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Benjamin Mélinand

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Météotsunamis, résonance de Proudman et effet Coriolis pour les équations des vagues

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Directeur de thèse :	M. Lannes David	DR CNRS (Université de Bordeaux)
Rapporteurs :	Mme Gallagher Isabelle	Professeur (Université Paris-Diderot)
	Mme Sulem Catherine	Professeur (University of Toronto)

Composition du jury :

Président du jury :	M. Rousset Frédéric	Professeur (Université Paris-Sud)
Directeur de thèse :	M. Lannes David	DR CNRS (Université de Bordeaux)
Rapporteurs :	Mme Gallagher Isabelle	Professeur (Université Paris-Diderot)
	Mme Sulem Catherine	Professeur (University of Toronto)
Examinateurs :	M. Chiron David	Maître de conférence (Université de Nice Sophia Antipolis)
	M. Colin Mathieu	Maître de conférence (Université de Bordeaux)
	M. Sueur Franck	Professeur (Université de Bordeaux)

Météotsunamis, résonance de Proudman et effet Coriolis pour les équations des vagues

Mélinand Benjamin *

*Institut de Mathématiques de Bordeaux UMR 5251, 351, cours de la Libération - F 33 405 TALENCE.
Email: benjamin.melinand@math.u-bordeaux.fr

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Résumé

Dans ce travail nous nous intéressons aux comportement de vagues soumises à l'action d'une pression atmosphérique non constante, un fond mobile et la force de Coriolis. Une première partie est dédiée à l'étude de la résonance de Proudman. Nous proposons une approche mathématique rigoureuse pour étudier ce phénomène. Nous commençons par démontrer un résultat d'existence locale dans un cadre irrotationnel sur les équations des vagues (appelées aussi formulation de Zakharov/Craig-Sulem). Puis, nous justifions différents modèles asymptotiques pour généraliser cette résonance dans diverses situations physiques. Nous proposons en particulier une étude détaillée dans des eaux profondes dans un régime linéaire. Nous étudions aussi la propagation de vagues dans des eaux profondes dans un régime faiblement non-linéaire grâce aux équations de Saut-Xu et nous proposons un schéma numérique pour résoudre ces équations.

Dans une deuxième partie, nous étudions l'effet de la force de Coriolis sur les vagues. Nous démontrons un résultat d'existence locale sur les équations Castro-Lannes, équations qui généralisent la formulation de Zakharov/Craig-Sulem dans un cadre rotationnel. Nous justifions ensuite différents modèles asympotiques dans des eaux peu profondes en présence de la force de Coriolis. En particulier, nous proposons une généralisation des équations de Boussinesq (modèle asymptotique dans un régime faiblement linéaire) lorsque la force de Coriolis n'est pas négligeable. Ces équations nous permettent ensuite de justifier mathématiquement les ondes de Poincaré puis l'équation d'Ostrovsky qui généralise l'équation de Korteweg-De-Vries en présence de la force de Coriolis.

Mots clés : résonance de Proudman, effet Coriolis, équations des vagues, modèles asymptotiques.

Abstract

In this work, we are interested in the evolution of water waves under the influence of a non constant atmospheric pressure, a moving bottom and a Coriolis forcing. In a first part, we study the Proudman resonance. We propose a mathematical approach to understand this phenomenon. First, we prove a local wellposedness result in a irrotational framework on the water waves equations (also called the Zakharov/Craig-Sulem formulation). Then, we fully justify different asymptotic models. In particular, we carefully study the Proudman resonance in deep water in the linear regime. Finally, we study the propagation of water waves in a weakly nonlinear regime thanks to the Saut-Xu equations and we propose a numerical scheme in order to solve these equations.

In a second part, we study the influence of a Coriolis forcing on water waves. We prove a local wellposedness result on the Castro-Lannes equations, which generalize the Zakharov/Craig-Sulem formulation in the rotational framework. Then, we fully justify different asymptotic models where we take into account a Coriolis forcing. In particular, we generalize the Boussinesq equations (asymptotic model in a weakly nonlinear regime) in this setting. Thanks to these equations, we justify the Poincaré waves and then the Ostrovsky equation, which generalize the Korteweg-De-Vries equation when a Coriolis forcing is taken into account.

Key words : Proudman resonance, Coriolis effect, water waves equations, asymptotic models.

Contents

Chapter 1

Introduction

Sommaire

L'objectif principal de ce manuscrit est d'analyser le comportement de vagues soumises à l'action de forces extérieures autres que la force de gravité. Ce premier chapitre a pour but de replacer le contexte et le cheminement du présent document. Il est suivi de quatre autres chapitres. Le deuxième chapitre s'intéresse à la propagation de vagues induites par des tempêtes ou des glissements de terrain sous-marins. Pendant de tels événements, il peut se produire une amplification appelée résonance de Proudman ([120]). Ce chapitre a pour but d'étudier cette résonance dans différentes situations. Il est tiré de l'article [102]. Le troisième chapitre étudie la propagation de vagues dans des eaux dites *profondes*, à savoir lorsque la longueur des vagues est comparable à la hauteur d'eau. Il a été écrit en collaboration avec Afaf Bouharguane (IMB). Nous donnons un schéma numérique basé sur les équations de Saut-Xu ([125]). Différentes simulations numériques sont proposées pour valider notre schéma numérique. Nous étudions aussi l'effet de fonds fortement oscillants sur la propagation d'une vague. Ce chapitre est tiré de l'article [24]. Les deux derniers chapitres de ce manuscrit ont pour but d'étudier l'effet de la force de Coriolis sur les vagues. Dans le quatrième chapitre, nous proposons une analyse mathématique de ce phénomène et nous justifions les équations de Saint-Venant dans le cas d'eaux dites *peu profondes*. Ce chapitre est tiré de l'article [101]. Enfin, dans le dernier chapitre, nous étudions l'effet de la force de Coriolis dans le régime Boussinesq. Ce régime est valide pour des vagues ayant une grande longueur d'onde et une petite amplitude. Nous montrons que l'équation de Korteweig de Vries et l'équation d'Ostrovsky sont des modèles raisonnables dans un tel régime. Ce chapitre est tiré de l'article [103].

1.1 Les équations d'Euler à surface libre

Dans ce manuscrit, nous nous intéressons à l'évolution d'un fluide soumis à son propre poids et à la force de Coriolis. Le fluide est supposé délimité par deux frontières : le fond et la surface (voir la figure 1.1). Nous supposons que ces deux frontières peuvent être paramétrées par des graphes et qu'elles peuvent évoluer au cours du temps. Tout effet de déferlement est donc exclu. Le fond est imperméable. La surface sépare notre fluide de l'air et nous supposons qu'il n'y a pas d'échange entre notre fluide et l'extérieur (pas d'évaporation par exemple). Le fluide est supposé parfait et homogène, c'est à dire que tous les effets de viscosité et de conductivité thermique sont négligés, que notre fluide a une masse volumique constante et qu'il vérifie le principe de conservation de la masse. En outre, tous les effets de tension de surface sont négligés. Enfin, la pression à la surface et l'évolution du fond sont connues. Pour modéliser mathématiquement un tel fluide, nous avons recours aux équations d'Euler à surface libre.

Dans la suite, nous notons d la dimension horizontale, $z \in \mathbb{R}$ la variable verticale et $X \in \mathbb{R}^d$ la variable horizontale : $X = (x, y)$ si $d = 2$. Pour des applications concrètes, d est égal à 1 ou 2. On note la masse volumique du fluide ρ, la surface $\zeta(t, X)$ et on paramètre le fond par $-H + b(t, X)$ où H représente la profondeur moyenne et b la variation du fond. Le domaine fluide à l'instant t est noté Ω_t

$$\Omega_t = \{(X, z),\ -H + b(t, X) < z < \zeta(t, X)\}. \tag{1.1}$$

La vitesse dans le fluide est notée $\mathbf{U} = \mathbf{U}(t, X, z)$, sa composante horizontale par $\mathbf{V}$ et sa composante verticale par w. On note $\mathcal{P} = \mathcal{P}(t, X, z)$ la pression dans le fluide et $P = P(t, X)$ la pression à la surface. Enfin, comme nous l'avons dit précédemment, notre fluide est soumis à la force de gravité $\boldsymbol{g} = -g\boldsymbol{e_z}$ et à la rotation de la Terre de vecteur rotation $\mathbf{f} = \frac{f}{2}\mathbf{e_z}$. Nous

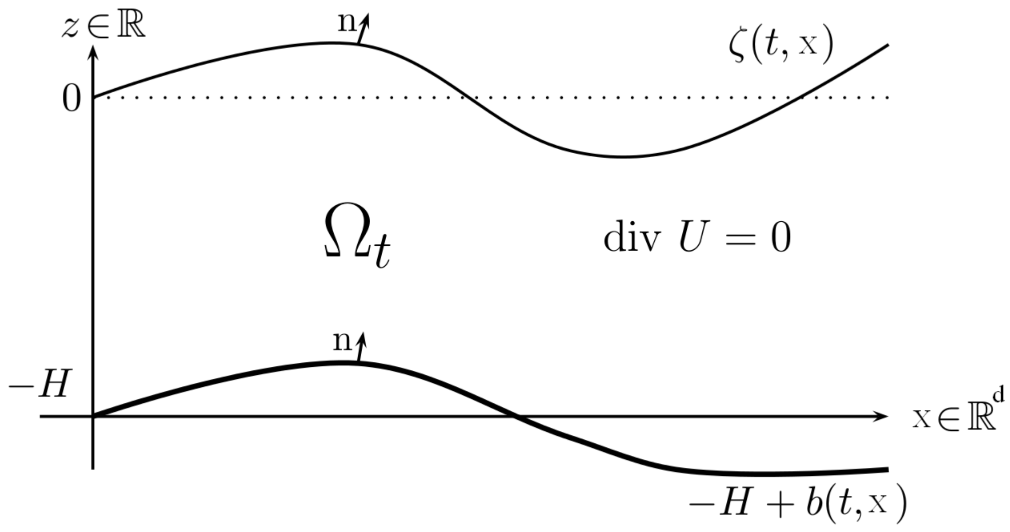

Figure 1.1: Notations

pouvons à présent donner les équations d'Euler à surface libre

$$\begin{cases} \partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_{X,z}) \mathbf{U} + \mathbf{f} \times \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} \mathcal{P} - g\mathbf{e_z} \text{ dans } \Omega_t, \\ \mathrm{div}_{X,z} \, \mathbf{U} = 0 \text{ dans } \Omega_t, \end{cases} \tag{1.2}$$

auxquelles nous rajoutons les conditions aux bords

$$\begin{cases} \partial_t \zeta - \underline{\mathrm{U}} \cdot N = 0, \\ \partial_t b - \mathrm{U}_b \cdot N_b = 0, \\ \mathcal{P}_{|z=\zeta} = P, \end{cases} \tag{1.3}$$

où $N = \begin{pmatrix} -\nabla_X \zeta \\ 1 \end{pmatrix}$ et $N_b = \begin{pmatrix} -\nabla_X b \\ 1 \end{pmatrix}$ sont les vecteurs normaux respectivement à la surface et au fond et $\underline{\mathbf{U}} = \begin{pmatrix} \underline{\mathbf{V}} \\ \underline{\mathrm{w}} \end{pmatrix} = \mathbf{U}_{|z=\zeta}$ et $\mathbf{U}_b = \begin{pmatrix} \mathbf{V}_b \\ \mathrm{w}_b \end{pmatrix} = \mathbf{U}_{|z=-H+b}$ sont les traces de la vitesse respectivement à la surface et au fond.

Rappelons que la première équation du sytème (1.2) correspond à l'équation de bilan de la quantité de mouvement et découle du principe fondamental de la dynamique. La deuxième équation du sytème (1.2) correspond à la conservation de la masse et à l'incompressibilité du fluide. Enfin, les deux premières conditions aux bords traduisent l'absence d'échange de matière entre notre fluide et l'extérieur. La troisième condition aux bords traduit la continuité à la surface entre la pression dans notre fluide et la pression extérieure (nous avons négligé la tension de surface). À ce stade nous avons donc un système d'équations sur la vitesse $\mathbf{U}$ et la surface libre ζ lesquelles sont alors, avec la pression $\mathcal{P}$, les inconnues du problème. En fait, la pression $\mathcal{P}$ n'est pas une vraie inconnue mais le multiplicateur de Lagrange associé à la contrainte d'incompressibilité.

