Rectification of a deep water model for surface gravity waves

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Abstract

In this work we discuss an approximate model for the propagation of deep irrotational water waves; specifically the model obtained when retaining only quadratic nonlinearities in the water waves system under the Zakharov-Craig-Sulem formulation. We argue that the initial-value problem associated with this system is most likely ill-posed in finite-regularity spaces, explaining spurious oscillations reported in numerical simulations for instance in [15], and thus agreeing with the conclusion of [4] although not on the proposed instability mechanism. We show that the system can be "rectified". Indeed, through the introduction of suitable regularizing operators we can recover well-posedness properties without sacrificing other desirable features such as a canonical Hamiltonian structure, cubic accuracy as an asymptotic model, and efficient numerical integration. Our study is supported with detailed and reproducible numerical simulations.

1 Introduction

1.1 Motivation

This study deals with models for the propagation of water waves, that is the motion of inviscid, incompressible, homogeneous and potential flows with a free surface under the influence of gravity. The derivation of simplified models in shallow water situations, with or without small-amplitude assumptions, has a rich history, including seminal works of Airy, Saint-Venant, Boussinesq, Rayleigh, Korteweg and de Vries among many others. The study of their rigorous justification has undergone a lot of activity in the last decade, providing a fairly comprehensive picture; see accounts in [18, 24, 13]. This was made possible in particular after the breakthrough of Alvarez-Samaniego and Lannes [3, 2] and Iguchi [17] concerning the "exact" water waves system. As a consequence of their analysis, the full rigorous justification of a given simplified model can be methodically inferred from (i) its consistency (namely that regular solutions to the model satisfy the water waves system up to a small remainder term) and (ii) the well-posedness of the associated initial-value problem with uniform control of the solutions. The first property is typically obtained from fairly standard elliptic estimates on a Laplace problem after suitable asymptotic expansions. The second one must be provided on a model-by-model basis, and standard energy estimates have proven to be fruitful on many shallow water models.

Comparatively, the literature on deep waters or infinite-depth models (based on small-steepness assumptions) is much less prolific, with however important exceptions such as [28, 6, 21, 10, 27, 1, 20, 2, 25, 9] (see also references therein for related works). In particular, to our knowledge, only one deep water model (without restricting to unidirectional or weakly transverse propagation) has been fully justified in spaces of finite regularity (and finite energy), in [25]. Yet this model involves

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a fairly artificial change of variables which breaks many features of the original water waves system (in particular Hamiltonian formulation, symmetry groups, preserved quantities) and prevents direct comparison with respect to "exact" solutions. All other models derived in the aforementioned works are only partially justified at best, because the well-posedness of their initial-value problem in finite-regularity spaces is open.

In this work, we consider arguably the simplest and most natural (nonlinear) model stemming from the Zakharov/Craig-Sulem formulation of the water waves system, namely

$$\text{(WW2)} \qquad \begin{cases} \partial_t \zeta - \mathsf{T}^\mu \cdot \nabla \psi + \epsilon \mathsf{T}^\mu \cdot \nabla (\zeta \mathsf{T}^\mu \cdot \nabla \psi) + \epsilon \nabla \cdot (\zeta \nabla \psi) = 0, \\ \partial_t \psi + \zeta + \frac{\epsilon}{2} \left(|\nabla \psi|^2 - (\mathsf{T}^\mu \cdot \nabla \psi)^2 \right) = 0. \end{cases}$$

Here, ζ (resp. ψ) represents the surface deformation (resp. the trace of the velocity potential at the surface) at time $t \in \mathbb{R}$ and horizontal location $\boldsymbol{x} \in \mathbb{R}^d$ (with $d \in \{1, 2\}$), and the Fourier multiplier[1] $\mathsf{T}^\mu \stackrel{\text{def}}{=} -\frac{\mathrm{i} D \tanh(\sqrt{\mu}|D|)}{|D|} = -\frac{\tanh(\sqrt{\mu}|D|)}{|D|} \nabla$ is the "Tiesz transform". Variables have been non-dimensionalized (*i.e.* rescaled). The dimensionless parameter μ is the shallowness parameter, and can take arbitrarily large values in this work, while ϵ represents the steepness of the waves, and is typically small. Indeed, the quadratic system (WW2) is constructed from the fully nonlinear water waves system (WW) by neglecting $\mathcal{O}(\epsilon^2)$ contributions; see details in Appendix A.

One asset of the quadratic system (WW2) with respect to competitors in aforementioned works is that it retains the canonical Hamiltonian structure of the fully nonlinear equations. Indeed (WW2) reads

$$\text{(1.1)} \qquad \begin{cases} \partial_t \zeta - \delta_\psi \mathcal{H}^\mu = 0, \\ \partial_t \psi + \delta_\zeta \mathcal{H}^\mu = 0, \end{cases}$$

with the functional

$$\text{(1.2)} \qquad \mathcal{H}^\mu(\zeta, \psi) \stackrel{\text{def}}{=} \frac{1}{2} \int_{\mathbb{R}^d} \zeta^2 + \psi \mathsf{T}^\mu \cdot \nabla \psi + \epsilon \zeta \left(|\nabla \psi|^2 - (\mathsf{T}^\mu \cdot \nabla \psi)^2 \right) \mathrm{d}\boldsymbol{x}.$$

It is also the first (nonlinear) system in a hierarchy which has been put forward in the seminal work of Craig and Sulem [12] as an efficient strategy for numerically computing, by means of Fourier pseudospectral methods, approximate solutions to the water waves system. This approach was subsequently implemented in [15] (among other works) where it is noted (page 90) that

> In the computations [...] it was observed that spurious oscillations can develop in the wave profile, due to the onset of an instability related to the growth of numerical errors at high wavenumbers.

A natural question arises as whether the cause of these instabilities is due to the numerical discretization or can be seen at the continuous level. In [4] the authors suggest, based on tailored numerical simulations and the analysis of a toy model, that (WW2) —in the unidimensional ($d = 1$) and infinite depth ($\mu = \infty$) situation— is ill-posed in some Sobolev spaces. While we agree with this general statement, the instability mechanism we describe (and validate through in-depth numerical investigation) in this work is quite different from that described in [4], as can be seen from comparing the toy model (B.1) in Appendix B with [4, (2.3)].

We also propose a rectification method which is essentially costless from the point of view of the numerical integration. Specifically, we consider

$$\text{(RWW2)} \qquad \begin{cases} \partial_t \zeta - \mathsf{T}^\mu \cdot \nabla \psi + \epsilon \mathsf{T}^\mu \cdot \nabla ((\mathsf{J}\zeta) \mathsf{T}^\mu \cdot \nabla \psi) + \epsilon \nabla \cdot ((\mathsf{J}\zeta) \nabla \psi) = 0, \\ \partial_t \psi + \zeta + \frac{\epsilon}{2} \mathsf{J} \left(|\nabla \psi|^2 - (\mathsf{T}^\mu \cdot \nabla \psi)^2 \right) = 0, \end{cases}$$

[1] Fourier multipliers associated with bounded symbols $\sigma \in L^\infty(\mathbb{R}^d)$ are defined from $L^2(\mathbb{R}^d)$ to itself by

$$\forall f \in L^2(\mathbb{R}^d), \qquad \widehat{\sigma(D) f}(\boldsymbol{\xi}) = \sigma(\boldsymbol{\xi}) \widehat{f}(\boldsymbol{\xi}).$$

Setting $\mu = \infty$, $\mathsf{T}^\infty = \frac{-\mathrm{i} D}{|D|}$ is the Riesz transform if $d = 2$, and the Hilbert transform if $d = 1$.

where $\mathsf{J} = J(D)$ is a Fourier multiplier which can be freely chosen in a space of regular "rectifiers" (see Definition 1.1), and can be thought as a (smoothed-out) low-pass filter. Hence our rectification method has similar flavor with, but is different from, standard de-aliasing techniques.

Some comments are in order. While it is obvious that embedding regularizing operators in System (WW2) allows to provide the well-posedness of the initial-value problem in Sobolev spaces, by means of the Picard–Lindelöf (or Cauchy–Lipschitz) theorem in Banach spaces, and while it is fairly easy to check that (RWW2) is indeed of semilinear nature, our result of *large time* existence and control of solutions relies on delicate energy estimates, making use in particular of the equivalent of Alinhac's good unknowns which are crucial in the analysis of the water waves systems; see [18, §4]. The aftermath is that it is possible to choose J so that considering (RWW2) rather than (WW2) does not deteriorate the precision of the system as an asymptotic model for the water waves system as $\epsilon \searrow 0$, while the control of solutions allows to fully justify (RWW2) on the relevant timescale.

Finally, let us express the perhaps counter-intuitive fact that introducing J in the first equation of (RWW2) provides the essential regularization, while we could actually set $\mathsf{J} = \mathrm{Id}$ in the second equation and preserve the large-time existence and control of solutions. This expressly invalidates the rationale for instabilities proposed in [4]. We choose to insert the rectifier in the second equation in order to preserve the aforementioned Hamiltonian structure. Indeed, when J is symmetric for the $L^2(\mathbb{R}^d)$ inner product, (RWW2) can be read as (1.1) with the modified functional

$$\mathcal{H}^\mu(\zeta,\psi) \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 + \psi \mathsf{T}^\mu\cdot\nabla\psi + \epsilon(\mathsf{J}\zeta)\left(|\nabla\psi|^2 - (\mathsf{T}^\mu\cdot\nabla\psi)^2\right)\mathrm{d}\boldsymbol{x}. \tag{1.3}$$

Hence by Noether's theorem or direct inspection, conserved quantities (invariants) of (RWW2) include the Hamiltonian (representing the total energy) $\mathcal{H}^\mu(\zeta,\psi)$, the excess of mass, $\int_{\mathbb{R}^d}\zeta\,\mathrm{d}\boldsymbol{x}$, and the horizontal impulse, $\int_{\mathbb{R}^d}\zeta\nabla\psi\,\mathrm{d}\boldsymbol{x}$.

1.2 Notations

Before stating our main result, let us introduce a few notations used throughout this work.

- We denote $C(\lambda_1,\dots,\lambda_N)$ a positive "constant" depending non-decreasingly on its variables. We write $a \lesssim b$ and sometimes $a = \mathcal{O}(b)$ if $a \le Cb$ where $C>0$ is a universal constant or its dependencies are obvious from the context. We write $a \approx b$ when $a \lesssim b$ and $b \lesssim a$.
- We denote the ceiling function as $\lceil x\rceil$ and the floor function as $\lfloor x\rfloor$ for any $x\in\mathbb{R}$.
- The space $L^\infty(\mathbb{R}^d)$ consists of all real-valued, essentially bounded, Lebesgue-measurable functions. We denote
$$\big|f\big|_{L^\infty} \stackrel{\text{def}}{=} \operatorname{ess\,sup}_{\boldsymbol{x}\in\mathbb{R}^d}|f(\boldsymbol{x})|.$$
- $L^2(\mathbb{R}^d)$ denotes the real-valued square-integrable functions, endowed with the topology associated with the inner product
$$\forall f,g\in L^2(\mathbb{R}^d),\quad \big(f,g\big)_{L^2} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} fg\,\mathrm{d}\boldsymbol{x}.$$
- For any $s\in\mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^d)$ is the space of tempered distributions such that
$$\big|f\big|_{H^s} \stackrel{\text{def}}{=} \big|\langle\cdot\rangle^s\widehat{f}\big|_{L^2} < \infty$$
where $\langle\boldsymbol{\xi}\rangle \stackrel{\text{def}}{=} (1+|\boldsymbol{\xi}|^2)^{1/2}$ and $\widehat{f}$ is the Fourier transform of f. Of course $H^s(\mathbb{R}^d)$ is endowed with the norm $|\cdot|_{H^s}$. We use without clarification that when $s=n\in\mathbb{N}$,
$$\big|f\big|_{H^n}^2 \approx \sum_{\alpha\in\mathbb{N}^d,\ |\alpha|\le n}\big|\partial^\alpha f\big|_{L^2}^2,$$

where for multi-indices $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we denote $|\alpha| = \sum_{i=1}^d \alpha_i$ and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$.

- For any $s \in \mathbb{R}$, the Beppo Levi space $\mathring{H}^s(\mathbb{R}^d)$ denotes the tempered distributions such that $\nabla f \in H^{s-1}(\mathbb{R}^d)^d$, endowed with the semi-norm $\big|f\big|_{\mathring{H}^s} \stackrel{\text{def}}{=} \big|\nabla f\big|_{H^{s-1}}$.
- For any $\mu > 0$, the operator $\mathfrak{P}^\mu : \mathring{H}^s(\mathbb{R}^d) \to H^{s-1/2}(\mathbb{R}^d)$ (for any $s \in \mathbb{R}$) is defined by

$$\widehat{\mathfrak{P}^\mu f} = \sqrt{|\cdot| \tanh(\sqrt{\mu}|\cdot|)}\, \widehat{f}.$$

We also recall the notation $\mathsf{T}^\mu \stackrel{\text{def}}{=} -\frac{\mathrm{i}D\tanh(\sqrt{\mu}|D|)}{|D|} : H^s(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$.

We now describe the class of operators (among which J in (RWW2) is chosen) which we consider in this work.

Definition 1.1 (Rectifiers). *Let $\mathsf{J} = J(D)$ with $J \in L^\infty(\mathbb{R}^d)$, real-valued and even.*

- *We say that J is* regularizing *of order $m \le 0$ if $\langle\cdot\rangle^{-m} J \in L^\infty(\mathbb{R}^d)$.*
- *We say that J is* regular *if J is regularizing of order -1 and, additionally, $\langle\cdot\rangle \nabla J \in L^\infty(\mathbb{R}^d)$.*
- *We say that J is* near-identity *of order $\ell \ge 0$ if $|\cdot|^{-\ell}(1-J) \in L^\infty(\mathbb{R}^d)$.*

Since its symbol is real-valued and Hermitian, the operator $\mathsf{J} = J(D)$ maps real-valued functions to real-valued functions, and is symmetric for the $L^2(\mathbb{R}^d)$ inner product. By Plancherel theorem, regularizing operators of order m satisfy for any $s \in \mathbb{R}$ and $f \in H^s(\mathbb{R}^d)$, $\mathsf{J}f \in H^{s-m}(\mathbb{R}^d)$ and

$$\big|\mathsf{J}f\big|_{H^{s-m}} \le \big|\langle\cdot\rangle^{-m} J\big|_{L^\infty} \big|f\big|_{H^s}.$$

Regularizing properties are essential to our well-posedness analysis. Assuming $\langle\cdot\rangle\nabla J \in L^\infty$ and in particular precluding ideal low-pass filters is necessary to the commutator estimate of Lemma 2.5, which is used in our "large time" well-posedness analysis. If J is near-identity of order $\ell \ge 0$, then we have for all $s \in \mathbb{R}$ and $f \in H^{s+\ell}(\mathbb{R}^d)$,

$$\big|f - \mathsf{J}f\big|_{H^s} \le \big||\cdot|^{-\ell}(1-J)\big|_{L^\infty} \big|f\big|_{H^{s+\ell}}.$$

This property is used in our consistency analysis. Together, the well-posedness and consistency analyses yield the complete rigorous justification of (RWW2) as an asymptotic model for the the water waves system. For our purposes, we find it desirable to have $\big||\cdot|^{-\ell}(1-J)\big|_{L^\infty}$ small and $\big|\langle\cdot\rangle^1 J\big|_{L^\infty}$ not too large. There is a competition between these two inclinations, which is easily seen by introducing a scaling parameter. Indeed, if we set $\mathsf{J}^\delta = J(\delta D)$ with $\delta \in (0,1]$, then

$$\big||\cdot|^{-\ell}(1-J(\delta\cdot))\big|_{L^\infty} = \delta^\ell \big||\cdot|^{-\ell}(1-J(\cdot))\big|_{L^\infty} \quad \text{and} \quad \big|\langle\cdot\rangle^1 J(\delta\cdot)\big|_{L^\infty} \le \delta^{-1}\big|\langle\cdot\rangle^1 J(\delta\cdot)\big|_{L^\infty}.$$

In our numerical simulations, we use the rectifiers, for $m \le 0$,

$$\mathsf{J}^\delta = \min(\{1, |\delta D|^m\}) \tag{1.4}$$

which are regularizing of order m, regular if $m \le 0$, and near-identity of order ℓ for any $\ell \ge 0$.

Finally we find it convenient to use the following norms on the symbols of the rectifiers $\mathsf{J} = J(D)$: for $s \in \mathbb{R}$ and $k \in \mathbb{N}$ (in fact $k \in \{0,1\}$ in this work) we denote

$$\mathcal{N}^s(J) \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\boldsymbol{\xi}\in\mathbb{R}^d} \langle\boldsymbol{\xi}\rangle^{-s}|J(\boldsymbol{\xi})|, \qquad \dot{\mathcal{N}}^s(J) \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{\boldsymbol{\xi}\in\mathbb{R}^d} |\boldsymbol{\xi}|^{-s}|J(\boldsymbol{\xi})|,$$

$$\mathcal{N}^s_k(J) \stackrel{\text{def}}{=} \max_{\beta\in\mathbb{N}^d,\ |\beta|\le k} \Big(\operatorname*{ess\,sup}_{\boldsymbol{\xi}\in\mathbb{R}^d} \langle\boldsymbol{\xi}\rangle^{|\beta|-s}|\partial^\beta J(\boldsymbol{\xi})|\Big), \quad \dot{\mathcal{N}}^s_k(J) \stackrel{\text{def}}{=} \max_{\beta\in\mathbb{N}^d,\ |\beta|\le k} \Big(\operatorname*{ess\,sup}_{\boldsymbol{\xi}\in\mathbb{R}^d} |\boldsymbol{\xi}|^{|\beta|-s}|J(\boldsymbol{\xi})|\Big).$$

1.3 Main results

We can now state our main results.

Theorem 1.2 (Well-posedness)**.** *Let $d \in \{1,2\}$, $t_0 > \frac{d}{2}$, $N \in \mathbb{N}$ with $N \geq t_0+2$, $C > 1$ and $M > 0$. Set $\mathsf{J}_0 = J_0(D)$ a regular rectifier. There exists $T_0 > 0$ such that for any $\mu \geq 1$ and $\epsilon > 0$, for any $(\zeta_0, \psi_0) \in H^N(\mathbb{R}^d) \times \mathring{H}^{N+\frac{1}{2}}(\mathbb{R}^d)$ such that*

$$0 < \epsilon M_0 \stackrel{\text{def}}{=} \epsilon\big(\big|\zeta_0\big|_{H^{\lceil t_0\rceil+2}} + \big|\mathfrak{P}^\mu\psi_0\big|_{H^{\lceil t_0\rceil+2}}\big) \leq M,$$

and for any $\delta \geq \epsilon M_0$, the following holds.

Defining $\mathsf{J} = J_0(\delta D)$, there exists a unique $(\zeta,\psi) \in \mathcal{C}([0,T_0/(\epsilon M_0)]; H^N(\mathbb{R}^d) \times \mathring{H}^{N+\frac{1}{2}}(\mathbb{R}^d))$ solution to (RWW2) *with initial data $(\zeta,\psi)\big|_{t=0} = (\zeta_0,\psi_0)$, and it satisfies*

$$\sup_{t\in[0,T_0/(\epsilon M_0)]} \big(\big|\zeta(t,\cdot)\big|^2_{H^N} + \big|\mathfrak{P}^\mu\psi(t,\cdot)\big|^2_{H^N}\big) \leq C\left(\big|\zeta_0\big|^2_{H^N} + \big|\mathfrak{P}^\mu\psi_0\big|^2_{H^N}\right).$$

Theorem 1.2 is a corollary of more precise results presented in Propositions 3.1 and 3.2, namely

- an unconditional well-posedness result for "small times", *i.e.* up to $t \approx \min(1/\epsilon, \delta/\epsilon)$;
- a conditional (fulfilled by small data) result for "large times", *i.e.* up to $t \approx \min(1/\epsilon, \delta/\epsilon^2)$.

We also prove in this work that if the rectifier J_0 is regularizing of order m with $m > \frac{3}{2} + \frac{d}{2}$, then the energy preservation provides the global-in-time well-posedness for sufficiently small initial; see the precise statement in Proposition 3.14.

Theorem 1.2 should be considered together with the following result, describing the precision (in the sense of consistency) of (RWW2) as an asymptotic model for the fully nonlinear water waves system, that is (WW) displayed in Appendix A.

Theorem 1.3 (Consistency)**.** *Let $d \in \{1,2\}$, $t_0 > d/2$, $s \geq 0$, $h_\star > 0$ and $M > 0$. There exists $C > 0$ such that for any $\epsilon > 0$, $\mu \geq 1$, $\mathsf{J}_0 = J_0(D)$ a rectifier near-identity of order $\ell \geq 0$ and for any $(\zeta,\psi) \in \mathcal{C}([0,T]; H^{\max(s+\ell+1,s+2,t_0+\frac{3}{2})}(\mathbb{R}^d) \times \mathring{H}^{\max(s+\ell+\frac{3}{2},t_0+1)}(\mathbb{R}^d))$ solution to* (RWW2)*, with $\mathsf{J} = \mathsf{J}_0(\delta D)$ and $\delta > 0$, with the property that on $[0,T]$*

$$1 + \tfrac{\epsilon}{\sqrt{\mu}}\zeta \geq h_\star, \qquad \epsilon M_0 \stackrel{\text{def}}{=} \epsilon\big(\big|\zeta\big|_{H^{t_0+\frac{3}{2}}} + \big|\mathfrak{P}^\mu\psi\big|_{H^{t_0+\frac{1}{2}}}\big) \leq M,$$

one has

$$\begin{cases} \partial_t\zeta - G^\mu[\epsilon\zeta]\psi = \epsilon\delta^\ell R_1 + \epsilon^2\widetilde{R}_1, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\epsilon}{2}\frac{(G^\mu[\epsilon\zeta]\psi + \epsilon\nabla\zeta\cdot\nabla\psi)^2}{1+\epsilon^2|\nabla\zeta|^2} = \epsilon\delta^\ell R_2 + \epsilon^2\widetilde{R}_2, \end{cases}$$

with G^μ the Dirichlet-to-Neumann operator defined and discussed in Appendix A, and

$$\begin{aligned} \big|R_1\big|_{H^s} + \big|R_2\big|_{H^{s+\frac{1}{2}}} &\leq C\,M_0\,\dot{\mathcal{N}}^\ell(1-J_0)\,\big(\big|\zeta\big|_{H^{s+\ell+1}} + \big|\nabla\psi\big|_{H^{s+\ell+\frac{1}{2}}}\big), \\ \big|\widetilde{R}_1\big|_{H^s} + \big|\widetilde{R}_2\big|_{H^{s+\frac{1}{2}}} &\leq C\,M_0^2\left(\big|\zeta\big|_{H^{s+2}} + \big|\mathfrak{P}^\mu\psi\big|_{H^{s+1}}\right). \end{aligned}$$

Thanks to the above well-posedness and consistency results, and, making use of the well-posedness and stability results on the water waves system obtained in [2], we can conclude with the full justification of (RWW2) regarding its ability to produce $\mathcal{O}(\epsilon^2)$ approximations to exact solutions of the water waves system (WW) in the relevant timescale, as stated below.

Theorem 1.4 (Convergence)**.** *There exists $p \in \mathbb{N}$ such that the following holds.*

Let $d \in \{1,2\}$, $t_0 > d/2$, $s \geq 0$, $h_\star > 0$, $M > 0$, and $\mathsf{J}_0 = J_0(D)$ a regular rectifier near-identity of order $\ell \geq 0$. There exists $C > 0$ and $T > 0$ such that for any $\epsilon > 0$ and $\mu \geq 1$, any $(\zeta_0, \psi_0) \in H^{s+a+p}(\mathbb{R}^d) \times \mathring{H}^{s+\ell+p+\frac{1}{2}}(\mathbb{R}^d)$ such that

$$1 + \tfrac{\epsilon}{\sqrt{\mu}}\zeta \geq h_\star, \qquad \epsilon M_0 \overset{\text{def}}{=} \epsilon\big(\big|\zeta_0\big|_{H^{s+\ell+p}} + \big|\mathfrak{P}^\mu \psi_0\big|_{H^{s+\ell+p}}\big) \leq M,$$

and for any $\delta \geq \epsilon M_0$, there exists

- *a unique $(\zeta, \psi) \in \mathcal{C}([0, T/(\epsilon M_0)]; H^{\lfloor s+\ell+p \rfloor}(\mathbb{R}^d) \times \mathring{H}^{\lfloor s+\ell+p \rfloor + \frac{1}{2}}(\mathbb{R}^d))$ solution to* (RWW2) *where $\mathsf{J} = J_0(\delta D)$ with initial data $(\zeta, \psi)\big|_{t=0} = (\zeta_0, \psi_0)$,*
- *a unique $(\zeta_{\rm ww}, \psi_{\rm ww}) \in \mathcal{C}([0, T/(\epsilon M_0)]; H^s(\mathbb{R}^d) \times \mathring{H}^{s+\frac{1}{2}}(\mathbb{R}^d))$ solution to the water waves system* (WW) *with initial data $(\zeta_{\rm ww}, \psi_{\rm ww})\big|_{t=0} = (\zeta_0, \psi_0)$;*

and moreover one has

$$\sup_{t \in [0, T/(\epsilon M_0)]} \big(\big|(\zeta - \zeta_{\rm ww})(t, \cdot)\big|_{H^s} + \big|(\mathfrak{P}^\mu \psi - \mathfrak{P}^\mu \psi_{\rm ww})(t, \cdot)\big|_{H^s}\big) \leq C\, M_0\, t\, \big(\delta^\ell (\epsilon M_0) + (\epsilon M_0)^2\big)\,.$$

1.4 Discussion and prospects

The instability mechanism Our motivation for introducing rectifiers in (RWW2) stems from the forthcoming Proposition 3.9, where we extract a "quasilinear structure" of the system, of the form

$$\begin{cases} \partial_t \dot\zeta - \mathsf{T}^\mu \cdot \nabla \dot\psi + \epsilon \nabla \cdot ((\mathsf{J}\dot\zeta) \nabla \psi) = \text{lower order terms}, \\ \partial_t \dot\psi + \mathfrak{a}^\mu[\epsilon \mathsf{J}\zeta, \epsilon \nabla \psi] \dot\zeta + \epsilon \mathsf{J}\Big(\nabla\psi \cdot \nabla \dot\psi\Big) = \text{lower order terms}, \end{cases}$$

where the "Rayleigh–Taylor" (see forthcoming Remark 3.3) operator is defined as

$$\mathfrak{a}^\mu[\epsilon \mathsf{J}\zeta, \epsilon\nabla\psi] f \overset{\text{def}}{=} f - \epsilon(\mathsf{T}^\mu \cdot \nabla \mathsf{J}\zeta)\mathsf{J}f - \epsilon^2 \mathsf{J}\Big((\mathsf{T}^\mu \cdot \nabla\psi)|D|\big\{(\mathsf{T}^\mu \cdot \nabla\psi)\mathsf{J}f\big\}\Big).$$

In the absence of rectifiers ($\mathsf{J} = \mathrm{Id}$), this quasilinear structure is of elliptic type (with the same nature as Cauchy–Riemann equations) because $(\mathfrak{a}^\mu[\epsilon\zeta, \epsilon\nabla\psi]f, f)_{L^2}$ can take arbitrarily large negative values for some smooth f with $\big\|f\big\|_{L^2} = 1$, as soon as $\epsilon \mathsf{T}^\mu \cdot \nabla\psi \neq 0$. As such, the instability mechanism we exhibit is strikingly similar to the well-studied Kelvin-Helmholtz instabilities (see *e.g.* [19, §4] and references therein). Notice however that in our case the instability mechanism does not relate to any physically relevant phenomenon, and that it is fully nonlinear in the sense that it cannot be revealed through linearization about equilibria. Our strategy for taming these instabilities also differs from works on Kelvin-Helmholtz instabilities: while we could (artificially) add surface tension contributions, it appears less invasive to incorporate rectifiers J as we do, from which the nature of the quasilinear structure turns to hyperbolic for sufficiently regular and small data, namely such that the Rayleigh–Taylor condition holds:

$$\forall f \in L^2(\mathbb{R}^d)\ ,\ (f, \mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta, \epsilon\nabla\psi]f)_{L^2} \geq \mathfrak{a}^\mu_\star \big|f\big|^2_{L^2},$$

with some $\mathfrak{a}^\mu_\star > 0$.

We investigate in more details our proposed instability mechanism through a toy model in Appendix B. Therein we prove local and global well-posedness results when rectifiers are sufficiently regularizing, as well as a strong ill-posedness result when rectifiers are not sufficiently regularizing.

We are unfortunately not able to prove the ill-posedness of the initial-value problem for (WW2). However, detailed numerical experiments provided in Section 5 fully support our proposed instability mechanism. Further insights on the involved scalings and recommendations for practical use in numerical simulations can be found in Section 6.

Other models with quadratic precision A key ingredient in our analysis is the observation that the instability mechanism stems from cubic terms (arising from the use of Alinhac's good unknowns, see again Remark 3.3) whereas (WW2) is a quadratic model. It is natural to ask whether it is possible to exhibit models with different sets of unknowns or with additional terms for which the initial-value problem is well-posed, without employing artificial regularizing operators. As discussed in the introduction, several models with quadratic precision have been proposed in the literature, among which only the one studied in [25] have been proved to be well-posed in finite-regularity spaces. Yet the latter model uses a change of variables for the unknown describing the surface deformation, which can be seen as undesirable. In Appendix C, we propose another model with quadratic precision using a relatable set of unknowns. Notice however that this system involves fully nonlinear operators, and is hence less suitable than (RWW2) for numerical simulations. Moreover, it appears that the algebra used in Appendix C cannot be applied to cubic or higher order models, while we do hope that the "rectifying" method introduced in this work can be extended to a general framework.

The infinite-depth situation Our study is restricted to the deep-water framework, in the sense that our results hold uniformly with respect to $\mu \in [1, +\infty)$, *finite*. A difficulty arises in the infinite-depth case (*i.e.* setting $\mu = \infty$, $\mathsf{T}^\infty \cdot \nabla = |D|$) due to the fact that the operator $\mathfrak{P}^\mu : \mathring{H}^s(\mathbb{R}^d) \to H^{s-1/2}(\mathbb{R}^d)$ is not bounded uniformly with respect to $\mu \geq 1$ (see Lemma 2.1), and hence Beppo Levi spaces are no longer suitable. A first remark is that our results continue to hold and extend straightforwardly to the infinite-depth case if we enforce $\psi \in H^s(\mathbb{R}^d)$ instead of $\psi \in \mathring{H}^s(\mathbb{R}^d)$. This was the choice made in [18] (see Remark 2.50 therein), but it can be considered as too restrictive. A bigger space would consist in using instead $\breve{H}^s(\mathbb{R}^d) = \dot{H}^{1/2}(\mathbb{R}^d) \cap \mathring{H}^s(\mathbb{R}^d)$ (when $s \geq 1/2$). Yet there is no fully accepted definition of the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^d)$. If $d = 2$ then we can define, as in [5], $\dot{H}^{1/2}(\mathbb{R}^2)$ as the space of tempered distributions f such that $\hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $|\cdot|^{1/2} f \in L^2(\mathbb{R}^2)$ (endowed with the corresponding norm). However when $d = 1$ the space defined as above is not a Hilbert space (see [5, Proposition 1.34]), and hence our results do not apply. It is possible to provide an alternative definition for $\dot{H}^{1/2}(\mathbb{R})$ as a Hilbert space when defined modulo polynomials (see *e.g.* [26]) or simply modulo constants (see [8]), but it is not straightforward to check that our proofs can be extended using the corresponding topologies. Finally, an alternative and physically sound strategy consists in constructing solutions to the infinite-depth system as limits of finite-depth solutions as $\mu \to \infty$.

The shallow-water situation In the opposite direction, let us discuss the shallow-water situation, namely $\mu \in (0, 1]$. Firstly, a rescaling must be performed so that (WW2) remains non-trivial as $\mu \searrow 0$ (see [2, Appendix A] for the water waves system). This amounts in replacing (WW2) with

$$\text{(WW2')} \qquad \begin{cases} \partial_t \zeta - \frac{1}{\sqrt{\mu}} \mathsf{T}^\mu \cdot \nabla \psi + \varepsilon \mathsf{T}^\mu \cdot \nabla (\zeta \mathsf{T}^\mu \cdot \nabla \psi) + \varepsilon \nabla \cdot (\zeta \nabla \psi) = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} \left(|\nabla \psi|^2 - (\mathsf{T}^\mu \cdot \nabla \psi)^2 \right) = 0. \end{cases}$$

One of the key ingredients in our analysis is Lemma 2.9, exhibiting "commutator" estimates on the operator $\psi \mapsto \mathsf{T}^\mu \cdot \nabla (\zeta \mathsf{T}^\mu \cdot \nabla \psi) + \nabla \cdot (\zeta \nabla \psi)$. These estimates do not hold uniformly with respect to $\mu \in (0, 1]$, since we have (for smooth data) $\mathsf{T}^\mu \cdot \nabla (\zeta \mathsf{T}^\mu \cdot \nabla \psi) = \mathcal{O}(\mu)$ as $\mu \searrow 0$. However, one should notice that formally setting $\mu = 0$ in (WW2') yields

$$\begin{cases} \partial_t \zeta + \nabla \cdot ((1 + \varepsilon \zeta) \nabla \psi) = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2} \left(|\nabla \psi|^2 \right) = 0. \end{cases}$$

Taking the gradient of the second equation, we obtain as expected the shallow-water system, which is a well-known example of symmetrizable hyperbolic systems. As such, its initial-value problem is

well-posed for $(\zeta, \nabla\psi) \in H^s(\mathbb{R}^d)^{1+d}$ for any $s > 1 + d/2$ under the non-cavitation condition; see *e.g.* [18]. Hence we believe that our results can be adapted to the shallow-water situation, although such a study should take into account the aforementioned commutator estimate in combination with the symmetric structure appearing in the shallow-water equation.

Aftermath The climax of our analysis, Theorem 1.4, is that the "rectified" model, (RWW2), is able to approximate solutions to the water waves system, (WW), with the desired cubic accuracy for well-chosen choices of rectifiers, and suitable values for the strength δ. Introducing rectifiers is essentially costless from the point of view of numerical integration when the (pseudo-)spectral schemes are employed. In some sense, the rectification strategy can be related to standard regularization strategies, either through artificial parabolic regularization or low-pass filters. The difference is that we are able to point out precisely where regularization should be introduced, and to measure the "cost" of this regularization. Once again, it should be pointed out that this cost is in fact immaterial from the point of view of asymptotic modeling. We believe our strategy can be adapted to the whole hierarchy of systems with arbitrary order put forward by Craig and Sulem in [12], which would provide a robust and efficient method to approximate solutions to the water waves system with arbitrary accuracy. We leave this topic for a further study.

1.5 Outline

Let us now describe the remaining content of this paper.

In Section 2 we introduce some important technical tools: product and commutator estimates and key results on the operators $\mathsf{T}^\mu \cdot \nabla$ and $\mathfrak{P}^\mu$.

Section 3 is dedicated to well-posedness results on the regularized system (RWW2). We first prove Theorem 1.2 as the consequence of two results:

- an unconditional "short-time" well-posedness result, Proposition 3.1, proved in Section 3.1;
- a conditional "large-time" well-posedness result, Proposition 3.2, which is proved in Section 3.2.

The latter may be considered as the main and most technical result of this work. Finally, we prove a global-in-time well-posedness result for sufficiently small data, Proposition 3.14, in Section 3.3.

The full justification of (RWW2) as a model for deep water waves is completed in Section 4, with the proof of Theorems 1.3 and 1.4.

