3 k \right) = \frac{1}{2} k, \tag{22}$$

and when $\frac{\varepsilon}{\mathrm{Ro}} = \mu$, we derived the KdV equation

$$\partial_\tau k + \frac{3}{2} k \partial_\xi k + \frac{1}{6} \partial_\xi^3 k = 0. \tag{23}$$

We would like to emphasize that we can weaken the assumption $\gamma = \mathcal{O}(\mu^2)$ into $\gamma = \mu$. In the following, we show this fact on the Ostrovsky equation. We consider the asymptotic regime

$$\mathcal{A}_{\mathrm{ostrov}} = \left\{ (\mu, \varepsilon, \gamma, \mathrm{Ro}), 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \gamma = \mu, \frac{\varepsilon}{\mathrm{Ro}} = \sqrt{\mu} \right\}.$$

Then we seek an approximate solution $(\zeta_{app}, u_{app}, v_{app})$ of the Boussinesq-Coriolis equations in the form

$$\begin{aligned}
\zeta_{app}(t,x,y) &= k(x-t, y, \mu t) + \mu \zeta_{(1)}(t,x,y,\mu t), \\
u_{app}(t,x,y) &= k(x-t, y, \mu t) + \mu u_{(1)}(t,x,y,\mu t), \\
v_{app}(t,x,y) &= \sqrt{\mu} v_{(1/2)}(t,x,y,\mu t) + \mu v_{(1)}(t,x,y,\mu t)
\end{aligned}$$

Plugging the ansatz into the Boussinesq-Coriolis equations, we obtain

$$\begin{cases} \partial_t \zeta_{app} + \nabla^\gamma \cdot \left([1 + \mu \zeta_{app}] \overline{\mathbf{V}}_{app} \right) = \mu R^1_{(1)} + \mu^{\frac{3}{2}} R_1, \\ \left(1 - \frac{\mu}{3} \nabla^\gamma \nabla^\gamma \cdot \right) \partial_t \overline{\mathbf{V}}_{app} + \nabla^\gamma \zeta_{app} + \mu \overline{\mathbf{V}}_{app} \cdot \nabla^\gamma \overline{\mathbf{V}}_{app} + \sqrt{\mu} \overline{\mathbf{V}}_{app}^\perp = \sqrt{\mu} R^2_{(1/2)} + \mu R^2_{(1)} + \mu^{\frac{3}{2}} R_2. \end{cases}$$

where

$$R^1_{(1)} = \partial_t \zeta_{(1)} + \partial_x u_{(1)} + \partial_\tau k + 2k \partial_\xi k,$$
$$R^2_{(1/2)} = \begin{pmatrix} 0 \\ \partial_t v_{(1/2)} + k \end{pmatrix} \text{ and } R^2_{(1)} = \begin{pmatrix} \partial_t u_{(1)} + \partial_x \zeta_{(1)} + \partial_\tau k + \frac{1}{3} \partial_\xi^3 k + k \partial_\xi k - v_{(1/2)} \\ \partial_t v_{(1)} + \partial_y k \end{pmatrix},$$

and where R_1, R_2 are remainders similar to the ones found in Sections 3.1 and 3.2. Then, using the same strategy than before, we impose that $R^1_{(1)} = 0$ and $R^2_{(1/2)} = R^2_{(1)} = 0$. We obtain

$$v_{(1/2)} = \partial_\xi^{-1} k + v^0_{(1/2)} - \partial_\xi^{-1} k^0,$$
$$v_{(1)} = \partial_\xi^{-1} \partial_y k + v^0_{(1)} - \partial_\xi^{-1} \partial_y k^0,$$

and, denoting $w_\pm = \zeta_{(1)} \pm u_{(1)}$, we get

$$(\partial_t + \partial_x) w_+ + \left(2\partial_\tau k + 3k \partial_\xi k + \frac{1}{3} \partial_\xi^3 k - \partial_\xi^{-1} k \right) (x - t, \tau) - F_0 = 0,$$
$$(\partial_t - \partial_x) w_- + \left(k \partial_\xi k - \frac{1}{3} \partial_\xi^3 k + \partial_\xi^{-1} k \right) (x - t, \tau) + F_0 = 0,$$

where $F_0 = v^0_{(1/2)} - \partial_\xi^{-1} k^0$. In order to avoid a linear growth (see Lemma 3.2), k must satisfies the Ostrovsky equation (22). Proceeding as in [16], we can generalize Theorem 3.9 in [16] to the asymptotic regime $\mathcal{A}_{\text{ostrov}}$. A solution of the Ostrovsky equation provides a $\mathcal{O}(\sqrt{\mu})$ approximation of the Boussinesq-Coriolis equations over a time $\mathcal{O}\left(\frac{1}{\mu}\right)$. We can proceed similarly for the KdV equation (23). Under the asymptotic regime

$$\mathcal{A}_{\text{KdV}} = \left\{ (\mu, \varepsilon, \gamma, \text{Ro}), 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \gamma = \mu, \frac{\varepsilon}{\text{Ro}} = \mu \right\}.$$

and with the ansatz

$$\zeta_{app}(t, x, y) = k(x - t, y, \mu t) + \mu \zeta_{(1)}(t, x, y, \mu t),$$
$$u_{app}(t, x, y) = k(x - t, y, \mu t) + \mu u_{(1)}(t, x, y, \mu t),$$
$$v_{app}(t, x, y) = \mu v_{(1)}(t, x, y, \mu t),$$

we can generalize Theorem 3.12 in [16] to the asymptotic regime $\mathcal{A}_{\text{KdV}}$. A solution of the KdV equation provides a $\mathcal{O}(\mu)$ approximation of the Boussinesq-Coriolis equations over a time $\mathcal{O}\left(\frac{1}{\mu}\right)$.

4.2 Conclusion

We summarize Section 3 and Subsection 4.1 by the following table. Notice that all of these models provide a $\mathcal{O}(\sqrt{\mu})$ approximation (at least) in the long wave regime ($\varepsilon = \mu$) of the Boussinesq-Coriolis equations over a time $\mathcal{O}\left(\frac{1}{\mu}\right)$.

γ \ $\frac{\varepsilon}{\text{Ro}}$	$\sqrt{\mu}$	μ
$\sqrt{\mu}$	Rotation-modified KP equation	KP equation
μ	Ostrovsky equation	KdV equation

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