Dans la suite, nous serons amenés à séparer deux situations. Premièrement, nous supposerons que la force de Coriolis et les effets dus à la vorticité sont négligeables. Cette hypothèse sera utilisée dans les deuxième et troisième chapitres de ce manuscrit. Nous verrons que cela simplifie les équations d'Euler. Puis, dans les chapitres quatre et cinq, nous ne négligerons ni la vorticité ni la force de Coriolis. Il est assez courant de négliger la vorticité en océanographie lorsque l'on s'intéresse à la propagation de vagues loin des côtes. En revanche, cette hypothèse n'est pas réaliste lorsque l'on souhaite prendre en compte la force de Coriolis. En effet, si nous prenons le rotationnel de la première équation de (1.2) et que nous notons $\boldsymbol{\omega} = \mathrm{rot}_{X,z}\mathbf{U}$, nous obtenons l'équation suivante

$$\partial_t \boldsymbol{\omega} + (\mathbf{U} \cdot \nabla_{X,z})\, \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla_{X,z})\, \mathbf{U} = f \partial_z \mathbf{U} \text{ dans } \Omega_t.$$

La force de Coriolis génère donc de la vorticité même si cette dernière est initialement nulle. Pour quantifier l'influence de la force de Coriolis sur les vagues, les physiciens utilisent le nombre de Rossby, noté Ro dans ce manuscrit. C'est un nombre sans dimension qui représente le rapport entre les forces d'inertie et l'effet de la rotation de la Terre :

$$\mathrm{Ro} = \frac{\mathrm{V}_0}{fL},$$

où V_0 est la vitesse typique horizontale, L la longueur typique de notre phénomène et f le paramètre de Coriolis valant $f = 2\omega \sin(\phi)$, où ω est la vitesse angulaire de notre référentiel tournant et ϕ la latitude. Pour la Terre, $\omega = 7,310^{-5} s^{-1}$. Ainsi, pour une latitude moyenne ϕ comprise entre 30 degrés et 60 degrés, le paramètre f est de l'ordre de $10^{-4} s^{-1}$. Nous voyons donc que, pour des vagues de longueurs d'onde de l'ordre du kilomètre avec une vitesse typique de $1 m.s^{-1}$, les effets de la force de Coriolis sont négligeables. Cependant, pour des vagues ayant une plus grande longue d'onde, cette simplification n'est pas réaliste. En particulier, pour une vague créée par un tsunami, la longueur d'onde typique peut atteindre la dizaine voire la centaine de kilomètres et la force de Coriolis a tendance à atténuer son amplitude. Nous étudions cet effet dans la section 4.5.4 lorsque $d = 2$ et dans la section 5.3.2 lorsque $d = 1$ (voir aussi [61], [93]).

1.1.1 Formulation de Zakharov/Craig-Sulem-Sulem, cas d'un fluide irrotationel

1.1.1.1 Les équations

Dans cette partie, comme dans les chapitres 2 et 3, nous supposons que la force de Coriolis et les effets de vorticité sont négligeables. Nous avons donc

$$\mathrm{rot}_{X,z}\, \mathbf{U} = 0 \text{ dans } \Omega_t. \tag{1.4}$$

Si, en outre, Ω_t est simplement connexe, $\mathbf{U}$ peut s'écrire sous la forme d'un gradient. Un moyen simple d'assurer que le domaine fluide est simplement connexe est de supposer la condition suivante

$$\exists\, h_{\min} > 0 \;,\; \zeta + H - b \geq h_{\min}. \tag{1.5}$$

Nous ferons cette hypothèse dans toute la suite du manuscrit. Nous avons donc l'existence d'un potentiel Φ, appelé potentiel des vitesses, tel que

$$\mathbf{U} = \nabla_{X,z} \Phi. \tag{1.6}$$

On peut alors réécrire les équations d'Euler (1.2) de la façon suivante

$$
\begin{cases} \partial_t \Phi + \dfrac{1}{2} |\nabla_{X,z}\Phi|^2 + gz = -\dfrac{1}{\rho} (\mathcal{P} - P) \;\text{ dans } \Omega_t, \\ \Delta_{X,z}\, \Phi = 0 \text{ dans } \Omega_t. \end{cases} \tag{1.7}
$$

La première équation s'appelle l'équation de Bernoulli. La deuxième équation montre que le potentiel des vitesses est une fonction harmonique dans le domaine fluide. Remarquons que nous avons implicitement supposé que le fluide est au repos à l'infini. Nos inconnues sont maintenant la surface ζ et le potentiel des vitesses Φ. Cependant, Φ est défini dans le domaine fluide Ω_t qui dépend de la surface ζ et qui varie au cours du temps. Cela complique l'analyse mathématique. Un moyen de fixer le domaine est de travailler en coordonnées lagrangiennes. Avec cette approche, divers auteurs ont montré des résultats d'existence locale (voir par exemple [39], [90], [42], [155], [156]). Une autre possibilité lorsque $d = 1$ est de voir le domaine fluide comme un domaine de $\mathbb{C}$ et les conditions d'incompressibilité et d'irrotationalité comme des conditions de Cauchy-Riemann sur le conjugué de la vitesse. Grâce à cette approche, Nalimov ([110]) a obtenu le premier résultat d'existence locale dans le cas $d = 1$, avec une profondeur infinie et des données initiales Sobolev assez petites. Yosihara ([152]) a ensuite étendu le résultat de Nalimov à un fond fixe fini et Wu ([150]) a obtenu un résultat d'existence locale dans le cas d'une profondeur infinie et de données initiales Sobolev quelconque. Wu a aussi traité le cas $d = 2$ ([151]) en utilisant l'analyse de Clifford. Nous n'allons pas nous baser sur ces approches mais plutôt utiliser l'approche de Zakharov [153]. Il remarqua que la connaissance de la trace à la surface du potentiel des vitesses suffit pour déterminer Φ. En effet, en notant $\Phi_{|z=\zeta} = \psi$ et en utilisant la deuxième équation des conditions aux bords (1.3) et la deuxième équation de (1.7), nous obtenons le problème de Laplace suivant

$$
\begin{cases} \Delta_{X,z}\Phi = 0 \text{ dans } \Omega_t, \\ \Phi_{|z=\zeta} = \psi \;,\; \sqrt{1 + |\nabla_X b|^2}\partial_{\mathbf{n}}\Phi_{|z=-H+b} = \partial_t b. \end{cases} \tag{1.8}
$$

Nous allons décomposer Φ en deux parties en séparant la contribution de la surface et du fond

$$
\Phi = \Phi^S + \Phi^B.
$$

Nous obtenons les deux problèmes de Laplace suivant

$$
\begin{cases} \Delta_{X,z}\Phi^S = 0 \text{ dans } \Omega_t, \\ {\Phi^S}_{|z=\zeta} = \psi \;,\; \sqrt{1 + |\nabla_X b|^2}\partial_{\mathbf{n}}{\Phi^S}_{|z=-H+b} = 0, \end{cases} \tag{1.9}
$$

et

$$
\begin{cases} \Delta_{X,z}\Phi^B = 0 \text{ dans } \Omega_t, \\ {\Phi^B}_{|z=\zeta} = 0 \;,\; \sqrt{1 + |\nabla_X b|^2}\partial_{\mathbf{n}}{\Phi^B}_{|z=-H+b} = \partial_t b. \end{cases} \tag{1.10}
$$

Notons ici que $\mathbf{n}$ est le vecteur normal normalisé qui pointe vers le haut

$$
\mathbf{n} = \begin{cases} \dfrac{1}{\sqrt{1 + |\nabla_X \zeta|^2}} N, \text{ à la surface}, \\ \dfrac{1}{\sqrt{1 + |\nabla_X b|^2}} N_b, \text{ au fond}. \end{cases} \tag{1.11}
$$

Nous étudions en détail ces deux problèmes de Laplace dans l'Appendice A. En particulier, nous montrons l'existence et l'unicité de Φ. Nos inconnues sont maintenant la trace du potentiel des vitesses ψ qui est définie sur $\mathbb{R}^d$ et la surface ζ. Ainsi, via le procédé de Zakharov, nous avons *fixé* le domaine de nos inconnues. Il s'agit alors de trouver deux équations d'évolution sur ζ et ψ. On obtient une première équation grâce à la première équation de (1.3). Nous introduisons les opérateurs suivants : l'opérateur de Dirichlet-Neumann

$$G[\zeta,b]:\psi\mapsto\sqrt{1+|\nabla_X\zeta|^2}\partial_{\mathbf{n}}\Phi^S{}_{|z=\zeta},\tag{1.12}$$

où Φ^S satisfait le système (1.9), et l'opérateur de Neumann-Neumann

$$G^{NN}[\zeta,b]:\partial_t b\mapsto\sqrt{1+|\nabla_X\zeta|^2}\partial_{\mathbf{n}}\Phi^B{}_{|z=\zeta},\tag{1.13}$$

où Φ^B satisfait (1.10). Nous pouvons alors reformuler la première équation de (1.3) en

$$\partial_t\zeta-G[\zeta,b](\psi)=G^{NN}[\zeta,b](\partial_t b).\tag{1.14}$$

Pour obtenir une deuxième équation d'évolution, nous prenons la trace de l'équation de Bernoulli à la surface. En utilisant la troisième équation de (1.3), nous pouvons alors supprimer la pression $\mathcal{P}$. Nous obtenons alors le système d'équations suivant, appelé équations des vagues,

$$\begin{cases}\partial_t\zeta-G[\zeta,b](\psi)=G^{NN}[\zeta,b](\partial_t b),\\ \partial_t\psi+g\zeta+\dfrac{1}{2}|\nabla_X\psi|^2-\dfrac{1}{2}\dfrac{\left(G[\zeta,b](\psi)+G^{NN}[\zeta,b](\partial_t b)+\nabla_X\zeta\cdot\nabla_X\psi\right)^2}{(1+|\nabla_X\zeta|^2)}=-\dfrac{P}{\rho}.\end{cases}\tag{1.15}$$

Le système (1.15) porte aussi le nom de formulation de Zakharov/Craig-Sulem-Sulem. En effet, en utilisant le travail de Zakharov, Craig, Sulem et Sulem ([47], [48]) ont obtenu ce sytème dans le cas d'un fond plat fixe et avec une pression constante à la surface. Le grand avantage de cette formulation est de ramener les équations d'Euler à surface libre à un système d'équations à la surface.

D'un point de vue mathématique, ces équations sont complètement non linéaires et liées à un problème de surface libre, ce qui rend leur étude délicate. Lannes ([78]) montre le premier résultat d'existence locale. Sa preuve repose sur un schéma de Nash-Moser. Puis, Iguchi ([70]) montre que l'on peut *quasilinéariser* et *symétriser* ces équations. Citons aussi les travaux d'Alazard, Burq et Zuily (voir par exemple [4], [5]) qui permettent d'améliorer la régularité minimale demandée pour la donnée initiale en utilisant le côté dispersif des équations des vagues dans le cas d'une profondeur finie (voir aussi l'article [52] de De Poyferré pour un critère d'explosion). Enfin, les travaux récents de Wang ([148], [147]) montrent que, pour $d=2$, il y a existence globale des équations des vagues pour des petites données initiales dans le cas d'un fond plat ($b=0$). Tous les résultats donnés précédemment font l'hypothèse d'une pression à la surface constante et d'un fond fixe. Iguchi ([71]) est le premier à étudier le cas d'un fond mobile pour modéliser la propagation d'une vague créée par une variation brusque de la bathymétrie. Dans la section 2.2, nous généralisons le résultat d'Iguchi ([71]) en ajoutant une pression non constante à la surface, en baissant la régularité minimale de la donnée initiale et en considérant des potentiels des vitesses plus généraux. Notons enfin le résultat de contrôle pour les équations des vagues [2] pour lequel Alazard, Baldi et Han-Kwan étudient le cas d'une pression non constante à la surface mais dans le cas périodique (voir aussi [1]).

Comme nous l'avons dit précédemment, Iguchi ([70]) prouve l'existence locale des équations des vagues en *quasilinéarisant* et en *symétrisant* les équations. Nous allons expliquer ce principe sur un exemple plus simple. On considère un système d'équations d'évolution quasilinéaire de la forme

$$\partial_t U + \mathcal{A}(U, \nabla_X)U = 0.$$

Si on peut trouver un opérateur matriciel $S(U, \nabla_X)$ défini positif tel que l'opérateur matriciel $S(U, \nabla_X)\mathcal{A}(U, \nabla_X)$ soit anti-symétrique, on dit que le système est symétrisable et on obtient l'existence locale en étudiant l'énergie $(S(U, \nabla_X)U, U)_{L^2}$. Maintenant, si nous avons un système complètement non linéaire

$$\partial_t U + f(U) = 0,$$

on dit que ce système peut être quasilinéarisé et symétrisé si, quand on dérive l'équation assez de fois, on obtient un système d'équations d'évolution sur U et ses dérivées qui est quasilinéaire symétrisable. Pour les équations des vagues, nous avons une difficulté supplémentaire. Comme nous avons un problème à surface libre, la bonne inconnue d'Alinhac va jouer un rôle crucial pour clore les estimations d'énergie. Cette notion a été introduite pour la première fois par Alinhac ([7]) dans le cadre d'ondes de raréfaction pour des systèmes hyperboliques quasilinéaires. Alinhac montre alors que pour pouvoir clore les estimations d'énergie pour un problème à surface libre quasilinéaire (il remarque qu'il manque une demi régularité pour clore ses estimations), il ne faut pas considérer des dérivées classiques pour nos inconnues, mais des dérivées adaptées à notre domaine mobile. Pour les équations des vagues, la notion de bonne inconnue d'Alinhac a été introduite par Alazard et Métivier ([6]). Elle avait cependant été utilisée par Lannes ([78]) et Iguchi ([70]) sans faire le lien avec les travaux d'Alinhac.