In Section 5 we report on numerical experiments validating the proposed instability mechanism and rectifying strategy.

A summary of our results and numerical investigation with some conclusions is provided in Section 6.

We recall some information on the water waves system and the Dirichlet-to-Neumann expansion in Appendix A.

A toy model for the proposed instability mechanism is introduced and studied in details in Appendix B.

An alternative deep water model with quadratic precision without involving regularizing operators is introduced and discussed in Appendix C.

2 Technical ingredients

Recall the notations $\mathsf{T}^\mu \overset{\text{def}}{=} -\frac{\tanh(\sqrt{\mu}|D|)}{|D|}\nabla$ and $\mathfrak{P}^\mu \overset{\text{def}}{=} \sqrt{|D|\tanh(\sqrt{\mu}|D|)}$.

Lemma 2.1. *Let $s \in \mathbb{R}$. For any $\mu \geq 1$ and any functions $f \in \mathring{H}^{s+1/2}(\mathbb{R}^d)$,*

$$\left|\nabla f\right|_{H^{s-1/2}} \leq \left|\mathfrak{P}^\mu f\right|_{H^s}.$$

Conversely, $\mathfrak{P}^\mu : \mathring{H}^{s+1/2}(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ *is well-defined and bounded, yet not uniformly with respect to* $\mu \geq 1$*:*

$$\big|\mathfrak{P}^\mu f\big|_{H^s} \leq (2\mu)^{\frac{1}{4}} \big|\nabla f\big|_{H^{s-1/2}}.$$

Moreover, for any $f \in H^{s+1/2}(\mathbb{R}^d)$*, one has the uniform bound*

$$\big|\mathfrak{P}^\mu f\big|_{H^s} \leq \big|f\big|_{H^{s+1/2}}.$$

For any $f \in \mathring{H}^{s+1}(\mathbb{R}^d)$

$$\big|\mathsf{T}^\mu \cdot \nabla f\big|_{H^s} \leq \big|\nabla f\big|_{H^s}.$$

The operators $\mathfrak{P}^\mu$ *and* $\mathsf{T}^\mu \cdot \nabla = (\mathfrak{P}^\mu)^2$ *are symmetric for the* $L^2(\mathbb{R}^d)$ *inner product, in particular for any* $f \in H^{1/2}(\mathbb{R}^d)$*,* $g \in \mathring{H}^1(\mathbb{R}^d)$*,*

$$(\mathsf{T}^\mu \cdot \nabla g, f)_{L^2} = (\mathfrak{P}^\mu g, \mathfrak{P}^\mu f)_{L^2}.$$

Proof. Results follow from Parseval's theorem and the fact that for any $\xi \geq 0$ and $\mu \geq 1$

$$\xi/\sqrt{1+\xi^2} \leq \tanh(\xi) \leq \tanh(\sqrt{\mu}\xi) \leq \min(1, \sqrt{\mu}\xi).$$

Only the first inequality requires explanation. It follows from $\tanh(\xi) = \frac{\sinh(\xi)}{\sqrt{1+\sinh(\xi)^2}}$, $\sinh(\xi) \geq \xi$ and the fact that $\xi \mapsto \xi/\sqrt{1+\xi^2}$ is increasing. □

The following product estimate is proved for instance in [16, Th. 8.3.1].

Lemma 2.2. *Let* $d \in \mathbb{N}^\star$*. Let* $s, s_1, s_2 \in \mathbb{R}$ *such that* $s \leq s_1$*,* $s \leq s_2$*,* $s_1 + s_2 \geq 0$ *and* $s_1 + s_2 > s + \frac{d}{2}$*. Then, there exists a constant* $C > 0$ *such that for all* $f \in H^{s_1}(\mathbb{R}^d)$ *and for all* $g \in H^{s_2}(\mathbb{R}^d)$*, we have* $fg \in H^s(\mathbb{R}^d)$ *and* [2]

$$\big|fg\big|_{H^s} \leq C\big|f\big|_{H^{s_1}}\big|g\big|_{H^{s_2}}.$$

In particular, for any $t_0 > d/2$*, and* $s \in [-t_0, t_0]$*, there exists* $C > 0$ *such that for all* $f \in H^s(\mathbb{R}^d)$ *and* $g \in H^{t_0}(\mathbb{R}^d)$*,* $fg \in H^s(\mathbb{R}^d)$ *and*

$$\big|fg\big|_{H^s} \leq C\,\big|f\big|_{H^s}\big|g\big|_{H^{t_0}}.$$

We infer the following useful "tame" product estimates.

Lemma 2.3. *Let* $d \in \mathbb{N}^\star$*. Let* $s, s_1, s_2, s_1', s_2' \in \mathbb{R}$ *be such that*

$$s_1 + s_2 = s_1' + s_2' > s + d/2, \quad s_1 + s_2 \geq 0, \quad s \leq s_1,\ s \leq s_2'.$$

Then there exists a constant $C > 0$ *such that for all* $f \in H^{s_1}(\mathbb{R}^d) \cap H^{s_1'}(\mathbb{R}^d)$*,* $g \in H^{s_2}(\mathbb{R}^d) \cap H^{s_2'}(\mathbb{R}^d)$*, we have* $fg \in H^s(\mathbb{R}^d)$ *and*

$$\big|fg\big|_{H^s} \leq C\big(\big|f\big|_{H^{s_1}}\big|g\big|_{H^{s_2}} + \big|f\big|_{H^{s_1'}}\big|g\big|_{H^{s_2'}}\big).$$

In particular, for any $t_0 > d/2$*,* $s \geq -t_0$*, there exists* $C > 0$ *such that for all* $f, g \in H^s(\mathbb{R}^d) \cap H^{t_0}(\mathbb{R}^d)$*,* $fg \in H^s(\mathbb{R}^d)$ *and*

$$\big|fg\big|_{H^s} \leq C\big(\big|f\big|_{H^s}\big|g\big|_{H^{t_0}} + \big|f\big|_{H^{t_0}}\big|g\big|_{H^s}\big).$$

[2] When $s < 0$ the product fg is well-defined as a tempered distribution as soon as $s_1 + s_2 \geq 0$. Indeed, we have

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^d), \qquad \langle u_1 u_2, \varphi\rangle := \langle u_1, u_2\varphi\rangle, \quad \text{and} \quad \big|\langle u_1 u_2, \varphi\rangle\big| \lesssim \big|u_1\big|_{H^{s_1}}\big|u_2\big|_{H^{s_2}}\big|\varphi\big|,$$

for $H^{s_1}(\mathbb{R}^d)' = H^{-s_1}(\mathbb{R}^d) \supset H^{s_2}(\mathbb{R}^d)$ and $\big|u_2\varphi\big|_{H^{s_2}} \lesssim \big|u_2\big|_{H^{s_2}}\big|\varphi\big|$ where the norm on φ involves a finite number (depending on s_2) of semi-norms. Another statement of Lemma 2.2 is that the pointwise multiplication, which is well-defined from $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ (say), extends as a continuous bilinear map from $H^{s_1}(\mathbb{R}^d) \times H^{s_2}(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$. Henceforth, we will use without mention the standard identification between functions and (tempered) distributions.

Proof. We consider several cases. If $s \leq s_2$, then Lemma 2.2 yields immediately

$$\big|fg\big|_{H^s} \lesssim \big|f\big|_{H^{s_1}} \big|g\big|_{H^{s_2}}.$$

Symmetrically, if $s \leq s_1'$, then

$$\big|fg\big|_{H^s} \lesssim \big|f\big|_{H^{s_1'}} \big|g\big|_{H^{s_2'}}.$$

Otherwise $s_1' < s \leq s_1$ and $s_2 < s \leq s_2'$. Assume moreover that $s \leq d/2$. Then denoting $s_1^\star = s$ and $s_2^\star = s_1 + s_2 - s = s_1' + s_2' - s$, we have $s_1' < s_1^\star = s \leq s_1$, hence $s_2 \leq s_2^\star < s_2'$, and $s_2^\star > d/2 \geq s$. Lemma 2.2 yields

$$\big|fg\big|_{H^s} \lesssim \big|f\big|_{H^{s_1^\star}} \big|g\big|_{H^{s_2^\star}}.$$

We conclude by the Sobolev interpolation

$$\big|f\big|_{H^{s_1^\star}} \big|g\big|_{H^{s_2^\star}} \leq \big|f\big|_{H^{s_1}}^{\theta} \big|f\big|_{H^{s_1'}}^{1-\theta} \big|g\big|_{H^{s_2}}^{\theta} \big|g\big|_{H^{s_2'}}^{1-\theta}$$

with $\theta = \frac{s_1^\star - s_1'}{s_1 - s_1'} = \frac{s_2' - s_2^\star}{s_2' - s_2}$, and then by Young's inequality. The first statement is proved when $s \leq d/2$. When $s > d/2$, we set $n \in \mathbb{N}$ such that $-d/2 \leq s - n \leq d/2$, and use that

$$\big|fg\big|_{H^s} \lesssim \big|fg\big|_{L^2} + \sum_{\alpha \in \mathbb{N}^d,\, |\alpha| = n} \big|\partial^\alpha (fg)\big|_{H^{s-n}}.$$

From the previously proved estimate we have

$$\big|fg\big|_{L^2} \lesssim \big|f\big|_{H^{s_1}} \big|g\big|_{H^{s_2}} + \big|f\big|_{H^{s_1'}} \big|g\big|_{H^{s_2'}}$$

and for any $\beta, \gamma \in \mathbb{N}^d$ such that $\beta + \gamma = \alpha$,

$$\big|(\partial^\beta f)(\partial^\gamma g)\big|_{H^{s-n}} \lesssim \big|\partial^\beta f\big|_{H^{s_1 - |\beta|}} \big|\partial^\gamma g\big|_{H^{s_2 - |\gamma|}} + \big|\partial^\beta f\big|_{H^{s_1' - |\beta|}} \big|\partial^\gamma g\big|_{H^{s_2' - |\gamma|}}.$$

The first statement then follows by Leibniz rule and triangular inequality. The second one is a particular case with $s_1 = s_2' = s$ and $s_2 = s_1' = t_0$. □

We now turn to commutator estimates involving operators of order 1.

Lemma 2.4. *Let $d \in \mathbb{N}^\star$, $t_0 > \frac{d}{2}$ and $-t_0 \leq s \leq t_0$. There exists a constant $C > 0$ such that for any $\sigma \in L^\infty_{\rm loc}(\mathbb{R}^d)$ such that $\nabla \sigma \in L^\infty(\mathbb{R}^d)$, then for any $f \in \mathring{H}^{t_0+1}(\mathbb{R}^d)$ and $g \in H^s(\mathbb{R}^d)$, one has*

$$\big|\big[\sigma(D), f\big] g\big|_{H^s} \leq C \big|\nabla \sigma\big|_{L^\infty} \big|\nabla f\big|_{H^{t_0}} \big|g\big|_{H^s}.$$

Proof. We have

$$\big|\big[\sigma(D), f\big] g\big|_{H^s} \lesssim \Big|\int_{\mathbb{R}^d} \langle \cdot \rangle^s |\sigma(\cdot) - \sigma(\boldsymbol{\eta})| |\widehat{f}(\cdot - \boldsymbol{\eta})| |\widehat{g}(\boldsymbol{\eta})| \, {\rm d}\boldsymbol{\eta}\Big|_{L^2}.$$

The result follows immediately from the inequality

$$|\sigma(\boldsymbol{\xi}) - \sigma(\boldsymbol{\eta})| \leq \big|\nabla \sigma\big|_{L^\infty} |\boldsymbol{\xi} - \boldsymbol{\eta}|$$

valid for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$, and an application of Lemma 2.2, with $s_1 = t_0$ and $s_2 = s$. □

We now consider commutator estimates involving operators of order 0.

Lemma 2.5. *Let $d \in \mathbb{N}^\star$, $t_0 > \frac{d}{2}$ and $-t_0 \leq s \leq t_0 + 1$. There exists a constant $C > 0$ such that the following holds. For any $\sigma \in L^\infty(\mathbb{R}^d)$ such that*

$$\widetilde{\mathcal{N}}_1^0(\sigma) = \max\left(\left\{\left|\sigma\right|_{L^\infty}, \operatorname*{ess\,sup}_{|\boldsymbol{\xi}|\geq 1} |\boldsymbol{\xi}||\nabla\sigma(\boldsymbol{\xi})|\right\}\right) < \infty$$

and for any $f \in H^{t_0+1}(\mathbb{R}^d)$ and $g \in H^{s-1}(\mathbb{R}^d)$, one has $\big[\sigma(D), f\big]g \in H^s(\mathbb{R}^d)$ and

$$\left|\big[\sigma(D), f\big]g\right|_{H^s} \leq C\widetilde{\mathcal{N}}_1^0(\sigma)\left|f\right|_{H^{t_0+1}}\left|g\right|_{H^{s-1}}.$$

If moreover $\nabla\sigma \in L^\infty(\mathbb{R}^d)$ then, recalling the notation $\mathcal{N}_1^0(\sigma) = \max\left(\left\{\left|\sigma\right|_{L^\infty}, \left|\langle\cdot\rangle\nabla\sigma\right|_{L^\infty}\right\}\right)$,

$$\left|\big[\sigma(D), f\big]g\right|_{H^s} \leq C\mathcal{N}_1^0(\sigma)\left|\nabla f\right|_{H^{t_0}}\left|g\right|_{H^{s-1}}.$$

Proof. We have

$$\left|\big[\sigma(D), f\big]g\right|_{H^s} \lesssim \left|\langle\cdot\rangle^s \int_{\mathbb{R}^d} \langle\boldsymbol{\eta}\rangle|\sigma(\cdot) - \sigma(\boldsymbol{\eta})||\widehat{f}(\cdot - \boldsymbol{\eta})|\langle\boldsymbol{\eta}\rangle^{-1}|\widehat{g}(\boldsymbol{\eta})|\,\mathrm{d}\boldsymbol{\eta}\right|_{L^2}.$$

- If $|\boldsymbol{\eta}| \geq 2|\boldsymbol{\xi}|$ and $|\boldsymbol{\eta}| \geq 2$, then $\langle\boldsymbol{\eta}\rangle \leq 2|\boldsymbol{\eta}| \leq 4|\boldsymbol{\xi} - \boldsymbol{\eta}|$ and hence
$$\langle\boldsymbol{\eta}\rangle|\sigma(\boldsymbol{\xi}) - \sigma(\boldsymbol{\eta})| \leq 8\left|\sigma\right|_{L^\infty}|\boldsymbol{\xi} - \boldsymbol{\eta}|.$$
- If $|\boldsymbol{\xi}| \geq |\boldsymbol{\eta}|/2$ and $|\boldsymbol{\eta}| \geq 2$, suitably selecting an integration path γ (with endpoints $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$) taking values in $\{\zeta \in \mathbb{R}^d,\ |\zeta| \geq |\boldsymbol{\eta}|/2\}$ we find that
$$\langle\boldsymbol{\eta}\rangle|\sigma(\boldsymbol{\xi}) - \sigma(\boldsymbol{\eta})| = \langle\boldsymbol{\eta}\rangle\left|\int_\gamma \nabla\sigma\cdot\mathrm{d}\boldsymbol{r}\right| \leq \langle\boldsymbol{\eta}\rangle|\gamma| \operatorname*{ess\,sup}_{\zeta\in\gamma}|\nabla\sigma(\zeta)| \leq 2\pi|\boldsymbol{\xi} - \boldsymbol{\eta}| \operatorname*{ess\,sup}_{|\zeta|\geq 1}|\zeta||\nabla\sigma(\zeta)|.$$
- Finally, if $|\boldsymbol{\eta}| \leq 2$, then we have immediately (almost everywhere)
$$\langle\boldsymbol{\eta}\rangle|\sigma(\boldsymbol{\xi}) - \sigma(\boldsymbol{\eta})| \leq 2\sqrt{5}\left|\sigma\right|_{L^\infty}$$
in the first case and
$$\langle\boldsymbol{\eta}\rangle|\sigma(\boldsymbol{\xi}) - \sigma(\boldsymbol{\eta})| \leq \sqrt{5}\left|\nabla\sigma\right|_{L^\infty}|\boldsymbol{\xi} - \boldsymbol{\eta}|$$
in the second case.

The desired result when $-t_0 \leq s \leq t_0$ follows from an application of Lemma 2.2, with $s_1 = t_0$ and $s_2 = s$. Moreover, by symmetry considerations, we have (almost everywhere)

$$\langle\boldsymbol{\xi}\rangle|\sigma(\boldsymbol{\xi}) - \sigma(\boldsymbol{\eta})| \lesssim \min(\langle\boldsymbol{\xi} - \boldsymbol{\eta}\rangle\widetilde{\mathcal{N}}_1^0(\sigma), |\boldsymbol{\xi} - \boldsymbol{\eta}|\mathcal{N}_1^0(\sigma))$$

which yields the desired result when $-t_0 \leq s-1 \leq t_0$, using Lemma 2.2 with $s_1 = t_0$ and $s_2 = s-1$. Since $t_0 \geq 1/2$, the proof is complete. □

The following Lemma is a direct application of Lemmas 2.4 and 2.5.

Lemma 2.6. *Let $d \in \mathbb{N}^\star$ and $t_0 > \frac{d}{2}$. There exists a constant $C > 0$ such that for any $\mu \geq 1$, any $g \in H^{-1/2}(\mathbb{R}^d)$ and any $f \in \mathring{H}^{t_0+1}(\mathbb{R}^d)$,*

$$\left|[|D|, f]g\right|_{H^{-1/2}} \leq C\left|\nabla f\right|_{H^{t_0}}\left|g\right|_{H^{-1/2}},$$
$$\left|[\mathsf{T}^\mu, f]g\right|_{H^{1/2}} \leq C\left|f\right|_{H^{t_0+1}}\left|g\right|_{H^{-1/2}}.$$

We now provide improved commutator estimates for specific operators.

Lemma 2.7. *Let $d \in \{1,2\}$ and $t_0 > d/2$. If $d = 1$, there exists $C > 0$ such that for any $s \geq 0$ and any $r \geq s - t_0$,*

$$\big||D|(f|D|g) + \partial_x(f\partial_x g)\big|_{H^s} \leq C\big|\partial_x f\big|_{H^{t_0+r}}\big|\partial_x g\big|_{H^{s-r}}.$$

If $d = 2$, for any $0 \leq s \leq t_0 + 1$ and $s - t_0 \leq r \leq 1$, there exists $C > 0$ such that

$$\big||D|(f|D|g) + \nabla\cdot(f\nabla g)\big|_{H^s} \leq C\big|\nabla f\big|_{H^{t_0+r}}\big|\nabla g\big|_{H^{s-r}}.$$

Moreover, for any $s \geq 0$ and $r \leq 1$, there exists $C > 0$ such that

$$\big||D|(f|D|g) + \nabla\cdot(f\nabla g)\big|_{H^s} \leq C\big|\nabla f\big|_{H^{t_0+r}}\big|\nabla g\big|_{H^{s-r}} + C\big|\nabla f\big|_{H^s}\big|\nabla g\big|_{H^{t_0}}.$$

All the constants C above are independent of f, g such that the right-hand side is finite.

Proof. The case $d = 1$ follows from [25, Lemma 3.1] and the identity

$$|D|(f|D|g) + \partial_x(f\partial_x g) = [\mathcal{H}, \partial_x f](|D|g) + [\mathcal{H}, f](\partial_x|D|g),$$

where $\mathcal{H} \overset{\text{def}}{=} -\frac{\partial_x}{|D|}$ is the Hilbert transform. We however provide the short proof for the sake of completeness. Let us denote $a = |D|(f|D|g) + \partial_x(f\partial_x g)$. One has for almost any $\xi \geq 0$,

$$\begin{aligned}\widehat{a}(\xi) &= \int_{\mathbb{R}} \big(|\xi||\xi-\eta| - \xi(\xi-\eta)\big)\widehat{f}(\eta)\widehat{g}(\xi-\eta)\,\mathrm{d}\eta \\ &= 2\int_{\xi}^{\infty} \xi(\eta-\xi)\widehat{f}(\eta)\widehat{g}(\xi-\eta)\,\mathrm{d}\eta,\end{aligned}$$

and hence, using $|\xi| \leq |\eta|$ and $|\eta - \xi| \leq |\eta|$, one has for any $r' \geq 0$

$$\langle\xi\rangle^s|\widehat{a}(\xi)| \leq 2\int_{\mathbb{R}} \langle\eta\rangle^{s+r'}|\eta||\widehat{f}(\eta)|\langle\xi-\eta\rangle^{-r'}|\xi-\eta||\widehat{g}(\xi-\eta)|\,\mathrm{d}\eta.$$

We conclude by Young's inequality and the fact that $\langle\cdot\rangle^{-t_0} \in L^2(\mathbb{R})$, setting $r' = r + t_0 - s$.

When $d = 2$, [25, Lemma 3.3] is not sufficient to our purpose due to the restriction $r + s \leq 1$. However its proof may be adapted as follows. Let us denote $a = |D|(f|D|g) + \nabla\cdot(f\nabla g)$, and set $s \geq 0$. One has (almost everywhere)

$$\begin{aligned}\widehat{a}(\boldsymbol{\xi}) &= \int_{\mathbb{R}^2} \big(|\boldsymbol{\xi}||\boldsymbol{\xi}-\boldsymbol{\eta}| - \boldsymbol{\xi}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\big)\widehat{f}(\boldsymbol{\eta})\widehat{g}(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} \\ &= \int_{|\boldsymbol{\eta}|\geq|\boldsymbol{\xi}|/2} \big(|\boldsymbol{\xi}||\boldsymbol{\xi}-\boldsymbol{\eta}| - \boldsymbol{\xi}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\big)\widehat{f}(\boldsymbol{\eta})\widehat{g}(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} \\ &\quad + \int_{|\boldsymbol{\eta}|<|\boldsymbol{\xi}|/2} \big(|\boldsymbol{\xi}||\boldsymbol{\xi}-\boldsymbol{\eta}| - \boldsymbol{\xi}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\big)\widehat{f}(\boldsymbol{\eta})\widehat{g}(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} \\ &= I_1 + I_2.\end{aligned}$$

For I_1, using $|\boldsymbol{\xi}| \leq 2|\boldsymbol{\eta}|$, there exists $C > 0$ depending uniquely on $s \geq 0$ such that

$$\langle\boldsymbol{\xi}\rangle^s|I_1| \leq C\int_{\mathbb{R}^2} \langle\boldsymbol{\eta}\rangle^s|\boldsymbol{\eta}||\widehat{f}|(\boldsymbol{\eta})|\boldsymbol{\xi}-\boldsymbol{\eta}||\widehat{g}|(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}.$$

and it follows by Young's inequality, and the fact that $\langle\cdot\rangle^{-t_0} \in L^2$, that

$$\big|\langle\boldsymbol{\xi}\rangle^s I_1\big|_{L^2} \leq C\big|\nabla f\big|_{H^s}\big|\nabla g\big|_{H^{t_0}}.$$

We consider now I_2. We use that $|\boldsymbol{\xi}||\boldsymbol{\xi}-\boldsymbol{\eta}| - \boldsymbol{\xi}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) = |\boldsymbol{\xi}||\boldsymbol{\xi}-\boldsymbol{\eta}|(1-\cos\alpha)$ with $\alpha \to 0$ as $|\boldsymbol{\eta}|/|\boldsymbol{\xi}| \to 0$. In fact, $|\tan\alpha| \le \frac{2}{\sqrt{3}}|\boldsymbol{\eta}|/|\boldsymbol{\xi}|$ so there exists $c>0$ such that $(1-\cos\alpha) \le c|\boldsymbol{\eta}|^2/|\boldsymbol{\xi}|^2$ as long as $|\boldsymbol{\eta}|/|\boldsymbol{\xi}| \le 1/2$. Hence

$$\langle\boldsymbol{\xi}\rangle^s |I_2| \lesssim \langle\boldsymbol{\xi}\rangle^s \int_{|\boldsymbol{\eta}|<|\boldsymbol{\xi}|/2} \frac{|\boldsymbol{\eta}|^2}{|\boldsymbol{\xi}|} |\widehat{f}|(\boldsymbol{\eta})|\boldsymbol{\xi}-\boldsymbol{\eta}||\widehat{g}|(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}.$$

When $|\boldsymbol{\xi}| \le 1$ we have (since $|\boldsymbol{\eta}|^2/|\boldsymbol{\xi}|^2 \le \frac{1}{2}|\boldsymbol{\eta}|/|\boldsymbol{\xi}|$)

$$\langle\boldsymbol{\xi}\rangle^s |I_2| \lesssim \int_{|\boldsymbol{\eta}|<|\boldsymbol{\xi}|/2} |\boldsymbol{\eta}||\widehat{f}|(\boldsymbol{\eta})|\boldsymbol{\xi}-\boldsymbol{\eta}||\widehat{g}|(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta},$$

and we have the same estimate as for I_1. When $|\boldsymbol{\xi}| \ge 1$, $1/|\boldsymbol{\xi}| \le 2/\langle\boldsymbol{\xi}\rangle$, and we have

$$\langle\boldsymbol{\xi}\rangle^s |I_2| \lesssim \langle\boldsymbol{\xi}\rangle^{s-1} \int_{|\boldsymbol{\eta}|<|\boldsymbol{\xi}|/2} |\boldsymbol{\eta}|^2|\widehat{f}|(\boldsymbol{\eta})|\boldsymbol{\xi}-\boldsymbol{\eta}||\widehat{g}|(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}.$$

Since $s-1 \ge -t_0$ ($d=2$ and $s\ge 0$), and using the first estimate of Lemma 2.3 with $s_1 = s-1$, $s_2 = t_0$, $s_1' = t_0 - 1 + r$, $s_2' = s-r$, we have, for any $r \le 1$,

$$\big|\langle\boldsymbol{\xi}\rangle^s I_2\big|_{L^2} \lesssim \big|\nabla f\big|_{H^s}\big|\nabla g\big|_{H^{t_0}} + \big|\nabla f\big|_{H^{t_0+r}}\big|\nabla g\big|_{H^{s-r}}.$$

We have proved the last estimate. When $0 \le s \le t_0+1$, we use that by the above analysis we have (almost everywhere)

$$\langle\boldsymbol{\xi}\rangle^s |\widehat{a}|(\boldsymbol{\xi}) \lesssim \langle\boldsymbol{\xi}\rangle^{s-1} \int_{\mathbb{R}} \langle\boldsymbol{\eta}\rangle|\boldsymbol{\eta}||\widehat{f}|(\boldsymbol{\eta})|\boldsymbol{\xi}-\boldsymbol{\eta}||\widehat{g}|(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}.$$

The result follows by Lemma 2.2, since $s_1 := t_0 + r - 1 \ge s-1$, $s_2 := s-r \ge s-1$, and $s_1 + s_2 = t_0 + s - 1 > 0$ (recall $t_0 > d/2 = 1$). This concludes the proof. □

Remark 2.8. *The result of Lemma 2.7 exhibits a regularizing effect of order* 2*, and in particular improves the naive result obtained when using Lemmas 2.2 and 2.4 on the identity*

$$|D|(f|D|g) + \nabla\cdot(f\nabla g) = [|D|, f]|D|g + (\nabla f)\cdot(\nabla g).$$

This turns out to be crucial in our analysis.

We deduce the following "finite-depth" version of Lemma 2.7.

Lemma 2.9. *Let $d \in \{1,2\}$ and $t_0 > \frac{d}{2}$. Let $s, r \in \mathbb{R}$ such that $0 \le s \le t_0+1$ and $s - t_0 \le r \le 1$. There exists $C>0$ such that for any $\mu \ge 1$,*

$$\big|\mathsf{T}^\mu\cdot\nabla(f\mathsf{T}^\mu\cdot\nabla g) + \nabla\cdot(f\nabla g)\big|_{H^s} \le C\left(\tfrac{1}{\sqrt{\mu}}\big|f\big|_{L^2} + \big|\nabla f\big|_{H^{t_0+r}}\right)\big|\nabla g\big|_{H^{s-r}}.$$

Moreover, for any $s \ge 0$ and $r \le 1$, there exists a constant $C>0$ such that

$$\big|\mathsf{T}^\mu\cdot\nabla(f\mathsf{T}^\mu\cdot\nabla g) + \nabla\cdot(f\nabla g)\big|_{H^s} \le C\big|\nabla f\big|_{H^{t_0+r}}\big|\nabla g\big|_{H^{s-r}} + C\left(\tfrac{1}{\sqrt{\mu}}\big|f\big|_{L^2} + \big|\nabla f\big|_{H^s}\right)\big|\nabla g\big|_{H^{t_0}}.$$

All the constants C above are independent of f, g such that the right-hand side is finite.

Proof. We start with the first estimate. We first note that

$$\begin{aligned}|D|\tanh(\sqrt{\mu}|D|)(f|D|\tanh(\sqrt{\mu}|D|)g) =& |D|(\tanh(\sqrt{\mu}|D|)-1)(f|D|\tanh(\sqrt{\mu}|D|)g)\\ &+ |D|(f|D|(\tanh(\sqrt{\mu}|D|)-1)g) + |D|(f|D|g).\end{aligned}$$

We have for any $r' \geq 0$ and $\mu \in [1, +\infty)$

$$\begin{aligned}\big|\sqrt{\mu}|D|(\tanh(\sqrt{\mu}|D|)-1)(f|D|\tanh(\sqrt{\mu}|D|)g)\big|_{H^s} &\leq C_{r',s}\big|f|D|\tanh(\sqrt{\mu}|D|)g\big|_{H^{\min(t_0-r',0)}}\\ &\leq C'_{r',s}\big|f\big|_{H^{r'}}\big|\nabla g\big|_{t_0-r'},\end{aligned}$$

where we used Lemma 2.2 and $C_{r',s}$, $C'_{r',s}$ depend uniquely on t_0, $r' \geq 0$ and s. Furthermore, one has for any $\mu \in [1, +\infty)$ and any $r' \in \mathbb{R}$

$$\begin{aligned}\big|\sqrt{\mu}|D|(f(\tanh(\sqrt{\mu}|D|)-1)|D|g)\big|_{H^s} &\leq C_{t_0,s}\sqrt{\mu}\big|\nabla f\big|_{H^s}\big|(\tanh(\sqrt{\mu}|D|)-1)|D|g\big|_{H^{\max(t_0,s)}}\\ &\quad + C_{t_0,s}\big|f\big|_{H^s}\big|\sqrt{\mu}|D|(\tanh(\sqrt{\mu}|D|)-1)|D|g\big|_{H^{\max(t_0,s)}}\\ &\leq C_{t_0,s}C''_{r',s}\big(\big|f\big|_{H^s}+\sqrt{\mu}\big|\nabla f\big|_{H^s}\big)\big|\nabla g\big|_{H^{t_0-r'}}\end{aligned}$$

where we used Lemma 2.2 and $C_{t_0,s}, C''_{r',s}$ depend uniquely on t_0, r' and s. The desired estimates now follow from the triangular inequality, the fact that for any $\theta \in \mathbb{R}$, $\big|f\big|_{H^\theta} \lesssim \big|f\big|_{L^2} + \big|\nabla f\big|_{H^{\theta-1}}$, the above estimates and Lemma 2.7. □

By a similar compensation mechanism as in Lemma 2.7, we infer the following result that allows us to define a quantity at low regularity that could not be defined if one only use the product estimate of Lemma 2.2.

Lemma 2.10. *Let $d \in \{1,2\}$ and $t_0 > d/2$. Set $\sigma \in [\frac{1}{2}, 1]$. The bilinear map*

$$B : (f,g) \in \mathring{H}^1(\mathbb{R}^d) \times \mathring{H}^1(\mathbb{R}^d) \mapsto (|D|f)(|D|g) - (\nabla f)\cdot(\nabla g) \in L^1(\mathbb{R}^d) \subset H^{-t_0}(\mathbb{R}^d)$$

extends as a continuous bilinear map from $\mathring{H}^\sigma(\mathbb{R}^d) \times \mathring{H}^\sigma(\mathbb{R}^d)$ to $H^{2\sigma-2-t_0}(\mathbb{R}^d)$.

Proof. The fact that B is defined follows from Lemma 2.2. We assume below that $f, g \in \mathring{H}^1(\mathbb{R}^d)$, and the result follows by a density argument from the desired estimate. One has (almost everywhere)

$$\begin{aligned}\widehat{B(f,g)}(\boldsymbol{\xi}) &= \int_{\mathbb{R}^d} \big(|\boldsymbol{\eta}||\boldsymbol{\xi}-\boldsymbol{\eta}| + \boldsymbol{\eta}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\big)\widehat{f}(\boldsymbol{\eta})\widehat{g}(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}\\ &= \int_{\Omega} \big(|\boldsymbol{\eta}||\boldsymbol{\xi}-\boldsymbol{\eta}| + \boldsymbol{\eta}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\big)\widehat{f}(\boldsymbol{\eta})\widehat{g}(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}\\ &\quad + \int_{\mathbb{R}^d\setminus\Omega} \big(|\boldsymbol{\eta}||\boldsymbol{\xi}-\boldsymbol{\eta}| + \boldsymbol{\eta}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\big)\widehat{f}(\boldsymbol{\eta})\widehat{g}(\boldsymbol{\xi}-\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}\\ &= I_1 + I_2.\end{aligned}$$

where we define $\Omega \overset{\text{def}}{=} \big\{(\boldsymbol{\xi},\boldsymbol{\eta}) \in \mathbb{R}^d \times \mathbb{R}^d,\ |\boldsymbol{\eta}| \leq 2\langle\boldsymbol{\xi}\rangle\big\}$. Using that on Ω, $\langle\boldsymbol{\xi}\rangle^{-s_0} \leq 3^{s_0}\langle\boldsymbol{\eta}\rangle^{-s_0}$ for any $s_0 \geq 0$, we have by Lemma 2.2 (with $s = -t_0$)

$$\big|\langle\boldsymbol{\xi}\rangle^{-s_0-t_0} I_1\big|_{L^2} \leq C\big|\nabla f\big|_{H^{s_1-s_0}}\big|\nabla g\big|_{H^{s_2}}$$

as soon as $s_1 + s_2 \geq 0$ and $s_1, s_2 \geq -t_0$. We choose $s_1 = -s_2 = s_0/2$ with $s_0 = 2(1-\sigma) \in [0, 2t_0]$.

On $\mathbb{R}^d \setminus \Omega$, we have $|\boldsymbol{\eta}| \geq 2\langle\boldsymbol{\xi}\rangle \geq 2\max(1, |\boldsymbol{\xi}|)$. Proceeding as in the proof of Lemma 2.7, we infer that there exists $c > 0$ such that for any $s_0 \in [0,2]$ when $d = 2$ (and for any $s_0 \geq 0$ when $d = 1$)

$$\big||\boldsymbol{\eta}||\boldsymbol{\xi}-\boldsymbol{\eta}| - \boldsymbol{\xi}\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})\big| \lesssim |\boldsymbol{\eta}||\boldsymbol{\xi}-\boldsymbol{\eta}|\frac{\langle\boldsymbol{\xi}\rangle^{s_0}}{\langle\boldsymbol{\eta}\rangle^{s_0}}.$$

Lemma 2.2 yields once again

$$\big|\langle\boldsymbol{\xi}\rangle^{-s_0-t_0} I_1\big|_{L^2} \leq C\big|\nabla f\big|_{H^{-s_0/2}}\big|\nabla g\big|_{H^{-s_0/2}}$$

when $s_0 \in [0, 2t_0]$ (and $s_0 \in [0,2]$ when $d = 2$). Setting one again $s_0 = 2(1-\sigma)$ proves the result. □

We deduce the following "finite-depth" version of Lemma 2.10, which allows (among other things) to define the second equations in (WW2) and (RWW2) when $\psi \in \mathring{H}^{1/2}(\mathbb{R}^d)$, corresponding to the natural energy space defined by the corresponding Hamiltonians, (1.2) and (1.3).