Pour finir sur cette partie, notons que nous avons réduit les équations d'Euler à surface libre à une équation à la surface. Nous sommes partis d'une solution d'Euler pour obtenir une solution des équations des vagues. On peut aussi faire le chemin inverse. Alazard, Burq et Zuily ([3]) ont montré que, à partir d'une solution des équations des vagues, on peut reconstruire la pression dans le domaine fluide et obtenir une solution des équations d'Euler.

1.1.1.2 Adimensionnement

Pour étudier la propagation de vagues, le système (1.15) s'avère en général trop compliqué. Il contient toute la dynamique du fluide. Nous allons donc simplifier des termes qui nous semblent insignifiants. Mais comment savoir lesquels ? Un moyen très simple et connu des physiciens est le principe d'adimensionnement. On adimensionne chaque variable du système par une quantité caractéristique. Dans notre cas nous avons 4 échelles caractéristiques : la longueur L_x des vagues que nous étudions, leur largeur L_y, leur amplitude a et la profondeur typique H de l'océan. Le fond doit aussi être adimensionné. Nous travaillons ici avec un fond mobile. Le but de ce manuscrit et en particulier le chapitre 2 est d'étudier la propagation de vagues créées par des glissements de terrain sous-marins. Notre fond b peut s'écrire de la forme $b(t, X) = b_0(X) + b_m(t, X)$ et nous avons deux échelles caractéristiques, l'amplitude du fond fixe a_{bott} et l'amplitude de notre glissement de terrain $a_{\mathrm{bott,m}}$. Ensuite, nous pouvons créer 5 paramètres sans dimension

$$\varepsilon = \frac{a}{H},\ \mu = \frac{H^2}{L_x^2},\ \gamma = \frac{L_x}{L_y},\ \beta = \frac{a_{\mathrm{bott}}}{H},\ \lambda = \frac{a_{\mathrm{bott,m}}}{a_{\mathrm{bott}}},$$

où ε est appelé paramètre de non-linearité, μ est appelé paramètre de faible profondeur, γ est appelé paramètre de transversalité, $\beta = \frac{a_{\mathrm{bott}}}{H}$ est appelé paramètre de bathymétrie et où λ compare l'amplitude du fond fixe à celle du fond mobile. Nous pouvons adimensionner les variables de nos équations

$$\begin{cases} x' = \dfrac{x}{L_x},\ y' = \dfrac{y}{L_y},\ z' = \dfrac{z}{H},\ \zeta' = \dfrac{\zeta}{a},\ b' = \dfrac{b}{a_{\text{bott}}},\ b'_0 = \dfrac{b_0}{a_{\text{bott}}},\ b'_m = \dfrac{b_m}{a_{\text{bott,m}}},\ t' = \dfrac{\sqrt{gH}}{L}t, \\ \left(\Phi^S\right)' = \dfrac{H}{aL\sqrt{gH}}\Phi^S,\ \left(\Phi^B\right)' = \dfrac{L}{Ha_{\text{bott,m}}\sqrt{gH}}\Phi^B,\ \psi' = \dfrac{H}{aL\sqrt{gH}}\psi,\ P' = \dfrac{P}{a\rho g}. \end{cases} \tag{1.16}$$

L'adimensionnalisation de Φ^S, ψ et t provient de la théorie linéaire (voir paragraphe **1.3.2** dans [80]). Nous expliquons l'adimensionnalisation de Φ^B dans le chapitre 2. Dans la suite, nous omettons les primes pour simplifier les notations. Nous notons

$$\nabla^{\mu,\gamma}_{X,z} = (\sqrt{\mu}\nabla^{\gamma}_X\ , \partial_z\ )^t\ ,\ \Delta^{\mu,\gamma}_{X,z}\ := \mu\Delta^{\gamma}_X\ + \partial_z^2\ ,\ \nabla^{\gamma}_X = (\partial_x, \gamma\partial_y)^t \text{ et } \Delta^{\gamma}_X = \partial_x^2 + \gamma\partial_y^2.$$

Les équations des vagues s'adimensionnent alors de la façon suivante

$$\begin{cases} \partial_t\zeta - \dfrac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\psi) = \dfrac{\beta\lambda}{\varepsilon}G^{NN}_\mu[\varepsilon\zeta,\beta b](\partial_t b), \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\nabla^{\gamma}_X\psi|^2 - \dfrac{\varepsilon}{2\mu}\dfrac{\left(G_\mu[\varepsilon\zeta,\beta b](\psi) + \frac{\lambda\beta\mu}{\varepsilon}G^{NN}_\mu[\varepsilon\zeta,\beta b](\partial_t b) + \mu\varepsilon\nabla^{\gamma}_X\zeta\cdot\nabla^{\gamma}_X\psi\right)^2}{(1+\varepsilon^2\mu|\nabla^{\gamma}_X\zeta|^2)} = -P, \end{cases} \tag{1.17}$$

où l'opérateur de Dirichlet-Neumann $G_\mu[\varepsilon\zeta,\beta b]$ est

$$G_\mu[\varepsilon\zeta,\beta b](\psi) := \sqrt{1+\varepsilon^2|\nabla^{\gamma}_X\zeta|^2}\partial_{\mathbf{n}}\Phi^S{}_{|z=\varepsilon\zeta}\ , \tag{1.18}$$

avec Φ^S qui satisfait

$$\begin{cases} \Delta^{\mu,\gamma}_{X,z}\Phi^S = 0 \text{ dans } \Omega_t\ , \\ \Phi^S{}_{|z=\varepsilon\zeta} = \psi\ ,\ \partial_{\mathbf{n}}\Phi^S{}_{|z=-1+\beta b} = 0, \end{cases} \tag{1.19}$$

et l'opérateur de Neumann-Neumann $G^{NN}_\mu[\varepsilon\zeta,\beta b]$ est

$$G^{NN}_\mu[\varepsilon\zeta,\beta b](\partial_t b) := \sqrt{1+\varepsilon^2|\nabla^{\gamma}_X\zeta|^2}\partial_{\mathbf{n}}\Phi^B{}_{|z=\varepsilon\zeta}\ , \tag{1.20}$$

où Φ^B satisfait

$$\begin{cases} \Delta^{\mu,\gamma}_{X,z}\Phi^B = 0 \text{ dans } \Omega_t\ , \\ \Phi^B{}_{|z=\varepsilon\zeta} = 0\ ,\ \sqrt{1+\beta^2|\nabla^{\gamma}_X b|^2}\partial_{\mathbf{n}}\Phi^B{}_{|z=-1+\beta b} = \partial_t b. \end{cases} \tag{1.21}$$

Notons que $\partial_{\mathbf{n}}$ est ici la dérivée *conormale* pointant vers le haut

$$\partial_{\mathbf{n}}\Phi = \mathbf{n}\cdot\begin{pmatrix}\sqrt{\mu}I_d & 0\\ 0 & 1\end{pmatrix}\nabla^{\mu,\gamma}_{X,z}\Phi_{|\partial\Omega}.$$

Dans [9], Alvarez-Samaniego et Lannes montrent l'existence locale du système (1.17) dans le cas d'une pression constante à la surface et d'un fond fixe. Leur temps d'existence est de la forme $\frac{T}{\max(\varepsilon,\beta)}$ où T est indépendant de μ, ε et β. Dans [70] et [71], Iguchi obtient aussi un temps d'existence indépendant de μ dans le cas d'un fond mobile ou fixe mais avec un adimensionnement un peu différent. Mésognon-Gireau ([105]) généralise le résultat de [9] et montre que, si l'on rajoute de la tension de surface, on peut avoir un temps d'existence de la forme $\frac{T}{\varepsilon}$. Notons que le rajout de la tension de surface est essentiel dans son travail et qu'il considère des coefficients de tension de surface qui ne sont pas aberrants physiquement. Dans la section 2.2, nous obtenons un temps d'existence de la forme $\frac{T}{\sqrt{\max(\varepsilon,\beta)}}$ (Théorème 2.2.4) pour le système (1.17) avec pression non constante à la surface et fond mobile.

Le temps d'existence que nous avons obtenu peut sembler moins intéressant que celui d'Alvarez-Samaniego et Lannes [9]. Pourtant il s'avère optimal si l'on considère des solutions fortes et si le fond est plat ($\beta = 0$) car nous avons un terme source de taille 1 et non de taille $\max(\varepsilon, \beta)$. Pour expliquer comment nous avons pu obtenir un tel temps d'existence et pourquoi il est optimal, prenons un modèle plus simple. Nous considérons le problème suivant défini sur $\mathbb{R}$:

$$\partial_t u + \varepsilon u \partial_x u + Lu = f,$$

avec L un opérateur anti-symétrique qui commute avec ∂_x, f un terme source d'énergie finie et ε un petit paramètre. Nous cherchons des solutions fortes à ce problème. Nous allons donc effectuer des estimations d'énergie. Si nous considérons l'énergie $E = E(t) = |u(t,\cdot)|_{H^2}^2$, alors nous pouvons montrer que

$$\frac{d}{dt} E \leq \varepsilon C E^{\frac{3}{2}} + |f|\, E^{\frac{1}{2}}.$$

Si f est identiquement nulle, nous pouvons obtenir un temps d'existence de la forme $\frac{T}{\varepsilon}$ qui correspond au temps donné par Alvarez-Samaniego et Lannes. On appelle souvent ce temps d'existence, le temps hyperbolique. Si maintenant f n'est pas petit, une estimation brutale nous donne un temps d'existence de l'ordre de 1 qui n'est pas très intéressant si l'on souhaite étudier notre phénomène sur de longues durées. Pour améliorer ce temps d'existence, on considére une nouvelle énergie $F = F(\tau) = \varepsilon E(\frac{\tau}{\sqrt{\varepsilon}})$. Nous pouvons alors montrer que

$$\frac{d}{d\tau} F \leq C(f) \left(F^{\frac{3}{2}} + F^{\frac{1}{2}} \right).$$

Ainsi, il existe un temps $T > 0$ et une constante $C > 0$,

$$\sup_{t \in \left[0, \frac{T}{\sqrt{\varepsilon}}\right]} \varepsilon E(t) = \sup_{\tau \in [0,T]} F(\tau) \leq C,$$

et notre solution u existe sur un temps $\frac{T}{\sqrt{\varepsilon}}$. Il y a cependant une contrepartie : l'énergie n'est pas bornée uniformément par rapport à ε sur ce temps. Cela jouera un rôle dans la justification de modèles asymptotiques (voir Section 2.3). Nous allons maintenant montrer que ce temps d'existence est optimal. Supposons pour simplifier que L est identiquement nulle, que l'on part d'une donnée initiale nulle et que f ne dépend pas du temps. Nous cherchons une solution forte à l'équation de Burgers 1d avec terme source

$$\begin{cases} \partial_t u + \varepsilon u \partial_x u = f, \\ u_{|t=0} = 0. \end{cases}$$

Un moyen de résoudre de manière exacte cette équation est la méthode des caractéristiques. Nous notons $\Phi_0^t(x)$ la solution de l'équation différentielle ordinaire suivante

$$\begin{cases} \frac{d}{dt} \Phi_0^t(x) = \varepsilon u(t, \Phi_0^t(x)), \\ \Phi_0^0(x) = x. \end{cases}$$

Nous avons donc

$$u(t, \Phi_0^t(x)) = \int_0^t f(\Phi_0^s(x))ds.$$

Cela caractérise la solution u. Nous voyons aussi que $\Phi_0^t(x)$ vérifie l'équation différentielle ordinaire suivante

$$\begin{cases} \dfrac{d^2}{dt^2}\Phi_0^t(x) = \varepsilon f(\Phi_0^t(x)), \\ \Phi_0^0(x) = x, \\ \left(\dfrac{d}{dt}\Phi_t^0(x)\right)_{|t=0} = 0. \end{cases}$$

Il est alors facile de montrer que l'on peut trouver des fonctions f telles que la solution de cette équation différentielle ordinaire existe sur $\left[0, \frac{T}{\sqrt{\varepsilon}}\right]$ (T indépendant de ε) et non pas sur $\left[0, \frac{T}{\varepsilon^\alpha}\right]$ pour $\alpha > \frac{1}{2}$. Au delà du temps $\left[0, \frac{T}{\sqrt{\varepsilon}}\right]$, les courbes caractéristiques se croisent et une solution forte ne peut plus continuer à exister. Le temps d'existence $\frac{T}{\sqrt{\varepsilon}}$ est donc un temps optimal dans ce contexte. Ainsi, le temps d'existence que nous trouvons au théorème 2.2.4 est un temps hyperbolique dans le cadre d'un système hyperbolique avec terme source.