Lemma 2.11. *Let $d \in \{1,2\}$ and $t_0 > d/2$. Set $\sigma \in [1/2, 1]$. For any $\mu \geq 1$, the bilinear map*

$$B^\mu : (f,g) \in \mathring{H}^1(\mathbb{R}^d) \times \mathring{H}^1(\mathbb{R}^d) \mapsto (\mathsf{T}^\mu \cdot \nabla f)(\mathsf{T}^\mu \cdot \nabla g) - (\nabla f) \cdot (\nabla g) \in L^1(\mathbb{R}^d) \subset H^{-t_0}(\mathbb{R}^d)$$

extends as a continuous bilinear map from $\mathring{H}^\sigma(\mathbb{R}^d) \times \mathring{H}^\sigma(\mathbb{R}^d)$ to $H^{2\sigma-2-t_0}(\mathbb{R}^d)$. Moreover there exists a constant $C > 0$, depending only on t_0 and σ, such that for any $f, g \in \mathring{H}^\sigma(\mathbb{R}^d)$,

$$\big|B^\mu(f,g)\big|_{H^{2\sigma-2-t_0}} \leq C \big|\nabla f\big|_{H^{\sigma-1}} \big|\nabla g\big|_{H^{\sigma-1}}.$$

Proof. By Lemma 2.10, we need to prove the corresponding result on

$$B^\mu - B : (f,g) \in \mathring{H}^1(\mathbb{R}^d) \times \mathring{H}^1(\mathbb{R}^d) \mapsto (\mathsf{T}^\mu \cdot \nabla f)(\mathsf{T}^\mu \cdot \nabla g) - (|D|f)(|D|g).$$

We rewrite

$$(B^\mu - B)(f,g) = (|D|(\tanh(\sqrt{\mu}|D| - 1)f)(\mathsf{T}^\mu \cdot \nabla g) + (|D|f)(|D|(\tanh(\sqrt{\mu}|D| - 1)g)$$

Yet since for any $s \in \mathbb{R}$ there exists $C > 0$ such that

$$\big|(|D|(\tanh(\sqrt{\mu}|D| - 1)f)\big|_{H^s} \leq \big|(|D|(\tanh(|D| - 1)f)\big|_{H^s} \leq C\big|\nabla f\big|_{H^{\sigma-1}}$$

and $\big|\mathsf{T}^\mu \cdot \nabla g\big|_{H^{\sigma-1}} \leq \big|\nabla g\big|_{H^{\sigma-1}}$ by Lemma 2.1, we infer immediately from Lemma 2.2 that

$$\big|(|D|(\tanh(\sqrt{\mu}|D| - 1)f)(\mathsf{T}^\mu \cdot \nabla g)\big|_{H^{2\sigma-2-t_0}} \lesssim C\big|\nabla f\big|_{H^{\sigma-1}} \big|\nabla g\big|_{H^{\sigma-1}}.$$

By similar considerations on the second term and triangular inequality, we obtain the desired estimate, and the proof is complete. □

3 Well-posedness results and proof of Theorem 1.2

In this section we prove several well-posedness results on the initial-value problem for (RWW2), which we rewrite below for the sake of readability :

$$\text{(RWW2)} \qquad \begin{cases} \partial_t \zeta - \mathsf{T}^\mu \cdot \nabla \psi + \epsilon \mathsf{T}^\mu \cdot \nabla((\mathsf{J}\zeta)\mathsf{T}^\mu \cdot \nabla\psi) + \epsilon \nabla \cdot ((\mathsf{J}\zeta)\nabla\psi) = 0, \\ \partial_t \psi + \zeta + \frac{\epsilon}{2}\mathsf{J}\left(|\nabla\psi|^2 - (\mathsf{T}^\mu \cdot \nabla\psi)^2\right) = 0. \end{cases}$$

We prove in particular Theorem 1.2, as a consequence of the following two results.

We start with the "small-time" existence and uniqueness of solutions expressing the semilinear nature of the system (for sufficiently regular data) as soon as J is regularizing of order -1.

Proposition 3.1. *Let $d \in \mathbb{N}^\star$, $t_0 > d/2$, $s \geq 0$ and $C > 1$. There exists $T_0 > 0$ such that for any $\mu \geq 1$, any rectifier $\mathsf{J} = J(D)$ be regularizing of order $-\max(1, t_0 + \frac{3}{2} - s)$ (see Definition 1.1) and any $(\zeta_0, \psi_0) \in H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d)$ the following holds. There exists $T^\star, T_\star \in (0, +\infty]$ and a unique $(\zeta, \psi) \in \mathcal{C}((-T_\star, T^\star); H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d))$ maximal solution to* (RWW2) *with initial data $(\zeta, \psi)\big|_{t=0} = (\zeta_0, \psi_0)$. Moreover, one has $(\partial_t\zeta, \partial_t\psi) \in \mathcal{C}((-T_\star, T^\star); H^{s-1/2}(\mathbb{R}^d) \times H^s(\mathbb{R}^d))$ and $\min(T_\star, T^\star) > T_1$ with*

$$T_1 \stackrel{\text{def}}{=} \frac{T_0}{\epsilon\big(\big|\zeta_0\big|_{H^{\min(s, t_0+\frac{1}{2})}} + \big|\nabla\psi_0\big|_{H^{\min(s-\frac{1}{2}, t_0)}}\big)\mathcal{N}^{-\max(1, t_0+\frac{3}{2}-s)}(J)}$$

and [3]

$$\max_{t\in[-T_1,T_1]}\left(\left|\zeta(t,\cdot)\right|_{H^s}^2+\left|\mathfrak{P}^\mu\psi(t,\cdot)\right|_{H^s}^2\right)\leq C\left(\left|\zeta_0\right|_{H^s}^2+\left|\mathfrak{P}^\mu\psi_0\right|_{H^s}^2\right),$$

where we recall that $\mathfrak{P}^\mu\stackrel{\text{def}}{=}\left(|D|\tanh(\sqrt{\mu}|D|)\right)^{1/2}$.

Then, we give a "large-time" result under some hyperbolic-type condition. First, we define the Rayleigh–Taylor operator $\mathfrak{a}^\mu$ as

$$\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]f\stackrel{\text{def}}{=}f-\epsilon(\mathsf{T}^\mu\cdot\nabla\mathsf{J}\zeta)\mathsf{J}f-\epsilon^2\mathsf{J}\Big((\mathsf{T}^\mu\cdot\nabla\psi)|D|\big\{(\mathsf{T}^\mu\cdot\nabla\psi)\mathsf{J}f\big\}\Big). \tag{3.1}$$

Then, we introduce the energy, for $N\in\mathbb{N}^\star$,

$$\begin{aligned}\mathcal{E}^N(\zeta,\psi)=&\sum_{\alpha\in\mathbb{N}^d,|\alpha|\leq N-1}\left(\left|\partial^\alpha\zeta\right|_{L^2}^2+\left|\mathfrak{P}^\mu\partial^\alpha\psi\right|_{L^2}^2\right)\\&+\sum_{\alpha\in\mathbb{N}^d,|\alpha|=N}\left(\zeta_{(\alpha)},\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]\zeta_{(\alpha)}\right)_{L^2}+\left|\mathfrak{P}^\mu\psi_{(\alpha)}\right|_{L^2}^2\end{aligned} \tag{3.2}$$

with $\zeta_{(\alpha)}\stackrel{\text{def}}{=}\partial^\alpha\zeta$ and $\psi_{(\alpha)}\stackrel{\text{def}}{=}\partial^\alpha\psi-\epsilon(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\partial^\alpha\zeta)$.

Proposition 3.2. *Let $d\in\{1,2\}$, $t_0>d/2$ and $N\in\mathbb{N}$ with $N\geq t_0+2$. Let $M_J>0$, $M_U>0$, $\mathfrak{a}^\mu_\star>0$ and $C>1$. There exists $T_0>0$ such that for any $\epsilon>0$, $\mu\geq1$, and any regular rectifier J (see Definition 1.1) satisfying $\left|J\right|_{L^\infty}+\left|\langle\boldsymbol{\xi}\rangle\nabla J\right|_{L^\infty}\leq M_J$, the following holds. For any $(\zeta_0,\psi_0)\in H^N(\mathbb{R}^d)\times\mathring{H}^{N+\frac12}(\mathbb{R}^d)$ satisfying*

$$0<\epsilon M_0\stackrel{\text{def}}{=}\epsilon\sqrt{\mathcal{E}^{\lceil t_0\rceil+2}(\zeta_0,\psi_0)}\leq M_U$$

and the Rayleigh–Taylor condition

$$\forall f\in L^2(\mathbb{R}^d)\ ,\ \left(f,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta_0,\epsilon\nabla\psi_0]f\right)_{L^2}\geq\mathfrak{a}^\mu_\star\left|f\right|_{L^2}^2, \tag{3.3}$$

the maximal solution to (RWW2) *with initial data $(\zeta,\psi)\left|_{t=0}\right.=(\zeta_0,\psi_0)$ satisfies $\min(T_\star,T^\star)>T_2$ where*

$$T_2\stackrel{\text{def}}{=}\frac{T_0}{\epsilon M_0+\epsilon^2M_0^2\dot{\mathcal{N}}^{-1/2}(J)^2},$$

and for any $0\leq|t|\leq T_2$ and any $N_\star\in\{\lceil t_0\rceil+2,N\}$,

$$\mathcal{E}^{N_\star}(\zeta(t,\cdot),\psi(t,\cdot))\leq C\,\mathcal{E}^{N_\star}(\zeta_0,\psi_0).$$

The scope of Propositions 3.1 and 3.2 is more easily seen when considering a family of regular Fourier multipliers $\mathsf{J}=J(\delta D)$ with $\delta\ll1$. As $\delta\searrow0$, the lower bounds on the time of existence is of magnitude at least $T_1\approx\delta\epsilon^{-1}$ (when s is sufficiently large) in Proposition 3.1, and of magnitude at least $T_2\approx\min(\epsilon^{-1},\delta\epsilon^{-2})$ in Proposition 3.2. Hence the first statement provides the unconditional small-time existence (and control) of solutions while the second statement provides a conditional large-time existence (and control) of solutions in the small steepness framework ($\epsilon\ll1$) and for weak rectification ($\delta\ll1$).

The first result ("small-time" existence) follows from standard techniques on semilinear dispersive equations. The proof of Proposition 3.1 is provided on Section 3.1, and a blow-up criterion is

[3] We also have

$$\max_{t\in[-T_1,T_1]}\left(\left|(\mathfrak{P}^\mu)^{-1}\nabla\zeta(t,\cdot)\right|_{H^{s+\frac12}}^2+\left|\nabla\psi(t,\cdot)\right|_{H^{s+\frac12}}^2\right)\leq C\left(\left|(\mathfrak{P}^\mu)^{-1}\nabla\zeta_0\right|_{H^{s+\frac12}}^2+\left|\nabla\psi_0\right|_{H^{s+\frac12}}^2\right),$$

which is not used afterward but provides an additional control when $\mu=\infty$.

stated in Corollary 3.7. The difficult part consists in proving Proposition 3.2, *i.e.* the "large-time" existence. This relies on careful *a priori* energy estimates on smooth solutions. We divide the proof in several parts: first in Section 3.2.1 we extract simple sets of equations satisfied by smooth solutions and their derivatives. Based on these equations, and assuming that a certain hyperbolicity criterion holds, we obtain in Section 3.2.2 energy estimates, that is a differential inequality satisfied by the functional $\mathcal{E}^N$. The completion of the proof of Proposition 3.2 is provided in Section 3.2.3.

An additional global-in-time well-posedness (for small initial data) result is stated and proved in Section 3.3, based on the low-regularity well-posedness result provided in Proposition 3.1, and the fact that the Hamiltonian function, (1.3), is an invariant quantity.

We conclude this introduction with several remarks, followed by the completion of the proof of Theorem 1.2, as a consequence of Propositions 3.1 and 3.2.

Remark 3.3. *The operator $\mathfrak{a}^\mu$ defined in* (3.1) *is a key ingredient of our analysis. Notice that the function*

$$a[\epsilon\zeta, \epsilon\nabla\psi] \stackrel{\text{def}}{=} 1 - \epsilon(\mathsf{T}^\mu \cdot \nabla\zeta)$$

is the $\mathcal{O}(\epsilon^2)$ approximation of the Rayleigh–Taylor coefficient appearing for instance in [18, (4.20), see also (4.27) and discussion in §4.3.5]. We claim that the last term in (3.1)*, which has no counterpart in the fully nonlinear water waves system, is responsible for the observed spurious oscillations in numerical simulations, as the consequence of the ill-posedness of the initial-value problem. Indeed, without any regularization (that is setting $\mathsf{J} = \mathrm{Id}$), the Rayleigh–Taylor condition* (3.3) *can never be satisfied unless $\epsilon\nabla\psi_0 = \mathbf{0}$.*

Another key ingredient of our "large-time" analysis is the use of $\psi_{(\alpha)}$ in lieu of $\partial^\alpha\psi$ when defining the energy functional $\mathcal{E}^N(\zeta,\psi)$. Setting $\mathsf{J} = \mathrm{Id}$, we recognize in $\psi_{(\alpha)}$ a $\mathcal{O}(\epsilon^2)$ approximation of Alinhac's good unknowns for the water waves system; see discussion in [18, §4.1]. Notice that using that the rectifier operators J are regularizing of order $-1/2$, we infer (in contrast with the water waves situation) that under the Rayleigh–Taylor condition,

$$\mathcal{E}^N(\zeta,\psi) \approx \big|\zeta\big|_{H^N} + \big|\mathfrak{P}^\mu\psi\big|_{H^N},$$

although the equivalence is not uniform with respect to J (consider $\mathsf{J} = J(\delta|D|)$ with $\delta \searrow 0$); see Lemma 3.11.

Remark 3.4. *We claim the second result holds for rectifiers J regularizing of order $-1/2$ and not -1. Yet in this case* (RWW2) *is of quasilinear nature and the well-posedness theory requires additional arguments which we decided to avoid for simplicity.*

Remark 3.5. *With a small adaptation of this work, one can consider non integer regularities $N = s \in \mathbb{R}$ with $s > \frac{d}{2} + 2$. Then we need to replace the functional* (3.2) *with*

$$\begin{aligned}\mathcal{E}^s(U) = \sum_{\alpha\in\mathbb{N}^d, |\alpha|\le\lceil s\rceil-1} \Big(\big|\partial^\alpha\zeta\big|_{L^2}^2 + \big|\mathfrak{P}^\mu\partial^\alpha\psi\big|_{L^2}^2\Big) \\ + \big(|D|^s\zeta, \mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta, \epsilon\nabla\psi]|D|^s\zeta\big)_{L^2} + \big|\mathfrak{P}^\mu\psi_{(s)}\big|_{L^2}^2\end{aligned}$$

with $\psi_{(s)} \stackrel{\text{def}}{=} |D|^s\psi - \epsilon(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}|D|^s\zeta)$.

Let us now show how Theorem 1.2, which we recall below for the convenience of the reader, follows from Proposition 3.1 and Proposition 3.2.

Theorem 3.6. *Let $d \in \{1,2\}$, $t_0 > \frac{d}{2}$, $N \in \mathbb{N}$ with $N \ge t_0 + 2$, $C > 1$ and $M > 0$. Set $\mathsf{J}_0 = J_0(D)$ a regular rectifier. There exists $T_0 > 0$ such that for any $\mu \ge 1$ and $\epsilon > 0$, for any $(\zeta_0,\psi_0) \in H^N(\mathbb{R}^d) \times \mathring{H}^{N+\frac12}(\mathbb{R}^d)$ such that*

$$0 < \epsilon M_0 \stackrel{\text{def}}{=} \epsilon\big(\big|\zeta_0\big|_{H^{\lceil t_0\rceil+2}} + \big|\mathfrak{P}^\mu\psi_0\big|_{H^{\lceil t_0\rceil+2}}\big) \le M,$$

and for any $\delta \geq \epsilon M_0$, the following holds.

Defining $\mathsf{J} = J_0(\delta D)$, there exists a unique $(\zeta, \psi) \in \mathcal{C}([0, T_0/(\epsilon M_0)]; H^N(\mathbb{R}^d) \times \mathring{H}^{N+\frac{1}{2}}(\mathbb{R}^d))$ solution to (RWW2) *with initial data $(\zeta, \psi)\big|_{t=0} = (\zeta_0, \psi_0)$, and it satisfies*

$$\sup_{t\in[0,T_0/(\epsilon M_0)]} \big(\big|\zeta(t,\cdot)\big|_{H^N}^2 + \big|\mathfrak{P}^\mu\psi(t,\cdot)\big|_{H^N}^2\big) \leq C\left(\big|\zeta_0\big|_{H^N}^2 + \big|\mathfrak{P}^\mu\psi_0\big|_{H^N}^2\right).$$

Proof of Theorem 1.2/3.6. For any $\delta > 0$, let$(\zeta^\delta, \psi^\delta) \in \mathcal{C}([0, T_\delta^\star); H^N(\mathbb{R}^d) \times \mathring{H}^{N+\frac{1}{2}}(\mathbb{R}^d))$ be the maximal solution for positive times (the result for negative times following from time-reversibility of the equations) to (RWW2), for $\mathsf{J} = J_0(\delta|D|)$, with initial data $(\zeta^\delta, \psi^\delta)\big|_{t=0} = (\zeta_0, \psi_0)$, provided by Proposition 3.1.

We start with some preliminary estimates in the case $\delta \in (0, 1]$. First by Lemma 2.1, the continuous Sobolev embedding $H^{t_0}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ and Lemma 2.2, there exists $C_1 > 0$, depending only on $t_0 > d/2$ such that, for all $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$,

$$\big|(\mathsf{T}^\mu \cdot \nabla \mathsf{J}\zeta)(\mathsf{J}f)\big|_{L^\infty} \leq C_1 \big|J_0\big|_{L^\infty}^2 \big|\zeta\big|_{H^{t_0+1}} \big|f\big|_{L^2},$$

and, for all $\psi \in \mathring{H}^{t_0+1}(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$,

$$\begin{aligned}
\big|\mathsf{J}\Big((\mathsf{T}^\mu \cdot \nabla\psi)|D|\big\{(\mathsf{T}^\mu \cdot \nabla\psi)\mathsf{J}f\big\}\Big)\big|_{L^2} &\leq \delta^{-1/2}\big|\langle\cdot\rangle^{1/2} J_0\big|_{L^\infty}\big|(\mathsf{T}^\mu \cdot \nabla\psi)|D|\big\{(\mathsf{T}^\mu \cdot \nabla\psi)\mathsf{J}f\big\}\big|_{H^{-1/2}} \\
&\leq C_1\delta^{-1/2}\big|\langle\cdot\rangle^{1/2} J_0\big|_{L^\infty}\big|\mathsf{T}^\mu \cdot \nabla\psi\big|_{H^{t_0}}\big|(\mathsf{T}^\mu \cdot \nabla\psi)(\mathsf{J}f)\big|_{H^{1/2}} \\
&\leq C_1^2\delta^{-1}\big|\langle\cdot\rangle^{1/2} J_0\big|_{L^\infty}^2\big|\mathsf{T}^\mu \cdot \nabla\psi\big|_{H^{t_0}}^2\big|f\big|_{L^2}.
\end{aligned}$$

It follows that there exists $M_1 > 0$ depending only on t_0 and $\big|\langle\cdot\rangle^{1/2} J_0\big|_{L^\infty}$ such that for any $(\zeta, \psi) \in H^{t_0+1}(\mathbb{R}^d) \times \mathring{H}^{t_0+1}(\mathbb{R}^d)$ satisfying

$$\epsilon\big|\zeta\big|_{H^{t_0+1}} + \epsilon^2\delta^{-1}\big|\mathsf{T}^\mu \cdot \nabla\psi\big|_{H^{t_0}}^2 \leq M_1, \tag{3.4}$$

the operator $\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta, \epsilon\nabla\psi]$ satisfies the Rayleigh–Taylor condition (3.3) with $\mathfrak{a}^\mu_\star = 1/2$. This holds in particular, using that $\delta \geq \epsilon M_0$ and Lemma 2.1, if

$$0 < \epsilon M_0 = \epsilon\big(\big|\zeta_0\big|_{H^{\lceil t_0\rceil+2}} + \big|\mathfrak{P}^\mu\psi_0\big|_{H^{\lceil t_0\rceil+2}}\big) \leq M_1.$$

Then, using as above Lemmas 2.1 and 2.2, we define $C_2 > 0$, depending only on $t_0 > d/2$, such that for all $\psi \in \mathring{H}^{t_0+1}(\mathbb{R}^d)$ and $f \in L^2(\mathbb{R}^d)$,

$$\big|(\mathsf{T}^\mu \cdot \nabla\psi)(\mathsf{J}f)\big|_{H^{1/2}} \leq C_2\delta^{-1/2}\big|\langle\cdot\rangle^{1/2} J_0\big|_{L^\infty}\big|\mathfrak{P}^\mu\psi\big|_{H^{t_0+1/2}}\big|f\big|_{L^2}.$$

Using this estimate on the second term of $\psi_{(\alpha)} \overset{\text{def}}{=} \partial^\alpha\psi - \epsilon(\mathsf{T}^\mu \cdot \nabla\psi)(\mathsf{J}\partial^\alpha\zeta)$, Lemma 2.1 as well as the above analysis on $\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta, \epsilon\nabla\psi]$, we infer that for any $C' > 1$, there exists $M' > 0$ depending only on $t_0 > d/2$, $\big|\langle\cdot\rangle^{1/2} J_0\big|_{L^\infty}$, $\big|J_0\big|_{L^\infty}$ and C' such that for any $(\zeta, \psi) \in H^N(\mathbb{R}^d) \times \mathring{H}^{N+1/2}(\mathbb{R}^d)$ satisfying

$$\epsilon\big|\zeta\big|_{H^{t_0+1}} \leq M' \qquad \text{and} \qquad \epsilon^2\delta^{-1}\big|\mathfrak{P}^\mu\psi\big|_{H^{t_0+1/2}}^2 \leq M', \tag{3.5}$$

then for any $N_\star \in \{\lceil t_0\rceil + 2, N\}$, we have

$$\frac{1}{C'}\mathcal{E}^{N_\star}(\zeta, \psi) \leq \big|\zeta\big|_{H^{N_\star}}^2 + \big|\mathfrak{P}^\mu\psi\big|_{H^{N_\star}}^2 \leq C'\mathcal{E}^{N_\star}(\zeta, \psi). \tag{3.6}$$

We can now prove the proposition. We define $C' = C^{1/3} > 1$ and we introduce $M' > 0$ accordingly so that (3.5) yields (3.6). We consider two cases. Firstly, if $\epsilon M_0 \geq \min(M_1, \frac{M'}{C})$ or

$\delta > 1$, then $\delta \geq \min(M_1, \frac{M'}{C}, 1) \stackrel{\rm def}{=} \delta_0$ where δ_0 depends only on $t_0 > d/2$, $\big|\langle\cdot\rangle^{1/2} J_0\big|_{L^\infty}$, $\big|J_0\big|_{L^\infty}$, and $C > 1$. Therefore we can simply use Proposition 3.1 with $s = N$, using that

$$\mathcal{N}^{-1}(J) = \big|\langle\cdot\rangle J_0(\delta\cdot)\big|_{L^\infty} \leq \big|\frac{\langle\cdot\rangle}{\langle\delta\cdot\rangle}\big|_{L^\infty} \big|\langle\delta\cdot\rangle J_0(\delta\cdot)\big|_{L^\infty} \leq \delta_0^{-1} \big|\langle\cdot\rangle J_0\big|_{L^\infty}.$$

Secondly, we assume that $\epsilon M_0 < \min(M_1, \frac{M'}{C})$ and $\delta \in (0, 1]$. Thanks to the preliminary estimates we can apply Proposition 3.2 with $M_U = \sqrt{C'} M$, $\mathfrak{a}^\mu_\star = 1/2$ and amplification factor $C'' \in (1, \frac{C}{(C')^2})$. Hence there exists $\widetilde{T}_0$, depending only on t_0, N, $\big|J_0\big|_{L^\infty}$, $\big|\langle\cdot\rangle \nabla J_0\big|_{L^\infty}$, M and C such that $T^\star_\delta > T_2$ where (using again that $\delta \geq \epsilon M_0$ and (3.6))

$$T_2 \stackrel{\rm def}{=} \frac{\widetilde{T}_0}{\epsilon\sqrt{\mathcal{E}^{\lceil t_0\rceil+2}(\zeta_0,\psi_0)} + \epsilon^2 \mathcal{E}^{\lceil t_0\rceil+2}(\zeta_0,\psi_0) \big|\big|\cdot|^{\frac12} J_0(\delta\cdot)\big|^2_{L^\infty}} \geq \frac{\widetilde{T}_0}{\epsilon M_0(\sqrt{C'} + C' \big|\big|\cdot|^{\frac12} J_0(\cdot)\big|^2_{L^\infty})},$$

and for any $0 \leq t \leq T_2$ and any $N_\star \in \{\lceil t_0\rceil + 2, N\}$,

$$\mathcal{E}^{N_\star}(\zeta^\delta(t,\cdot), \psi^\delta(t,\cdot)) \leq C''\, \mathcal{E}^{N_\star}(\zeta_0,\psi_0). \tag{3.7}$$

The above estimate provides the desired control provided that (3.5) and hence (3.6) holds. Using that $C''(C')^2 < C$, $\epsilon M_0 C \leq M'$ and $\delta \geq \epsilon M_0$, we infer from (3.7) with $N_\star = \lceil t_0\rceil + 2$ and the continuity of $(\zeta^\delta(t,\cdot), \mathfrak{P}^\mu \psi^\delta(t,\cdot))$ in $H^{\lceil t_0\rceil+2}(\mathbb{R}^d)^2$ that

$$I \stackrel{\rm def}{=} \Big\{ t \in [0, T_2] \ : \ \big|\zeta^\delta(t,\cdot)\big|^2_{H^{\lceil t_0\rceil+2}} + \big|\mathfrak{P}^\mu \psi^\delta(t,\cdot)\big|^2_{H^{\lceil t_0\rceil+2}} \leq C\big(\big|\zeta_0\big|^2_{H^{\lceil t_0\rceil+2}} + \big|\mathfrak{P}^\mu \psi_0\big|^2_{H^{\lceil t_0\rceil+2}}\big) \Big\}$$

is an open subset of $[0, T_2]$. Since it is also closed and non-empty, we have $I = [0, T_2]$. In particular we have that (3.6) holds on $[0, T_2]$, and hence (3.7) with $N_\star = N$ provides the desired control, which concludes the proof. □

3.1 Short-time well-posedness. Proof of Proposition 3.1

We now prove Proposition 3.1. We write (RWW2) as

$$\partial_t U + \mathsf{L} U + N(U) \ = \ 0 \tag{3.8}$$

where $U = (\zeta, \psi)^\top$,

$$\mathsf{L} = \begin{pmatrix} 0 & -\mathsf{T}^\mu \cdot \nabla \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(\mathfrak{P}^\mu)^2 \\ 1 & 0 \end{pmatrix}$$

and

$$N(U) = \begin{pmatrix} \epsilon \mathsf{T}^\mu \cdot \nabla((\mathsf{J}\zeta)\mathsf{T}^\mu \cdot \nabla\psi) + \epsilon \nabla \cdot ((\mathsf{J}\zeta)\nabla\psi) \\ \frac{\epsilon}{2}\mathsf{J}\left(|\nabla\psi|^2 - (\mathsf{T}^\mu \cdot \nabla\psi)^2\right) \end{pmatrix}.$$

Let $X^s \stackrel{\rm def}{=} H^s(\mathbb{R}^d) \times (\mathring{H}^{s+1/2}(\mathbb{R}^d)/\mathbb{R})$ be the Hilbert space (see [18, Proposition 2.3] concerning the quotient space $\mathring{H}^{s+1/2}(\mathbb{R}^d)/\mathbb{R}$) endowed with the inner product

$$\big(\ (\zeta_1,\psi_1)\ ,\ (\zeta_2,\psi_2)\ \big)_{X^s} \stackrel{\rm def}{=} \int_{\mathbb{R}^d} (\Lambda^s\zeta_1)(\Lambda^s\zeta_2) + (\Lambda^s \mathfrak{P}^\mu \psi_1)(\Lambda^s \mathfrak{P}^\mu \psi_2)\, {\rm d}\boldsymbol{x}.$$

We denote $|\cdot|_{X^s}$ the norm associated with the inner product $(\cdot,\cdot)_{X^s}$. Using Lemma 2.1, one easily checks that the (unbounded) operator $\mathsf{i}\mathsf{L}$ with domain $X^{s+1/2}$ is self-adjoint on X^s. Hence by Stone's theorem (see *e.g.* [23, Theorem 10.8]) L generates a strongly continuous group of unitary operators on $\big(X^s, \big|\cdot\big|_{X^s}\big)$, which we denote $e^{t\mathsf{L}}$.

By Lemma 2.9, Lemma 2.11 (when $0 \leq s < \frac{1}{2}$) and Lemma 2.3, we have that there exists $C_1 > 0$ and $C_2 > 0$ depending only on s and $t_0 > d/2$ such that for any $U \in H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d)$,

$$\big|N(U)\big|_{H^s \times H^{s+1/2}} \leq \begin{cases} \epsilon C_1 \mathcal{N}^{s-t_0-\frac{3}{2}}(J) \big|U\big|^2_{H^s \times \mathring{H}^{s+1/2}} & \text{if } 0 \leq s \leq t_0 + \frac{1}{2}, \\ \epsilon C_1 \mathcal{N}^{-1}(J) \big|U\big|_{H^{t_0+\frac{1}{2}} \times \mathring{H}^{t_0+1}} \big|U\big|_{H^s \times \mathring{H}^{s+1/2}} & \text{if } s \geq t_0 + \frac{1}{2}, \end{cases}$$

and for any $U, V \in H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d)$

$$\big|N(U) - N(V)\big|_{H^s \times H^{s+1/2}} \leq \epsilon C_2 \mathcal{N}^{-\max(1, t_0+\frac{3}{2}-s)}(J) \big(\big|U+V\big|_{H^s \times \mathring{H}^{s+\frac{1}{2}}}\big) \big|U-V\big|_{H^s \times \mathring{H}^{s+1/2}}.$$

Recall (see Lemma 2.1) that $\big(X^s, |\cdot|_{X^s}\big)$ is equivalent to $H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d)/\mathbb{R}$ and that for any $\mu \geq 1$ and any $(f, g) \in H^s(\mathbb{R}^d) \times H^{s+1/2}(\mathbb{R}^d)$,

$$\big|f\big|^2_{H^s} + \big|\nabla g\big|^2_{H^{s-1/2}} \;\leq\; \big|(f,g)\big|^2_{X^s} = \big|f\big|^2_{H^s} + \big|\mathfrak{P}^\mu g\big|^2_{H^s} \;\leq\; \big|f\big|^2_{H^s} + \big|g\big|^2_{H^{s+1/2}}.$$

We shall then apply Banach fixed-point theorem on the Duhamel formula

$$U = G_{U(0)}(U), \qquad G_{U_0} : U \;\mapsto\; \big(t \mapsto e^{-t\,\mathsf{L}} U_0 - \int_0^t e^{-(t-\tau)\,\mathsf{L}} N(U(\tau,\cdot))\,\mathrm{d}\tau\big). \tag{3.9}$$

To this aim, we define for any $M > 0$ and any $T > 0$ (determined later on) the set

$$\mathcal{X}^s_{T,M} := \Big\{ U \in \mathcal{C}([-T,T]; X^s) \;:\; \max_{t \in [-T,T]} \big|U(t,\cdot)\big|^2_{X^s} \leq M \Big\}.$$

From the above, we find that for any $U \in \mathcal{X}^s_{T,M}$ and $U_0 \in X^s$, $G_{U_0}(U) \in \mathcal{C}([-T,T]; X^s)$ and

$$\Big| \int_0^t e^{-(t-\tau)\,\mathsf{L}} N(U(\tau,\cdot))\,\mathrm{d}\tau \Big|_{X^s} \leq \epsilon |t|\, C_1\, C_J\, M,$$

where we denote here and thereafter $C_J = \mathcal{N}^{-\max(1, t_0+\frac{3}{2}-s)}(J)$; and for any $U, V \in \mathcal{X}^s_T$, one has (by the triangular inequality)

$$\Big| \int_0^t e^{-(t-\tau)\,\mathsf{L}} \big(N(U) - N(V)\big)(\tau,\cdot)\,\mathrm{d}\tau \Big|_{X^s} \leq 2\epsilon |t|\, C_2\, C_J\, \sqrt{M} \big|(U-V)(\tau,\cdot)\big|_{X^s}.$$

Hence, choosing M and M' such that $0 < M' < M$ and defining T as

$$T = \min\Big(\tfrac{\sqrt{M} - \sqrt{M'}}{\epsilon\, C_1\, C_J\, M} \;,\; \tfrac{1}{3\epsilon\, C_2\, C_J\, \sqrt{M}} \Big), \tag{3.10}$$

we find that G_{U_0} defines a contraction mapping in $\mathcal{X}_{T,M}$ for any U_0 satisfying $\big|U_0\big|^2_{X^s} \leq M$.

This proves the existence and uniqueness of a solution in $\mathcal{X}_{T,M}$ to (3.9) with $U(0) = (\zeta_0, \psi_0)^\top$. We deduce the uniqueness in $\mathcal{X}_T \overset{\text{def}}{=} \mathcal{C}([-T,T]; X^s)$ from a standard continuity argument, and one easily checks the equivalence between $U \in \mathcal{X}_T$ satisfying (3.9) and $U \in \mathcal{X}_T$ satisfying (3.8). Up to now the second component of the solution as well as the second equation of (3.8) are defined up to an additive constant. Requiring additionally that (3.8) holds in $\mathcal{C}([-T,T]; H^{s-1/2}(\mathbb{R}^d) \times H^s(\mathbb{R}^d))$ uniquely determines these constants —and hence $U \in \mathcal{C}([-T,T]; H^s(\mathbb{R}^d) \times \mathring{H}^{s+\frac{1}{2}}(\mathbb{R}^d))$— and we have $\partial_t U \in \mathcal{C}([-T,T]; H^{s-1/2}(\mathbb{R}^d) \times H^s(\mathbb{R}^d))$ by the above estimates for $N(\cdot)$ and straightforward bounds on L.