1.1.2 Formulation de Castro-Lannes, cas avec vorticité

1.1.2.1 Les équations

Dans cette partie, comme dans les chapitres 4 et 5, nous prenons en compte la vorticité. Nous supposons aussi que le fond est fixe. Nous devons résoudre le système (1.2) avec les conditions aux bords (1.3). Nos inconnues sont la vitesse $\mathbf{U}$ et la surface ζ. Comme dans le cas précédent, la vitesse est définie dans le domaine fluide (1.1) qui évolue au cours du temps. Divers résultats d'existence locale ont été montrés en utilisant une approche Lagrangienne ([39], [90], [42], [155], [156]). Une autre solution consiste à redresser le domaine grâce à un difféomorphisme (voir par exemple [96]). Nous avons vu dans le cas irrotationnel que nous pouvons réduire les équations d'Euler à surface libre à un système d'équations à la surface. À cause de la vorticité, ce ne sera pas possible ici. Cependant nous pouvons essayer de réduire au maximum le nombre d'inconnues définies dans le domaine fluide. En adoptant ce point de vue, Castro et Lannes ([34]) ont proposé une généralisation de la formulation de Zakharov/Craig-Sulem-Sulem pour des fluides avec vorticité. Nous suivons leur approche dans ce manuscrit. En prenant la trace à la surface de la première équation de (1.2) et en utilisant la première équation de (1.3), nous obtenons l'équation

$$\partial_t \underline{\mathrm{U}} + (\underline{\mathrm{V}} \cdot \nabla_X) \underline{\mathrm{U}} + f \begin{pmatrix} \underline{\mathrm{V}}^\perp \\ 0 \end{pmatrix} = - \begin{pmatrix} \nabla_X P \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ g \end{pmatrix} - (\partial_z \mathcal{P})_{|z=\zeta} N, \tag{1.22}$$

où si $\mathbf{V} = \begin{pmatrix} \mathrm{V}_1 \\ \mathrm{V}_2 \end{pmatrix} \in \mathbb{R}^2$, nous définissons $\mathbf{V}^\perp = \begin{pmatrix} -\mathrm{V}_2 \\ \mathrm{V}_1 \end{pmatrix}$. On supprime alors le terme de pression $\partial_z \mathcal{P}_{|z=\zeta} N$ en prenant le produit vectoriel de l'équation précédente avec N. Nous introduisons la notation suivante :

$$\mathrm{U}_{/\!/} = \underline{\mathrm{V}} + \underline{\mathrm{w}} \nabla_X \zeta.$$

Notons que

$$\underline{\mathrm{U}} \times N = \begin{pmatrix} -\mathrm{U}_{/\!/}^{\perp} \\ -\mathrm{U}_{/\!/}^{\perp} \cdot \nabla_X \zeta \end{pmatrix}.$$

La quantité $\mathrm{U}_{/\!/}$ définit donc entièrement $\underline{\mathrm{U}} \times N^{\mu}$ et réciproquement. Ainsi en prenant le produit vectoriel de (1.22) avec N nous obtenons

$$\partial_t \mathrm{U}_{/\!/} + g\nabla_X \zeta + \frac{1}{2}\nabla_X \left|\mathrm{U}_{/\!/}\right|^2 - \frac{1}{2}\nabla_X \left[\left(1 + |\nabla_X \zeta|^2\right) \underline{\mathrm{w}}^2\right] + \left(\nabla_X^{\perp} \cdot \mathrm{U}_{/\!/} + f\right) \underline{\mathrm{V}}^{\perp} = -\nabla_X P. \quad (1.23)$$

Remarquons que dans le cas irrotationnel $\mathrm{U}_{/\!/} = \nabla_X (\Phi_{|z=\zeta})$ si $\mathrm{U} = \nabla_{X,z}\Phi$. De plus, si nous notons la vorticité $\boldsymbol{\omega} = \mathrm{rot}_{X,z}\mathbf{U}$ et $\underline{\boldsymbol{\omega}} = \boldsymbol{\omega}_{|z=\zeta}$, nous avons la formule suivante

$$\nabla_X^{\perp} \cdot \mathrm{U}_{/\!/} = \underline{\boldsymbol{\omega}} \cdot N.$$

La quantité $\nabla_X^{\perp} \cdot \mathrm{U}_{/\!/}$ est donc entièrement déterminé par $\boldsymbol{\omega}$ et ζ. Fort de ce constat nous décomposons $\mathrm{U}_{/\!/}$ en deux parties :

$$\mathrm{U}_{/\!/} = \nabla_X \Delta_X^{-1} \nabla_X \cdot \mathrm{U}_{/\!/} + \nabla_X^{\perp} \Delta_X^{-1} \nabla_X^{\perp} \cdot \mathrm{U}_{/\!/}. \quad (1.24)$$

Ces opérateurs seront définis rigoureusement dans le chapitre 4. Nous posons

$$\psi := \Delta_X^{-1} \nabla_X \cdot \mathrm{U}_{/\!/}$$

et en appliquant l'opérateur $\Delta_X^{-1}\nabla_X \cdot$ à (1.23), nous avons

$$\partial_t \psi + \zeta + \frac{1}{2}\left|\mathrm{U}_{/\!/}\right|^2 - \frac{1}{2}\left(1 + |\nabla_X \zeta|^2\right)\underline{\mathrm{w}}^2 + \Delta_X^{-1}\nabla_X \cdot \left[\left(\underline{\boldsymbol{\omega}} \cdot N + f\right) \underline{\mathrm{V}}^{\perp}\right] = -P. \quad (1.25)$$

Nous obtenons donc le système suivant appelé équations des vagues ou formulation de Castro-Lannes. C'est un système de trois équations avec les trois inconnues $(\zeta, \psi, \boldsymbol{\omega})$,

$$\begin{cases} \partial_t \zeta - \underline{\mathrm{U}} \cdot N = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}\left|\mathrm{U}_{/\!/}\right|^2 - \frac{1}{2}\left(1 + |\nabla_X \zeta|^2\right) \underline{\mathrm{w}}^2 + \Delta_X^{-1}\nabla_X \cdot \left[\left(\underline{\boldsymbol{\omega}} \cdot N + f\right) \underline{\mathbf{V}}^{\perp}\right] = -P, \\ \partial_t \boldsymbol{\omega} + \left(\mathbf{U} \cdot \nabla_{X,z}\right) \boldsymbol{\omega} = \left(\boldsymbol{\omega} \cdot \nabla_{X,z}\right) \mathbf{U} + f\partial_z \mathbf{U} \text{ , dans } \Omega_t, \end{cases} \quad (1.26)$$

où $\mathbf{U} := \mathbf{U}[\zeta, b](\psi, \boldsymbol{\omega}) = (\mathbf{V}, \mathrm{w})^t$ est l'unique solution du problème divergence-rotationnel suivant

$$\begin{cases} \mathrm{curl}_{X,z}\, \mathbf{U} = \boldsymbol{\omega} \text{ dans } \Omega_t, \\ \mathrm{div}_{X,z}\, \mathbf{U} = 0 \text{ dans } \Omega_t, \\ \mathrm{U}_{/\!/} = \nabla_X \psi + \Delta_X^{-1} \nabla_X^{\perp} \left(\underline{\boldsymbol{\omega}} \cdot N\right), \\ \mathrm{U}_b \cdot N_b = 0. \end{cases} \quad (1.27)$$

Dans la section 4.2, nous étudions le problème divergence-rotationnel (1.27). Nous montrons alors qu'il est bien défini. Dans [34], Castro et Lannes prouvent l'existence locale du système (1.26) dans le cas d'un fond plat, d'une pression constante à la surface et sans la force de Coriolis. Dans la section 4.3, nous traitons le cas général.

1.1.2.2 Adimensionnement

Nous pouvons procéder à un adimensionnement similaire au cas irrotationnel. La pression $\mathcal{P}$ peut être décomposée en une contribution surfacique et une pression interne

$$\mathcal{P}(t,X,z) = P(t,X) + P_{\text{ref}} + \widetilde{\mathcal{P}}(t,X,z),$$

avec $\widetilde{\mathcal{P}}_{|z=\zeta} = 0$. Nous adimensionnons les variables de la manière suivante

$$\begin{cases} x' = \dfrac{x}{L_x},\ y' = \dfrac{y}{L_y},\ z' = \dfrac{z}{H},\ \zeta' = \dfrac{\zeta}{a},\ b' = \dfrac{b}{a_{\text{bott}}},\ t' = \dfrac{\sqrt{gH}}{L}t, \\ \mathbf{V}' = \sqrt{\dfrac{H}{g}}\dfrac{\mathbf{V}}{a},\ \mathrm{w}' = H\sqrt{\dfrac{H}{g}}\dfrac{\mathrm{w}}{aL_x},\ P' = \dfrac{P}{\rho g a} \text{ and } \widetilde{\mathcal{P}}' = \dfrac{\widetilde{\mathcal{P}}}{\rho g H} \end{cases} \tag{1.28}$$

et nous avons les paramètres sans dimension suivants

$$\varepsilon = \frac{a}{H},\ \beta = \frac{a_{\text{bott}}}{H},\ \mu = \frac{H^2}{L_x^2},\ \gamma = \frac{L_x}{L_y} \text{ et } \mathrm{Ro} = \frac{a}{fL_x}\sqrt{\frac{g}{H}}.$$

Notre adimensionnement est basé sur une étude linéaire de ces équations (voir par exemple [34] et [80]). Puisque nous nous intéressons à des effets surfaciques, nous avons utilisé la quantité a comme distance caractéristique dans l'adimensionnement des vitesses horizontale et verticale. De même, pour créer le nombre de Rossby, nous avons pris la vitesse horizontale surfacique caractéristique $a\sqrt{\frac{g}{H}}$ comme vitesse horizontale typique. Cet adimensionnement diffère de celui fréquemment utilisé pour étudier les fluides géophysiques pour lequel la vitesse horizontale typique considérée est $\sqrt{gH}$ (voir par exemple [117] ou [37] pour un tel adimensionnement). Il reste à adimensionner la vorticité $\boldsymbol{\omega} = \mathrm{curl}_{X,z}\mathbf{U}$. Nous allons nous placer dans le cas d'un écoulement faiblement cisaillé (voir par exemple [139], [121], [33]). Pour comprendre cette hypothèse, analysons de plus près les conséquences de notre adimensionnement quand nous supposons que μ est petit. En utilisant la condition d'incompressibilité, nous avons dans le domaine fluide

$$\mu\partial_{x'}\mathrm{V}'_x + \gamma\mu\partial_{y'}\mathrm{V}'_y + \partial_{z'}\mathrm{w}' = 0.$$

On introduit alors les notations suivantes

$$\nabla^{\mu,\gamma}_{X',z'} = \begin{pmatrix} \sqrt{\mu}\nabla^{\gamma}_{X'} \\ \partial_{z'} \end{pmatrix},\ \mathrm{curl}^{\mu,\gamma}_{X',z'} = \nabla^{\mu,\gamma}_{X',z'}\times,\ \mathrm{div}^{\mu,\gamma}_{X',z'} = \nabla^{\mu,\gamma}_{X',z'}\cdot \tag{1.29}$$

et

$$\begin{aligned} &\mathbf{U}^{\mu} = \begin{pmatrix} \sqrt{\mu}\mathbf{V}' \\ \mathrm{w}' \end{pmatrix},\ \underline{\mathrm{U}}^{\mu} = \mathbf{U}^{\mu}_{|z'=\varepsilon\zeta'},\ \mathrm{U}^{\mu}_b = \mathbf{U}^{\mu}_{|z'=-1+\beta b'}, \\ &N^{\mu,\gamma} = \begin{pmatrix} -\varepsilon\sqrt{\mu}\nabla^{\gamma}_{X'}\zeta \\ 1 \end{pmatrix},\ N^{\mu}_b = \begin{pmatrix} -\beta\sqrt{\mu}\nabla^{\gamma}_{X'}b \\ 1 \end{pmatrix}. \end{aligned} \tag{1.30}$$

Ainsi, on peut réécrire la condition d'incompressibilité en

$$\nabla^{\mu,\gamma}_{X',z'}\cdot\mathbf{U}^{\mu} = \mu\nabla^{\gamma}_{X'}\cdot\mathbf{V} + \partial_{z'}\mathrm{w} = 0.$$

De plus, la deuxième équation de (1.3) (le fond est fixe ici) nous donne que

$$-\beta\mu\mathbf{V}'_b\cdot\nabla^{\gamma}_{X'}b' + \mathrm{w}'_b = 0.$$

Nous voyons alors que la vitesse verticale w' est d'ordre $\mathcal{O}(\mu)$ si $\mathbf{V}'$ est d'ordre $\mathcal{O}_\mu(1)$. De plus, nous avons

$$\mathrm{curl}^{\mu,\gamma}_{X',z'}\mathbf{U}^\mu = \frac{H^2}{aL_x}\sqrt{\frac{H}{g}}\boldsymbol{\omega}.$$

Nous pouvons être tentés d'adimensionner $\boldsymbol{\omega}$ de la façon suivante

$$\boldsymbol{\omega}' = \frac{H^2}{aL_x}\sqrt{\frac{H}{g}}\boldsymbol{\omega}.$$

En prenant la partie horizontale de la vorticité, noté $\boldsymbol{\omega}_h'$, nous obtenons alors que

$$\boldsymbol{\omega}_h' = \mu\partial_{z'}\mathbf{V}^\perp - \sqrt{\mu}(\nabla^\gamma_{X'})^\perp \mathrm{w}.$$

Nous voyons que $\boldsymbol{\omega}_h'$ est d'ordre $\mathcal{O}(\mu)$ si $\partial_{z'}\mathbf{V}'$ est d'ordre $\mathcal{O}_\mu(1)$. L'hypothèse d'un écoulement faiblement cisaillé consiste précisement à supposer que $\partial_{z'}\mathbf{V}'$ est d'ordre $\mathcal{O}_\mu(1)$ (voir [139], [121]). La vorticité est alors de l'ordre de $\mathcal{O}(\mu)$. Remarquons que, dans le cas irrotationnel, cette hypothèse est automatiquement vérifiée. Nous adimensionnons alors $\boldsymbol{\omega}$ de la manière suivante

$$\boldsymbol{\omega}' = \frac{L_x}{a}\sqrt{\frac{H}{g}}\boldsymbol{\omega},$$

de sorte que

$$\mathrm{curl}^{\mu,\gamma}_{X',z'}\mathbf{U}^\mu = \mu\boldsymbol{\omega}'.$$

Nous omettons les primes dans la suite. Les équations des vagues deviennent

$$\begin{cases}\partial_t\zeta - \dfrac{1}{\mu}\underline{\mathrm{U}}^\mu\cdot N^{\mu,\gamma} = 0,\\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}\left|\mathrm{U}^\mu_{/\!/}\right|^2 - \dfrac{\varepsilon}{2\mu}\left(1+\varepsilon^2\mu\left|\nabla^\gamma_X\zeta\right|^2\right)\underline{\mathrm{w}}^2 + \varepsilon\Delta_X^{-1}\nabla_X\cdot\left[\left(\underline{\omega}\cdot N^{\mu,\gamma} + \dfrac{1}{\mathrm{Ro}}\right)\underline{\mathrm{V}}^\perp\right] = -P,\\ \partial_t\boldsymbol{\omega} + \dfrac{\varepsilon}{\mu}\left(\mathbf{U}^\mu\cdot\nabla^{\mu,\gamma}_{X,z}\right)\boldsymbol{\omega} = \dfrac{\varepsilon}{\mu}\left(\boldsymbol{\omega}\cdot\nabla^{\mu,\gamma}_{X,z}\right)\mathbf{U}^\mu + \dfrac{\varepsilon}{\mu\mathrm{Ro}}\partial_z\mathbf{U}^\mu, \text{ dans } \Omega_t,\end{cases} \tag{1.31}$$

où $\mathbf{U}^\mu := \mathbf{U}^\mu[\varepsilon\zeta,\beta b](\psi,\boldsymbol{\omega})$ est solution de

$$\begin{cases}\mathrm{curl}^{\mu,\gamma}_{X,z}\,\mathbf{U}^\mu = \mu\boldsymbol{\omega} \text{ dans } \Omega_t,\\ \mathrm{div}^{\mu,\gamma}_{X,z}\,\mathbf{U}^\mu = 0 \text{ dans } \Omega_t,\\ \mathrm{U}^\mu_{/\!/} = \nabla^\gamma\psi + \nabla^{\gamma\,\perp}_X\left(\Delta^\gamma_X\right)^{-1}\left(\underline{\omega}\cdot N^{\mu,\gamma}\right),\\ \mathrm{U}^\mu_b\cdot N^{\mu,\gamma}_b = 0.\end{cases} \tag{1.32}$$

Dans [34], Castro et Lannes montrent l'existence locale du système (1.31) dans le cas d'un fond plat, d'une pression constante à la surface et sans la force de Coriolis. Ils obtiennent un temps d'existence de la forme $\frac{T}{\varepsilon}$. Dans la section 4.3, nous traitons le cas d'un fond non plat et nous obtenons un temps d'existence de la forme $\frac{T}{\max(\varepsilon,\beta,\frac{\varepsilon}{\mathrm{Ro}})}$ dans le cas où P est constante (Théorème 4.3.6). Nous donnons aussi un temps d'existence dans le cas où P n'est pas constante.

1.2 Modèles asymptotiques pour les équations des vagues

Les systèmes (1.17) et (1.31) sont en général trop compliqués pour étudier la propagation de vagues. Nous allons donc les simplifier en supposant que certains des paramètres sans dimension ε, β, γ, μ et $\frac{\varepsilon}{\mathrm{Ro}}$ sont petits. On parle alors de *régime asymptotique*. Nous donnons dans la suite un état de l'art de différents modèles asymptotiques que nous étudions dans ce manuscrit. Notons que dans cette partie, sauf mention du contraire, la pression à la surface est supposée constante et le fond fixe.

1.2.1 Régime d'eaux peu profondes

1.2.1.1 Équations de Saint-Venant

Lorsque μ est petit, on parle de régime d'eaux *peu profondes*. Ce régime est bien connu des physiciens. Il revient à supposer que la longueur typique de notre phénomène L_x est très grande devant la hauteur d'eau typique H. On peut alors simplifier les équations des vagues en ne négligeant que les termes d'ordre $\mathcal{O}(\mu)$. On obtient les équations dites de Saint-Venant (ou "Nonlinear Shallow Water equations" en anglais). C'est un système d'équations sur la vitesse horizontale moyennée sur la profondeur, noté $\overline{\mathbf{V}}$, et la surface ζ. Dans le cas irrotationnel, Barré de Saint Venant dérive ces équations pour la première fois en 1871 ([13], [12]). Leur justification mathématique est obtenue un siècle plus tard par Ovsjannikov ([115], [116]) puis par Kano et Nishida ([74]). Notons que, dans ces travaux, les auteurs utilisent des données initiales analytiques. Alvarez-Samaniego et Lannes ([9]) et Iguchi ([70]) traitent des données initiales avec régularité Sobolev. Iguchi ([71]) considère aussi le cas d'un fond mobile. Enfin, Castro et Lannes ([34]) justifient ces équations pour un écoulement faiblement cisaillé dans le cas où le fond est plat. Dans la section 4.5, nous généralisons ce dernier résultat en ajoutant la force de Coriolis, un fond non plat et une pression non constante à la surface.

1.2.1.2 Équations de Green-Naghdi

En pratique, les équations de Saint-Venant ne sont pas adaptées à tous les types de phénomènes liés aux eaux peu profondes. Ce sont des équations non dispersives (au moins pour $d = 1$) contrairement aux équations des vagues. Pour gagner en précision, nous allons alors un cran plus loin dans le développement en ne négligeant que les termes d'ordre $\mathcal{O}(\mu^2)$ dans les équations des vagues. On obtient les équations de Green-Naghdi, dérivées pour la première fois par Green et Naghdi en 1976 dans le cas irrotationnel ([62]). On parle aussi des équations de Serre, dérivées vingt ans plus tôt par Serre ([131]) dans le cas où $d = 1$. La première justification des équations de Green-Naghdi est faite par Makarenko ([94]) pour des données analytiques, dans le cas où $d = 1$ et pour un fond plat et sur un temps T indépendant de μ. Li ([88]) généralise ce résultat à des données à régularité Sobolev et sur un temps T indépendant de μ. Enfin, Alvarez-Samaniego et Lannes ([9], [81]) traitent le cas $d = 1$ et 2 avec fond quelconque grâce à un schéma de Nash-Moser pour prouver l'existence locale des équations de Green-Naghdi et justifient les équations de Green-Naghdi sur un temps $\frac{T}{\mu}$. Notons aussi les travaux de Israwi ([72]) qui permettent de se passer du schéma de Nash-Moser lorsque $d = 1$. Dans [33], Castro et Lannes étendent les équations de Green-Naghdi pour $d = 1$ (vorticité scalaire) et $d = 2$ pour un écoulement faiblement cisaillé lorsque β de l'ordre de $\sqrt{\mu}$. Dans la section 5.4, nous généralisons cette dérivation en ajoutant la force de Coriolis et en nous plaçant dans le cas où γ est de l'ordre $\mathcal{O}(\mu^2)$ et β de l'ordre $\mathcal{O}(\mu)$. Précisons que les résultats de la section 5.4 et de [33] ne sont que des dérivations. À notre connaissance, il n'y a aucune justification complète des équations de Green-Naghdi ou de Serre dans le cas d'un fluide faiblement cisaillé. Il manque une preuve du caractère bien posé de ce système pour pouvoir totalement le justifier.

1.2.2 Régime faiblement non linéaire

Dans cette sous-section nous parlerons seulement du cas irrotationnel.

1.2.2.1 Équations linéaires des vagues

Lorsque ε est petit et μ de l'ordre de 1, on parle d'un régime *faiblement non linéaire* pour des eaux dites *profondes*. Si ε est très petit, un premier modèle asymptotique simple consiste à supposer que tous les termes d'ordre $\mathcal{O}(\varepsilon)$ sont négligeables. On obtient les équations linéaires des vagues. Ce modèle fait apparaître l'opérateur non local $G_\mu[0, \beta b]$. On peut alors se demander si des effets dispersifs sont possibles sur ce modèle. Dans la section 2.3.3, nous montrons que, pour un fond plat et lorsque $d = 1$, ce modèle est bien un système dispersif et nous donnons des estimations de décroissance de la norme L^∞ (Proposition 2.3.15). Notons que Mésognon-Gireau ([106]) généralise ce résultat en améliorant la régularité demandée pour la condition initiale et que Bulut ([31]) traite le cas d'une profondeur infinie.

1.2.2.2 Équations de Saut-Xu

En pratique, les équations linéaires des vagues sont trop simplistes dans de nombreuses situations. Si nous souhaitons être plus précis, nous allons un cran plus loin dans le développement en ne négligeant uniquement que les termes d'ordre $\mathcal{O}(\varepsilon^2)$ dans les équations des vagues. Le premier modèle asymptotique obtenu dans un tel régime a été découvert par Matsuno pour $d = 1$ et pour un fond plat ([97]) et un fond faiblement variable ([98]) puis dans le cas où $d = 2$ pour des vagues faiblement transverses (γ de l'ordre de ε) et un fond plat ([99]). Choi ([38]) étend le résultat de [99] à des vagues quelconques (voir aussi le papier de Smith [132]). Bonneton et Lannes ([22]) donnent une formulation des équations de Matsuno lorsque $d = 1$ et $d = 2$ dans le cas d'un fond non plat de faible amplitude (β d'ordre $\mathcal{O}(\varepsilon)$). Les inconnues de ces équations sont la vitesse horizontale à la surface et la surface ζ. Notons que tous les résultats précédents ne sont que des dérivations. À notre connaissance, nous se savons toujours pas montrer si les équations de Matsuno sont bien posées (voir l'article de Ambrose, Bona et Nicholls [10] à ce sujet). Pour pallier cette difficulté, Saut et Xu ([125]) développent un modèle équivalent aux équations de Matsuno (avec la même précision) dans le cas d'un fond plat et ils montrent l'existence locale de leur système. Ainsi, en utilisant les résulats d'Alvarez-Samaniego et Lannes ([9]), ils justifient mathématiquement leur système comme un modèle asymptotique des équations des vagues sur un temps $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$ avec une précision de l'ordre de $\mathcal{O}\left(\varepsilon\right)$. Dans le chapitre 3, nous généralisons leur résultat dans le cas où $d = 1$ en rajoutant un fond (Système (3.5)) et nous proposons un schéma numérique pour résoudre les équations de Saut-Xu (Section 3.3) . Notre schéma est basé sur un splitting entre les termes locaux et non locaux. Nous justifions la convergence de ce dernier (Théorème 3.4.6) et nous l'utilisons pour étudier le comportement d'un soliton de KdV lorsque le paramètre μ augmente (Sous-section 3.5.3) et l'effet d'homogénéisation d'un fond fortement oscillant sur la propagation des vagues (Sous-section 3.5.4).

1.2.3 Régime d'ondes longues

1.2.3.1 Équation des ondes

Lorsque ε et μ sont petits et du même ordre, on parle de régime *d'ondes longues*. Un premier modèle consiste à ne négliger que les termes d'ordre $\mathcal{O}(\max(\varepsilon, \mu))$. Dans le cas irrotationnel, on obtient une équation des ondes satisfaite par la surface ζ. Lagrange ([77]) est le premier à dériver cette équation dans le cas d'un fond plat. Dans le chapitre 2, nous proposons différents modèles pour ce régime lorsque $d = 1$ (avec en plus γ de l'ordre de $\mathcal{O}(\varepsilon)$). Dans le cas d'un écoulement faiblement cisaillé, si nous supposons que $\frac{\varepsilon}{\mathrm{Ro}}$ est de l'ordre de 1 et que β est de l'ordre de $\mathcal{O}(\varepsilon)$, nous obtenons un système d'équations sur la vitesse moyennée sur la profondeur $\overline{\mathbf{V}}$ et la surface ζ. Ces équations jouent un rôle important dans la littérature physique puisqu'elles permettent de modéliser les ondes de Poincaré, appelées aussi ondes de Sverdrup ([134]), et les ondes de Kelvin. Majda étudie ces ondes dans [93] (voir aussi [117], [61] et [84]). Dans les sections 4.5.4 (pour $d = 2$) et 5.3.2 (pour $d = 1$), nous montrons que ces équations sont dispersives et nous justifions mathématiquement les équations linéaires des vagues comme modèle asymptotique des équations des vagues.