There remains to prove the lower bound on the maximal time of existence. We focus first on the case $s > t_0 + \frac{1}{2}$. From the above (and in particular uniqueness) we can define a maximal time

of existence and maximal solutions $U=(\zeta,\psi)\in\mathcal{C}((-T_\star,T^\star);H^s(\mathbb{R}^d)\times\mathring{H}^{s+\frac12}(\mathbb{R}^d))$. On $(-T_\star,T^\star)$, we have by the above estimates

$$\big|U(t,\cdot)\big|_{X^s}\leq\big|U(0,\cdot)\big|_{X^s}+|t|\,(\epsilon\,C_1\,C_J)\,\big|U(t,\cdot)\big|_{H^{t_0+\frac12}\times\mathring{H}^{t_0+\frac12}}\big|U(t,\cdot)\big|_{X^s}\tag{3.11}$$

with $C_J=\mathcal{N}^{-1}(J)$, and (augmenting $C_1>0$ if necessary) the same estimate replacing s with $t_0+\frac12$. Hence defining $T_1>0$ such that

$$1+C\,T_1\,(\epsilon\,C_1\,C_J)\big|U(0,\cdot)\big|_{X^{t_0+\frac12}}=C^{1/3},$$

and

$$I_s\stackrel{\text{def}}{=}\Big\{t\in[-T_1,T_1]\cap(-T_\star,T^\star)\ :\ \forall\tau\in[0,t],\ \big|U(\tau,\cdot)\big|^2_{X^s}\leq C\,\big|U(0,\cdot)\big|^2_{X^s}\Big\},$$

we infer (since $C^{2/3}<C$) that $I_s\cap I_{t_0+\frac12}$ is an open subset of $[-T_1,T_1]\cap(-T_\star,T^\star)$. Since it is also closed and non-empty, $I_s\cap I_{t_0+\frac12}=[-T_1,T_1]\cap(-T_\star,T^\star)$, and hence (arguing as in Corollary 3.7) $\min(T_\star,T^\star)>T_1$. If now $s\leq t_0+\frac12$, taking $M=C\big|U_0\big|^2_{X^s}$ and $M'=\big|U_0\big|^2_{X^s}$ (if $U_0\neq(0,0)$ in which case the result is trivial) in (3.10) provides immediately the corresponding lower bound for $T_\star$ and $T^\star$. Gathering the two previous results, we find that there exists $T_0>0$ depending only on C and C_1, such that

$$\min(T_\star,T^\star)\geq\frac{T_0}{\epsilon\big(\big|\zeta_0\big|_{H^{\min(s,t_0+\frac12)}}+\big|\mathfrak{P}^\mu\psi_0\big|_{H^{\min(s,t_0+\frac12)}}\big)\,\mathcal{N}^{-\max(1,t_0+\frac32-s)}(J)}.$$

A subtlety arises as $\big|\nabla\psi_0\big|_{H^{s-\frac12}}$ does not control $\big|\mathfrak{P}^\mu\psi_0\big|_{H^s}$ with a uniform bound with respect to $\mu\geq1$ (see Lemma 2.1). Yet the desired result follows from the following additional ingredient. Applying $\widetilde{\mathfrak{P}^\mu}\stackrel{\text{def}}{=}\frac{\sqrt{|D|}}{\sqrt{\tanh(\sqrt{\mu}|D|)}\langle D\rangle^{1/2}}$ to both equations in (RWW2) and following the above arguments but with a careful use of Lemma 2.9, we infer that there exists $\widetilde{C}_1>0$ depending only on s and $s_\star\stackrel{\text{def}}{=}\min(s,t_0+\frac12)$ such that for any $t\in[0,T^\star)$

$$\begin{aligned}\big|U(t,\cdot)\big|_{X^s}&\leq\big|U(0,\cdot)\big|_{X^s}+|t|\,(\epsilon\,\widetilde{C}_1\,C_J)\,\big|U(t,\cdot)\big|_{\widetilde{X}^{s_\star}}\big|U(t,\cdot)\big|_{X^s},\\ \big|U(t,\cdot)\big|_{\widetilde{X}^{s_\star}}&\leq\big|U(0,\cdot)\big|_{\widetilde{X}^{s_\star}}+|t|\,(\epsilon\,\widetilde{C}_1\,C_J)\,\big|U(t,\cdot)\big|^2_{\widetilde{X}^{s_\star}},\end{aligned}$$

where we define $\big|U\big|_{\widetilde{X}^{s_\star}}\stackrel{\text{def}}{=}\big|\widetilde{\mathfrak{P}^\mu}U\big|_{X^{s_\star}}$, and notice that

$$\frac{1}{\sqrt{\mu}}\big|\zeta\big|^2_{H^{s_\star-\frac12}}+\big|\nabla\zeta\big|^2_{H^{s_\star-1}}+\big|\nabla\psi\big|^2_{H^{s_\star-\frac12}}\lesssim\big|(\zeta,\psi)\big|^2_{\widetilde{X}^{s_\star}}\lesssim\big|\zeta\big|^2_{H^{s_\star}}+\big|\nabla\psi\big|^2_{H^{s_\star-\frac12}}.$$

Hence proceeding as previously we infer first the control of $\big|U(t,\cdot)\big|_{\widetilde{X}^{s_\star}}$, then the control of $\big|U(t,\cdot)\big|_{X^s}$ (with the amplification factor $C>1$), for $t\in[-\widetilde{T}_1,\widetilde{T}_1]$ with $\widetilde{T}_1=\widetilde{T}_0/(\varepsilon\big|U\big|_{\widetilde{X}^{s_\star}}\mathcal{N}^{-s_\star+t_0+\frac32}(J))$ where $\widetilde{T}_0$ depends only on C and $\widetilde{C}_1$. The proof of Proposition 3.1 is now complete.

Corollary 3.7. *Under the assumptions of Proposition 3.1, denote $T^\star(\zeta_0,\psi_0;s)\in(0,+\infty]$ the maximal time of existence associated with initial data $(\zeta_0,\psi_0)\in H^s(\mathbb{R}^d)\times\mathring{H}^{s+1/2}(\mathbb{R}^d)$ and index $s\geq0$, defined as the supremum of $T>0$ such that there exists $(\zeta,\psi)\in\mathcal{C}([0,T];H^s(\mathbb{R}^d)\times\mathring{H}^{s+1/2}(\mathbb{R}^d))$ solution to* (RWW2) *with initial data $(\zeta,\psi)\big|_{t=0}=(\zeta_0,\psi_0)$.*

If $(\zeta_0,\psi_0)\in H^s(\mathbb{R}^d)\times\mathring{H}^{s+1/2}(\mathbb{R}^d)$ with $s>\frac{d}{2}+\frac12$, then $T^\star(\zeta_0,\psi_0;s)=T^\star(\zeta_0,\psi_0;s')$ for any $s'\in\big(\frac{d}{2}+\frac12,s\big]$ and one has the blowup criterion

$$T^\star(\zeta_0,\psi_0;s)<\infty\quad\Longrightarrow\quad\forall s'\in\left(\frac{d}{2}+\frac12,s\right],\quad\big|\zeta(t,\cdot)\big|_{H^{s'}}+\big|\mathfrak{P}^\mu\psi(t,\cdot)\big|_{H^{s'-1/2}}\to\infty\ \ \text{as } t\nearrow T^\star.$$

The corresponding result also holds for the (negative) minimal time of existence.

Proof. Let s and s' such that $s \geq s' > \frac{d}{2} + \frac{1}{2}$, and $(\zeta_0, \psi_0) \in H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d)$. Let denote for simplicity $T^\star_{s_\star} \stackrel{\text{def}}{=} T^\star(\zeta_0, \psi_0; s_\star)$ for $s_\star \in \{s, s'\}$. From Proposition 3.1 and Lemma 2.1 we have (reasoning by contradiction and using a suitable sequence of times approaching $T^\star_s$)

$$T^\star_s < \infty \quad \Longrightarrow \quad \big|\zeta(t,\cdot)\big|_{H^s} + \big|\mathfrak{P}^\mu \psi(t,\cdot)\big|_{H^{s-1/2}} \to \infty \ \text{ as } t \nearrow T^\star_s \,.$$

From the uniqueness in Proposition 3.1, we have obviously $T^\star_s \leq T^\star_{s'}$ and there remains to prove that $T^\star_s \geq T^\star_{s'}$. We argue by contradiction and assume $T^\star_s < T^\star_{s'}$ (and in particular $T^\star_s < \infty$). Thus $(\zeta, \psi) \in \mathcal{C}([0, T^\star_s]; H^{s'}(\mathbb{R}^d) \times \mathring{H}^{s'+1/2}(\mathbb{R}^d))$. Set $M' \stackrel{\text{def}}{=} \max_{t \in [0, T^\star_s]} \big(\big|\zeta(t,\cdot)\big|_{H^{s'}} + \big|\mathfrak{P}^\mu \psi(t,\cdot)\big|_{H^{s'}} \big)$. We use once again the tame estimates (3.11) obtained in the proof of Proposition 3.1, and Lemma 2.1: there exists $C > 0$, depending on ϵ, s', s and J such that for any $t \in [0, T^\star_s)$,

$$\big|(\zeta, \mathfrak{P}^\mu \psi)(t,\cdot)\big|_{H^s \times H^s} \leq \big|(\zeta, \mathfrak{P}^\mu \psi)(0,\cdot)\big|_{H^s \times H^s} + |t|\, C\, M' \, \big|(\zeta, \mathfrak{P}^\mu \psi)(t,\cdot)\big|_{H^s \times H^s}.$$

Grönwall's inequality provides the desired contradiction. □

Remark 3.8. *In order to certify that the initial-value problem for* (RWW2) *is (unconditionally and locally-in-time) well-posed in $H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d)$ in the sense of Hadamard, one should discuss the regularity of the solution map*

$$\Phi : (\zeta_0, \psi_0) \in H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d) \mapsto (\zeta, \psi) \in \mathcal{C}([-T, T]; H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d)).$$

The proof of Proposition 3.1 readily shows that the solution map is Lipschitz from any ball of $H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d)$ to $\mathcal{C}([-T, T]; H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d))$ (with T sufficiently small), and the estimates therein allow to prove that the solution map Φ is in fact analytic (and hence infinitely differentiable), in the sense that for any $U_0 \stackrel{\text{def}}{=} (\zeta_0, \psi_0) \in H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d)$ in a given ball and restricting to $T > 0$ sufficiently small we can write

$$\Phi(U_0) = \sum_{k=1}^{\infty} \Phi_k(U_0, \cdots, U_0)$$

where the operators $\Phi_k : \big(H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d)\big)^k \to \mathcal{C}([-T, T]; H^s(\mathbb{R}^d) \times \mathring{H}^{s+1/2}(\mathbb{R}^d))$ are continuous k-multilinear and the series is normally convergent.

3.2 Large-time well-posedness; proof of Proposition 3.2

In this section we provide the proof of Proposition 3.2. It follows from suitable energy estimates on smooth solutions to (RWW2). Here and henceforth, we refer to (ζ, ψ) as a smooth (local-in-time) solution to (RWW2) when there exists an interval $I \subset \mathbb{R}$ such that

$$\forall N \in \mathbb{N}, \quad (\zeta, \psi) \in \mathcal{C}^1(I; H^N(\mathbb{R}^d) \times \mathring{H}^{N+\frac{1}{2}}(\mathbb{R}^d))$$

and (RWW2) holds for any $t \in I$. The existence of smooth solutions (for smooth initial data) follows from Proposition 3.1 and Corollary 3.7. In Section 3.2.1 we extract a "quasilinear structure"[4] of the system, which is then used in Section 3.2.2 to infer the control of suitable energy functionals. The completion of the proof is postponed to Section 3.2.3.

[4] As mentioned in the introduction and proved in Proposition 3.1, the nature of (RWW2) is in fact semilinear. We refer to the structure of (3.14)-(3.15) as quasilinear in the sense that we will refuse to make use of the full regularization effects of J but will rather obtain improved energy estimates using the skew-symmetry of the leading-order contributions of the system. The system is genuinely quasilinear if J is regularizing of order $-m$ with $m \in [\frac{1}{2}, 1)$ but not regularizing of order -1.

3.2.1 Quasilinearization

Proposition 3.9. *Let $d \in \{1,2\}$, $t_0 > \frac{d}{2}$, $N \in \mathbb{N}$ with $N \geq t_0+2$. There exists $C>0$ such that for any $\epsilon \geq 0$, $\mu \geq 1$, J regular rectifier, and (ζ,ψ) smooth solution to* (RWW2)*, the following holds. Let $\alpha \in \mathbb{N}^d$ a multi-index. If $|\alpha| \leq N-1$, we have*

$$\partial_t \partial^\alpha \zeta - \mathsf{T}^\mu \cdot \nabla \partial^\alpha \psi = \epsilon R_1^{(\alpha)}, \tag{3.12}$$

$$\partial_t \partial^\alpha \psi + \partial^\alpha \zeta = \epsilon R_2^{(\alpha)}, \tag{3.13}$$

with

$$\begin{aligned}
\big|R_1^{(\alpha)}\big|_{L^2} &\leq C\mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0}}\big|\zeta\big|_{H^N} + \mathcal{N}^0(J)\big|\zeta\big|_{H^{t_0+1}}\big|\nabla\psi\big|_{H^{N-1}},\\
\big|R_2^{(\alpha)}\big|_{L^2} &\leq C\mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0}}\big|\nabla\psi\big|_{H^{N-1}},
\end{aligned}$$

whereas if $|\alpha| = N$

$$\partial_t \zeta_{(\alpha)} - \mathsf{T}^\mu \cdot \nabla \psi_{(\alpha)} + \epsilon \nabla \cdot ((\mathsf{J}\zeta_{(\alpha)})\nabla\psi) = \epsilon \widetilde{R}_1^{(\alpha)}, \tag{3.14}$$

$$\partial_t \psi_{(\alpha)} + \mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta, \epsilon\nabla\psi]\zeta_{(\alpha)} + \epsilon\mathsf{J}\left(\nabla\psi\cdot\nabla\psi_{(\alpha)}\right) = \epsilon\widetilde{R}_2^{(\alpha)} + \epsilon^2\widetilde{R}_3^{(\alpha)}, \tag{3.15}$$

with $\zeta_{(\alpha)} \overset{\text{def}}{=} \partial^\alpha\zeta$ and $\psi_{(\alpha)} \overset{\text{def}}{=} \partial^\alpha\psi - \epsilon(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\partial^\alpha\zeta)$, $\mathfrak{a}^\mu$ defined in (3.1) *and*

$$\begin{aligned}
\big|\widetilde{R}_1^{(\alpha)}\big|_{L^2} &\leq C\left(\mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0+1}}\big|\zeta\big|_{H^N} + \mathcal{N}^0(J)\big|\zeta\big|_{H^{t_0+2}}\big|\nabla\psi\big|_{H^{N-1}}\right),\\
\big|\widetilde{R}_2^{(\alpha)}\big|_{H^{\frac12}} &\leq C\left(\mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0}}\big|\nabla\psi\big|_{H^{N-\frac12}} + \dot{\mathcal{N}}_1^0(J)\big|\nabla\psi\big|_{H^{t_0+1}}\big|\psi_{(\alpha)}\big|_{H^{\frac12}}\right),\\
\big|\widetilde{R}_3^{(\alpha)}\big|_{H^{\frac12}} &\leq C\mathcal{N}^{-\frac12}(J)\dot{\mathcal{N}}_1^0(J)\big|\nabla\psi\big|_{H^{t_0}}\left(\big|\nabla\psi\big|_{H^{t_0+1}}\big|\zeta\big|_{H^N} + \big|\zeta\big|_{H^{t_0+2}}\big|\nabla\psi\big|_{H^{N-1}}\right).
\end{aligned}$$

Proof. We first focus on the first equation. By the second estimate in Lemma 2.9 (with $r=0$) and the fact that $\big\|\mathsf{J}\big\|_{H^s\to H^s} = \big|J\big|_{L^\infty} = \mathcal{N}^0(J)$ and (see Lemma 2.1) $\big\|\mathsf{T}^\mu\big\|_{H^s\to H^s} = 1$ for any $s\in\mathbb{R}$ and $\mu>0$, we get

$$\big|\mathsf{T}^\mu\cdot\nabla((\mathsf{J}\zeta)\mathsf{T}^\mu\cdot\nabla\psi) + \nabla\cdot((\mathsf{J}\zeta)\nabla\psi)\big|_{H^{N-1}} \lesssim \mathcal{N}^0(J)\big|\zeta\big|_{H^{t_0+1}}\big|\nabla\psi\big|_{H^{N-1}} + \mathcal{N}^0(J)\big|\zeta\big|_{H^N}\big|\nabla\psi\big|_{H^{t_0}}.$$

This provides the estimate for $R_1^{(\alpha)}$ for $|\alpha|\leq N-1$. We consider now the case $|\alpha|=N$. We differentiate α times the first equation of (RWW2). We get

$$\partial_t\partial^\alpha\zeta - \mathsf{T}^\mu\cdot\nabla\partial^\alpha\psi + \epsilon\mathsf{T}^\mu\cdot\nabla((\mathsf{J}\partial^\alpha\zeta)\mathsf{T}^\mu\cdot\nabla\psi) + \epsilon\nabla\cdot((\mathsf{J}\partial^\alpha\zeta)\nabla\psi) = \epsilon\sum_{\substack{\beta+\gamma=\alpha\\ 0\leq|\beta|\leq N-1}} A_{(\beta,\gamma)} \overset{\text{def}}{=} \epsilon\widetilde{R}_1^{(\alpha)}$$

where

$$A_{(\beta,\gamma)} = C(\beta,\gamma)\left(\mathsf{T}^\mu\cdot\nabla((\mathsf{J}\partial^\beta\zeta)\mathsf{T}^\mu\cdot\nabla\partial^\gamma\psi) + \nabla\cdot((\mathsf{J}\partial^\beta\zeta)\nabla\partial^\gamma\psi)\right).$$

If $|\beta|=0$ or $|\beta|=1$ using the first estimate in Lemma 2.9 (with $s=0$ and $r=1-|\beta|$), we get

$$\big|A_{(\beta,\gamma)}\big|_{L^2} \lesssim \big|\mathsf{J}\partial^\beta\zeta\big|_{H^{t_0+2-|\beta|}}\big|\partial^\gamma\nabla\psi\big|_{H^{|\beta|-1}} \lesssim \mathcal{N}^0(J)\big|\zeta\big|_{H^{t_0+2}}\big|\nabla\psi\big|_{H^{N-1}}.$$

If $2\leq|\beta|\leq N-1$, we obtain by the triangular inequality and the product estimate in Lemma 2.3 with $s_1 = N-|\beta|$, $s_2 = t_0+1-|\gamma|$, $s_1' = t_0+2-|\beta|$, $s_2' = N-1-|\gamma|$,

$$\begin{aligned}
\big|A_{(\beta,\gamma)}\big|_{L^2} &\leq \big|(\mathsf{J}\partial^\beta\zeta)(\mathsf{T}^\mu\cdot\nabla\partial^\gamma\psi)\big|_{H^1} + \big|(\mathsf{J}\partial^\beta\zeta)(\nabla\partial^\gamma\psi)\big|_{H^1}\\
&\lesssim \big|\mathsf{J}\zeta\big|_{H^N}\left(\big|\mathsf{T}^\mu\cdot\nabla\psi\big|_{H^{t_0+1}} + \big|\nabla\psi\big|_{H^{t_0+1}}\right) + \big|\mathsf{J}\zeta\big|_{H^{t_0+2}}\left(\big|\mathsf{T}^\mu\cdot\nabla\psi\big|_{H^{N-1}} + \big|\nabla\psi\big|_{H^{N-1}}\right)\\
&\lesssim \mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0+1}}\big|\zeta\big|_{H^N} + \mathcal{N}^0(J)\big|\zeta\big|_{H^{t_0+2}}\big|\nabla\psi\big|_{H^{N-1}}.
\end{aligned}$$

This concludes the estimate for $\widetilde{R}_1^{(\alpha)}$.

We now focus on the second equation of (RWW2). First we notice that, by Lemma 2.3,

$$\big|\mathsf{J}\big(|\nabla\psi|^2-(\mathsf{T}^\mu\cdot\nabla\psi)^2\big)\big|_{H^{N-1}}\lesssim\mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0}}\big|\nabla\psi\big|_{H^{N-1}}.$$

This provides the estimate for $R_2^{(\alpha)}$ for $|\alpha|\leq N-1$. Now we consider the case $|\alpha|=N$. Differentiating α times the second equation of (RWW2), we get

$$\partial_t\partial^\alpha\psi+\partial^\alpha\zeta+\epsilon\mathsf{J}B_{(\alpha)}=\epsilon\sum_{\substack{\beta+\gamma=\alpha\\1\leq|\beta|\leq N-1}}\mathsf{J}B_{(\beta,\gamma)}\stackrel{\text{def}}{=}\epsilon\widetilde{R}_{2,i}^{(\alpha)}$$

with

$$B_{(\alpha)}\stackrel{\text{def}}{=}\nabla\psi\cdot(\nabla\partial^\alpha\psi)-(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{T}^\mu\cdot\nabla\partial^\alpha\psi)$$

and

$$B_{(\beta,\gamma)}\stackrel{\text{def}}{=}C(\beta,\gamma)\left((\nabla\partial^\beta\psi)\cdot(\nabla\partial^\gamma\psi)-(\mathsf{T}^\mu\cdot\nabla\partial^\beta\psi)(\mathsf{T}^\mu\cdot\nabla\partial^\gamma\psi)\right).$$

Then, using the unknown $\psi_{(\alpha)}\stackrel{\text{def}}{=}\partial^\alpha\psi-\epsilon(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\partial^\alpha\zeta)$, we can rewrite the previous equation as

$$\partial_t\psi_{(\alpha)}+\zeta_{(\alpha)}+\epsilon(\mathsf{T}^\mu\cdot\nabla\partial_t\psi)(\mathsf{J}\zeta_{(\alpha)})+\epsilon(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\partial_t\zeta_{(\alpha)})+\epsilon\mathsf{J}B_{(\alpha)}=\epsilon\widetilde{R}_{2,i}^{(\alpha)}$$

and using (3.14) and reorganizing terms,

$$\begin{aligned}\partial_t\psi_{(\alpha)}+\tilde{\mathfrak{a}}[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]\zeta_{(\alpha)}+\epsilon\mathsf{J}\big((\nabla\psi)\cdot(\nabla\psi_{(\alpha)})-(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{T}^\mu\cdot\nabla\psi_{(\alpha)})\big)\\+\epsilon(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\mathsf{T}^\mu\cdot\nabla\psi_{(\alpha)})=\epsilon\widetilde{R}_{2,i}^{(\alpha)}-\epsilon^2(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\widetilde{R}_1^{(\alpha)})\end{aligned}$$

where

$$\begin{aligned}\tilde{\mathfrak{a}}[\epsilon\mathsf{J}\zeta,\psi]\zeta_{(\alpha)}=&\zeta_{(\alpha)}+\epsilon(\mathsf{T}^\mu\cdot\nabla\partial_t\psi)(\mathsf{J}\zeta_{(\alpha)})-\epsilon^2(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\nabla\cdot((\mathsf{J}\zeta_{(\alpha)})\nabla\psi))\\&+\epsilon^2\mathsf{J}\big((\nabla\psi)\cdot\nabla\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\}\big)-\epsilon^2\mathsf{J}\big((\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{T}^\mu\cdot\nabla\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\})\big)\\=&\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\psi]\zeta_{(\alpha)}+\epsilon(\mathsf{T}^\mu\cdot\nabla(\partial_t\psi+\zeta))(\mathsf{J}\zeta_{(\alpha)})-\epsilon^2(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\nabla\cdot((\mathsf{J}\zeta_{(\alpha)})\nabla\psi))\\&+\epsilon^2\mathsf{J}\big((\nabla\psi)\cdot\nabla\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\}\big)\\&-\epsilon^2\mathsf{J}\big((\mathsf{T}^\mu\cdot\nabla\psi)\big((\mathsf{T}^\mu\cdot\nabla-|D|)\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\}\big)\big).\end{aligned}$$

Let us estimate each of the contributions.

- We get from the second product estimates of Lemma 2.3 (with $s=\frac12$, $s_1=N-\frac12-|\beta|$, $s_2=t_0+1-|\gamma|$, $s_1'=t_0+1-|\beta|$ $s_2'=N-\frac12-|\gamma|$) and Lemma 2.1

$$\big|\widetilde{R}_{2,i}^{(\alpha)}\big|_{H^{\frac12}}\leq\mathcal{N}^0(J)\sum_{\substack{\beta+\gamma=\alpha\\1\leq|\beta|\leq N-1}}\big|B_{(\beta,\gamma)}\big|_{H^{\frac12}}\lesssim\mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0+1}}\big|\nabla\psi\big|_{H^{N-\frac12}}.$$

- Denoting

$$\widetilde{R}_{2,ii}^{(\alpha)}\stackrel{\text{def}}{=}-\mathsf{J}\big((\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{T}^\mu\cdot\nabla\psi_{(\alpha)})\big)+(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\mathsf{T}^\mu\cdot\nabla\psi_{(\alpha)})$$

and by the commutator estimate in Lemma 2.5 with $s=1/2$ and Lemma 2.1, we have

$$\big|\widetilde{R}_{2,ii}^{(\alpha)}\big|_{H^{1/2}}\lesssim\dot{\mathcal{N}}_1^0(J)\big|\nabla\psi\big|_{H^{t_0+1}}\big|\nabla\psi_{(\alpha)}\big|_{H^{-1/2}}.$$

- We have by Lemma 2.2 and Lemma 2.1

$$\big|(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\widetilde{R}_1^{(\alpha)})\big|_{H^{1/2}}\lesssim\big|\mathsf{T}^\mu\cdot\nabla\psi\big|_{H^{t_0}}\big|\mathsf{J}\widetilde{R}_1^{(\alpha)}\big|_{H^{1/2}}\leq\big|\nabla\psi\big|_{H^{t_0}}\big|\mathsf{J}\widetilde{R}_1^{(\alpha)}\big|_{H^{1/2}}$$

and by the previously obtained estimate on $\widetilde{R}_1^{(\alpha)}$, and that $\big\|\mathsf{J}\big\|_{L^2\to H^{1/2}}=\mathcal{N}^{-\frac12}(J)$,

$$\big|\mathsf{J}\widetilde{R}_1^{(\alpha)}\big|_{H^{1/2}}\lesssim\mathcal{N}^{-\frac12}(J)\big(\mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0+1}}\big|\zeta\big|_{H^N}+\mathcal{N}^0(J)\big|\zeta\big|_{H^{t_0+2}}\big|\nabla\psi\big|_{H^{N-1}}\big).$$

- Rewriting

$$\begin{aligned}\widetilde{R}_{3,i}^{(\alpha)}&\stackrel{\text{def}}{=}\mathsf{J}\big((\nabla\psi)\cdot(\nabla\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\})\big)-(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\nabla\cdot((\mathsf{J}\zeta_{(\alpha)})\nabla\psi))\\&=\big[\mathsf{J},\mathsf{T}^\mu\cdot\nabla\psi\big]\big((\nabla\psi)\cdot(\mathsf{J}\nabla\zeta_{(\alpha)})\big)\\&\qquad+\mathsf{J}\big((\nabla\psi)\cdot((\mathsf{J}\zeta_{(\alpha)})(\nabla\mathsf{T}^\mu\cdot\nabla\psi))\big)-(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\{(\mathsf{J}\zeta_{(\alpha)})(\Delta\psi)\}),\end{aligned}$$

we find using as above Lemma 2.5 for the first contribution and Lemma 2.2 for the other ones

$$\big|\widetilde{R}_{3,i}^{(\alpha)}\big|_{H^{1/2}}\lesssim\dot{\mathcal{N}}_1^0(J)\big|\nabla\psi\big|_{H^{t_0+1}}\big|\nabla\psi\big|_{H^{t_0}}\big|\mathsf{J}\nabla\zeta_{(\alpha)}\big|_{H^{-1/2}}+\mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0+1}}\big|\nabla\psi\big|_{H^{t_0}}\big|\mathsf{J}\zeta_{(\alpha)}\big|_{H^{1/2}}.$$

- Then, we easily find, since for any $\xi\in\mathbb{R}^+$, $|\tanh(\sqrt{\mu}|\xi|)-1|\leq|\tanh(|\xi|)-1|\lesssim\langle\xi\rangle^{-3/2}$, that

$$\begin{aligned}\widetilde{R}_{3,ii}&\stackrel{\text{def}}{=}-\mathsf{J}\big((\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{T}^\mu\cdot\nabla\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\})\big)+\mathsf{J}\big((\mathsf{T}^\mu\cdot\nabla\psi)(|D|\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\})\big)\\&=-\mathsf{J}\big((\mathsf{T}^\mu\cdot\nabla\psi)(|D|(\tanh(\sqrt{\mu}|D|)-1)\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\})\big)\end{aligned}$$

can be estimated as

$$\big|\widetilde{R}_{3,ii}^{(\alpha)}\big|_{H^{1/2}}\lesssim\mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0}}^2\big|\mathsf{J}\zeta_{(\alpha)}\big|_{L^2}.$$

- Finally, we set

$$\begin{aligned}\widetilde{R}_{3,iii}^{(\alpha)}&\stackrel{\text{def}}{=}\big(\mathsf{T}^\mu\cdot\nabla(\partial_t\psi+\zeta)\big)(\mathsf{J}\zeta_{(\alpha)})\\&=-\frac{\epsilon}{2}\big(\mathsf{T}^\mu\cdot\nabla\mathsf{J}\left(|\nabla\psi|^2-(\mathsf{T}^\mu\cdot\nabla\psi)^2\right)\big)(\mathsf{J}\zeta_{(\alpha)})\end{aligned}$$

(where we used the first equation of (RWW2)) and by Lemma 2.2

$$\big|\widetilde{R}_{3,iii}^{(\alpha)}\big|_{H^{1/2}}\lesssim\epsilon\mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0+1}}\big|\nabla\psi\big|_{H^{t_0}}\big|\mathsf{J}\zeta_{(\alpha)}\big|_{H^{1/2}}.$$

We conclude combining the above estimates and the fact that $\big|\mathsf{J}\zeta_{(\alpha)}\big|_{H^{1/2}}\leq\mathcal{N}^{-\frac12}(J)\big|\zeta\big|_{H^N}$. □

Remark 3.10. *Several contributions in $\widetilde{R}_3^{(\alpha)}$ (but also in $\widetilde{R}_2^{(\alpha)}$ via the control of $\big|\nabla\psi\big|_{H^{N-\frac12}}$) require the regularizing properties of J to be controlled. A more careful analysis shows that most but not all contributions could be tackled through additional terms on the Rayleigh–Taylor operator. Shortly put, if we set $\mathsf{J}=\mathrm{Id}$, then Proposition 3.9 holds with suitable estimates on the remainders (*i.e. *controlled by the energy functional) and replacing $\mathfrak{a}^\mu[\epsilon\zeta,\epsilon\nabla\psi]\zeta_{(\alpha)}$ with $\widetilde{\mathfrak{a}^\mu}[\epsilon\zeta,\epsilon\nabla\psi]\zeta_{(\alpha)}+\mathfrak{b}_\alpha(\zeta,\psi)$ where*

$$\begin{aligned}\widetilde{\mathfrak{a}^\mu}[\epsilon\zeta,\epsilon\nabla\psi]\zeta=&\Big(1+\epsilon(\mathsf{T}^\mu\cdot\partial_t\psi)+\epsilon^2(\nabla\psi)\cdot(\nabla\mathsf{T}^\mu\cdot\nabla\psi)-\epsilon^2(\mathsf{T}^\mu\cdot\nabla\psi)(\Delta\psi)\Big)\,\zeta_{(\alpha)}\\&-\epsilon^2\Big((\mathsf{T}^\mu\cdot\nabla\psi)|D|\big\{(\mathsf{T}^\mu\cdot\nabla\psi)\zeta_{(\alpha)}\big\}\Big)\end{aligned}\tag{3.16}$$

and

$$\mathfrak{b}_\alpha(\zeta,\psi)=\epsilon\sum_{|\beta|=1,\ \beta\leq\alpha}\binom{\alpha}{\beta}\Big((\mathsf{T}^\mu\cdot\nabla\partial^\beta\psi)(\mathsf{T}^\mu\cdot\nabla\partial^{\alpha-\beta}\psi)+\epsilon(\mathsf{T}^\mu\cdot\nabla\psi)\mathsf{T}^\mu\cdot\nabla\big((\mathsf{T}^\mu\cdot\partial^\beta\nabla\psi)(\partial^{\alpha-\beta}\zeta)\big)\Big).\tag{3.17}$$

The contribution $\mathfrak{b}_\alpha(\zeta,\psi)$ is "bad" in the sense that it cannot be estimated as an order-zero term and has no particular structure (when tested against against $\mathsf{T}^\mu\cdot\nabla\psi_{(\alpha)}$).

3.2.2 Energy estimates

This section is devoted to the proof of energy estimates on smooth solutions to (RWW2). We start with some elementary results which provide useful tools for the comparison of the energy functional, $\mathcal{E}^N(\zeta,\psi)$, defined in (3.2), and suitable norms of (ζ,ψ).

Lemma 3.11. *Let $d\in\{1,2\}$, $t_0>\frac{d}{2}$ and $N\in\mathbb{N}$ with $N\geq t_0+\frac{3}{2}$. There exists $C>0$ such that for any $\epsilon\geq 0$, $\mu\geq 1$, $\zeta\in H^N(\mathbb{R}^d)$ and $\psi\in \mathring{H}^{N+\frac12}(\mathbb{R}^d)$, and $\alpha\in\mathbb{N}^d$ with $|\alpha|=N$,*

$$\begin{aligned}
&\big|\nabla\psi\big|_{H^{t_0}}\leq\big|\mathfrak{P}^\mu\psi\big|_{H^{t_0+1/2}}\leq\big|\mathfrak{P}^\mu\psi\big|_{H^{N-1}},\\
&\big|\nabla\psi\big|_{H^{N-1}}\leq C\big(\big|\mathfrak{P}^\mu\psi\big|_{H^{N-1}}+\sup_{|\beta|=N}\big|\mathfrak{P}^\mu\psi_{(\beta)}\big|_{L^2}+\epsilon\big|\nabla\psi\big|_{H^{t_0}}\mathcal{N}^0(J)\big|\zeta\big|_{H^N}\big),\\
&\big|\psi_{(\alpha)}\big|_{H^{\frac12}}\leq C\big(\big|\mathfrak{P}^\mu\psi\big|_{H^{N-1}}+\sup_{|\beta|=N}\big|\mathfrak{P}^\mu\psi_{(\beta)}\big|_{L^2}+\epsilon\big|\nabla\psi\big|_{H^{t_0}}\mathcal{N}^0(J)\big|\zeta\big|_{H^N}\big),\\
&\big|\partial^\alpha\psi\big|_{H^{\frac12}}\leq C\big(\big|\mathfrak{P}^\mu\psi_{(\alpha)}\big|_{L^2}+\epsilon\big|\nabla\psi\big|_{H^{t_0}}\mathcal{N}^{-\frac12}(J)\big|\zeta\big|_{H^N}\big),
\end{aligned}$$

where we recall that $\psi_{(\alpha)}\overset{\text{def}}{=}\partial^\alpha\psi-\epsilon(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\partial^\alpha\zeta)$.