1.2.3.2 Équation de Boussinesq

Les modèles donnés dans le paragraphe précédent sont des modèles linéaires donc souvent trop simplistes. Si nous allons un cran plus loin dans le développement en ne négligeant que les termes d'ordre $\mathcal{O}(\max(\varepsilon, \mu)^2)$ dans les équations des vagues, nous obtenons les équations dites de Boussinesq quand β est de l'ordre $\mathcal{O}(\mu)$ et de Boussinesq-Peregrine dans le cas où β est de l'ordre de 1. Dans le cas irrotationnel, Boussinesq les dérive pour la première fois ([25], [26]) dans le cas d'un fond plat et lorsque $d = 1$. Puis Peregrine ([119]) les généralise pour un fond quelconque. Notons aussi le livre de Whitham [149] qui propose un reformulation de ces équations qui est plus agréable pour prouver l'existence locale. Les premiers travaux pour justifier mathématiquement les équations de Boussinesq sont dus à Craig [43] dans le cas d'un fond plat, lorsque $d = 1$, pour de petites données initiales et sur un temps long ($\frac{T}{\mu}$ indépendant de μ) et à Nishida et Kano [75] pour des données initiales quelconques lorsque $d = 1$ et sur un temps court (T indépendant de μ). Alvarez-Samaniego et Lannes ([9]) généralisent ces résultats pour des données initiales quelconques, lorsque $d = 1$ et $d = 2$, et justifient que les équations de Boussinesq (pour β de l'ordre $\mathcal{O}(\mu)$) approchent les équations des vagues avec une précision en $\mathcal{O}(\mu)$ sur un temps $\frac{T}{\mu}$. Mésognon-Gireau ([104]) étudie le cas où β n'est pas de l'ordre de μ (fond avec une grande amplitude) et justifie sur un temps $\frac{T}{\mu}$ des équations de Boussinesq-Peregrine modifiées ayant la même précision que les équations de Boussinesq-Peregrine. Citons aussi les nombreux résultats sur les système Boussinesq (voir par exemple [112], [17], [18], [19], [127], [126]), qui sont des équations modifiées ayant la même précision que les équations de Boussinesq. Dans la section 5.2, nous généralisons et justifions les équations de Boussinesq à un écoulement faiblement cisaillé en présence de Coriolis et pour γ de l'ordre de $\mathcal{O}(\mu^2)$. Nous montrons l'existence de solutions pour les équations de Boussinesq sur un temps $\frac{T}{\max\left(\varepsilon, \beta, \frac{\varepsilon\sqrt{\mu}}{\mathrm{Ro}}\right)}$ (Théorème 5.2.15).

1.2.3.3 Équations de KdV et d'Ostrovsky

Le régime *d'ondes longues* (on parle aussi de régime Boussinesq) permet de dériver des équations scalaires. Nous supposons dans ce paragraphe que γ et β sont de l'ordre de $\mathcal{O}(\mu^2)$. Dans le cas irrotationnel, divers auteurs montrent que l'équation de Korteweig de Vries est un bon modèle asymptotique des équations des vagues ([43], [75], [129] et [19]). Nous justifions dans la sous-section 5.3.4, qu'en présence de la force de Coriolis, l'équation de Korteweig de Vries approche les équations des vagues sur un temps $\frac{T}{\mu}$ avec une précision de l'ordre de $\mathcal{O}(\mu)$, si $\frac{\varepsilon}{\mathrm{Ro}}$ est de l'ordre de $\mathcal{O}(\mu)$. Dans le cas où $\frac{\varepsilon}{\mathrm{Ro}}$ est de l'ordre de $\mathcal{O}(\sqrt{\mu})$, Germain et Renouard [60] montrent formellement que l'équation d'Ostrovsky

$$\partial_\xi \left(\partial_\tau f + \frac{3}{2} f \partial_\xi f + \frac{1}{6} \partial_\xi^3 f \right) = \frac{1}{2} f \tag{1.33}$$

est un bon modèle asymptotique. Cette équation, dérivée pour la première fois par Ostrovsky ([113]), généralise l'équation de Korteweig de Vries dans le cas d'un force de Coriolis relativement faible. Dans la sous-section 5.3.3, nous justifions mathématiquement l'équation d'Ostrovsky sur un temps $\frac{T}{\sqrt{\mu}}$ avec une précision de l'ordre de $\mathcal{O}(\mu)$ lorsque $\frac{\varepsilon}{\mathrm{Ro}}$ est de l'ordre de $\mathcal{O}(\sqrt{\mu})$. Ce résultat est le premier travail qui justifie mathématiquement léquation d'Ostrovsky. Notons cependant que nous ne justifions pas cette équation sur un temps $\frac{T}{\mu}$ (voir Section 1.5.3 et Chapitre 5).

1.3 Résonance de Proudman

Dans cette section nous nous intéressons à la résonance dite de Proudman mise en évidence par ce dernier en 1929 ([120]). Elle correspond à une élévation localisée du niveau des eaux due aux déplacements de perturbations atmosphériques. Ce manuscrit, et en particulier le chapitre 2, a pour but d'étudier en détail cette résonance grâce à des modèles asymptotiques des équations des vagues.

1.3.1 Mécanisme de la résonance de Proudman

Commençons par expliquer la résonance de Proudman dans un cas très simple. Considérons une dépression atmosphérique (une tempête, un grain orageux ou une bombe météorologique) ou un anti-cyclone se déplacant à une vitesse fixe U au dessus d'un océan initialement au repos. Nous supposons que la longueur de la perturbation météorologique est très grande devant la hauteur d'eau moyenne (μ petit) et que l'amplitude des vagues est très faible devant la hauteur d'eau moyenne (ε petit). Nous nous intéressons au cas où $d = 1$ (pas d'effets transverses). Un modèle simple pour étudier la propagation des vagues dans un tel régime est l'équation des ondes 1d satisfaite par la surface ζ

$$\partial_t^2 \zeta - \partial_x^2 \zeta = \partial_x^2 P(t,x),$$

où P est la pression à la surface modélisant notre perturbation atmosphérique. Comme nous nous intéressons à une perturbation se déplacant à une vitesse fixe U, nous pouvons écrire P de la forme $P(t,x) = P(x - Ut)$. Nous résolvons explicitement cette équation par la formulation de Duhamel et nous obtenons la solution particulière suivante

$$\zeta(t,x) = \begin{cases} \dfrac{1}{1-U^2} \left(P(x-t) - P(x-Ut) \right) \text{ si } U \neq 1, \\ -\dfrac{t}{2} P'(x-t) \text{ si } U = 1. \end{cases} \tag{1.34}$$

Nous voyons donc que si la vitesse de la tempête est proche de 1, une amplification importante est possible. Notons que la vitesse 1 provient de l'adimensionnement et correspond à la vitesse typique $\sqrt{gH}$.

Historiquement, Proudman ([120]) met en avant cette amplification pour montrer que les vents et les variations atmosphériques peuvent être à l'origine de vagues de grandes tailles malgré leur faible puissance. Rabinovich, Vilibić, Montserrat et coauteurs (voir par exemple [109], [144] ou [73]) reprennent les travaux de Proudman pour expliquer les météotsunamis (ou tsunamis météorologiques). Dans [109], les auteurs montrent que la résonance de Proudman est le principal mécanisme à l'origine des météotsunamis.

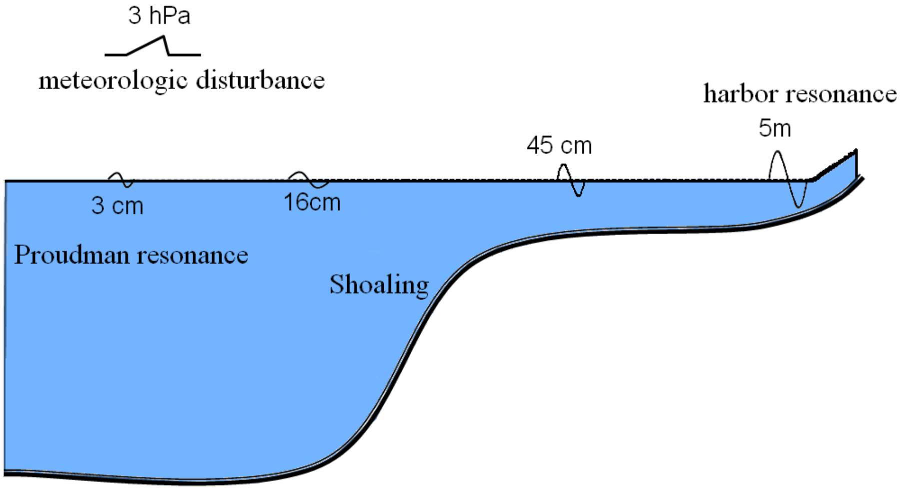

Figure 1.2: mécanismes d'un météotsunami

1.3.2 Tsunamis météorologiques

Les tsunamis météorologiques sont des phénomènes assez rares. Ils sont le résultat de trois mécanismes bien distincts. Ils se forment à cause d'une très large perturbation atmosphérique (de l'ordre de la dizaine voire de la centaine de kilomètres). Le premier mécanisme est la résonance de Proudman qui se déroule loin des côtes. À cause de cette résonance, des vagues de plus en plus hautes se forment. Elles se déplacent à la même vitesse que la perturbation atmosphérique. Quand ces vagues arrivent près des côtes, un deuxième mécanisme entre en scène, le *shoaling*. La longueur d'onde des vagues diminue et leur amplitude augmente du fait de la remontée de la bathymétrie. Cet effet est bien connu des océanographes et ne se limite pas qu'aux météotsunamis. Après le *shoaling*, les vagues générées sont souvent relativement peu destructrices. Un troisième effet peut cependant se produire. Si les vagues rentrent dans un port ou une baie (donc un domaine semi-clos), une résonance est possible : la résonance de Helmholtz (voir par exemple [108]). Cette résonance est basée sur les modes de vibrations propres du domaine semi-clos. Si la fréquence des vagues correspond à ces modes propres, leurs amplitudes sont amplifiées. Ainsi, un météotsunami est en général la combinaison de trois phénomènes : la résonance de Proudman, le *shoaling* et la résonance de Helmholtz, d'où sa rareté. La figure 1.2 résume ces trois mécanismes en donnant un ordre de grandeur typique de l'amplitude des vagues à chaque étape. Pour une étude plus détaillée sur les météotsunamis nous référons à [109] et [86].

Comme la résonance de Helmholtz est souvent nécessaire pour obtenir un météotsunami, les océanographes ont répertorié les lieux comportant un risque. Dans la suite, nous détaillons quelques tsunamis météorologiques qui se sont produits ces dernières années.

La baie de Nagasaki est un des lieux les plus connus pour ses seiches générées par des perturbations atmosphériques. Les habitants parlent de *abiki* pour qualifier ce phénomène. Le plus important *abiki* enregistré depuis 100 ans s'est produit en 1979. Des vagues atteignant 5 mètres de haut ont déferlé sur les côtes de Nagasaki. Aucune alerte tsunami n'a été déclenchée car l'origine de ce tsunami n'était pas sismique mais météorologique. Quelques heures plus tôt une immense tempête s'était formée en mer de Chine. Bien que nous n'ayons pas d'estimations précises, de nombreuses simulations numériques ont montré rétrospectivement que cette tempête faisait approximativement 200 kilomètres de long sur 300 kilomètres de large créant une variation de pression de 3 hectopascal. Elle s'est propagée pendant environ 3 heures jusqu'à atteindre la baie de Nagasaki et a multiplié l'amplitude des vagues créées par 5. Ce météotsunami sert aujourd'hui de référence. Il est étudié en détail dans [67] (voir aussi [109]).

Un autre lieu bien connu pour ses seiches destructrices est le port de Ciutadella (île Baléares, mer méditerranée). Les habitants parlent de *rissaga*. En 2006, une importante élévation du niveau des eaux s'est produite, provoquant la destruction d'une grande partie du port de Ciutadella. Des vagues de 5 mètres de haut ont été observées. Ce phénomène semble provenir de la propagation d'un énorme anti-cyclone. Une augmentation brusque de la pression atmosphérique de 10 hectopascal a été mesurée au moment du phénomène. Pour plus d'informations voir par exemple [73] (voir aussi [109]).