Proof. The first inequality has been proved in Lemma 2.1. Then the following holds for any $\beta\in\mathbb{N}^d$ with $|\beta|=N$:

$$\big|\partial^\beta\psi\big|_{L^2}\leq\big|\tfrac{(1+|D|^2)^{1/4}}{|D|}\mathfrak{P}^\mu\partial^\beta\psi\big|_{L^2}\leq\big|(\tfrac{1}{|D|}+\tfrac{1}{(1+|D|^2)^{1/4}})\mathfrak{P}^\mu\partial^\beta\psi\big|_{L^2}\leq\big|\mathfrak{P}^\mu\psi\big|_{H^{N-1}}+\big|\mathfrak{P}^\mu\partial^\beta\psi\big|_{H^{-1/2}}$$

and, since $\partial^\beta\psi=\psi_{(\beta)}+\epsilon(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\partial^\beta\zeta)$,

$$\big|\mathfrak{P}^\mu\partial^\beta\psi\big|_{H^{-1/2}}\leq\big|\mathfrak{P}\psi_{(\beta)}\big|_{H^{-1/2}}+\epsilon\big|(\mathsf{T}^\mu\cdot\nabla\psi)\mathsf{J}\partial^\beta\zeta\big|_{L^2}\lesssim\big|\mathfrak{P}\psi_{(\beta)}\big|_{L^2}+\epsilon\big|\nabla\psi\big|_{H^{t_0}}\mathcal{N}^0(J)\big|\partial^\beta\zeta\big|_{L^2}$$

where we use Lemma 2.1 and the continuous Sobolev embedding $H^{t_0}(\mathbb{R}^d)\subset L^\infty(\mathbb{R}^d)$. The control of $\big|\partial^\beta\psi\big|_{L^2}$, and hence $\big|\nabla\psi\big|_{H^{N-1}}$ using the first inequality, immediately follows. We infer the same control on $\big|\psi_{(\alpha)}\big|_{L^2}$ which, combined with $\big|\nabla\psi_{(\alpha)}\big|_{H^{-1/2}}\leq\big|\mathfrak{P}^\mu\psi_{(\alpha)}\big|_{L^2}$ (see Lemma 2.1), yields the third inequality. Finally,

$$\big|\nabla\partial^\alpha\psi\big|_{H^{-1/2}}\leq\big|\mathfrak{P}^\mu\partial^\alpha\psi\big|_{L^2}\leq\big|\mathfrak{P}^\mu\psi_{(\alpha)}\big|_{L^2}+\epsilon\big|(\mathsf{T}^\mu\cdot\nabla\psi)\mathsf{J}\partial^\alpha\zeta\big|_{H^{1/2}}$$

and, by the product estimate in Lemma 2.2

$$\big|(\mathsf{T}^\mu\cdot\nabla\psi)\mathsf{J}\partial^\alpha\zeta\big|_{H^{1/2}}\lesssim\big|\nabla\psi\big|_{H^{t_0}}\big|\mathsf{J}\zeta\big|_{H^{N+1/2}}\leq\big|\nabla\psi\big|_{H^{t_0}}\mathcal{N}^{-\frac12}(J)\big|\zeta\big|_{H^N},$$

and the fourth inequality follows. □

Now we are ready to prove the following key estimates.

Proposition 3.12. *Let $d\in\{1,2\}$, $t_0>\frac{d}{2}$, $N\in\mathbb{N}$ with $N\geq t_0+2$, $\mathfrak{a}^\mu_\star>0$, $M_J,M_U>0$. There exists $C>0$ such that for any $\epsilon\geq 0$, $\mu\geq 1$, regular rectifier J (see Definition 1.1) with $\mathcal{N}^0_1(J)\leq M_J$, and any (ζ,ψ) smooth solution to* (RWW2) *on the time interval $I\subset\mathbb{R}$ and satisfying for any $t\in I$*

$$\epsilon M(t)\overset{\text{def}}{=}\epsilon\big(\big|\zeta(t,\cdot)\big|_{H^{t_0+2}}+\big|\nabla\psi(t,\cdot)\big|_{H^{t_0+1}}\big)\leq M_U$$

and the Rayleigh–Taylor condition (with $\mathfrak{a}^\mu$ defined in (3.1)*)*

$$\forall f\in L^2(\mathbb{R}^d)\ ,\ (f,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta(t,\cdot),\epsilon\nabla\psi(t,\cdot)]f)_{L^2}\geq\mathfrak{a}^\mu_\star\big|f\big|^2_{L^2},$$

one has for any $t\in I$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^N(\zeta(t,\cdot),\psi(t,\cdot))\ \leq\ C\left(\epsilon M(t)+\dot{\mathcal{N}}^{-\frac12}(J)^2\epsilon^2M(t)^2\right)\mathcal{E}^N(\zeta(t,\cdot),\psi(t,\cdot)),$$

where the functional $\mathcal{E}^N$ is defined in (3.2).

Proof. In this proof, we denote by C a constant which depends uniquely on N and t_0, and by C_M a constant which depends additionally and non-decreasingly on M_U and M_J. They vary from line to line. Let $\alpha \in \mathbb{N}^d$ and consider Proposition 3.9.

When $|\alpha| \leq N-1$, we sum the $L^2(\mathbb{R}^d)$ inner product of (3.12) against $\partial^\alpha \zeta$ and the $L^2(\mathbb{R}^d)$ inner product of (3.13) in with $\mathsf{T}^\mu \cdot \nabla \partial^\alpha \psi$. It follows, using Cauchy-Schwarz inequality and Lemma 2.1,

$$\begin{aligned}(3.18)\quad &\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big|\partial^\alpha\zeta\big|_{L^2}^2 + \big|\partial^\alpha \mathfrak{P}^\mu \psi\big|_{L^2}^2\Big)\\ &\qquad \leq \epsilon\, C\mathcal{N}^0(J)\,\Big(\big|\nabla\psi\big|_{H^{t_0+1}}\big|\zeta\big|_{H^N}^2 + \big|\zeta\big|_{H^{t_0+2}}\big|\zeta\big|_{H^N}\big|\nabla\psi\big|_{H^{N-1}} + \big|\nabla\psi\big|_{H^{t_0+1}}\big|\nabla\psi\big|_{H^{N-1}}^2\Big).\end{aligned}$$

When $|\alpha| = N$, we sum the $L^2(\mathbb{R}^d)$ inner product of (3.14) against $\mathfrak{a}^\mu[\epsilon \mathsf{J}\zeta, \epsilon\nabla\psi]\zeta_{(\alpha)}$ and the $L^2(\mathbb{R}^d)$ inner product of (3.15) with $\mathsf{T}^\mu \cdot \nabla \psi_{(\alpha)}$, where we recall the notations $\zeta_{(\alpha)} \overset{\text{def}}{=} \partial^\alpha \zeta$ and $\psi_{(\alpha)} \overset{\text{def}}{=} \partial^\alpha \psi - \epsilon(\mathsf{T}^\mu \cdot \nabla\psi)(\mathsf{J}\zeta_{(\alpha)})$. This yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\big(\zeta_{(\alpha)}, \mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]\zeta_{(\alpha)}\big)_{L^2} + \big|\mathfrak{P}^\mu\psi_{(\alpha)}\big|_{L^2}^2\Big)\\ (3.19a)\qquad &= \tfrac{1}{2}\big(\zeta_{(\alpha)}, [\partial_t, \mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]]\,\zeta_{(\alpha)}\big)_{L^2} + \tfrac{\epsilon}{2}\big(\,[\mathsf{J}, \mathsf{T}^\mu\cdot\nabla\mathsf{J}\zeta]\,\partial_t\zeta_{(\alpha)}, \zeta_{(\alpha)}\big)_{L^2}\\ (3.19b)\qquad &\quad - \epsilon\big(\nabla\cdot((\mathsf{J}\zeta_{(\alpha)})\nabla\psi), \mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]\zeta_{(\alpha)}\big)_{L^2} - \epsilon\big(\mathsf{J}\left(\nabla\psi\cdot\nabla\psi_{(\alpha)}\right), \mathsf{T}^\mu\cdot\nabla\psi_{(\alpha)}\big)_{L^2}\\ (3.19c)\qquad &\quad + \epsilon\big(\widetilde{R}_1^{(\alpha)}, \mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]\zeta_{(\alpha)}\big)_{L^2} + \epsilon\big(\widetilde{R}_2^{(\alpha)} + \epsilon\widetilde{R}_3^{(\alpha)}, \mathsf{T}^\mu\cdot\nabla\psi_{(\alpha)}\big)_{L^2}.\end{aligned}$$

We now estimate each term on the right-hand side of (3.19).

Contributions in (3.19a). By direct inspection, we find

$$\begin{aligned}[\partial_t, \mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]]\,\zeta_{(\alpha)} &= \epsilon(\mathsf{T}^\mu\cdot\nabla\mathsf{J}\partial_t\zeta)\mathsf{J}\zeta_{(\alpha)}\\ &\quad - \epsilon^2\mathsf{J}\Big((\mathsf{T}^\mu\cdot\nabla\partial_t\psi)|D|\big\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\big\} - \epsilon^2\mathsf{J}(\mathsf{T}^\mu\cdot\nabla\psi)|D|\big\{(\mathsf{T}^\mu\cdot\nabla\partial_t\psi)(\mathsf{J}\zeta_{(\alpha)})\big\}\Big)\end{aligned}$$

and hence, using triangular inequality and product estimates in Lemma 2.2,

$$\big|[\partial_t, \mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]]\,\zeta_{(\alpha)}\big|_{L^2} \lesssim \epsilon\,\mathcal{N}^0(J)^2\big|\partial_t\zeta\big|_{H^{t_0+1}}\big|\zeta_{(\alpha)}\big|_{L^2} + \epsilon^2\,\mathcal{N}^{-1/2}(J)^2\big|\partial_t\nabla\psi\big|_{H^{t_0}}\big|\nabla\psi\big|_{H^{t_0}}\big|\zeta_{(\alpha)}\big|_{L^2}.$$

Using the equations (RWW2), the first inequality in Lemma 2.9 (with $s = t_0+1$, $r=1$) we have

$$\big|\partial_t\zeta\big|_{H^{t_0+1}} \leq \big|\nabla\psi\big|_{H^{t_0+1}} + \epsilon\, C\mathcal{N}^0(J)\big|\zeta\big|_{H^{t_0+2}}\big|\nabla\psi\big|_{H^{t_0}},$$

and by the product estimate in Lemma 2.2 we infer

$$\big|\partial_t\nabla\psi\big|_{H^{t_0}} \leq \big|\zeta\big|_{H^{t_0+1}} + \epsilon\, C\mathcal{N}^0(J)\big|\nabla\psi\big|_{H^{t_0}}\big|\nabla\psi\big|_{H^{t_0+1}}.$$

By Cauchy-Schwarz inequality and collecting the above, we find

$$\begin{aligned}(3.20)\quad \big|\big(\zeta_{(\alpha)}, [\partial_t, \mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]]\,\zeta_{(\alpha)}\big)_{L^2}\big| &\leq \epsilon C_M\big|\nabla\psi\big|_{H^{t_0+1}}\big|\zeta_{(\alpha)}\big|_{L^2}^2\\ &\quad + \epsilon^2\, C\mathcal{N}^{-1/2}(J)^2\big(\big|\zeta\big|_{H^{t_0+2}} + \big|\nabla\psi\big|_{H^{t_0+1}}\big)^2\big|\zeta_{(\alpha)}\big|_{L^2}^2.\end{aligned}$$

For the second term in (3.19a), we have by Lemma 2.5 with $s=0$,

$$\big|[\mathsf{J}, \mathsf{T}^\mu\cdot\nabla\mathsf{J}\zeta]\,\partial_t\zeta_{(\alpha)}\big|_{L^2} \leq C\mathcal{N}_1^0(J)\big|\nabla\mathsf{J}\zeta\big|_{H^{t_0+1}}\big|\partial_t\zeta_{(\alpha)}\big|_{H^{-1}}$$

which yields, proceeding as above but with the second inequality in Lemma 2.9 (with $s = N-1$)

$$(3.21)\qquad \big|\tfrac{\epsilon}{2}\big(\,[\mathsf{J}, \mathsf{T}^\mu\cdot\nabla\mathsf{J}\zeta]\,\partial_t\zeta_{(\alpha)}, \zeta_{(\alpha)}\big)_{L^2}\big| \leq \epsilon\, C_M\big|\zeta\big|_{H^{t_0+2}}\big(\big|\zeta\big|_{H^N} + \big|\nabla\psi\big|_{H^{N-1}}\big)\big|\zeta_{(\alpha)}\big|_{L^2}.$$

Contributions in (3.19b). Using integration by parts and that J is self-adjoint, we find

$$
\begin{aligned}
&\big(\nabla\cdot((\mathsf{J}\zeta_{(\alpha)})\nabla\psi),\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]\zeta_{(\alpha)}\big)_{L^2}\\
&\quad=\Big(\nabla\cdot((\mathsf{J}\zeta_{(\alpha)})\nabla\psi),\zeta_{(\alpha)}-\epsilon(\mathsf{T}^\mu\cdot\nabla\mathsf{J}\zeta)\mathsf{J}\zeta_{(\alpha)}-\epsilon^2\mathsf{J}\big((\mathsf{T}^\mu\cdot\nabla\psi)|D|\big\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\big\}\big)\Big)_{L^2}\\
&\quad=\tfrac12\big((\mathsf{J}\zeta_{(\alpha)})\Delta\psi,\zeta_{(\alpha)}\big)_{L^2}-\tfrac12\big([\mathsf{J},\nabla\psi]\nabla\zeta_{(\alpha)},\zeta_{(\alpha)}\big)_{L^2}\\
&\qquad-\tfrac{\epsilon}{2}\big((\mathsf{J}\zeta_{(\alpha)})\Delta\psi,(\mathsf{T}^\mu\cdot\nabla\mathsf{J}\zeta)\mathsf{J}\zeta_{(\alpha)}\big)_{L^2}+\tfrac{\epsilon}{2}\big((\mathsf{J}\zeta_{(\alpha)})\nabla\psi,(\mathsf{J}\zeta_{(\alpha)})\nabla(\mathsf{T}^\mu\cdot\nabla\mathsf{J}\zeta)\big)_{L^2}\\
&\qquad-\tfrac{\epsilon^2}{2}\big((\mathsf{J}\zeta_{(\alpha)})\Delta\psi,\mathsf{J}\big((\mathsf{T}^\mu\cdot\nabla\psi)|D|\big\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\big\}\big)\big)_{L^2}\\
&\qquad+\tfrac{\epsilon^2}{2}\big((\mathsf{J}\zeta_{(\alpha)})\nabla\psi,\mathsf{J}\big(\{\nabla(\mathsf{T}^\mu\cdot\nabla\psi)\}|D|\big\{(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{J}\zeta_{(\alpha)})\big\}\big)\big)_{L^2}\\
&\qquad+\tfrac{\epsilon^2}{2}\big((\mathsf{J}\zeta_{(\alpha)})\nabla\psi,\mathsf{J}\big((\mathsf{T}^\mu\cdot\nabla\psi)|D|\big\{\{\nabla(\mathsf{T}^\mu\cdot\nabla\psi)\}(\mathsf{J}\zeta_{(\alpha)})\big\}\big)\big)_{L^2}\\
&\qquad-\tfrac{\epsilon^2}{2}\big(\zeta_{(\alpha)},\mathsf{J}\big[\mathsf{J}\big((\mathsf{T}^\mu\cdot\nabla\psi)|D|\big\{(\mathsf{T}^\mu\cdot\nabla\psi)\bullet\big\}\big),\nabla\psi\big]\cdot\nabla\mathsf{J}\zeta_{(\alpha)}\big)_{L^2}.
\end{aligned}
$$

The first two contributions are easily estimated, using Lemma 2.5 with $s=0$, as

$$\big|\star\big|\le C\mathcal{N}_1^0(J)\big|\nabla\psi\big|_{H^{t_0+1}}\big|\zeta_{(\alpha)}\big|_{L^2}^2.$$

The next two are straightforward by the continuous embedding $H^{t_0}(\mathbb{R}^d)\subset L^\infty(\mathbb{R}^d)$:

$$\big|\star\big|\le\epsilon\,C\,\mathcal{N}^0(J)^3\,\big|\nabla\psi\big|_{H^{t_0+1}}\big|\zeta\big|_{H^{t_0+2}}\big|\zeta_{(\alpha)}\big|_{L^2}^2.$$

For the next three, we use the regularizing effect of J and obtain by a repeated use of the product estimates in Lemma 2.2

$$\big|\star\big|\le\epsilon^2\,C\,\mathcal{N}^0(J)\mathcal{N}^{-1/2}(J)^2\,\big|\nabla\psi\big|_{H^{t_0+1}}^3\big|\zeta_{(\alpha)}\big|_{L^2}^2.$$

Finally, for the last term, we decompose

$$
\begin{aligned}
\mathsf{J}\big[\mathsf{J}\big((\mathsf{T}^\mu\cdot\nabla\psi)|D|\big\{(\mathsf{T}^\mu\cdot\nabla\psi)\bullet\big\}\big),\nabla\psi\big]\nabla\mathsf{J}\zeta_{(\alpha)}&=\mathsf{J}^2\big((\mathsf{T}^\mu\cdot\nabla\psi)\big[|D|,\nabla\psi\big]\big\{(\mathsf{T}^\mu\cdot\nabla\psi)\nabla\mathsf{J}\zeta_{(\alpha)}\big\}\big)\\
&\quad+\mathsf{J}\big[\mathsf{J},\nabla\psi\big]\big((\mathsf{T}^\mu\cdot\nabla\psi)|D|\big\{(\mathsf{T}^\mu\cdot\nabla\psi)\nabla\mathsf{J}\zeta_{(\alpha)}\big\}\big).
\end{aligned}
$$

We estimate the right-hand side in $L^2(\mathbb{R}^d)$ thanks the regularizing effect of J, and the commutator estimates in Lemma 2.4 (see also Lemma 2.6) and Lemma 2.5 with $s=-1/2$, and again product estimates in Lemma 2.2, which yields by Cauchy-Schwarz inequality

$$\big|\star\big|\le\epsilon^2\,C\,(\mathcal{N}^0(J)+\mathcal{N}_1^0(J))\mathcal{N}^{-1/2}(J)^2\,\big|\nabla\psi\big|_{H^{t_0+1}}^3\big|\zeta_{(\alpha)}\big|_{L^2}^2.$$

Collecting the above, we find

(3.22)
$$\big|\epsilon\big(\nabla\cdot((\mathsf{J}\zeta_{(\alpha)})\nabla\psi),\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]\zeta_{(\alpha)}\big)_{L^2}\big|\le\epsilon\,C_M\big|\nabla\psi\big|_{H^{t_0+1}}\Big(1+\epsilon^2\mathcal{N}^{-1/2}(J)^2\big|\nabla\psi\big|_{H^{t_0+1}}^2\Big)\big|\zeta_{(\alpha)}\big|_{L^2}^2.$$

Next, we have

$$\big(\mathsf{J}\left(\nabla\psi\cdot\nabla\psi_{(\alpha)}\right),\mathsf{T}^\mu\cdot\nabla\psi_{(\alpha)}\big)_{L^2}=\tfrac12\big([\mathsf{J},\nabla\psi]\cdot\nabla\psi_{(\alpha)},\mathsf{T}^\mu\cdot\nabla\psi_{(\alpha)}\big)_{L^2}-\tfrac12\big([\mathsf{T}^\mu,\nabla\psi]\cdot\mathsf{J}\nabla\psi_{(\alpha)},\nabla\psi_{(\alpha)}\big)_{L^2}.$$

Using Lemma 2.1 and the commutator estimates in Lemma 2.5 with $s=1/2$ (see also Lemma 2.6) we find

$$\big|\epsilon\big(\mathsf{J}\left(\nabla\psi\cdot\nabla\psi_{(\alpha)}\right),\mathsf{T}^\mu\cdot\nabla\psi_{(\alpha)}\big)_{L^2}\big|\le\epsilon\,C\big(\mathcal{N}_1^0(J)+\mathcal{N}^0(J)\big)\big|\nabla\psi\big|_{H^{t_0+1}}\big|\mathfrak{P}^\mu\psi_{(\alpha)}\big|_{L^2}^2.\tag{3.23}$$

Contributions in (3.19c). We have by Lemma 2.2,

$$\big|\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]\zeta_{(\alpha)}\big|_{L^2}\le\big|\zeta_{(\alpha)}\big|_{L^2}\big(1+\epsilon\mathcal{N}^0(J)^2\big|\zeta\big|_{H^{t_0+1}}+\epsilon^2\mathcal{N}^{-\frac12}(J)^2\big|\nabla\psi\big|_{H^{t_0}}^2\big)$$

and hence by the estimate for $\widetilde{R}_1^{(\alpha)} \in L^2(\mathbb{R}^d)$ displayed in Proposition 3.9,

$$\big|\epsilon\big(\widetilde{R}_1^{(\alpha)}, \mathfrak{a}^\mu[\epsilon \mathsf{J}\zeta, \epsilon\nabla\psi]\zeta_{(\alpha)}\big)_{L^2}\big| \le \epsilon\, C_M \big|\zeta\big|_{H^N}\big(\big|\nabla\psi\big|_{H^{t_0+1}}\big|\zeta\big|_{H^N} + \big|\zeta\big|_{H^{t_0+2}}\big|\nabla\psi\big|_{H^{N-1}}\big). \tag{3.24}$$

Finally, using Lemma 2.1 and the estimates for $\widetilde{R}_2^{(\alpha)}, \widetilde{R}_3^{(\alpha)} \in H^{1/2}(\mathbb{R}^d)$ in Proposition 3.9,

$$\begin{aligned}\big|\big(\epsilon\widetilde{R}_2^{(\alpha)} + \epsilon^2\widetilde{R}_3^{(\alpha)}, \mathsf{T}^\mu\cdot\nabla\psi_{(\alpha)}\big)_{L^2}\big| &\le \epsilon\, C_M\Big(\big(\big|\zeta\big|_{H^{t_0+2}} + \big|\nabla\psi\big|_{H^{t_0+1}}\big)\big(\big|\nabla\psi\big|_{H^{N-\frac12}} + \big|\psi_{(\alpha)}\big|_{H^{\frac12}}\big)\\ &\quad + \epsilon\mathcal{N}^{-\frac12}(J)\big|\nabla\psi\big|_{H^{t_0}}\big(\big|\zeta\big|_{H^{t_0+2}} + \big|\nabla\psi\big|_{H^{t_0+1}}\big)\big(\big|\nabla\psi\big|_{H^{N-1}} + \big|\zeta\big|_{H^N}\big)\Big) \times \big|\mathfrak{P}^\mu\psi_{(\alpha)}\big|_{L^2}.\end{aligned} \tag{3.25}$$

Finally we note that by Lemma 3.11,

$$\begin{aligned}\big|\nabla\psi\big|_{H^{N-1}} + \sup_{|\beta|=N}\big|\psi_{(\beta)}\big|_{H^{\frac12}} &\le C\,\big(1+\epsilon\big|\nabla\psi\big|_{H^{t_0}}\mathcal{N}^0(J)\big)\,\mathcal{E}^N(\zeta,\psi)^{1/2} \le C_M\mathcal{E}^N(\zeta,\psi)^{1/2},\\ \big|\nabla\psi\big|_{H^{N-\frac12}} &\le C\,\big(1+\epsilon\big|\nabla\psi\big|_{H^{t_0}}\mathcal{N}^{-\frac12}(J)\big)\mathcal{E}^N(\zeta,\psi)^{1/2},\end{aligned}$$

and by (3.3)

$$\big|\mathfrak{P}^\mu\psi_{(\alpha)}\big|_{L^2} \le \mathcal{E}^N(U)^{1/2}, \quad \big|\zeta_{(\alpha)}\big|_{L^2} \le \big(\tfrac{1}{\mathfrak{a}_\star^\mu}\mathcal{E}^N(U)\big)^{1/2}, \qquad \big|\zeta\big|_{H^N} \le \big(\max(1,\tfrac{1}{\mathfrak{a}_\star^\mu})\mathcal{E}^N(U)\big)^{1/2}.$$

The result is now a direct consequence (3.18) and (3.19) with (3.20)–(3.25) and since

$$\mathcal{N}^{-\frac12}(J) \le \dot{\mathcal{N}}^{-\frac12}(J) + \mathcal{N}^0(J).$$

The proof is complete. □

We conclude this section with the following result showing that the Rayleigh–Taylor condition, (3.3), propagates in time.

Lemma 3.13. *Let $d \in \{1,2\}$, $t_0 > \frac{d}{2}$, $N \in \mathbb{N}$ with $N \ge t_0+2$, $M_J, M_U > 0$. There exists $C > 0$ such that for any $\epsilon \ge 0$, $\mu \ge 1$, rectifier J regularizing of order $-1/2$ with $\mathcal{N}^0(J) \le M_J$, and any (ζ,ψ) smooth solution to* (RWW2) *on the time interval $I \subset \mathbb{R}$ and satisfying*

$$\epsilon M \stackrel{\text{def}}{=} \sup_{t\in I}\big(\epsilon\big|\zeta(t,\cdot)\big|_{H^{t_0+2}} + \epsilon\big|\nabla\psi(t,\cdot)\big|_{H^{t_0+1}}\big) \le M_U$$

one has for any $f \in L^2(\mathbb{R}^d)$ and any $t,t' \in I$,

$$\Big|\,(f,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta(t,\cdot),\epsilon\nabla\psi(t,\cdot)]f)_{L^2} - (f,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta(t',\cdot),\epsilon\nabla\psi(t',\cdot)]f)_{L^2}\,\Big| \le K\,|t-t'|\,\big|f\big|_{L^2}^2$$

with $K = C\times\big(\epsilon M + \dot{\mathcal{N}}^{-\frac12}(J)^2(\epsilon M)^2\big)$.

Proof. We have

$$(f,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta(t,\cdot),\epsilon\nabla\psi(t,\cdot)]f)_{L^2} = (f,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta(t',\cdot),\epsilon\nabla\psi(t',\cdot)]f)_{L^2} + \int_{t'}^t (f,[\partial_t,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta,\epsilon\nabla\psi]]f)_{L^2}\,\mathrm{d}\tau \tag{3.26}$$

and the result readily follows from the estimate (3.20), and $\mathcal{N}^{-\frac12}(J) \le \dot{\mathcal{N}}^{-\frac12}(J) + \mathcal{N}^0(J)$. □

3.2.3 Proof of Proposition 3.2

We shall now complete the proof of Proposition 3.2. By Proposition 3.1 and Corollary 3.7, we have the existence and uniqueness of $(\zeta,\psi)\in\mathcal{C}((-T_\star,T^\star);H^N(\mathbb{R}^d)\times\mathring{H}^{N+1/2}(\mathbb{R}^d))$ maximal solution to (RWW2) with initial data $(\zeta,\psi)\big|_{t=0}=(\zeta_0,\psi_0)\in H^N(\mathbb{R}^d)\times\mathring{H}^{N+1/2}(\mathbb{R}^d)$, thus we only need to show that the solution is estimated as in the proposition on the prescribed time interval.

To this aim, we first consider $\chi:\mathbb{R}^d\to\mathbb{R}^+$ a smooth cut-off function (radial, infinitely differentiable, with compact support, and such that $\chi(\boldsymbol{\xi})=1$ for $|\boldsymbol{\xi}|\leq 1$), and define for $n\in\mathbb{N}$, $(\zeta_0^n,\psi_0^n)\overset{\text{def}}{=}(\chi(\frac{D}{2^n})\zeta_0,\chi(\frac{D}{2^n})\psi_0)$. By construction, $(\zeta_0^n,\psi_0^n)\in\bigcap_{s\in\mathbb{N}}(H^s(\mathbb{R}^d)\times\mathring{H}^{s+\frac12}(\mathbb{R}^d))$ and $(\zeta_0^n,\psi_0^n)\to(\zeta_0,\psi_0)$ in $H^N(\mathbb{R}^d)\times\mathring{H}^{N+1/2}(\mathbb{R}^d)$ as $n\to\infty$. By Proposition 3.1, and the blowup alternative in Corollary 3.7, we have for any $n\in\mathbb{N}$ the existence and uniqueness of $T_\star^n$, $T_n^\star>0$ and $(\zeta^n,\psi^n)\in\bigcap_{s\in\mathbb{N}}\mathcal{C}((-T_\star^n,T_n^\star);H^s(\mathbb{R}^d)\times\mathring{H}^{s+1/2}(\mathbb{R}^d))$ maximal solution to (RWW2) with initial data $(\zeta^n,\psi^n)\big|_{t=0}=(\zeta_0^n,\psi_0^n)$. Let us first remark that for any $f\in L^2$,

$$\begin{aligned}(3.27)\quad &\big|\big(f,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta_0^n,\epsilon\nabla\psi_0^n]f\big)_{L^2}-\big(f,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta_0,\epsilon\nabla\psi_0]f\big)_{L^2}\big|\\ &\leq C_0\left(\epsilon\mathcal{N}^0(J)^2\big|\zeta_0^n-\zeta_0\big|_{H^{t_0+1}}+\epsilon^2\mathcal{N}^{-\frac12}(J)^2\big|\nabla\psi_0^n+\nabla\psi_0\big|_{H^{t_0}}\big|\nabla\psi_0^n-\nabla\psi_0\big|_{H^{t_0}}\right)\times\big|f\big|_{L^2}^2,\end{aligned}$$

where we used Lemma 2.1 and product estimates in Lemma 2.2, and C_0 depends uniquely on t_0. Hence by restricting to $n\geq n_\star$ with $n_\star$ sufficiently large, we have that (ζ_0^n,ψ_0^n) satisfy (3.3) with coercivity factor $\mathfrak{a}_\star^\mu/2$ (say). Similarly, augmenting $n_\star$ if necessary, we have for any $n\geq n_\star$

$$\begin{aligned}&\big|\zeta_0^n\big|_{H^{t_0+\frac12}}\leq 2\big|\zeta_0\big|_{H^{t_0+\frac12}},&&\big|\mathfrak{P}^\mu\psi_0^n\big|_{H^{t_0+\frac12}}\leq 2\big|\mathfrak{P}^\mu\psi_0\big|_{H^{t_0+\frac12}},\\ &\mathcal{E}^N(\zeta_0^n,\psi_0^n)\leq\sqrt{C}\mathcal{E}^N(\zeta_0,\psi_0),&&\mathcal{E}^{\lceil t_0\rceil+2}(\zeta_0^n,\psi_0^n)\leq\sqrt{C}\mathcal{E}^{\lceil t_0\rceil+2}(\zeta_0,\psi_0),\end{aligned}$$

where C is prescribed in the assumptions of Proposition 3.2. In the rest of the proof, we assume that $n\geq n_\star$. We introduce now $C_1>1$ to be defined later, and the interval $I^n\subset\mathbb{R}^+$ as the set of $T\geq 0$ such that,

$$(3.28)\qquad \forall t\in[-T,T],\quad \big|\zeta^n(t,\cdot)\big|_{H^{t_0+2}}^2+\big|\nabla\psi^n(t,\cdot)\big|_{H^{t_0+1}}^2\leq C_1\mathcal{E}^{\lceil t_0\rceil+2}(\zeta_0,\psi_0)=C_1M_0^2.$$

We need some preliminary estimates. Assume that (ζ^n,ψ^n) is defined on $[-T,T]$ for some $T>0$. We claim the following.

(a) Let $t\in[-T,T]$. Assume that $(f,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta^n(t,\cdot),\epsilon\nabla\psi^n(t,\cdot)]f)_{L^2}\geq(\mathfrak{a}_\star^\mu/3)\big|f\big|_{L^2}^2$ for any $f\in L^2$. There exists $C_{t_0}>0$, depending only on t_0, such that

$$\big|\nabla\psi^n(t,\cdot)\big|_{H^{t_0}}^2\leq C_{t_0}\mathcal{E}^{\lceil t_0\rceil+2}(\zeta^n(t,\cdot),\psi^n(t,\cdot)).$$

Furthermore, if $\big|\nabla\psi^n(t,\cdot)\big|_{H^{t_0}}^2\leq C_{t_0}CM_0^2$, there exists $\tilde{C}_{t_0}>0$, depending only on t_0, such that

$$\begin{aligned}\big|\zeta^n(t,\cdot)\big|_{H^{t_0+2}}^2+\big|\nabla\psi^n(t,\cdot)\big|_{H^{t_0+1}}^2\leq\tilde{C}_{t_0}\left(1+\tfrac{3}{\mathfrak{a}_\star^\mu}\right)\Big(1+\mathcal{N}^0(J)\sqrt{C_{t_0}C}M_U\Big)\\ \times\mathcal{E}^{\lceil t_0\rceil+2}(\zeta^n(t,\cdot),\psi^n(t,\cdot)),\end{aligned}$$

and, there exists $C_2>0$, depending only on t_0, N, $3/\mathfrak{a}_\star^\mu$, $\sqrt{C}M_U$ and $\dot{\mathcal{N}}^{-\frac12}(J)$ such that for any $t\in[-T,T]$ and any $N_\star\in\{\lceil t_0\rceil+2,N\}$,

$$\tfrac{1}{C_2}\mathcal{E}^{N_\star}(\zeta^n(t,\cdot),\psi^n(t,\cdot))\leq\big|\zeta^n(t,\cdot)\big|_{H^{N_\star}}^2+\big|\mathfrak{P}^\mu\zeta^n(t,\cdot)\big|_{H^{N_\star}}^2\leq C_2\mathcal{E}^{N_\star}(\zeta^n(t,\cdot),\psi^n(t,\cdot)).$$

(b) Assume that $\big|\zeta^n(t,\cdot)\big|_{H^{t_0+2}}^2+\big|\nabla\psi^n(t,\cdot)\big|_{H^{t_0+1}}^2\leq C_1M_0^2$ for any $t\in[-T,T]$. Then there exists $C_3>0$, depending only on t_0, $\mathcal{N}^0(J)$ and $\sqrt{C_1}M_U$ such that, for any $t\in[-T,T]$ and $f\in L^2$,

$$(f,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta^n(t,\cdot),\epsilon\nabla\psi^n(t,\cdot)]f)_{L^2}\geq\frac{\mathfrak{a}_\star^\mu}{2}\big|f\big|_{L^2}^2-C_3\times\big(\sqrt{C_1}\epsilon M_0+\dot{\mathcal{N}}^{-\frac12}(J)^2C_1(\epsilon M_0)^2\big)|t|\big|f\big|_{L^2}^2.$$

Furthermore, if $(f,\mathfrak{a}^\mu[\epsilon \mathsf{J}\zeta^n(t,\cdot),\epsilon\nabla\psi^n(t,\cdot)]f)_{L^2} \geq (\mathfrak{a}^\mu_\star/3)\big|f\big|^2_{L^2}$ for any $t\in[-T,T]$ and any $f\in L^2$, there exists $C_4>0$, depending only on t_0, N, $\mathfrak{a}^\mu_\star$, $\mathcal{N}^0_1(J)$ and $\sqrt{C_1}M_U$, such that, for any $t\in[-T,T]$ and any $N_\star\in\{\lceil t_0\rceil+2,N\}$,

$$\mathcal{E}^{N_\star}(\zeta^n(t,\cdot),\psi^n(t,\cdot)) \leq \exp\big(C_4\big(\sqrt{C_1}\epsilon M_0+\dot{\mathcal{N}}^{-\frac{1}{2}}(J)^2C_1(\epsilon M_0)^2\big)|t|\big)\sqrt{C}\mathcal{E}^{N_\star}(\zeta_0,\psi_0).$$

Estimates (a) follow from Lemma 3.11 and the definition of the energy in (3.2). Estimates (b) follow from Lemma 3.13 and Proposition 3.12 (using that (ζ^n,ψ^n) is smooth). Using the previous notations, we can now define C_1 such that

$$C_1 = 2C\tilde{C}_{t_0}\left(1+\frac{3}{\mathfrak{a}^\mu_\star}\right)\left(1+\mathcal{N}^0(J)\sqrt{C_{t_0}C}M_U\right).$$

With such definition of C_1, we have $0\in I^n$ from Estimates (a). We now introduce T_2 as the largest time such that

$$C_4(\sqrt{C_1}\epsilon M_0+\dot{\mathcal{N}}^{-\frac{1}{2}}(J)^2C_1(\epsilon M_0)^2)T_2\leq \ln\big(C^{1/2}\big) \text{ and } C_3\big(\sqrt{C_1}\epsilon M_0+\dot{\mathcal{N}}^{-\frac{1}{2}}(J)^2C_1(\epsilon M_0)^2\big)T_2\leq\frac{\mathfrak{a}^\mu_\star}{6}.$$

We claim that $T^n\overset{\text{def}}{=}\max(I^n)\geq T_2$. We argue by contradiction and assume that $T_2>T^n$. Notice that $T^n>0$ and is well-defined (that is $\sup(I^n)\in I^n$ when $\sup(I^n)<\infty$) using the continuity in time of the solution with arbitrary smoothness in space, provided by Corollary 3.7 and the prescribed bound on $(\zeta^n(t,\cdot),\psi^n(t,\cdot))\in H^{t_0+2}\times\mathring{H}^{t_0+2}$. Using Estimates (b) and then (a), we get successively that for any $t\in[0,T^n]$ and any $f\in L^2$,

$$(f,\mathfrak{a}^\mu[\epsilon\mathsf{J}\zeta^n(t,\cdot),\epsilon\nabla\psi^n(t,\cdot)]f)_{L^2}\geq\frac{\mathfrak{a}^\mu_\star}{3}\big|f\big|^2_{L^2},\qquad \mathcal{E}^{\lceil t_0\rceil+2}(\zeta^n(t,\cdot),\psi^n(t,\cdot))\leq C\mathcal{E}^{\lceil t_0\rceil+2}(\zeta_0,\psi_0),$$

$$\big|\nabla\psi^n(t,\cdot)\big|^2_{H^{t_0}}\leq CC_{t_0}M_0^2,\quad\text{and}\qquad \big|\zeta^n(t,\cdot)\big|^2_{H^{t_0+2}}+\big|\nabla\psi^n(t,\cdot)\big|^2_{H^{t_0+1}}\leq\frac{1}{2}C_1M_0^2.$$

Using again Corollary 3.7 we obtain a time $T>T^n$ such that $T\in I^n$, which is a contradiction.