Un exemple intéressant est la série de deux météotsunamis en 1954 dans le lac Michigan. Des vagues de 3 mètres de haut ont été observées sur les berges du lac. Ce phénomène est étudié en détail dans [14]. Notons que ces deux météotsunamis ont pour particularité de n'être que d'origine météorologique (il n'y a pas eu de résonance de Helmholtz). Citons enfin le météotsunami de 1978 à Vela Luka (Croatie, mer Adriatique) qui est considéré comme le plus grand météotsunami d'Europe. Des vagues de 6 mètres de haut ont été observées dans la baie de Vela Luka. Voir [145] pour une étude détaillée.

1.3.3 Tsunamis par glissements de terrain

Dans ce manuscrit, et en particulier dans le chapitre 2, nous étudions un autre phénomène océanographique : les tsunamis par glissements de terrain sous-marins. Leurs formations sont assez similaires aux météotsunamis. Un glissement de terrain sous-marin peut amplifier la vague qu'il crée si sa vitesse est proche de $\sqrt{gH}$ où H est la profondeur d'eau typique. On observe une amplification similaire à la résonance de Proudman. Dans le chapitre 2, nous comparons les tsunamis météorologiques et les tsunamis par glissement de terrain. Nous donnons deux différences entre ces deux phénomènes : la durée et la taille. Alors qu'une perturbation atmosphérique peut se propager jusqu'aux côtes sur plusieurs heures sans trop se déformer, un glissement de terrain sous-marin se propage au maximum pendant une dizaine de minutes. De plus, un tsunami par glissement de terrain a tendance à avoir une amplitude typique en haute mer beaucoup plus importante qu'un météotsunami. Les tsunamis par glissement de terrain sont très similaires aux tsunamis classiques. Une perturbation assez rapide du fond crée une grande vague qui se propage ensuite jusqu'au côtes. Pour plus de détail sur les tsunamis par glissements de terrain voir par exemple [86].

La taille du glissement de terrain est un paramètre important pour créer un tsunami. Les tsunamis par glissement de terrain sous-marin sont donc assez rares. Ils proviennent en général de l'accumulation de sédiments sur des centaines voire des milliers d'années. Diverses études montrent par exemple que des glissements de terrain ont pu avoir eu lieu en mer du Nord au large de la Norvège il y a plus de 8000 ans (voir par exemple [30] pour une étude détaillée). Ils sont communément appelés glissements de terrain de Storegga dans la littérature physique. Citons aussi la zone de fracture d'Owen (frontière de plaque entre l'Arabie et l'Inde) qui est considéré comme un lieu propice à de potentiels glissements de terrain destructeurs (voir par exemple [122]).

Les glissements de terrain sous-marins sont parfois un apport supplémentaire d'énergie et amplifient une vague déjà existante comme par exemple lors du tsunami en Papouasie-Nouvelle-Guinée de 1998. Un tremblement de terre de magnitude 7.1 en plus de générer un tsunami, déstabilisa les fonds marins des environs et créa une série de glissements de terrain sous-marins. Ces glissements de terrain amplifièrent alors les vagues créées par le tremblement de terre (voir par exemple [137] pour une étude détaillée). Le tsunami de 2011 dans la région de Tohoku au Japon semble aussi avoir été amplifié par un glissement de terrain sous-marin (voir [136]), expliquant ainsi la catastrophe de Fukushima. L'amplification due au glissement de terrain aurait fourni suffisamment d'énergie aux vagues pour qu'elles puissent passer par dessus la digue qui protégeait la centrale nucléaire.

1.3.4 Résultats obtenus

Toutes les études menées sur les météotsunamis et les tsunamis par glissement de terrain sous-marin reposent sur l'hypothèse que la longueur d'onde de la perturbation est très grande devant la hauteur d'eau (μ petit) et utilisent les équations de Saint-Venant comme modèle mathématique. Dans la sous-section 2.3.3, nous proposons une étude de la résonance de Proudman sans cette hypothèse. Nous montrons qu'une résonance est encore possible dans des eaux profondes (μ de l'ordre de 1) mais avec un facteur d'amplification moins important que dans le cas d'eaux peu profondes. Supposer que μ n'est pas petit ne nous permet plus de négliger les termes dispersifs des équations des vagues. Ces derniers perturbent la résonance et tendent à l'atténuer. Ce phénomène de résonance en eau profonde n'avait pas été mise en évidence auparavant. Dans la sous-section 2.3.2.2, nous proposons une étude de la résonance de Proudman quand le fond n'est plus supposé relativement plat (β petit). Nous montrons que, pour des fonds tendant suffisament rapidement vers un fond plat, une résonance est encore possible. De plus, même si le travail originel de Proudman [120] prend en compte l'effet Coriolis, il est souvent admis par la communauté physique que ces effets sont négligeables (voir par exemple [144]). Dans les sous-sections 4.5.4 et 5.3.2, nous étudions la résonance de Proudman quand l'effet Coriolis n'est pas négligeable. Dans la sous-section 4.5.4, nous traitons le cas $d = 2$ et nous montrons que la résonance de Proudman n'est pas possible à cause des effets dispersifs dus à la force de Coriolis. Dans la sous-section 5.3.2, nous étudions le cas $d = 1$. Nous montrons qu'une résonance est possible pour un profil de pression particulier mais par pour des tempêtes se déplaçant à une vitesse fixe.

1.4 La résonance de Proudman au delà du cadre classique, effets nonlinéaires et dispersion non locale

L'étude de la résonance de Proudman que nous proposons dans ce manuscrit est essentiellement linéaire. Pourtant, des effets non linéaires peuvent se produire lors de la propagation de vagues, en particulier à l'approche des côtes. On peut alors distinguer deux situations. Soit les vagues que nous étudions sont très longues et nous sommes dans un régime d'eaux peu profondes, soit la longueur typique de nos vagues est comparable à la profondeur d'eau et nous sommes dans un régime d'eaux profondes. Lorsque que nous travaillons dans un régime d'eaux peu profondes, nous avons de nombreux modèles asymptotiques à notre disposition (voir Section 1.2). Pour étudier les effets non linéaires sur la résonance de Proudman, les équations de Saint-Venant fournissent une bonne approximation. Pour obtenir des informations quantitatives sur ces équations, nous allons alors adopter un point de vue numérique. Vilibic ([144]) propose une étude numérique de la résonance de Proudman grâce à ces équations. Il montre alors que les effets non linéaires ne perturbent pas la résonance (voir en particulier la figure 5 dans [144]). Nous donnons un aperçu des travaux de [144] dans la section 2.3.4.1 en utilisant un schéma numérique basé sur les travaux de Bouchut ([23]).

Lorsque nous sommes dans un régime d'eaux profondes, nous avons vu à la section 1.2 que les équations de Saut-Xu peuvent être utilisées pour prendre en compte les effets non linéaires et la forte dispersion (de caractère non local) propre à ce régime. Ainsi, afin d'obtenir des résultats qualitatifs, nous avons développé un schéma numérique pour ces équations. Il est étudié au chapitre 3. Notre schéma est basé sur un splitting entre les termes locaux et non locaux (voir Section 3.3) et nous justifions la convergence de ce dernier (Théorème 3.4.6). Nous l'utilisons alors pour étudier la résonance de Proudman dans un régime d'eaux profondes (voir Section 2.3.4.2). Il s'agit d'un régime pertinent physiquement (surtout pour les tsunamis générés par glissement de terrain). Nous montrons que, malgré les effets non linéaires, une amplification est encore possible. Ce phénomène de résonance en eau profonde n'avait pas été mise en évidence auparavant.

1.5 Perspectives de recherches

Dans cette partie nous proposons différentes perspectives de recherches qui pourraient prolonger ce manuscrit.

1.5.1 Vers un temps d'existence plus long dans le cas irrotationnel

Dans [105], Mésognon-Gireau donne un temps d'existence pour les équations des vagues (1.17) (cas irrotationnel) de la forme $\frac{T}{\varepsilon}$ dans le cas d'un fond fixe de grande amplitude (différent de [9]), d'une pression constante à la surface et avec tension de surface. La méthode utilisée pour obtenir ce résultat est inspirée de Bresch-Métivier [28]. Dans la section 2.2, nous donnons un temps d'existence pour (1.17) de la forme $\frac{T}{\sqrt{\max(\varepsilon,\beta)}}$ (Théorème 2.2.4). Ainsi, en utilisant les travaux de Mésognon-Gireau, nous pouvons espérer améliorer notre temps d'existence en $\frac{T}{\sqrt{\varepsilon}}$. Cette conjecture est motivée par le fait suivant : une solution $(\zeta, \overline{V})$ assez régulière des équations de Saint-Venant

$$\begin{cases} \partial_t \zeta + \nabla_X \left([1 + \varepsilon\zeta - \beta b] \overline{V} \right) = \partial_t b, \\ \partial_t \overline{V} + \varepsilon \left(\overline{V} \cdot \nabla_X \right) \overline{V} + \nabla\zeta = -\nabla_X P, \end{cases}$$

existe sur un intervalle de temps de taille $\frac{1}{\sqrt{\varepsilon}}$. La preuve de ce fait repose sur des estimations d'énergies des dérivées temporelles de nos inconnues. Nous commençons par changer l'échelle de la variable temporelle. Nous posons $\tau = \frac{t}{\sqrt{\varepsilon}}$ et nous obtenons

$$\begin{cases} \partial_\tau \zeta + \dfrac{1}{\sqrt{\varepsilon}} \nabla_X \left([1 + \varepsilon\zeta - \beta b] \overline{V} \right) = \partial_\tau b(\sqrt{\varepsilon}\tau, \cdot), \\ \partial_\tau \overline{V} + \sqrt{\varepsilon} \left(\overline{V} \cdot \nabla_X \right) \overline{V} + \dfrac{1}{\sqrt{\varepsilon}} \nabla\zeta = -\dfrac{1}{\sqrt{\varepsilon}} \nabla_X P(\sqrt{\varepsilon}\tau, \cdot). \end{cases}$$

Nous remarquons alors que l'on peut appliquer la méthode fournie par Bresch et Métivier [28]. Nous définissons l'énergie

$$\mathcal{E}^N(\zeta, \overline{V}) = \sum_{|(\alpha,k)| \leq N} \left| (\sqrt{\varepsilon})^k \partial_t^k \partial^\alpha \zeta \right|_{L^2}^2 + \left([1 + \varepsilon\zeta - \beta b] (\sqrt{\varepsilon})^k \partial_t^k \partial^\alpha \overline{V}, (\sqrt{\varepsilon})^k \partial_t^k \partial^\alpha \overline{V} \right)_{L^2},$$

et nous montrons que

$$\frac{d}{dt} \mathcal{E}^N(\zeta, \overline{V}) \leq C(P, b) \left(\sqrt{\varepsilon} \mathcal{E}^N(\zeta, \overline{V})^{\frac{3}{2}} + \mathcal{E}^N(\zeta, \overline{V}) + \frac{1}{\varepsilon} \sqrt{\mathcal{E}^N(\zeta, \overline{V})} \right).$$

Ainsi en utilisant la stratégie énoncée à la fin de la sous-section 1.1.1.2, nous obtenons que $(\zeta, \overline{V})$ existe sur un intervalle de temps de taille $\frac{1}{\sqrt{\varepsilon}}$. Nous espérons reproduire cette méthode sur le système complet (1.17) dans le futur. Rappelons cependant qu'en contrepartie de ce temps d'existence, l'énergie $\mathcal{E}^N(\zeta, \overline{V})$ n'est plus bornée uniformément par rapport au paramètre ε.

1.5.2 Dispersion pour les équations linéaires des vagues

Dans la section 2.3.3, Nous montrons que les équations linéaires des vagues sont des équations dispersives dans le cas où $d = 1$ et nous donnons des estimations de décroissances L^∞ (Proposition 2.3.15). En collaboration avec Maxime Gazeau (university of Toronto), nous voudrions étendre ce résultat à des fonds periodiques. Toute l'étude de la section 2.3.3 repose sur des méthodes de phases stationnaires et la transformée de Fourier. Dans le cas périodique, la transformée de Bloch généralise la transformée de Fourier et toutes les méthodes de phases stationnaires sont encore applicables. Nous voulons nous inspirer des méthodes de Cuccagna sur l'équation de Schrödinger à potentiel périodique ([49, 50]) pour obtenir un estimation de dispersion de type L^∞ dans ce contexte.

1.5.3 Vers un temps d'existence plus long dans le cas rotationnel

Dans la section 4.3, nous donnons un temps d'existence de la forme $\frac{T}{\max(\varepsilon, \beta, \frac{\varepsilon}{\mathrm{Ro}})}$ pour le système (1.31) dans le cas où P est constante (Théorème 4.3.6). Dans la section 5.2, nous étudions les équations de Boussinesq avec vorticité et nous montrons l'existence de solutions sur un temps $\frac{T}{\max(\varepsilon, \beta, \frac{\varepsilon\sqrt{\mu}}{\mathrm{Ro}})}$ (Proposition 5.2.15). Nous espérons améliorer le théorème 4.3.6 et obtenir une temps d'existence similaire aux équations de Boussinesq. Cela nous permettrait de justifier l'équation d'Ostrovsky sur un temps plus long $\left(\text{en } \frac{T}{\mu} \text{ au lieu de } \frac{T}{\sqrt{\mu}}, \text{ voir Théorème 5.3.10}\right)$.