It remains to pass to the limit. Thanks to Remark 3.8, there exists $T>0$, depending uniquely on t_0, N, ϵ, μ, $\mathcal{N}^{-1}(J)$ and a prescribed bound on the initial data, such that if $(\tilde{\zeta}^n_0,\tilde{\psi}^n_0)\to(\tilde{\zeta}_0,\tilde{\psi}_0)$ in $H^N(\mathbb{R}^d)\times\mathring{H}^{N+1/2}(\mathbb{R}^d)$, then the emerging solution is defined on the time interval $[-T,T]$ and one has $\Phi((\tilde{\zeta}^n_0,\tilde{\psi}^n_0))\to\Phi((\tilde{\zeta}_0,\tilde{\psi}_0))$ in $\mathcal{C}([-T,T];H^N(\mathbb{R}^d)\times\mathring{H}^{N+1/2}(\mathbb{R}^d))$. Notice that we have a uniform bound for $(\zeta^n,\psi^n)\in\mathcal{C}([-T_2,T_2];H^N(\mathbb{R}^d)\times\mathring{H}^{N+1/2}(\mathbb{R}^d))$, by (3.28) together with the last inequality of Estimates (a). This provides a lower bound on $T>0$ which allows to show after a finite number of iterations (and thanks to the uniqueness of the solution to the Cauchy problem) that $(\zeta^n,\psi^n)\to(\zeta,\psi)$ in $\mathcal{C}([-T_2,T_2];H^N(\mathbb{R}^d)\times\mathring{H}^{N+1/2}(\mathbb{R}^d))$. In particular, (ζ,ψ) satisfy the desired energy control on $[-T_2,T_2]$ thanks to the second inequality of Estimates (b). The proof of Proposition 3.2 is complete.

3.3 Global-in-time well-posedness

We conclude this section with the following result showing the global-in-time existence of solutions for sufficiently small initial data.

Proposition 3.14. *Let $m<-(\frac{3}{2}+\frac{d}{2})$ and $C>1$. There exists $\epsilon_0>0$ such that for any $\mathsf{J}=J(D)$ regularizing of order m (see Definition 1.1), any $\mu\geq1$, $\epsilon>0$ and $(\zeta_0,\psi_0)\in L^2(\mathbb{R}^d)\times\mathring{H}^{1/2}(\mathbb{R}^d)$ such that*

$$\epsilon\big|\langle\cdot\rangle^{-m}J\big|_{L^\infty}\big(\big|\zeta_0\big|^2_{L^2}+\big|\mathfrak{P}^\mu\psi_0\big|^2_{L^2}\big)^{1/2}\leq\epsilon_0, \tag{3.29}$$

there exists a unique $(\zeta,\psi)\in\mathcal{C}(\mathbb{R};L^2(\mathbb{R}^d)\times\mathring{H}^{1/2}(\mathbb{R}^d))$ *solution to* (RWW2) *with initial data* $(\zeta,\psi)\,|_{t=0}=(\zeta_0,\psi_0)$*, and moreover for any* $t\in\mathbb{R}$*,*

(3.30)
$$\tfrac{1}{C}\left(\tfrac12\big|\zeta(t,\cdot)\big|_{L^2}^2+\tfrac12\big|\mathfrak{P}^\mu\psi(t,\cdot)\big|_{L^2}^2\right)\le\mathcal{H}^\mu(\zeta(t,\cdot),\psi(t,\cdot))=\mathcal{H}^\mu(\zeta_0,\psi_0)\le C\left(\tfrac12\big|\zeta_0\big|_{L^2}^2+\tfrac12\big|\mathfrak{P}^\mu\psi_0\big|_{L^2}^2\right)$$

where we recall that

$$\mathcal{H}^\mu(\zeta,\psi)\overset{\text{def}}{=}\frac12\int_{\mathbb{R}^d}\zeta^2+(\mathfrak{P}^\mu\psi)^2+\epsilon(\mathsf{J}\zeta)\left(|\nabla\psi|^2-(\mathsf{T}^\mu\cdot\nabla\psi)^2\right)\mathrm{d}\boldsymbol{x}.$$

Remark 3.15. *In the second equation of* (RWW2)*, the meaning of quadratic terms for low-regularity functions* $\psi\in\mathring{H}^{1/2}(\mathbb{R}^d)$ *is clarified by Lemma 2.11.*

Proof. First we recall that $\mathcal{H}^\mu(\zeta,\psi)$ is an invariant of (RWW2), in the sense that for any (ζ,ψ) smooth solutions to (RWW2) defined on a time interval $I\subset\mathbb{R}$, $t\mapsto\mathcal{H}^\mu(\zeta(t,\cdot),\psi(t,\cdot))$ is constant on I. This is readily checked by computing

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^\mu(\zeta(t,\cdot),\psi(t,\cdot))=\big(\zeta,\partial_t\zeta\big)_{L^2}+\big(\mathfrak{P}^\mu\psi,\partial_t\mathfrak{P}^\mu\psi\big)_{L^2}+\frac12\big(\epsilon(\mathsf{J}\partial_t\zeta)\left(|\nabla\psi|^2-(\mathsf{T}^\mu\cdot\nabla\psi)^2\right)\big)_{L^2}\\+\big((\epsilon\mathsf{J}\zeta)\left((\nabla\psi\cdot\nabla\partial_t\psi)-(\mathsf{T}^\mu\cdot\nabla\psi)(\mathsf{T}^\mu\cdot\nabla\partial_t\psi)\right)\big)_{L^2},\end{aligned}$$

replacing time derivatives with the formula provided by (RWW2) and using suitable integration by parts or Parseval's theorem (see Lemma 2.1) to infer that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^\mu(\zeta(t,\cdot),\psi(t,\cdot))=0$. The result for $(\zeta,\psi)\in\mathcal{C}((-T_\star,T^\star);L^2(\mathbb{R}^d)\times\mathring{H}^{1/2}(\mathbb{R}^d))$ maximal solution with initial data $(\zeta,\psi)\,|_{t=0}=(\zeta_0,\psi_0)$, defined by Proposition 3.1, follows by the density of the Schwartz space into Sobolev spaces, and the continuity of the solution map (recall Remark 3.8).

Then we remark that, by Lemma 2.11 and Cauchy-Schwarz inequality, and then Lemma 2.1 and Young's inequality, we have for any $(\zeta,\psi)\in L^2(\mathbb{R}^d)\times\mathring{H}^{1/2}(\mathbb{R}^d)$,

$$\begin{aligned}\left|\mathcal{H}^\mu(\zeta,\psi)-\tfrac12\big|\zeta\big|_{L^2}^2-\tfrac12\big|\mathfrak{P}^\mu\psi\big|_{L^2}^2\right|&\le\epsilon C_m\big|\zeta\big|_{L^2}\left(\big|\langle\cdot\rangle^{-m}J\big|_{L^\infty}\big|\nabla\psi\big|_{H^{-1/2}}^2\right)\\&\le\epsilon C_m\big|\langle\cdot\rangle^{-m}J\big|_{L^\infty}\left(\big|\zeta\big|_{L^2}^2+\big|\mathfrak{P}^\mu\psi\big|_{L^2}^2\right)^{3/2},\end{aligned}$$

where C_m depends only on $m<-(1+\frac{d}{2})$. By choosing $\epsilon_0>0$ such that $1+\epsilon_0C_mC^2<C$, we infer that for all initial data satisfying (3.29), $\big\{t\in[0,T^\star),\ (3.30)\text{ holds}\big\}$ is an open subset of $[0,T^\star)$. Since it is also closed and non-empty, we obtain that (3.30) holds on $[0,T^\star)$. If $T^\star<\infty$, we may use Proposition 3.1 with an "initial" time sufficiently close to $T^\star$ and the control provided by (3.30) to construct $\tilde{T}^\star>T^\star$ and $(\tilde\zeta,\tilde\psi)\in\mathcal{C}([0,\tilde{T}^\star];L^2(\mathbb{R}^d)\times\mathring{H}^{1/2}(\mathbb{R}^d))$ satisfying $(\tilde\zeta,\tilde\psi)\big|_{[0,T^\star)}=(\zeta,\psi)\big|_{[0,T^\star)}$ and (RWW2) on the time interval $[0,\widetilde{T}^\star]$. This brings the desired contradiction, and proves that $T^\star=+\infty$. Symmetrically, $-T_\star=-\infty$ and the proof is complete. □

4 Full justification; proof of Theorems 1.3 and 1.4

In this section we complete the full justification (RWW2) as provided in Theorems 1.3 and 1.4. We start by measuring the "cost" of introducing a near-identity rectifier in the system (WW2).

Proposition 4.1. *Let* $d\in\{1,2\}$*,* $t_0>\frac{d}{2}$*, and* $s\ge0$*. There exists* $C>0$ *such that for any* $\epsilon\ge0$*,* $\mu\ge1$*,* $\mathsf{J}=J(D)$ *a rectifier near-identity of order* $\ell\ge0$ *(see Definition 1.1) and any* $(\zeta,\psi)\in\mathcal{C}([0,T];H^{\max(s+\ell+1,t_0+\frac12)}(\mathbb{R}^d)\times\mathring{H}^{\max(s+\ell+\frac32,t_0+1)}(\mathbb{R}^d))$ *solution to* (RWW2)*, one has*

$$\begin{cases}\partial_t\zeta-\mathsf{T}^\mu\cdot\nabla\psi+\epsilon\mathsf{T}^\mu\cdot\nabla(\zeta\mathsf{T}^\mu\cdot\nabla\psi)+\epsilon\nabla\cdot(\zeta\nabla\psi)=\epsilon R_1,\\ \partial_t\psi+\zeta+\frac{\epsilon}{2}\left(|\nabla\psi|^2-(\mathsf{T}^\mu\cdot\nabla\psi)^2\right)=\epsilon R_2,\end{cases}$$

with (recalling the notation $\dot{\mathcal{N}}^a(1-J) = \big|\,|\cdot|^{-a}(1-J)\big|_{L^\infty}$*)*

$$\big|R_1\big|_{H^s} + \big|R_2\big|_{H^{s+\frac12}} \le C\dot{\mathcal{N}}^\ell(1-J)\Big(\big(|\zeta|_{H^{t_0+\frac12}} + |\nabla\psi|_{H^{t_0}}\big)\big(|\zeta|_{H^{s+\ell+1}} + |\nabla\psi|_{H^{s+\ell+\frac12}}\big)\Big).$$

In particular, if $\mathsf{J} = J_0(\delta D)$ *with* $\delta > 0$*, then*

$$\big|R_1\big|_{H^s} + \big|R_2\big|_{H^{s+\frac12}} \le \delta^\ell\dot{\mathcal{N}}^\ell(1-J_0)\Big(\big(|\zeta|_{H^{t_0+\frac12}} + |\nabla\psi|_{H^{t_0}}\big)\big(|\zeta|_{H^{s+\ell+1}} + |\nabla\psi|_{H^{s+\ell+\frac12}}\big)\Big).$$

Proof. One has immediately

$$\begin{cases} R_1 = \mathsf{T}^\mu\cdot\nabla((\zeta-\mathsf{J}\zeta)\mathsf{T}^\mu\cdot\nabla\psi) + \nabla\cdot((\zeta-\mathsf{J}\zeta)\nabla\psi), \\ R_2 = \frac12(\mathrm{Id}-\mathsf{J})\left(|\nabla\psi|^2 - (\mathsf{T}^\mu\cdot\nabla\psi)^2\right). \end{cases}$$

By the second estimate of Lemma 2.9 with $r=0$, we find

$$\big|R_1\big|_{H^s} \lesssim \big|\zeta-\mathsf{J}\zeta\big|_{H^{s+1}}\big|\nabla\psi\big|_{H^{t_0}} + \big|\zeta-\mathsf{J}\zeta\big|_{H^{t_0+1}}\big|\nabla\psi\big|_{H^s}.$$

Now, by Parseval's theorem, for any $\sigma\in\mathbb{R}$ and $\ell\ge0$

$$\big|\zeta-\mathsf{J}\zeta\big|_{H^\sigma} \le \dot{\mathcal{N}}^a(1-J)\big|\zeta\big|_{H^{\sigma+a}}.$$

For any $s > t_0$, since $\ell\ge0$, we can put $\theta = \frac{s-t_0}{s+a+\frac12-t_0}\in(0,1)$ and infer by interpolation and Young's inequalities,

$$\begin{aligned}\big|\zeta\big|_{H^{t_0+a+1}}\big|\nabla\psi\big|_{H^s} &\lesssim \big|\zeta\big|^\theta_{H^{t_0+\frac12}}\big|\zeta\big|^{1-\theta}_{H^{s+a+1}}\big|\nabla\psi\big|^\theta_{H^{s+a+\frac12}}\big|\nabla\psi\big|^{1-\theta}_{H^{t_0}} \\ &\lesssim \big|\zeta\big|_{H^{t_0+\frac12}}\big|\nabla\psi\big|_{H^{s+a+\frac12}} + \big|\zeta\big|_{H^{s+a+1}}\big|\nabla\psi\big|_{H^{t_0}}.\end{aligned}$$

This provides the desired estimate for $\big|R_1\big|_{H^s}$. Then, we have

$$\big|R_2\big|_{H^{s+\frac12}} \le \frac12\dot{\mathcal{N}}^a(1-J)\big||\nabla\psi|^2 - (\mathsf{T}^\mu\cdot\nabla\psi)^2\big|_{H^{s+a+\frac12}}$$

and product estimates (Lemma 2.3) yield

$$\big||\nabla\psi|^2 - (\mathsf{T}^\mu\cdot\nabla\psi)^2\big|_{H^{s+a+\frac12}} \lesssim \big|\nabla\psi\big|_{H^{t_0}}\big|\nabla\psi\big|_{H^{s+a+\frac12}}.$$

This concludes the proof. □

Proposition 4.1 should be articulated with the following consistency result with respect to the water waves system.

Proposition 4.2. *Let* $d\in\{1,2\}$*,* $t_0 > d/2$*,* $s\ge0$*,* $h_\star>0$ *and* $M>0$*. There exists* $C>0$ *such that for any* $\epsilon>0$*,* $\mu\ge1$*, and for any* $(\zeta,\psi)\in\mathcal{C}([0,T];H^{\max(s+2,t_0+\frac32)}(\mathbb{R}^d)\times\mathring{H}^{\max(s+\frac32,t_0+1)}(\mathbb{R}^d))$ *solution to* (WW2) *(that is* (RWW2) *with* $\mathsf{J}=\mathrm{Id}$*), and such that on* $[0,T]$

$$1+\tfrac{\epsilon}{\sqrt{\mu}}\zeta \ge h_\star, \qquad \epsilon M_0 \overset{\text{def}}{=} \epsilon\left(\big|\zeta\big|_{H^{t_0+\frac32}} + \big|\mathfrak{P}^\mu\psi\big|_{H^{t_0+\frac12}}\right) \le M,$$

one has

$$\begin{cases} \partial_t\zeta - \frac{1}{\sqrt{\mu}}G^\mu[\epsilon\zeta]\psi = \epsilon^2\widetilde{R}_1 \\ \partial_t\psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\epsilon}{2}\frac{\left(\frac{1}{\sqrt{\mu}}G^\mu[\epsilon\zeta]\psi + \epsilon\nabla\zeta\cdot\nabla\psi\right)^2}{1+\epsilon^2|\nabla\zeta|^2} = \epsilon^2\widetilde{R}_2 \end{cases}$$

with G^μ *the Dirichlet-to-Neumann operator defined and discussed in Appendix A, and*

$$\big|\widetilde{R}_1\big|_{H^s} + \big|\widetilde{R}_2\big|_{H^{s+\frac12}} \le CM_0^2\Big(\big|\zeta\big|_{H^{s+2}} + \big|\mathfrak{P}^\mu\psi\big|_{H^{s+1}}\Big).$$

Proof. One has

$$\begin{aligned}
\epsilon^2\widetilde{R}_1 &= -\frac{1}{\sqrt{\mu}}G^\mu[\epsilon\zeta]\psi + \mathsf{T}^\mu\cdot\nabla\psi - \epsilon\mathsf{T}^\mu\cdot\nabla(\zeta\mathsf{T}^\mu\cdot\nabla\psi) - \epsilon\nabla\cdot(\zeta\nabla\psi),\\
\epsilon^2\widetilde{R}_2 &= -\frac{\epsilon}{2}\frac{\left(\frac{1}{\sqrt{\mu}}G^\mu[\epsilon\zeta]\psi + \epsilon\nabla\zeta\cdot\nabla\psi\right)^2}{1+\epsilon^2|\nabla\zeta|^2} + \frac{\epsilon}{2}(\mathsf{T}^\mu\cdot\nabla\psi)^2\\
&= -\frac{\epsilon}{2}\frac{\left(\frac{1}{\sqrt{\mu}}G^\mu[\epsilon\zeta]\psi + \epsilon\nabla\zeta\cdot\nabla\psi - \mathsf{T}^\mu\cdot\nabla\psi\right)\left(\frac{1}{\sqrt{\mu}}G^\mu[\epsilon\zeta]\psi + \epsilon\nabla\zeta\cdot\nabla\psi + \mathsf{T}^\mu\cdot\nabla\psi\right)}{1+\epsilon^2|\nabla\zeta|^2}\\
&\quad + \frac{\epsilon^3}{2}\frac{|\nabla\zeta|^2(\mathsf{T}^\mu\cdot\nabla\psi)^2}{1+\epsilon^2|\nabla\zeta|^2}.
\end{aligned}$$

By Proposition A.1, we have

$$\big|\widetilde{R}_1\big|_{H^s} \le C\big(\tfrac{1}{h_\star},M\big)\big|\zeta\big|_{H^{t_0+1}}\left(\big|\zeta\big|_{H^{t_0+1}}\big|\mathfrak{P}^\mu\psi\big|_{H^{s+\frac12}} + \big|\zeta\big|_{H^{s+\frac32}}\big|\mathfrak{P}^\mu\psi\big|_{H^{t_0}}\right).$$

Then, by Moser tame estimates (see for instance [18, Prop. B.4]) we have for any $F\in H^{s+\frac12}(\mathbb{R}^d)$,

$$\big|\tfrac{F}{1+\epsilon^2|\nabla\zeta|^2}\big|_{H^{s+1/2}} \le C\left(\epsilon\big|\nabla\zeta\big|_{H^{t_0}}\right)\left(\big|F\big|_{H^{s+\frac12}} + \epsilon\big|F\big|_{H^{t_0}}\big|\nabla\zeta\big|_{H^{s+\frac12}}\right)$$

and, using both Proposition A.1 and Lemma 2.9 (with $r=0$), and the triangular inequality, we get for any $\sigma\ge 0$

$$\begin{aligned}
\big|\tfrac{1}{\sqrt{\mu}}G^\mu[\epsilon\zeta]\psi - \mathsf{T}^\mu\cdot\nabla\psi\big|_{H^\sigma} \le \epsilon^2 C\big(\tfrac{1}{h_\star},M\big)\big|\zeta\big|_{H^{t_0+1}}&\left(\big|\zeta\big|_{H^{t_0+1}}\big|\mathfrak{P}^\mu\psi\big|_{H^{\sigma+\frac12}} + \big|\zeta\big|_{H^{\sigma+\frac32}}\big|\mathfrak{P}^\mu\psi\big|_{H^{t_0}}\right)\\
&+ \epsilon\big|\zeta\big|_{H^{\sigma+1}}\big|\nabla\psi\big|_{H^{t_0}} + \epsilon\big|\zeta\big|_{H^{t_0+1}}\big|\nabla\psi\big|_{H^\sigma}.
\end{aligned}$$

The result follows from the above with $\sigma\in\{s+\frac12,t_0\}$, Lemma 2.1, the triangular inequality and tame product estimates, Lemma 2.3. □

Propositions 4.1 and 4.2 immediately yield Theorem 1.3.

Then, Theorem 1.4 is a direct consequence of Theorem 1.3, Theorem 1.2 and the well-posedness and stability of the water waves system provided in [2]. The proof is almost identical to [2, Theorem 6.5] (in fact (WW2) arises explicitly in (6.13) therein). Specifically, one infers the existence and uniqueness of a solution to (WW) on the appropriate time interval from Theorem 5.1 and Proposition 5.1 in [2], solutions to (RWW2) are provided by Theorem 1.2 and the control of the difference stems from Theorem 1.3 and appropriate energy estimates.

5 Numerical study

In this section we perform numerical experiments illustrating our results, and enlightening features of our rectification procedure. Let us first describe the numerical method we employ in order to produce approximate solutions to systems (WW2) and (RWW2) (restricted here to horizontal dimension $d=1$). The figures are obtained using a code written in the Julia programming language [7], and specifically the package written by the first author and P. Navaro and available at https://github.com/WaterWavesModels/WaterWaves1D.jl/.

Numerical scheme We use Fourier (pseudo-)spectral methods for the spatial discretization. We shortly describe the principles thereafter, and let the reader refer to *e.g.* [29] for more details. A periodic[5] function with period $P = 2L$ is approximated as the superposition of an even number, N, of monochromatic waves:

$$f(x) \approx \sum_{k=-N/2}^{N/2-1} \widehat{f}_k e^{\mathrm{i}\frac{\pi}{L}kx}$$

where we refer to $(\widehat{f}_k)_{k=-N/2,\dots,N/2-1}$ as the discrete Fourier coefficients. In practice, we can efficiently compute the discrete Fourier coefficients from the values of the function at regularly spaced collocation points, $(f(x_n))_{n=0,\dots,N-1}$ where $x_n = -L + 2\frac{n}{N}L$ through the Fast Fourier Transform (FFT), and conversely recover values at collocation points from discrete Fourier coefficients through the inverse Fast Fourier Transform (IFFT). This allows to perform efficiently (and without introducing approximations except for rounding errors) the action of Fourier multiplication operators through pointwise multiplication on discrete Fourier coefficients, and the action of multiplication in the physical space at collocation point. However the latter operation cannot be performed exactly due to aliasing effects, stemming from the fact that $x \mapsto 1$ and $x \mapsto e^{\mathrm{i}\frac{\pi}{L}Nx}$ are indistinguishable in our collocation grid. Aliasing effects are known to possibly generate spurious numerical instabilities. In this case one customarily compute a Galerkin approximation (since the error of the approximation is orthogonal all considered monochromatic waves) of products by setting first a sufficient number (since our problem includes only quadratic nonlinearities, we use Orszag's 3/2 rule [22]) of discrete Fourier coefficients to zero. Hence in practice we solve, unless otherwise stated,

$$\left\{\begin{array}{l} \partial_t\zeta - \mathsf{T}^\mu\cdot\nabla\psi + \frac{\epsilon}{\sqrt{\mu}}\Pi\mathsf{T}^\mu\cdot\nabla((\mathsf{J}\zeta)\mathsf{T}^\mu\cdot\nabla\psi) + \frac{\epsilon}{\sqrt{\mu}}\Pi\nabla\cdot((\mathsf{J}\zeta)\nabla\psi) = 0, \\ \partial_t\psi + \zeta + \frac{\epsilon}{2\sqrt{\mu}}\Pi\mathsf{J}\left(|\nabla\psi|^2 - (\mathsf{T}^\mu\cdot\nabla\psi)^2\right) = 0, \end{array}\right. \tag{5.1}$$

where Π is the "dealiasing" projection operator onto $(2L)$-periodic functions with all but the first $\lfloor 2N/3\rfloor$ discrete Fourier coefficients equal to zero, and initial data $\zeta_0 = \zeta(t=0,\cdot)$, $\psi_0 = \psi(t=0,\cdot)$ such that $\Pi\zeta_0 = \zeta_0$ and $\Pi\psi_0 = \psi_0$. As for the time integration, we use the standard explicit fourth order Runge-Kutta method. Since our problem is stiff (that is involves unbounded operators before discretization), a stability condition on the time step must be secured to avoid spurious numerical instabilities. By Dahlquist's stability theory (see [29, Chapter 10]), and given the nature and order of the operators at stake, we expect that it is sufficient to enforce $\triangle t \leq C\triangle x^{1/2}$ with $\triangle t$ the time step, $\triangle x = L/N$, and C a sufficiently small constant. In practice we validate that the time-discretization induces no spurious result by checking that the observations are unchanged when varying the time step. Unless otherwise noted, we use $\triangle t = 0.001$ in reported numerical experiments. It is not the place to provide a full validation of such a standard numerical scheme. Let us just report that in the experiments corresponding to Figure 7a, the total energy (that is $\mathcal{H}^\mu$) is relatively preserved up to $1.56\,10^{-6}$ if $\triangle t = 0.1$, $1.93\,10^{-11}$ if $\triangle t = 0.01$, and $9.13\,10^{-16}$ if $\triangle t = 0.001$; consistently with the expected accumulated error of order $\mathcal{O}(\triangle t^4)$.

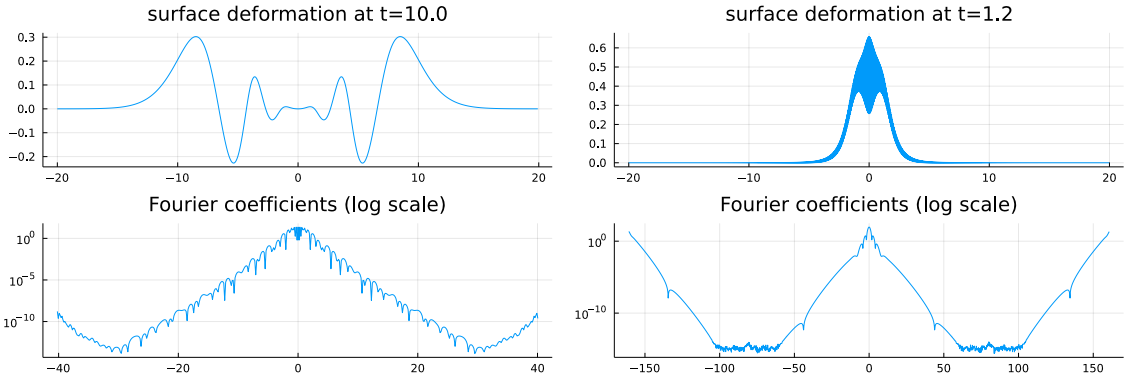
Instabilities without rectification First we exhibit instabilities occurring when the equations are not regularized, that is considering (5.1) with $\mathsf{J} = \mathrm{Id}$, with or without the dealiasing operator, Π. We use for initial data

$$\zeta_0(x) = \zeta(t=0,x) = \exp(-|x|^2) \quad \text{and} \quad \psi_0'(x) = \psi'(t=0,x) = 0 \quad (x\in(-L,L)), \tag{5.2}$$

and set $\mu = 1$, $\epsilon = 0.1$, and $L = 20$. The upshot is that instabilities are observed as soon as a sufficient number of Fourier modes are included, with or without dealiasing. In Figure 1 (without dealiasing) and Figure 2 (with dealiasing) we plot the surface deformation, $\zeta(t,x_n)$ at the final

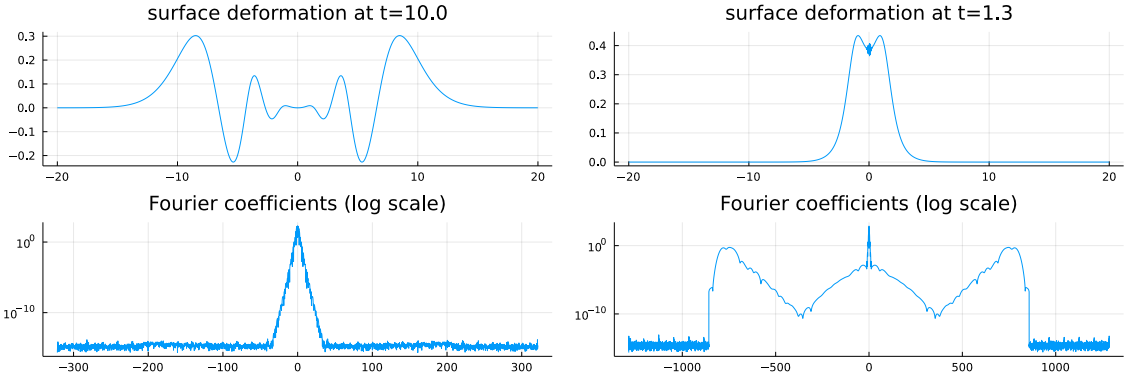
[5] In the foregoing numerical experiments we use rapidly decaying initial data and L large, so that the periodic problem, $x \in L\mathbb{T}$, provides a close approximation of the real-line problem, $x\in\mathbb{R}$, at least for sufficiently small times.

time of computation and at collocation points in the top panels, as well as the associated (modulus of) discrete Fourier coefficients, $\widehat{\zeta_k}(t)$ in the bottom panels. In the left panels we report situations where the number of Fourier modes is too small to notice the rise of discrete Fourier coefficients with large wavenumbers, despite an inflection about extreme values, at least up to the final computation time $t = 10$. In the right panel, we add more modes (with all other parameters kept identical) and observe that the large-wavenumbers component quickly grows to plotting accuracy and, as a consequence, is clearly visible on the top-right panel. The solution breaks down after just a few more time steps. The instability occurs more rapidly as more modes are added and does not depend on sufficiently small values of the time step. These observations are consistent with the ones already reported in [15] for instance, and with the conjecture that the initial-value problem associated with the continous (*i.e.* before discretization) system is ill-posed in finite-regularity spaces. Incidentally, let us clarify that the initial-value problem associated with the continuous problem is well-posed in the analytic framework (see [9] in the inifinite-layer case) covering our initial data, and that what we observe in Figure 1 and Figure 2 is the growth of the spurious large-wavenumbers component generated by machine-precision rounding errors.



(a) $N = 2^9$ modes, at time $t = 10$. (b) $N = 2^{11}$ modes, at time $t = 1.2$.

Figure 1: Time evolution of smooth initial data (5.2), without dealiasing and rectifiers.



(a) $N = 2^{12}$ modes, at time $t = 10$. (b) $N = 2^{14}$ modes, at time $t = 1.3$.

Figure 2: Time evolution of smooth initial data (5.2), with dealiasing and without regularization.

Description of the instability mechanism In order to describe features of the reported instability mechanism, we set up additional numerical experiments. Motivated by the analysis of the toy model in Appendix B, we set

$$\zeta_0(x) = \zeta(t=0,x) = 0 \quad \text{and} \quad \psi_0'(x) = \psi'(t=0,x) = \left(\sin(x) + \frac{\sin(Kx)}{K^2}\right)\exp(-|x|^2). \tag{5.3}$$

Similar initial data were also experimented in [4]. As we will see in Appendix B, solutions to the toy model (B.1) satisfy the following behavior: the flow is stable if $\epsilon^2 K J^2(K) \ll 1$ while the component of the flow corresponding to wavenumbers about K suffers from exponential-in-time amplification (for sufficiently small times, that is before high-high frequency interactions occur) characterized by a multiplicative factor taking the form $a(t) = \exp(C\,t\,K^{1/2}(\epsilon^2 K J^2(K))^{1/2})$ if $\epsilon^2 K J^2(K) \gg 1$. Of course our overly simplified toy model does not take into account many ingredients in (RWW2): the presence of a $\mathcal{O}(\epsilon)$ term in the Rayleigh–Taylor operator $\mathfrak{a}^\mu$ (see (3.1) and Remark 3.3), the possible catastrophes on the timescale $\mathcal{O}(1/\epsilon)$ due to nonlinear (quadratic) interactions and most importantly the production of harmonics through nonlinear interactions. These harmonics with wavenumbers nK ($n \in \mathbb{N}$) will suffer from exponential amplification with larger rate (for instance n-times larger if $\mathsf{J} = \mathrm{Id}$, by the above expression of a), yet they are (at least for small times) orders of magnitude smaller. When $\mathsf{J} = \mathrm{Id}$, back-of-the-envelope calculations indicate that all harmonics are to be amplified up to order-one at roughly the same time.[6] Moreover even for (ϵ, K) in the "stable regime", $\epsilon^2 K J^2(K) \ll 1$, some harmonics (ϵ, nK) lie in the "unstable regime". Lastly, from the numerical point of view, the spurious contribution of machine precision rounding errors at very large wavenumbers is sufficient to trigger instabilities when the number of modes N is large, as shown in Figure 2b. Based on these considerations we conjecture that (numerical) blowup will occur *before* time $T^\star$ with

$$T^\star \propto (\epsilon K |J(K)|)^{-1} \qquad \text{if } \epsilon^2 K J^2(K) \gg 1. \tag{5.4}$$

Notice that we have here implicitly assumed that the time interval for $\mathcal{O}(1)$ (or rather $\mathcal{O}(K^{-2})$, discarding the corresponding logarithmic factor) amplification due to low-high frequency interactions is much larger than the time needed for subsequent high-high frequency interactions to result in the break-down of the solution; see however the discussion of the "intermediate regime" below.

In the following experiments we set $N = 2^{14}$ and again $L = 20$. Hence wavenumbers (after dealiasing) span the interval $(-\frac{2}{3}\frac{\pi}{2}\frac{N}{L}, \frac{2}{3}\frac{\pi}{2}\frac{N}{L}) \approx (-858, 858)$. In order to examine the validity of (5.4), we solve (5.1) with $\mathsf{J} = \mathrm{Id}$, $\mu = 1$ and

(a) for several values of ϵ in the interval $[0.1, 1]$ and $K \in \{100, 200, 400, 800\}$; and

(b) for several values of K in the interval $[100, 800]$ and $\epsilon \in \{0.15, 0.2\}$.

For each numerical experiment, we compute the numerical blowup time as the last time for which the produced solution does not involve NaN (as Not a Number) values. Alternatively, we could as in [4] use an amplification threshold; this would provide very similar results since blowup occurs extremely rapidly after the solution has been amplified multiple times in our numerical experiments.

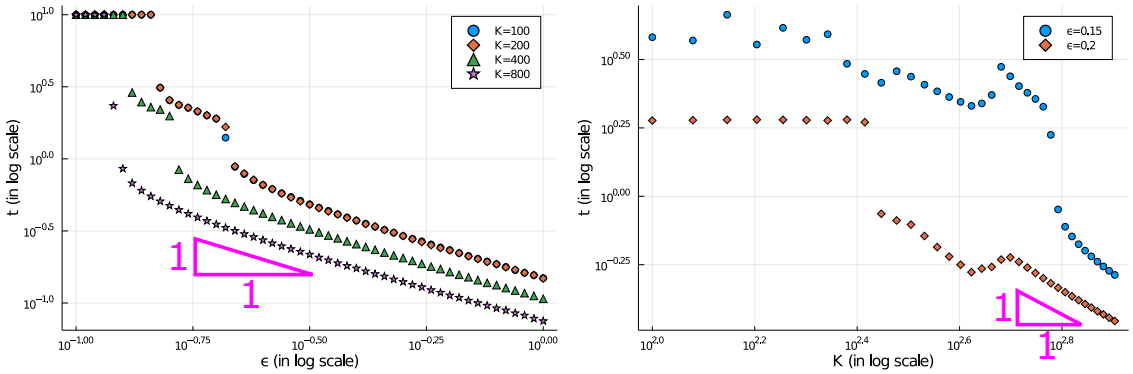
We report the resulting numerical blowup times in Figure 3a for experiments (a) and in Figure 3b experiments (a), in log-log scale. We explain below that the results are consistent with (5.4) and the description above.

In Figure 3a we clearly see three regimes depending on the values of ϵ. For small values of ϵ, the blowup times are announced as the final time of computation (that is $t = 10$), which means

[6] This is corroborated by numerical simulations, as shown Figure 4. Let us point out however that the number of harmonics lying in the interval of wavenumbers spanned by discrete Fourier modes does influence the blowup time (which decreases as N grows), even when such blowup cannot be attributed to the amplification of rounding errors (as in Figure 2b). Similar considerations were reported in [4]; see the dependency with respect to the "filter parameter" P therein.

that numerical blowup did not occur. For large values, the blowup time decays proportionally with $C_K\epsilon^{-1}$ as expected, although the prefactor C_K does not appear to scale proportionally to K^{-1} (we explore and explain this behavior through experiments (b) in Figure 3b) . Moreover, the threshold above which the numerical experiments lie in the unstable regime is roughly consistent with the expected $\epsilon^2 K \approx 1$. We observe a different behavior for intermediate values of ϵ. In this "intermediate regime" we observe, when monitoring the time-evolution, the quick amplification of the large-wavenumbers component of the solution according to our scenario, which then stagnates for some time, before eventually triggering the blowup. We have no explanation for this behavior. Finally let us comment that the figures for $K = 100$ and $K = 200$ are almost superimposed because for such values the main source of instability stems from the amplification of the largest wavenumbers rounding errors (as in Figure 2b) and is hence independent of K.

In Figure 3b we also observe distinct regimes: for K sufficiently small the blowup time does not depend strongly on K. This blowup occurs due to the amplification of the largest-wavenumbers rounding errors as in Figure 2b, which is only mildly influenced by the flow at smaller wavenumbers. For K large enough the blowup time behaves proportionally with K^{-1} in some ranges, with a visible inflection about $K = 450$ when $\epsilon = 0.2$. This phenomenon has been described in footnote 6: past this threshold there is only one possible harmonic in the computed wavenumbers, that is $2K \notin (-\frac{2}{3}\frac{\pi}{2}\frac{N}{L}, \frac{2}{3}\frac{\pi}{2}\frac{N}{L})$. The presence of the second harmonic for lower values of K enhances the instability amplification and eventually the blowup mechanism, yet we observe that it still occurs roughly with the predicted rate, that is proportionally with K^{-1}. If we decide to constraint (as in [4]) the dealiasing cut-off Π to be proportional to K, then the inflection disappears (not shown).



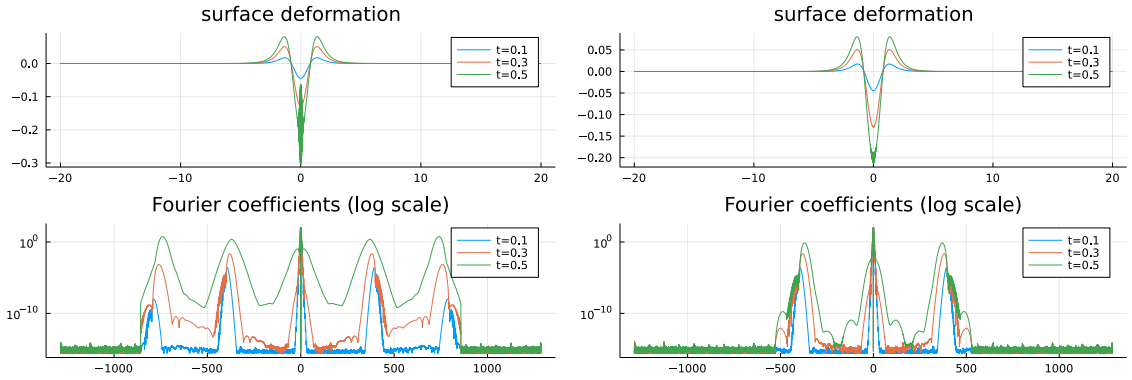
(a) Blowup times with respect to ϵ, in log-log scale. (b) Blowup times with respect to K, in log-log scale.

Figure 3: Numerically computed blowup times of solutions with initial data (5.3).

We further investigate this matter in Figure 4. Here we plot the solution of the previous experiment with $\epsilon = 0.2$ and $K = 400$ at several times: again the surface deformation at the top panels and the modulus of discrete Fourier coefficients in the bottom panels. In the left panels we set Π as the standard dealiasing projection, while in the right panels this low-pass filter is set to retain only one harmonic. We do observe that the instability mechanism is enhanced in the first case (with two harmonics computed), but its features and scales remain the same: the harmonics are amplified so as to become order-one at roughly the same time. All these numerical experiments support our toy model (B.1) as a simplistic model yet capable of providing a fairly accurate description of the blowup mechanism.

Stabilization through rectifiers Next we experiment the effect of introducing the operator J in (5.1). In order to discuss the role of the order of J as a regularizing operator, we set

$$\mathsf{J} = J(\delta D) \quad \text{with } J(\boldsymbol{k}) = \min(\{1, |\boldsymbol{k}|^m\}), \tag{5.5}$$



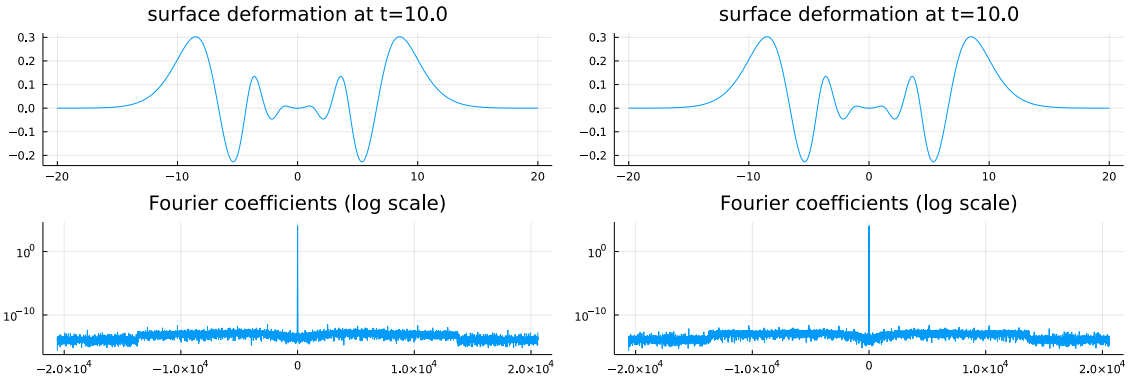
(a) With Orszag's 3/2 dealiasing.

(b) With stronger dealiasing.

Figure 4: Time evolution of the solution with initial data (5.3). $K = 400$, $\epsilon = 0.2$.

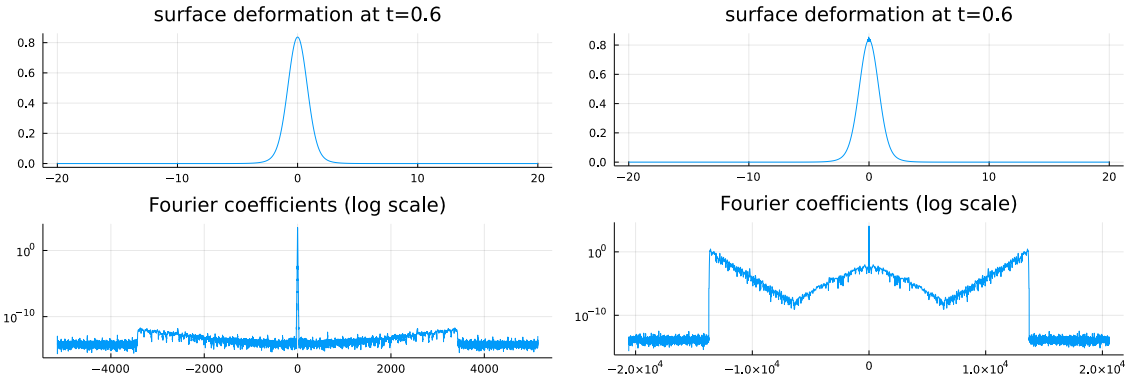
where we set $\delta = 0.01$ for now, and m is a parameter which will vary. We have that J is a regularizing operator of order m, in the sense of Definition 1.1, and is regular when $m \in (-\infty, -1]$. Hence our results, and in particular Theorem 1.2, are valid when $m \in (-\infty, -1]$, although we expect that setting to $m \in (-\infty, -1/2]$ is sufficient to secure well-posedness (see Remark 3.4). Let us comment that we have *not* witnessed any undesirable instabilities when setting J as an ideal low-pass filter, *i.e.* setting $m = -\infty$, despite the lack of regularity of the corresponding symbol. We reproduce the experiment of Figure 2 —that is setting initial data (5.2), $\mu = 1$, $\epsilon = 0.1$, and half-length $L = 20$— but with J as above, $\delta = 0.01$, and $m \in \{-1, -1/2, -1/4\}$. We have seen in Figure 2 that setting $m = 0$, the time-evolution generates instabilities by amplifying large-wavenumbers rounding errors. In Figure 5 we plot the surface deformation in the top panels and modulus of discrete Fourier coefficients in the bottom panel at time $t = 10$, with $m = -1$ (left) and $m = -1/2$ (right), and using $N = 2^{18}$ modes (since the experiments are computationally more demanding, we use the time step $\triangle t = 0.01$). There is no sign of instability. In Figure 6 we show the same results for $m = -1/4$. On the right panel we show the result with $N = 2^{18}$ modes, and clear signs of large-wavenumbers instabilities are visible at time $t = 0.6$ (the solutions breaks after a few more time steps). We use only $N = 2^{16}$ modes on the left panel and instabilities are tamed (yet still present and more easily witnessed at time $t = 1$, not shown). This shows that the phenomenon strongly depends on the number of computed modes, N, and suggests that the initial-value problem associated with (5.1) is ill-posed for J as above when $m = -1/4$. Hence these numerical experiments fully support our analysis as far as the order of the rectifier J as a regularizing operator is concerned.

The role of the strength of rectifiers Now we shall study the role of the strength of rectifiers, that is we set J as in (5.5) with $m = -1$ and vary the parameter $\delta > 0$. In Figure 7 we use again the initial data (5.2) and set $\mu = 1$ and $\epsilon = 0.1$, half-length $L = 20$ and $N = 2^{14}$ modes. We plot again the surface deformation in the top panels and the modulus of discrete Fourier coefficients in the bottom panel, at times $t = 2$ and $t = 10$. The left panel corresponds to the case $\delta = 0.01$ and the right panel to $\delta = 0.002$. The former shows no sign of instability of the large-wavenumber modes, despite a minor amplification of machine epsilon rounding errors. Adding more modes does not change the picture. In the latter we see a clear amplification of large-wavenumber modes, with a maximum about the wavenumber $\frac{2\pi}{2L}k \approx 1/\delta$. Yet the amplification arises at early times, and apparently remains stable for all times. Notice the amplification of intermediate wavenumbers is very sensitive to the parameter δ: for $\delta = 0.0025$ the amplification is barely noticeable (not shown). It is, however, stable with respect to parameters of the numerical scheme: augmenting the number of modes up to at least $N = 2^{20}$ does not generate additional instabilities. For values below $\delta = 0.001$,



(a) J of order -1. (b) J of order $-1/2$.

Figure 5: Time evolution of smooth initial data (5.2), with a rectifier of order $m \in \{-1, -1/2\}$.



(a) J of order $-1/4$, $N = 2^{16}$ modes. (b) J of order $-1/4$, $N = 2^{18}$ modes.

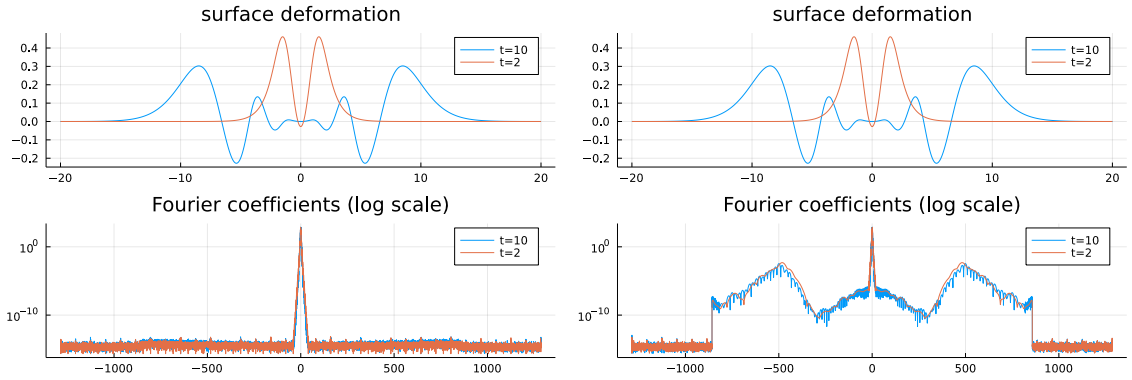
Figure 6: Time evolution of smooth initial data (5.2), with a rectifier of order $m = -1/4$.

the amplification does not reach a stable regime, and the solution breaks before $t = 2$. Without dealiasing, the solution breaks even for large values of δ. We reproduced the phenomenon using other values of μ, namely $\mu = 10$ and $\mu = \infty$ (not shown). Importantly, the same behavior is also reproduced when using less regular initial data, specifically

$$\zeta_0(x) = \zeta(t = 0, x) = \exp(-|x|^p) \quad \text{and} \quad \psi_0'(x) = \psi'(t = 0, x) = 0 \quad (x \in (-L, L)), \tag{5.6}$$

with $p = 1$ and $p = 3$. Again, we do not display the results as they are very similar.The outcome of these numerical experiments is again consistent with our analysis, and specifically Proposition 3.2. It can be elucidated as follows: for small values of δ, the solution quickly violates the Rayleigh–Taylor condition (3.3) and enters a regime where the large-wavenumbers component is amplified.

A critical strength of rectifiers In the foregoing numerical experiments, we observed mainly two scenarios depending on values of δ when the rectifier is sufficiently regularizing: for large values of δ the numerical solution seems to exist and remain regular for all times, while for small values the numerical solution rapidly breaks. We now investigate the transition between these two scenarios, conjecturing that for fixed initial data and time $T > 0$, there exists a unique "critical value" $\delta_c(T) \geq 0$ such that for any $\delta > \delta_c$ (resp. $\delta < \delta_c$), the maximal time of existence of the solution is greater than T (resp. smaller than T). Computing numerical blowups as described above



(a) $N = 2^{14}$ modes, $\delta = 0.01$. (b) $N = 2^{14}$ modes, $\delta = 0.002$.

Figure 7: Time evolution of smooth initial data varying the strength of an admissible rectifier.

and using the dichotomy method, one may provide a range for this critical value, δ_c. The result of such experiments for $T = 10$ and $T = 2$, using different values of ϵ (while $\mu = 1$) is reproduced in Figure 8. We used $N = 2^{20}$ modes (although the figures for larger values of ϵ are identical for a smaller number of modes, *e.g.* $N = 2^{16}$ when $\epsilon \geq 0.04$) and $\triangle t = 0.005$. We observe that the transition zone is extremely narrow, in particular for small values of ϵ, and that the critical value behaves asymptotically proportionally to ϵ^2. This behavior is fully consistent with our result in Proposition 3.2 and the instability mechanism described by our toy model in Appendix B, noticing that the Rayleigh–Taylor condition $\mathfrak{a}^\mu > 0$ (see (3.1)) is satisfied (and the lower-bound for the time of existence is large) for sufficiently regular data and $\epsilon, \epsilon^2\delta^{-1}$ sufficiently small; see (3.4).

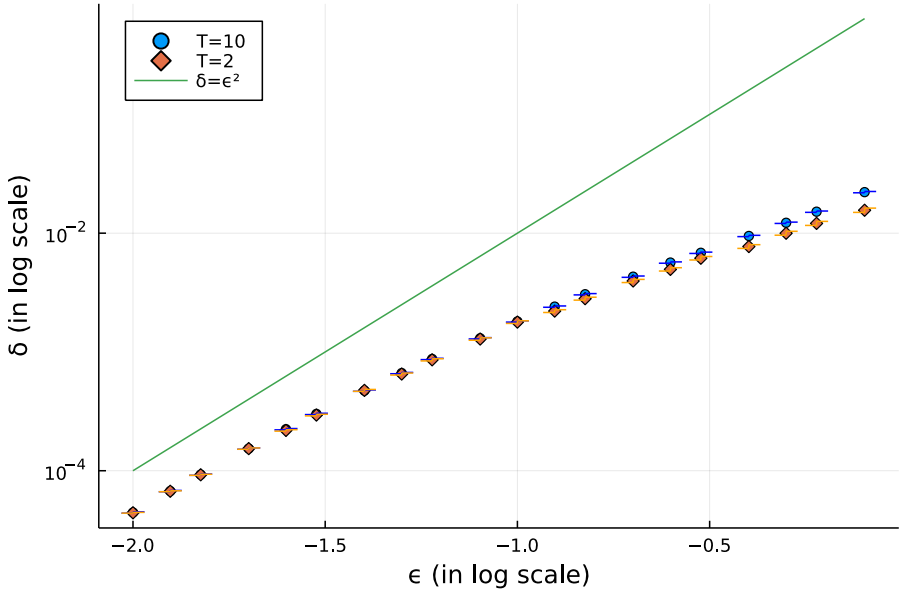
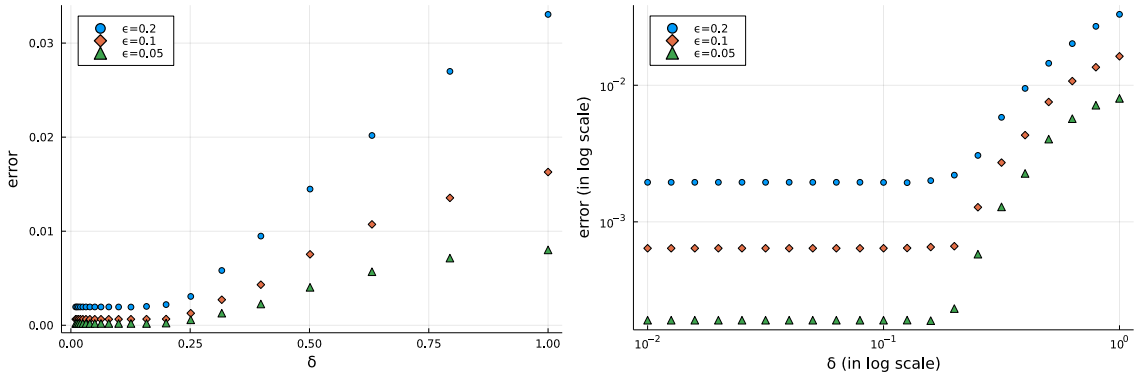


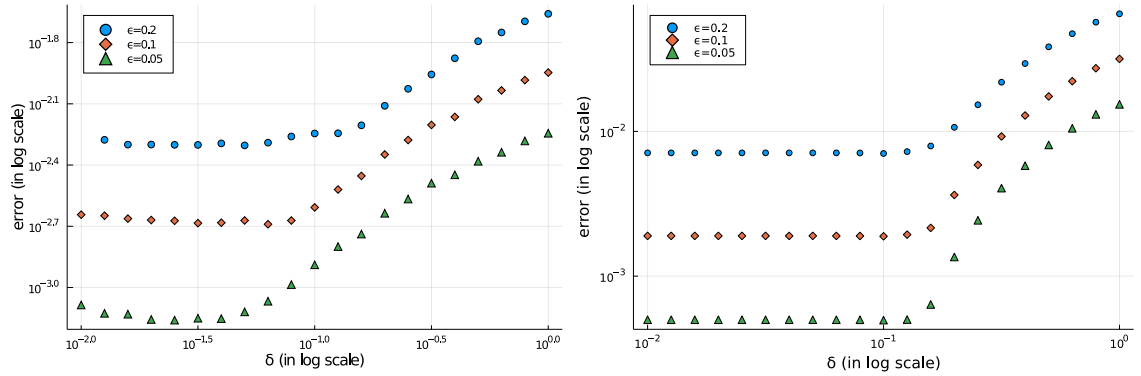
Figure 8: Critical value $\delta_c(T)$ ($T \in \{2, 10\}$) as a function of ϵ.
Horizontal bars frame the range, while markers locate the geometric mean.

The cost of rectifiers Lastly we turn to the study of the precision of our rectified model, depending on the strength of the involved rectifier. We compare the solutions to (5.1) with the solution to the water waves system, for initial data (5.6) with $p \in \{1, 2, 3\}$, and rectifier J set as (5.5) with $m = -1$ and varying $\delta \in (0.01, 1)$. We set $\mu = 1$, $\epsilon \in \{0.05, 0.1, 0.2\}$. The resulting errors for $p = 2$ is shown Figure 9, while the results for $p = 1$ and $p = 3$ are shown Figure 10. Each time, we plot the "error" defined as the ℓ^∞ norm of the difference between the solution (more precisely the surface elevation) of the model and the solution of the water-waves system at time $t = 10$. The water waves system is computed using the strategy based on conformal variables described in [14, 11] among others (and implemented in the aforementioned Julia package). We use $N = 2^{12}$ modes, so that the numerical scheme for (5.1) would converge (when $p = 2$) even without regularizing. However, augmenting the number of modes modifies the error only after several significant digits, showing that the error originates from the model, and not from the spatial discretization. Similarly, diminishing the time step (which is set to $\triangle t = 0.01$) does not modify the first significant digits.



(a) Linear scale. (b) Log scale.

Figure 9: Error of the model as a function of δ for smooth initial data, (5.2).



(a) $p = 1$. (b) $p = 3$.

Figure 10: Error of the model as a function of δ for non-smooth initial data, (5.6).

We observe that for δ sufficiently small, and in fact way above the "critical value" determined above, the error stagnates at a value which scales proportionally to ϵ^2. This means that the source of the error, as predicted in Theorem 1.3 and Theorem 1.4, originates mostly from the

consistency of the original model (without rectification) with respect to the water-waves system (see Proposition 4.2). On the contrary, for larger values of δ, the main error originates from the presence of the rectifier J. Consistently , for fixed value of δ, it scales proportionally to ϵ (see Proposition 4.1). Its behavior with respect to δ depends strongly on the regularity of the initial data (and hence of the solutions): one can observe that for smooth data ($p = 2$) the threshold above which the contribution of the rectifier J becomes dominant is almost independent of ϵ, while it strongly depends on ϵ when $p = 1$. This is of course consistent with our results, noticing that J set as (5.5) is near-identity of any order $\ell \geq 0$ (recall Definition 1.1), and hence the size of the remainder terms in Proposition 4.1 strongly depends on the regularity of the data. An important observation is that, even for non-regular data (when $p = 1$), it is possible to choose δ such that the (5.1) provides a stable solution for all (computed) times and the presence of J does not induce any noticeable additional errors. In other words, introducing a suitable rectifier is able to regularize the system while being harmless from the point view of the accuracy of the model.

6 Conclusion and recommendations

In this work we argue, through a combination of theoretical results and in-depth numerical simulations, that the initial-value problem for a standard model for the propagation of water waves with small steepness, (WW2), is ill-posed in finite-regularity spaces. We propose an instability mechanism, which can be roughly described as follows: cubic (low-low-high) nonlinear interactions trigger an arbitrarily fast amplification of the high-frequency component of the flow, with a scaling similar to the Cauchy–Riemann equations.

From the numerical point of view, assuming the use of Fourier pseudo-spectral methods, these instabilities will arise even starting from smooth data (amplifying machine precision rounding errors) as soon as a sufficient number of modes are employed.

Among the possible artificial regularizations, we advocate the use of suitable Fourier multipliers (which we name "rectifiers") in (RWW2). Introducing these rectifiers is harmless from the point of view of numerical discretization, again assuming the use of Fourier pseudo-spectral methods. Moreover, we prove that it is possible to choose such Fourier multipliers in a harmless manner, that is without damaging the accuracy of the involved model while ensuring the well-posedness of the initial-value problem for large times.

These two inclinations are competing, as the first one advocates the use of close-to-identity rectifiers, while the second-one demands the decay of the symbol of the Fourier multiplier. It is convenient to control the behavior of the rectifier concerning these matters by introducing a parameter, and set the rectifier $\mathsf{J} = J_0(\delta D)$ where $\mathsf{J}_0 = J_0(D)$ is a prescribed admissible rectifier (see Definition 1.1) and $\delta > 0$ a parameter describing the "strength" of the rectifier.

The scalings we exhibit are as follows. The initial-value problem associated with (RWW2) is well-posed in suitable (finite-regularity) functional spaces on the natural time interval $[-T, T]$ with $T \gtrsim \epsilon^{-1}$ provided that $\delta \geq \epsilon$. We also argue that no instability occurs provided that $\epsilon^2\delta^{-1}$ is sufficiently small. The precision of the model (RWW2) as an approximation of the water-waves system (WW) is of order $\mathcal{O}(\epsilon^2 + \epsilon\delta^\ell)$ where $\ell \geq 0$ can be arbitrarily large for suitable choices of rectifiers (we advocate (1.4) with $m = -1$). Notice however that our result involves a loss of $\ell + p$ (with p some constant) derivatives between the control of the solutions and the control of the error.

These scalings advocate for the choice of $\delta \propto \epsilon$ (here and above, ϵ should in fact be replaced with ϵM_0 where M_0 is the size of the initial data measured in a relevant functional space). Yet in practice one can simply set δ by trial and error, choosing δ sufficiently large so that spurious instabilities are not produced, yet sufficiently small so that the outcome of the numerical simulation does not depend up to the desired accuracy on δ.

A The water waves system

Let us recall a few facts on the water waves system, describing the motion of inviscid, incompressible and homogeneous fluids with a free surface. Under the assumption of potential flow, and assuming that the bottom is flat, the evolution equations can be rewritten (with dimensionless variables set accordingly to the deep water regime, following the convention of [2, Appendix A] with $\nu = \frac{1}{\sqrt{\mu}}$) as

$$\text{(WW)} \qquad \begin{cases} \partial_t \zeta - G^\mu[\epsilon\zeta]\psi = 0 \\ \partial_t \psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\epsilon}{2}\frac{(G^\mu[\epsilon\zeta]\psi + \epsilon\nabla\zeta\cdot\nabla\psi)^2}{1+\epsilon^2|\nabla\zeta|^2} = 0, \end{cases}$$

where $(\zeta, \psi) : (t, \boldsymbol{x}) \in \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^2$ (with $d \in \{1,2\}$), $\epsilon, \mu > 0$ and the Dirichlet-to-Neumann operator, G^μ, is defined for sufficiently nice functions as

$$G^\mu[\epsilon\zeta]\psi \overset{\text{def}}{=} (\partial_z \Phi)\big|_{z=\epsilon\zeta} - \epsilon\nabla\zeta\cdot(\nabla_{\boldsymbol{x}}\Phi)\big|_{z=\epsilon\zeta}$$

with Φ being the solution to the Laplace problem

$$\begin{cases} \Delta_{\boldsymbol{x}}\Phi + \partial_z^2\Phi = 0 \quad \text{ on } \{(\boldsymbol{x}, z) \in \mathbb{R}^{d+1},\ -\sqrt{\mu} < z < \epsilon\zeta(\boldsymbol{x})\}, \\ \Phi_{|z=\epsilon\zeta} = \psi, \quad \partial_z\Phi_{|z=-\sqrt{\mu}} = 0. \end{cases}$$

Basic properties about the Dirichlet-to-Neumann operator can be found in [18, Chapter 3] or [2, Section 3]. The only result that we use in this paper is the following asymptotic expansion adapted from Proposition 3.9 and Proposition 3.3 in [2]. Recall that $\mathsf{T}^\mu = -\frac{\tanh(\sqrt{\mu}|D|)}{|D|}\nabla$ and $\mathfrak{P}^\mu \overset{\text{def}}{=} \sqrt{|D|\tanh(\sqrt{\mu}|D|)}$.

Proposition A.1. *Let $d \in \{1,2\}$, $s \geq 0$, $t_0 > \frac{d}{2}$ and $h_\star > 0$ and $M > 0$. There exists $C > 0$ such that for any $\epsilon > 0$, $\mu \geq 1$ and $\zeta \in H^{\max(s+\frac{3}{2}, t_0+2)}$ such that*

$$1 + \tfrac{\epsilon}{\sqrt{\mu}}\zeta \geq h_\star, \qquad \big|\epsilon\zeta\big|_{H^{t_0+2}} \leq M,$$

and for any $\psi \in \mathring{H}^{1+s}(\mathbb{R}^d)$, we have $G^\mu[\epsilon\zeta]\psi \in H^s(\mathbb{R}^d)$ and

$$G^\mu[\epsilon\zeta]\psi - \mathsf{T}^\mu\cdot\nabla\psi - \epsilon\left(\mathsf{T}^\mu\cdot\nabla\left(\zeta\mathsf{T}^\mu\cdot\nabla\psi\right) + \nabla\cdot(\zeta\nabla\psi)\right) = \epsilon^2 R$$

where

$$\big|R\big|_{H^s} \leq C\big|\zeta\big|_{H^{t_0+1}}\left(\big|\zeta\big|_{H^{t_0+1}}\big|\mathfrak{P}^\mu\psi\big|_{H^{s+\frac{1}{2}}} + \big|\zeta\big|_{H^{s+\frac{3}{2}}}\big|\mathfrak{P}^\mu\psi\big|_{H^{t_0}}\right).$$

The asymptotic expansion above provides the foundation for the rigorous justification of (WW2) as an asymptotic model for (WW) with precision $\mathcal{O}(\epsilon^2)$, at least in the sense of consistency. The precise statement is displayed in Proposition 4.2.

B A toy model

In order to describe the high-frequency stability/instability mechanism which we exhibit in this work, we propose the following toy model

$$\text{(B.1)} \qquad \begin{cases} \partial_t\zeta - \mathsf{T}^\mu\cdot\nabla\psi = 0, \\ \partial_t\psi + \zeta - \epsilon^2\alpha[\psi]\mathsf{J}^2|D|\zeta = 0, \end{cases}$$

where we recall that $\mathsf{T}^\mu \overset{\text{def}}{=} -\frac{\mathrm{i}D\tanh(\sqrt{\mu}|D|)}{|D|}$ with $\mu \geq 1$, $\mathsf{J} = J(|D|)$ is a Fourier multiplier, and we set

$$\alpha[\psi] \overset{\text{def}}{=} \int (\mathsf{T}^\mu\cdot\nabla\psi)^2\,\mathrm{d}\boldsymbol{x}.$$

System (B.1) is inspired by Proposition 3.9 and specifically (3.14)–(3.15). It mimics the (possible) destabilization effect on the high-frequency component of solutions to (RWW2) induced by the low-high frequency interactions stemming from the operator $\mathfrak{a}^\mu[\epsilon \mathsf{J}\zeta, \epsilon\nabla\psi]$ defined in (3.1), while disregarding other contributions such as advection terms, bounded operators in $\mathfrak{a}^\mu$, and the choice of Alinhac's good unknowns.

In the following discussion, we consider the $(2\pi\mathbb{Z})^d$-periodic framework for convenience, although our the results can be adapted to the full space framework, $\mathbb{R}^d$. We can hence rewrite (B.1), using the decomposition in Fourier series, as an infinite system of ordinary differential equations. Specifically, we have for sufficiently regular real-valued solutions to (B.1) defined on $(2\pi\mathbb{T})^d$,

$$\forall \boldsymbol{k} \in \mathbb{Z}^d, \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\zeta}_{\boldsymbol{k}} - \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}|\widehat{\psi}_{\boldsymbol{k}} = 0, \\ \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\psi}_{\boldsymbol{k}} + \Big(1 - \epsilon^2\alpha[\psi]J(|\boldsymbol{k}|)^2|\boldsymbol{k}|\Big)\widehat{\zeta}_{\boldsymbol{k}} = 0, \end{cases} \tag{B.2}$$

with

$$\alpha[\psi] = (2\pi)^d \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (\tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k}}|)^2.$$

Here, we assumed $\mathsf{J} = J(\delta|D|)$, and set the convention

$$\widehat{\zeta}_{\boldsymbol{k}} = \frac{1}{(2\pi)^d}\int_{(2\pi\mathbb{T})^d} \zeta(\boldsymbol{x}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x} \quad ; \quad \zeta(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \widehat{\zeta}_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}.$$

In the following, we denote

$$\ell^{2,s} \stackrel{\text{def}}{=} \big\{ a \stackrel{\text{def}}{=} (a_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^d},\ a_{-\boldsymbol{k}} = \overline{a_{\boldsymbol{k}}},\ |a|^2_{\ell^{2,s}} \stackrel{\text{def}}{=} \textstyle\sum_{\boldsymbol{k}\in\mathbb{Z}^d}(1+|\boldsymbol{k}|^2)^s|a_{\boldsymbol{k}}|^2 < \infty \big\}$$

and $H^s((2\pi\mathbb{T})^d)$ the space of $(2\pi\mathbb{Z})^d$-periodic distributions such that $\widehat{\zeta} \in \ell^{2,s}$, endowed with the norm $|\zeta|_{H^s} \stackrel{\text{def}}{=} |\widehat{\zeta}|_{\ell^{2,s}}$, and $L^2((2\pi\mathbb{T})^d)$ the $(2\pi\mathbb{Z})^d$-periodic real-valued square-integrable functions.

In (B.2), the coupling between each mode arises only through the coefficient $\alpha[\psi]$. For the sake of the discussion, let us first fix α as a constant. Then the system is explicitly solvable, and we observe that a plane wave solution with wave vector $\boldsymbol{k}$ is stable if and only if $1 - \alpha\epsilon^2 J(|\boldsymbol{k}|)^2|\boldsymbol{k}| > 0$. In the opposite case $1 - \alpha\epsilon^2 J(|\boldsymbol{k}|)^2|\boldsymbol{k}| < 0$, the mode experiences an exponential growth with rate $\big(\tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}|(\alpha\epsilon^2 J(|\boldsymbol{k}|)^2|\boldsymbol{k}| - 1)\big)^{1/2}$ (whereas the growth is linear in the critical case, $\alpha\epsilon^2 J(|\boldsymbol{k}|)^2|\boldsymbol{k}| = 1$). This is the case in particular for sufficiently large $|\boldsymbol{k}|$ if $\alpha > 0$ and $\mathsf{J} = \mathrm{Id}$. Moreover in that case, since the growth rate is unbounded, the initial-value problem for the dynamical system (B.2) is strongly ill-posed in any polynomially weighted $\ell^{2,s}$ spaces. On the contrary, if $\alpha\epsilon^2 \limsup_{k\to\infty}(kJ^2(k)) < 1$, then the system is (globally) well-posed in $\ell^{2,s} \times \ell^{2,s+\frac{1}{2}}$.

The following propositions show that this naive analysis describes fairly well the behavior of the toy model.

Proposition B.1 (Local well-posedness in the sub-critical case)**.** *Let $J : \mathbb{N} \to \mathbb{R}$ be such that*

$$kJ^2(k) \to 0 \quad \text{as } k \to \infty. \tag{B.3}$$

Let $M \geq 0$. Then there exists $T_0 > 0$ such that for any $\mu \geq 1$, any $s \geq 3/4$ and any initial data $(\zeta_0, \psi_0) \in H^s((2\pi\mathbb{T})^d) \times H^{s+\frac{1}{2}}((2\pi\mathbb{T})^d)$ such that

$$\epsilon|\zeta_0|_{H^{\frac{1}{2}}} + \epsilon|\nabla\psi_0|_{L^2} \leq M,$$

there exists a unique $(\zeta, \psi) \in \mathcal{C}([-T, T]; H^s((2\pi\mathbb{T})^d) \times H^{s+\frac{1}{2}}((2\pi\mathbb{T})^d))$ solution to (B.1) *satisfying $(\zeta, \psi)|_{t=0} = (\zeta_0, \psi_0)$, where*

$$T = \frac{T_0}{\big(\epsilon|\zeta_0|_{H^{\frac{3}{4}}} + \epsilon|\nabla\psi_0|_{H^{\frac{1}{4}}}\big)^2}.$$

Moreover, if we assume $\sup_{k\in\mathbb{N}} k^{3/2}J^2(k) < \infty$*, then the above holds with* $s \geq 1/2$ *and*

$$T = \frac{T_0}{\big(\epsilon\big|\zeta_0\big|_{H^{\frac{1}{2}}} + \epsilon\big|\nabla\psi_0\big|_{L^2}\big)^2}.$$

Proof. Let us first consider the case where only a finite number of modes, $\widehat{\zeta}_{\boldsymbol{k},0}, \widehat{\psi}_{\boldsymbol{k},0}$, are non-zero. Then (B.2) is a finite system of ordinary differential equations. Cauchy-Lipschitz theorem applies, and defines uniquely a maximal local-in-time solution. On the maximal time of existence, we denote

$$a_{\boldsymbol{k}}(t) = 1 - \epsilon^2\alpha(t)J(|\boldsymbol{k}|)^2|\boldsymbol{k}|, \quad \alpha(t) = (2\pi)^d \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (\tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k}}(t)|)^2,$$

and (we assume henceforth that $\widehat{\psi}_{\boldsymbol{k},0} = 0$, the general case follows by subtracting to ψ its mean)

$$\mathcal{E}_s(t) \stackrel{\text{def}}{=} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \langle\boldsymbol{k}\rangle^{2s}\Big(|\widehat{\zeta}_{\boldsymbol{k}}|^2(t) + \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k},0}|^2(t)\Big) \approx \big|\widehat{\zeta}\big|^2_{\ell^{2,s}} + \big|\widehat{\psi}\big|^2_{\ell^{2,s+\frac{1}{2}}}.$$

In the following, we set $T \in (0, +\infty]$ the maximal value such that

$$\forall t \in (-T, T), \quad \mathcal{E}_{1/2}(t) \leq 4\mathcal{E}_{1/2}(0), \quad \mathcal{E}_{3/4}(t) \leq 2\mathcal{E}_{3/4}(0).$$

In particular, we have $\alpha(t) \leq 4(2\pi)^d\mathcal{E}_{1/2}(0)$ for $t \in (-T, T)$, and by (B.3) we can choose $k_\star \in \mathbb{N}$ such that

$$\forall k \geq k_\star, \qquad 4(2\pi)^d\epsilon^2\mathcal{E}_{1/2}(0)\, kJ^2(k) \leq 1/2$$

and infer

$$\begin{cases} 1/2 \leq a_{\boldsymbol{k}}(t) \leq 1 & \text{if } |\boldsymbol{k}| \geq k_\star, \\ 0 \leq 1 - a_{\boldsymbol{k}}(t) \leq 4(2\pi)^d\epsilon^2\mathcal{E}_{1/2}(0)\max_{k\in\mathbb{N}} kJ^2(k) & \text{if } |\boldsymbol{k}| < k_\star. \end{cases}$$

We first estimate the low-frequency components. By (B.2) and since $\tanh(\sqrt{\mu}|\boldsymbol{k}|) \leq 1$ we infer

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big(|\widehat{\zeta}_{\boldsymbol{k}}|^2(t) + \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k}}|^2(t)\big) \leq 2|\boldsymbol{k}||a_{\boldsymbol{k}}(t) - 1||\widehat{\zeta}_{\boldsymbol{k}}|(t)|\widehat{\psi}_{\boldsymbol{k}}|(t)$$

and hence, using Gronwall's inequality, we find that there exists $C_1 > 0$, depending only on $\max_{k\in\mathbb{N}} kJ^2(k)$ and $k_\star$ (and hence $\epsilon^2\mathcal{E}_{1/2}(0)$), such that for any $|\boldsymbol{k}| < k_\star$, and $t \in (-T, T)$,

$$|\widehat{\zeta}_{\boldsymbol{k}}|^2(t) + \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k}}|^2(t) \leq \big(|\widehat{\zeta}_{\boldsymbol{k},0}|^2 + \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k},0}|^2\big)e^{C_1\,\epsilon^2\mathcal{E}_{1/2}(0)\,k_\star^{1/2}\,|t|}.$$

In particular, restricting to $t \in (-T_1, T_1)$ with $T_1 = \ln(\frac{3}{2})(C_1\,\epsilon^2\mathcal{E}_{1/2}(0)\,k_\star^{1/2})^{-1}$ if necessary, we find that for any $s \in \mathbb{R}$,

$$\sum_{|\boldsymbol{k}|<k_\star} \langle\boldsymbol{k}\rangle^s\Big(|\widehat{\zeta}_{\boldsymbol{k}}|^2 + \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k}}|^2\Big)(t) \leq \frac{3}{2}\sum_{|\boldsymbol{k}|<k_\star} \langle\boldsymbol{k}\rangle^s\Big(|\widehat{\zeta}_{\boldsymbol{k},0}|^2 + \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k},0}|^2\Big).$$

We now consider the high-frequency components. Testing the first equation in (B.2) with $a_{\boldsymbol{k}}\overline{\widehat{\zeta}_{\boldsymbol{k}}}$ and the second equation with $\tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}|\overline{\widehat{\psi}_{\boldsymbol{k}}}$, we find

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(a_{\boldsymbol{k}}|\widehat{\zeta}_{\boldsymbol{k}}|^2(t) + \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k}}|^2(t)\Big) = a_{\boldsymbol{k}}'(t)|\widehat{\zeta}_{\boldsymbol{k}}|^2(t) = -\epsilon^2\alpha'(t)J(|\boldsymbol{k}|)^2|\boldsymbol{k}||\widehat{\zeta}_{\boldsymbol{k}}|^2(t),$$

with

$$\alpha'(t) = -(2\pi)^d \sum_{\boldsymbol{k}\in\mathbb{Z}^d} 2(\tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}|)^2 a_{\boldsymbol{k}}(t)\Re(\widehat{\zeta}_{\boldsymbol{k}}\overline{\widehat{\psi}_{\boldsymbol{k}}})(t).$$

Hence, since $\tanh(\sqrt{\mu}|\boldsymbol{k}|) \leq 1$ and using Cauchy-Schwarz inequality, we have for all $t \in (-T, T)$

$$|\alpha'(t)| \leq (2\pi)^d \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |a_{\boldsymbol{k}}(t)||\boldsymbol{k}|^{3/2} \left(|\widehat{\zeta}_{\boldsymbol{k}}|^2(t) + \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k}}|^2(t) \right) \leq C_2 \mathcal{E}_{3/4}(0)$$

where C_2 depends uniquely on $\epsilon^2 \mathcal{E}_{1/2}(0) \max_{k\in\mathbb{N}} kJ^2(k)$. In particular, since $1/2 \leq a_{\boldsymbol{k}}(t) \leq 1$ when $|\boldsymbol{k}| \geq k_\star$ and restricting to $t \in (-T_2, T_2)$ with $T_2 = \ln(\frac{5}{4})(C_2\, \epsilon^2 \mathcal{E}_{3/4}(0)\, 2 \max_{k\in\mathbb{N}} kJ^2(k))^{-1}$ if necessary, we have for any $s \in \mathbb{R}$,

$$\sum_{|\boldsymbol{k}|\geq k_\star} \langle \boldsymbol{k}\rangle^s \left(|\widehat{\zeta}_{\boldsymbol{k}}|^2(t) + \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k}}|^2(t) \right) \leq \frac{5}{2} \sum_{|\boldsymbol{k}|\geq k_\star} \langle \boldsymbol{k}\rangle^s \left(|\widehat{\zeta}_{\boldsymbol{k},0}|^2 + \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k},0}|^2 \right).$$

Combining the above, we find that for any $s \in \mathbb{R}$, and $t \in (-T_\star, T_\star)$ with $t_\star = \min(T, T_1, T_2)$,

$$\mathcal{E}_s(t) \leq \frac{5}{2} \mathcal{E}_s(0). \tag{B.4}$$

The standard continuity argument shows that $T \geq \min(T_1, T_2)$, $(\widehat{\zeta}_{\boldsymbol{k}}, \widehat{\psi}_{\boldsymbol{k}}) \in \mathcal{C}([-T, T]; \ell^{2,s} \times \ell^{2s+\frac{1}{2}})$ and satisfies (B.4) on $[-T, T]$.

By using these uniform bounds one easily obtains the corresponding result for general initial data (that is with an infinite number of non-zero modes, $\widehat{\zeta}_{\boldsymbol{k},0}, \widehat{\psi}_{\boldsymbol{k},0}$) by truncating Fourier modes and taking the limit. This concludes the first part of the proof.

The second part is immediate recalling the inequality valid for any $\boldsymbol{k} \in \mathbb{Z}^d$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \big(|\widehat{\zeta}_{\boldsymbol{k}}|^2 + \tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}_{\boldsymbol{k}}|^2 \big)(t) \leq 2|\boldsymbol{k}||a_{\boldsymbol{k}}(t) - 1||\widehat{\zeta}_{\boldsymbol{k}}|(t)|\widehat{\psi}_{\boldsymbol{k}}|(t)$$

and the fact that for any t in the maximal time interval,

$$|\boldsymbol{k}|^{1/2}|a_{\boldsymbol{k}}(t) - 1| = \epsilon^2 |\alpha(t)| J(|\boldsymbol{k}|)^2 |\boldsymbol{k}|^{3/2} \leq C_0 \epsilon^2 \mathcal{E}_{1/2}(t)$$

with C_0 depending uniquely on $\sup_{k\in\mathbb{N}} J(k)^2 k^{3/2}$. The proof is complete. □

Proposition B.2 (Conditional well-posedness in the critical case)**.** *Let $J : \mathbb{N} \to \mathbb{R}$ be such that*

$$M_J \stackrel{\text{def}}{=} \limsup_{k\to\infty} (kJ^2(k)) < \infty. \tag{B.5}$$

There exists $T_0 > 0$ and $M_0 > 0$ such for any $\mu \geq 1$, any $s \geq 3/4$, and any initial data $(\zeta_0, \psi_0) \in H^s((2\pi\mathbb{T})^d) \times H^{s+\frac{1}{2}}((2\pi\mathbb{T})^d)$ such that

$$\big(\epsilon^2 \big|\zeta_0\big|_{H^{\frac{1}{2}}}^2 + \epsilon^2 \big|\nabla\psi_0\big|_{L^2}^2 \big) M_J \leq M_0,$$

there exists a unique $(\zeta, \psi) \in \mathcal{C}([-T, T]; H^s((2\pi\mathbb{T})^d) \times H^{s+\frac{1}{2}}((2\pi\mathbb{T})^d))$ solution to (B.1) *satisfying $(\zeta, \psi)\,|_{t=0} = (\zeta_0, \psi_0)$, where*

$$T = \frac{T_0}{\big(\epsilon \big|\zeta_0\big|_{H^{\frac{3}{4}}} + \epsilon \big|\nabla\psi_0\big|_{H^{\frac{1}{4}}} \big)^2}.$$

Proof. The proof of Proposition B.2 is a direct adaptation of the proof of Proposition B.1, and is left to the reader. □

Proposition B.3 (Ill-posedness in the super-critical case)**.** *Let $J : \mathbb{N} \to \mathbb{R}$ be such that*

$$\limsup_{k\to\infty} (kJ^2(k)) = \infty. \tag{B.6}$$

For all $\epsilon > 0$, there exists $(\zeta_0^n, \psi_0^n)_{n\in\mathbb{N}} \in \mathcal{C}^\infty((2\pi\mathbb{T})^d)^{\mathbb{N}}$ as well as $(T_n)_{n\in\mathbb{N}} \in (\mathbb{R}^+)^{\mathbb{N}}$ such that

$$\forall s \in \mathbb{R}, \qquad \big|\zeta_0^n\big|_{H^s} + \big|\psi_0^n\big|_{H^s} \to 0 \quad \text{and} \quad T_n \searrow 0 \quad (n \to \infty),$$

and for any $\mu \geq 1$ the solution (ζ^n, ψ^n) to (B.1) *with $(\zeta^n, \psi^n)\,|_{t=0} = (\zeta_0^n, \psi_0^n)$ satisfies*

$$\forall s' \in \mathbb{R}, \qquad \big|\psi^n(t,\cdot)\big|_{H^{s'}} \to \infty \quad (t \nearrow T_n).$$

The same result holds backwards in time.

Proof. We use the initial data

$$\zeta_0^n(\boldsymbol{x}) = 0 \qquad ; \qquad \psi_0^n(\boldsymbol{x}) = b_n \cos(\boldsymbol{k}_0 \cdot \boldsymbol{x}) + c_n \cos(\boldsymbol{k}_n \cdot \boldsymbol{x}), \tag{B.7}$$

with $\boldsymbol{k}_0 \in \mathbb{Z}^d$ such that $k_0 \overset{\text{def}}{=} |\boldsymbol{k}_0|$ satisfies $k_0 \mathsf{J}^2(k_0) \neq 0$, $\boldsymbol{k}_n \in \mathbb{Z}^d$ such that $|\boldsymbol{k}_n| = k_n$ where $(k_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ is a sequence such that $k_n \to \infty$ and $k_n\ J^2(k_n) \to \infty$ as $n \to \infty$, and $b_n, c_n > 0$ will be determined later on.

Since (B.2) is a finite system of ordinary differential equations, the Cauchy-Lipschitz theorem applies, and defines uniquely maximal local-in-time solutions, which we denote $\widehat{\zeta}^n = (\widehat{\zeta}^n_{\pm\boldsymbol{k}_0}, \widehat{\zeta}^n_{\pm\boldsymbol{k}_n})$ and $\widehat{\psi}^n = (\widehat{\psi}^n_{\pm\boldsymbol{k}_0}, \widehat{\psi}^n_{\pm\boldsymbol{k}_n})$, on the maximal (forward-in-time) interval $I^n = [0, T^n_\star)$. On the maximal time of existence, we denote for $\boldsymbol{k} \in \{\pm\boldsymbol{k}_0, \pm\boldsymbol{k}_n\}$

$$a^n_{\boldsymbol{k}}(t) = 1 - \epsilon^2 \alpha^n(t) J(|\boldsymbol{k}|)^2 |\boldsymbol{k}|, \quad \alpha^n_{\boldsymbol{k}}(t) = (2\pi)^d (\tanh(\sqrt{\mu}|\boldsymbol{k}|)|\boldsymbol{k}||\widehat{\psi}^n_{\boldsymbol{k}}(t)|)^2.$$

Notice that for any $t \in I^n$, $\alpha^n_{\boldsymbol{k}}(t) \geq 0$ and

$$\alpha^n(t) = 2\alpha^n_{\boldsymbol{k}_0}(t) + 2\alpha^n_{\boldsymbol{k}_n}(t).$$

We define $\alpha_0 = \frac{1}{4}(2\pi)^d (\tanh(\sqrt{\mu}k_0)k_0)^2$. Henceforth we assume that $b_n > 0$ is such that

$$\tfrac{1}{4} b_n^2 \alpha_0 \epsilon^2 J(k_n)^2 k_n > 1 \quad \text{and} \quad b_n^2 < \tfrac{2}{5}(\alpha_0 \epsilon^2 J(k_0)^2 k_0)^{-1}. \tag{B.8}$$

Step 1: exponential growth.
We have the following controls.

(a) Assume there exists $t_0 \in I^n$ such that $\alpha^n_{\boldsymbol{k}_n}(t_0) \geq \frac{1}{4} b_n^2 \alpha_0$ with $\widehat{\zeta}^n_{\boldsymbol{k}_n}(t_0) \geq 0$ and $\widehat{\psi}^n_{\boldsymbol{k}_n}(t_0) > 0$. Then $\widehat{\zeta}^n_{\boldsymbol{k}_n}$ and $\widehat{\psi}^n_{\boldsymbol{k}_n}$ are increasing on $I^n \cap [t_0, +\infty)$ and for any $t \in I^n \cap [t_0, +\infty)$,

$$\widehat{\psi}^n_{\boldsymbol{k}_n}(t) \geq \frac{\widehat{\psi}^n_{\boldsymbol{k}_n}(t_0)}{2} \exp\big(\sqrt{\tanh(\sqrt{\mu}k_n)k_n}(t - t_0)\big), \tag{B.9}$$

(b) Assume there exists $t_0 \in I^n \cap [0, k_0^{-\frac{1}{2}}]$ such that $\alpha^n_{\boldsymbol{k}_n}(t) \leq \frac{1}{4} b_n^2 \alpha_0$ holds for any $t \in [0, t_0]$. Then, $\widehat{\zeta}^n_{\boldsymbol{k}_n}$ and $\widehat{\psi}^n_{\boldsymbol{k}_n}$ are increasing on $[0, t_0]$, and for any $t \in [0, t_0]$,

$$\widehat{\psi}^n_{\boldsymbol{k}_n}(t) \geq \frac{c_n}{4} \exp\big(\sqrt{\tanh(\sqrt{\mu}k_n)k_n}\, t\big). \tag{B.10}$$

First we prove controls (a). Since $\alpha^n(t_0) \geq 2\alpha^n_{\boldsymbol{k}_n}(t_0) \geq \frac{1}{2} b_n^2 \alpha_0$ and using the first assumption in (B.8), we get that $a^n_{\boldsymbol{k}_n}(t_0) < -1$. By continuity, for t close enough to t_0, we have on $[t_0, t]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\zeta}^n_{\boldsymbol{k}_n} = \tanh(\sqrt{\mu}k_n)k_n \widehat{\psi}^n_{\boldsymbol{k}_n} \quad ; \quad \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\psi}^n_{\boldsymbol{k}_n} \geq \widehat{\zeta}^n_{\boldsymbol{k}_n}$$

from which we infer that $\widehat{\zeta}^n_{\boldsymbol{k}_n}$ and $\widehat{\psi}^n_{\boldsymbol{k}_n}$ (and hence $\alpha^n_{\boldsymbol{k}_n}$) are increasing and (considering the differential inequalities for $\sqrt{\tanh(\sqrt{\mu}k_n)k_n}\widehat{\psi}^n_{\boldsymbol{k}_n} \pm \widehat{\zeta}^n_{\boldsymbol{k}_n}$)

$$\widehat{\psi}^n_{\boldsymbol{k}_n}(t) \geq \widehat{\psi}^n_{\boldsymbol{k}_n}(t_0)\cosh(\sqrt{\tanh(k_n)k_n}(t-t_0)) \geq \frac{\widehat{\psi}^n_{\boldsymbol{k}_n}(t_0)}{2}\exp(\sqrt{\tanh(k_n)k_n}(t-t_0)).$$

Since $\alpha^n_{\boldsymbol{k}_n}$ is increasing, continuity arguments show that the above holds on $I^n \cap [t_0, +\infty)$.

We now prove controls (b). We note that, using the second assumption in (B.8) and since $\alpha^n_{\boldsymbol{k}_n}(0) \leq \frac{1}{4}b_n^2\alpha_0$,

$$\epsilon^2\alpha^n(0)J(k_0)^2k_0 = \epsilon^2(2b_n^2\alpha_0 + 2\alpha^n_{\boldsymbol{k}_n}(0))J(k_0)^2k_0 < 1.$$

By continuity, we have $a^n_{\boldsymbol{k}_0} \in [0,1]$ on $[0,t]$ for t close enough to 0, and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\zeta}^n_{\boldsymbol{k}_0} = \tanh(\sqrt{\mu}k_0)k_0\widehat{\psi}^n_{\boldsymbol{k}_0} \quad ; \quad 0 \geq \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\psi}^n_{\boldsymbol{k}_0} \geq -\widehat{\zeta}^n_{\boldsymbol{k}_0}.$$

Restricting to $t \leq t_0^{-1/2}$, we infer that $\widehat{\zeta}^n_{\boldsymbol{k}_0}$ is increasing, $\widehat{\psi}^n_{\boldsymbol{k}_0}$ (and hence $\alpha^n_{\boldsymbol{k}_0}$) is decreasing, and

$$\widehat{\psi}^n_{\boldsymbol{k}_0}(t) \leq \frac{b_n}{2}, \quad 0 \leq \widehat{\zeta}^n_{\boldsymbol{k}_0}(t) \leq tk_0\frac{b_n}{2}, \qquad \widehat{\psi}^n_{\boldsymbol{k}_0}(t) \geq \frac{b_n}{2} - \frac{b_n}{2}\frac{k_0t^2}{2} \geq \frac{b_n}{4}.$$

In particular, $\alpha^n_{\boldsymbol{k}_0}(t) \leq b_n^2\alpha_0$ so that, using the assumption $\alpha^n_{\boldsymbol{k}_n}(t) \leq \frac{1}{4}b_n^2\alpha_0$ and (B.8), we infer

$$\epsilon^2\alpha^n(t)J(k_0)^2k_0 \leq \tfrac{5}{2}\epsilon^2\alpha_0b_n^2J(k_0)^2k_0 < 1.$$

Hence by continuity we find that the above holds on $[0,t_0]$. Notice now that since $\alpha^n_{\boldsymbol{k}_0}(t) \geq \frac{1}{4}\alpha_0b_n^2$, we have for any $t \in [0,t_0]$,

$$\alpha^n(t) \geq 2\alpha^n_{\boldsymbol{k}_0}(t) \geq \tfrac{1}{2}b_n^2\alpha_0.$$

We can then use similar arguments than in the proof of control (a) to infer that $\widehat{\zeta}^n_{\boldsymbol{k}_n}$ and $\widehat{\psi}^n_{\boldsymbol{k}_n}$ are increasing and the desired lower bound on $\widehat{\psi}^n_{\boldsymbol{k}_n}(t)$.

Gathering controls (a) and (b), and arguing on the sign of $\alpha^n_{\boldsymbol{k}_n}(0) - \frac{1}{4}b_n^2\alpha_0$, we find that under the assumption (B.8), there holds for any $t \in I^n \cap [0, k_0^{-\frac{1}{2}}]$,

$$\widehat{\psi}^n_{\boldsymbol{k}_n}(t) \geq \frac{c_n}{8}\exp\left(\sqrt{\tanh(\sqrt{\mu}k_n)k_n}t\right), \tag{B.11}$$

and hence, using the identity $\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\zeta}^n_{\boldsymbol{k}_n} = \tanh(\sqrt{\mu}k_n)k_n\widehat{\psi}^n_{\boldsymbol{k}_n}$,

$$\widehat{\zeta}^n_{\boldsymbol{k}_n}(t) \geq \frac{c_n}{8}\sqrt{\tanh(\sqrt{\mu}k_n)k_n}\left(\exp\left(\sqrt{\tanh(\sqrt{\mu}k_n)k_n}t\right) - 1\right). \tag{B.12}$$

Step 2: blowup.

We set $b_n = (\frac{1}{8}\epsilon^2\alpha_0J(k_n)^2k_n)^{-1/2}$ so that Condition (B.8) holds for n sufficiently large, and we define $c_n = 8\exp(-k_n^{1/4})$. We also consider n sufficiently large so that $\sqrt{\tanh(\sqrt{\mu}k_n)} \geq \frac{1}{2}$. Then (B.11)-(B.12) yields, for any $t \in I^n \cap [0, k_0^{-\frac{1}{2}}]$,

$$\widehat{\psi}^n_{\boldsymbol{k}_n}(t) \geq \exp(\frac{1}{2}k_n^{1/2}t - k_n^{1/4}), \quad \widehat{\zeta}^n_{\boldsymbol{k}_n}(t) \geq \frac{1}{2}k_n^{1/2}\exp(-k_n^{1/4})\Big(\exp(\frac{1}{2}k_n^{1/2}t) - 1\Big).$$

Then, one has

$$a^n_{\boldsymbol{k}_n}(t) \leq 1 - \epsilon^2\alpha^n(t)J(k_n)^2k_n \leq 1 - 2\epsilon^2\alpha^n_{\boldsymbol{k}_n}(t)J(k_n)^2k_n \leq 1 - \epsilon^2(2\pi)^d\tfrac{1}{8}J(k_n)^2k_n^3|\widehat{\psi}^n_{\boldsymbol{k}_n}(t)|^2.$$

Hence for n sufficiently large, we have $2k_n^{-1/4} < k_0^{-\frac{1}{2}}$ and for any $t \in I^n \cap [2k_n^{-1/4}, k_0^{-\frac{1}{2}}]$, there holds $\widehat{\psi}^n_{\boldsymbol{k}_n}(t) \geq 1$, $\widehat{\zeta}^n_{\boldsymbol{k}_n}(t) \geq \frac{1}{4}k_n^{1/2}$ and $a^n_{\boldsymbol{k}_n}(t) \leq -\widehat{\psi}^n_{\boldsymbol{k}_n}(t)^2$, from which we infer

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\psi}^n_{\boldsymbol{k}_n} \geq (\widehat{\psi}^n_{\boldsymbol{k}_n})^2\widehat{\zeta}^n_{\boldsymbol{k}_n} \geq \tfrac{1}{4}k_n^{1/2}(\widehat{\psi}^n_{\boldsymbol{k}_n})^2.$$

This yields the desired blowup in time $T^n_\star \leq 2k_n^{-1/4} + 4k_n^{-1/2}$.

The blowup for negative time follows from time-reversibility: $(t, \zeta_0, \psi_0) \leftarrow (-t, \zeta_0, -\psi_0)$. □

Remark B.4. *The ill-posedness result of Proposition B.3 is stronger than the corresponding ones obtained in [4], since the latter holds in a fixed functional space $H^s((2\pi\mathbb{T})^d)$ subject to the restriction $s \in [0, 2)$ in Theorem 3.1 and $s \in [0, 3)$ in Theorem 3.2. Our stronger result is made possible using that in our toy model the coupling between Fourier modes are weak. A direct inspection of the proof shows that our ill-posedness holds in fact for initial data in the Gevrey-σ^{-1} class for any $\sigma \in [0, \frac{1}{2})$ and for any $\sigma \in [0, 1)$ if $\mathsf{J} = \mathrm{Id}$).*

C A new model with quadratic precision

In this section, we introduce a new model for deep water waves which is consistent of order $\mathcal{O}(\epsilon^2)$ and does not rely on regularization operators as in (RWW2). We introduce the variable

$$\boldsymbol{v} = \nabla\psi - \epsilon(\mathsf{T}^\mu \cdot \nabla\psi)(\nabla\zeta) \tag{C.1}$$

representing (an approximation to) the horizontal velocity of the fluid at the free surface. Plugging the above identity in (WW2) and neglecting terms of order $\mathcal{O}(\epsilon^2)$ yields

$$\begin{cases} \partial_t\zeta - \mathsf{T}^\mu \cdot \boldsymbol{v} + \epsilon\mathsf{T}^\mu \cdot (\zeta\nabla(\mathsf{T}^\mu \cdot \boldsymbol{v})) + \epsilon\nabla \cdot (\zeta\boldsymbol{v}) = \mathcal{O}(\epsilon^2), \\ \partial_t\big(\boldsymbol{v} + \epsilon(\mathsf{T}^\mu \cdot \boldsymbol{v})(\nabla\zeta)\big) + \nabla\zeta + \frac{\epsilon}{2}\nabla\big(|\boldsymbol{v}|^2 - (\mathsf{T}^\mu \cdot \boldsymbol{v})^2\big) = \mathcal{O}(\epsilon^2). \end{cases} \tag{C.2}$$

Using that solutions to (C.2) satisfy

$$\partial_t\big((\mathsf{T}^\mu \cdot \boldsymbol{v})(\nabla\zeta)\big) = (\mathsf{T}^\mu \cdot \boldsymbol{v})(\nabla\mathsf{T}^\mu \cdot \boldsymbol{v}) - (\mathsf{T}^\mu \cdot \nabla\zeta)(\nabla\zeta) + \mathcal{O}(\epsilon),$$

we obtain the so-called Matsuno system, in the formulation of [20, 25], and justified rigorously (assuming the existence of solutions) in [2, Theorem 6.5]:

$$\begin{cases} \partial_t\zeta - \mathsf{T}^\mu \cdot \boldsymbol{v} + \epsilon\mathsf{T}^\mu \cdot (\zeta\nabla(\mathsf{T}^\mu \cdot \boldsymbol{v})) + \epsilon\nabla \cdot (\zeta\boldsymbol{v}) = \mathcal{O}(\epsilon^2), \\ \partial_t\boldsymbol{v} + \nabla\zeta + \frac{\epsilon}{2}\nabla\big(|\boldsymbol{v}|^2\big) - \epsilon(\mathsf{T}^\mu \cdot \nabla\zeta)(\nabla\zeta) = \mathcal{O}(\epsilon^2). \end{cases} \tag{C.3}$$

The well-posedness of the Cauchy problem for (C.3) (in functional spaces of finite regularity) is still an open problem to the authors' knowledge, and we have not been able to find a good structure allowing energy estimates in the spirit of this work. Hence we propose the following modification. Using that (C.1) yields $\operatorname{rot}\boldsymbol{v} = \mathcal{O}(\epsilon)$ (when $d = 2$) and $\exp(1 + X) = 1 + X + \mathcal{O}(X^2)$, we have

$$\begin{cases} \partial_t\zeta - \mathsf{T}^\mu \cdot \boldsymbol{v} + \epsilon\mathsf{T}^\mu \cdot (\zeta G_0^\mu\boldsymbol{v}) + \epsilon\nabla \cdot (\zeta\boldsymbol{v}) = \mathcal{O}(\epsilon^2), \\ \partial_t\boldsymbol{v} + \exp(-\epsilon G_0^\mu\zeta)\nabla\zeta + \epsilon(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \mathcal{O}(\epsilon^2). \end{cases} \tag{C.4}$$

where $G_0^\mu \overset{\text{def}}{=} |D|\tanh(\sqrt{\mu}|D|)$ (recall $\mathsf{T}^\mu \overset{\text{def}}{=} -\frac{\tanh(\sqrt{\mu}|D|)}{|D|}\nabla$).

The consistency of (C.4) with respect to (C.3) is easily obtained, using in particular Moser estimates (see for instance [18, Proposition B.2]): for any $s > t_0$,

$$\big|\exp(-\epsilon G_0^\mu\zeta) - \big(1 - \epsilon G_0^\mu\zeta\big)\big|_{H^s} \leq C\epsilon^2\big|\nabla\zeta\big|^2_{H^s}$$

where C depends uniquely on s and $\big|\nabla\zeta\big|_{H^{t_0}}$.

We would like to quickly explain why the structure of (C.4) is favorable. First, we can prove the following Lemma, using the same techniques as in Lemmas 2.7 and 2.9 (see also [25, Lemma 3.3]).

Lemma C.1. *Let $d \in \{1,2\}$ and $t_0 > \frac{d}{2}$. Let $s \geq 0$. There exists $C > 0$ such that*

$$\left| -\frac{\nabla}{|D|} \cdot (\zeta |D| \boldsymbol{v}) + \zeta \nabla \cdot \boldsymbol{v} \right|_{H^s} \leq C \big|\zeta\big|_{H^s} \big|\boldsymbol{v}\big|_{H^{t_0+1}} + C \big|\zeta\big|_{H^{t_0+\frac{3}{2}}} \big|\boldsymbol{v}\big|_{H^{s-\frac{1}{2}}}.$$

Moreover, there exists a constant $C > 0$ such that for any $\mu \geq 1$,

$$\big|\mathsf{T}^\mu \cdot (\zeta G_0^\mu \boldsymbol{v}) + \zeta \nabla \cdot \boldsymbol{v}\big|_{H^s} \leq C \big|\zeta\big|_{H^s} \big|\boldsymbol{v}\big|_{H^{t_0+1}} + C \big|\zeta\big|_{H^{t_0+\frac{3}{2}}} \big|\boldsymbol{v}\big|_{H^{s-\frac{1}{2}}}.$$

We can then extract the quasilinear structure of the first equation: after differentiating α times with $\alpha \in \mathbb{N}^d$, $|\alpha| = N \in \mathbb{N}$ and $N > \frac{d}{2} + \frac{3}{2}$, we obtain

$$\partial_t \partial^\alpha \zeta - \mathsf{T}^\mu \cdot \partial^\alpha \boldsymbol{v} + \epsilon \nabla \zeta \cdot \partial^\alpha \boldsymbol{v} + \epsilon \boldsymbol{v} \cdot \nabla \partial^\alpha \zeta = \epsilon\, r.$$

As for the second equation, the chain rule, standard commutator estimates and the smoothing effects of the operator $\tanh(\sqrt{\mu}|D|) - \mathrm{Id}$ yield

$$\partial_t \partial^\alpha \boldsymbol{v} + \exp(-\epsilon G_0^\mu \zeta)\big(\partial^\alpha \nabla \zeta - \epsilon (\nabla \zeta)(\tfrac{|D|}{\tanh(\sqrt{\mu}|D|)} \partial^\alpha \zeta)\big) + \epsilon (\boldsymbol{v} \cdot \nabla) \partial^\alpha \boldsymbol{v} = \epsilon\, \boldsymbol{r}.$$

Above, $(r, \boldsymbol{r})$ are order-zero terms in the sense that they are bounded in $H^{1/2}(\mathbb{R}^d) \times L^2(\mathbb{R}^d)^d$ if $(\zeta, \boldsymbol{v}) \in H^{N+\frac{1}{2}}(\mathbb{R}^d) \times H^N(\mathbb{R}^d)^d$. The symmetric structure is now visible, and *a priori* energy estimates in $H^{N+\frac{1}{2}}(\mathbb{R}^d) \times H^N(\mathbb{R}^d)^d$ when $N > \frac{d}{2} + \frac{3}{2}$ are obtained when testing the first equation with $\frac{|D|}{\tanh(\sqrt{\mu}|D|)} \partial^\alpha \zeta$ and the second one with $\exp(\epsilon G_0^\mu \zeta) \partial^\alpha \boldsymbol{v}$.

Remark C.2. *While* (C.4) *is attractive as a model with quadratic precision in the deep-water regime with good theoretical properties (unlike* (C.3)*) and without resorting to rectifiers as in* (RWW2) *or nonphysical change of variables as in [25] (therein, the surface deformation is no longer an unknown), we would like to point out some shortcomings. Firstly, the appearance of the exponential nonlinearity prevents the use of spectral schemes with perfect dealiasing. Unlike* (RWW2)*, we have lost the many important features of the original water waves system (variational structure, group symmetries, preserved quantities). Finally it is clear that this exponential term, which allows to "symmetrize" the system, arises from purely artificial algebra. From our calculations, it does not appear to be possible to extend the strategy to corresponding systems with cubic nonlinearities.*

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