Mais le temps d'existence que nous donnons dépend du paramètre de Coriolis. Dans l'optique de justifier des modèles asymptotiques sur des temps plus longs, il pourrait être intéressant de trouver un temps d'existence pour le système (1.31) qui soit indépendant du paramètre de Coriolis. En collaboration avec Stefano Scrobogna (IMB), nous voulons nous attaquer à cette problématique. Pour commencer, nous souhaitons obtenir un temps d'existence indépendant du paramètre de Coriolis sur les équations de Navier-Stokes à surface libre sans fond (profondeur infinie) et avec une viscosité ν non nulle fixe,

$$\begin{cases} \partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_{X,z}) \mathbf{U} + \mathbf{f} \times \mathbf{U} - \nu \Delta_{X,z} \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} \mathcal{P} - g \mathbf{e_z} \text{ dans } \{z < \zeta(t, X)\}, \\ \mathrm{div}_{X,z} \mathbf{U} = 0 \text{ dans } \{z < \zeta(t, X)\}, \end{cases} \tag{1.35}$$

auxquelles nous rajoutons les conditions aux bords suivantes

$$\begin{cases} \partial_t \zeta - \underline{\mathrm{U}} \cdot N = 0, \text{ on } z = \zeta, \\ \mathcal{P} N - \nu \left(\nabla \mathbf{U} + \nabla \mathbf{U}^t\right) N = g \zeta N, \text{ on } z = \zeta. \end{cases} \tag{1.36}$$

Un premier travail a été fait par Thai ([140]) pour des données initiales petites. Nous voulons traiter le cas de données initiales quelconques.

Ce travail nous permettrait aussi d'étudier les fluides géophysiques dans le cas d'une surface libre et de justifier le modèle quasi-géostrophique dans ce contexte (voir par exemple [37] ou [59] dans le cas d'une surface plate fixe).

Chapter 2

A mathematical study of meteorological and landslide tsunamis : The Proudman resonance

Sommaire

Ce chapitre est basé sur l'article [102]. Nous avons ajouté la sous-section 2.2.5 où nous expliquons le lien entre la pression et le coefficient de Rayleigh-Taylor. Nous avons aussi ajouté la sous-section 2.3.4 où nous étudions la résonance de Proudman dans le cas non linéaire. Nous avons amélioré le résultat de la sous-section 2.2.6 en prenant ψ dans un espace de type Beppo Levi (et non Sobolev). Enfin, nous avons étoffé la sous-section 2.3.2.2 et nous avons prouvé qu'une résonance peut se produire dans le cas d'un fond non plat pour des glissements de terrain qui se déplacent à vitesse 1.

2.1 Introduction

2.1.1 Presentation of the problem

In this chapter, we want to understand the Proudman resonance. It is a resonant respond in shallow waters of a water body on a traveling atmospheric disturbance when the speed of the disturbance is close to the typical water wave velocity. We show here that the same kind of resonance exists for landslide tsunamis and we propose a mathematical approach to investigate these phenomena based on the derivation, justification and analysis of relevant asymptotic models. This approach allows us to investigate more complex phenomena that are not dealt with in the physics literature such as the influence of a variable bottom or the generalization of the Proudman resonance in deeper waters.

A tsunami is popularly an elevation of the sea level due to an earthquake. However, tsunamis induced by seismic sources represent only 80 % of the tsunamis. 6% are due to landslides and 3% to meteorological effects ([86]). Big traveling storms for instance can give energy to the sea and lead to an elevation of the surface. In some cases, this amplification is important and this phenomenon is called the Proudman resonance ([120]) in the physics literature. Similarly, submarine landslides can significantly increase the level of the sea and we talk about landslide tsunamis. In this chapter, we study mathematically these two phenomena.

We model the sea by an irrotational and incompressible ideal fluid bounded from below by the seabed and from above by a free surface. We suppose that the seabed and the surface are graphs above the still water level. We model an underwater landslide by a moving seabed (moving bottom) and the meteorological effects by a non constant pressure at the surface (air-pressure disturbance). Therefore, we suppose that $b(t,X) = b_0(X) + b_m(t,X)$, where b_0 represents a fixed bottom and b_m the variation of the bottom because of the landslide. Similarly, the pressure at the surface is of the form $P + P_{\text{ref}}$, where P_{ref} is a constant which represents the pressure far from the meteorological disturbance, and $P(t,X)$ models the meteorological disturbance (we assume that the pressure at the surface is known). We denote by d the horizontal dimension, which is equal to 1 or 2. $X \in \mathbb{R}^d$ stands for the horizontal variable and $z \in \mathbb{R}$ is the vertical variable. H is the typical water depth. The water occupies a moving domain $\Omega_t := \{(X,z) \in \mathbb{R}^{d+1} \ , \ -H + b(t,X) < z < \zeta(t,X)\}$. The water is homogeneous (constant density ρ), inviscid, irrotational with no surface tension. We denote by $\mathbf{U}$ the velocity and Φ the velocity potential. We have $\mathbf{U} = \nabla_{X,z}\Phi$. The law governing the irrotational fluids is the Bernoulli law

$$\partial_t \Phi + \frac{1}{2}|\nabla_{X,z}\Phi|^2 + gz = \frac{1}{\rho}\left(P_{\text{ref}} - \mathcal{P}\right) \text{ in } \Omega_t, \tag{2.1}$$

where $\mathcal{P}$ is the pressure in the fluid domain. Changing Φ if necessary, it is possible to assume that $P_{\text{ref}} = 0$. Furthermore, the incompressibility of the fluid implies that

$$\Delta_{X,z}\Phi = 0 \text{ in } \Omega_t. \tag{2.2}$$

We suppose also that the fluid particles do not cross the bottom or the surface. We denote by $\mathbf{n}$ the unit normal vector, pointing upward and $\partial_{\mathbf{n}}$ the upward normal derivative. Then, the boundary conditions are

$$\partial_t \zeta - \sqrt{1+|\nabla\zeta|^2}\partial_{\mathbf{n}}\Phi = 0 \text{ on } \{z = \zeta(t,X)\}, \tag{2.3}$$

and

$$\partial_t b - \sqrt{1+|\nabla b|^2}\partial_{\mathbf{n}}\Phi = 0 \text{ on } \{z = -H + b(t,X)\}. \tag{2.4}$$

In 1968, Zakharov (see [153]) showed that the water waves problem is a Hamiltonian system and that ψ, the trace of the velocity potential at the surface ($\psi = \Phi_{|z=\zeta}$), and the surface ζ are canonical variables. Then, Craig, Sulem and Sulem (see [47] and [48]) formulate this remark into a system of two non local equations. We follow their construction to formulate our problem. Using the fact that Φ satisfies (2.2) and (2.4), we can characterize Φ thanks to ζ and $\psi = \Phi_{|z=\zeta}$

$$\begin{cases} \Delta_{X,z}\Phi = 0 \text{ in } \Omega_t, \\ \Phi_{|z=\zeta} = \psi \ , \ \sqrt{1+|\nabla b|^2}\partial_{\mathbf{n}}\Phi_{|z=-H+b} = \partial_t b. \end{cases} \tag{2.5}$$

We decompose this equation in two parts, the surface contribution and the bottom contribution

$$\Phi = \Phi^S + \Phi^B,$$

such that

$$\begin{cases} \Delta_{X,z}\Phi^S = 0 \text{ in } \Omega_t, \\ {\Phi^S}_{|z=\zeta} = \psi \ , \ \sqrt{1+|\nabla b|^2}\partial_{\mathbf{n}}{\Phi^S}_{|z=-H+b} = 0, \end{cases} \tag{2.6}$$

and

$$\begin{cases} \Delta_{X,z}\Phi^B = 0 \text{ in } \Omega_t, \\ {\Phi^B}_{|z=\zeta} = 0 \ , \ \sqrt{1+|\nabla b|^2}\partial_{\mathbf{n}}{\Phi^B}_{|z=-H+b} = \partial_t b. \end{cases} \tag{2.7}$$

In the purpose of expressing (2.3) with ζ and ψ, we introduce two operators. The first one is the Dirichlet-Neumann operator

$$G[\zeta,b] : \psi \mapsto \sqrt{1+|\nabla\zeta|^2}\partial_{\mathbf{n}}{\Phi^S}_{|z=\zeta}, \tag{2.8}$$

where Φ^S satisfies (2.6). The second one is the Neumann-Neumann operator

$$G^{NN}[\zeta,b] : \partial_t b \mapsto \sqrt{1+|\nabla\zeta|^2}\partial_{\mathbf{n}}{\Phi^B}_{|z=\zeta}, \tag{2.9}$$

where Φ^B satisfies (2.7). Then, we can reformulate (2.3) as

$$\partial_t\zeta - G[\zeta,b](\psi) = G^{NN}[\zeta,b](\partial_t b). \tag{2.10}$$

Furthermore thanks to the chain rule, we can express $(\partial_t\Phi)_{|z=\zeta}$, $(\nabla_{X,z}\Phi)_{|z=\zeta}$ and $(\partial_z\Phi)_{|z=\zeta}$ in terms of ψ, ζ, $G[\zeta,b](\psi)$ and $G^{NN}[\zeta,b](\partial_t b)$. Then, we take the trace at the surface of (2.1) (since there is no surface tension we have $\mathcal{P}_{|z=\zeta} = P$) and we obtain a system of two scalar equations that reduces to the standard Zakharov/Craig-Sulem formulation when $\partial_t b = 0$ and $P = 0$,

$$\begin{cases} \partial_t \zeta - G[\zeta, b](\psi) = G^{NN}[\zeta, b](\partial_t b), \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{\left(G[\zeta, b](\psi) + G^{NN}[\zeta, b](\partial_t b) + \nabla\zeta\cdot\nabla\psi\right)^2}{(1+|\nabla\zeta|^2)} = -\frac{P}{\rho}. \end{cases} \tag{2.11}$$

In the following, we work with a nondimensionalized version of the water waves equations with small parameters ε, β and μ (see section 2.2.1). The wellposedness of the water waves problem with a constant pressure and a fixed bottom was studied by many people. Wu ([150] and [151]) proved it in the case of an infinite depth without nondimensionalization. Then, Lannes ([78]) treated the case of a finite bottom without nondimensionalization, Iguchi ([70]) proved a local wellposedness result for μ small enough in order to justify shallow water approximations for water waves, and Alvarez-Samaniego and Lannes ([9]) showed, in the case of the nondimensionalized equations, that we can find an existence time $T = \frac{T_0}{\max(\varepsilon,\beta)}$ where T_0 does not depend on ε, β and μ. More recently, Mésognon-Gireau ([105]) improved the result of Lannes and Alvarez-Samaniego and proved that if we add enough surface tension we can find an existence time $T = \frac{T_0}{\varepsilon}$ where T_0 does not depend on ε and μ. Iguchi ([71]) studied the case of a moving bottom in order to justify asymptotic models for tsunamis. Finally, Alazard, Burq and Zuily study the optimal regularity for the initial data ([4]) and more recently, Alazard,Baldi and Han-Kwan ([2]) show that a well-chosen non constant external pressure can create any small amplitude two-dimensional gravity-capillary water waves (see also [1]).

We organize this chapter in two sections. Firstly in Section 2.2, we prove two local existence theorems for the water waves problem with a moving bottom and a non constant pressure at the surface by differentiating and "quasilinearizing" the water waves equations and we pay attention to the dependence of the time of existence and the size of the solution with respect to the parameters ε, β, λ and μ. This theorem extends the result of Iguchi ([71]) and Lannes (Chapter 4 in [80]). We also prove that the water waves problem can be viewed as a Hamiltonian system. All this part use results about elliptic problems, the Dirichlet-Neumann and the Neumann-Neumann operators that can be found in Appendix A. Secondly in Section 2.3, we justify some linear asymptotic models and study the Proudman resonance. First, in Section 2.3.1 we study the case of small topography variations in shallow waters, approximation used in the Physics literature to investigate the Proudman resonance; then in Section 2.3.2 we derive a model when the topography is not small in the shallow water approximation; and in Section 2.3.3 we study the linear water waves equations in order to extend the Proudman resonance in deep water with a small fixed topography.

2.1.2 Notations for this chapter

A good framework for the velocity in the Euler equations is the Sobolev spaces H^s. But we do not work with $\mathbf{U}$ but with ψ, the trace of Φ, and $\mathbf{U} = \nabla_{X,z}\Phi$. It will be too restrictive to take ψ in a Sobolev space. A good idea is to work with the Beppo Levi spaces (see [53]). For $s \geq 0$, the Beppo Levi spaces are

$$\dot{H}^s(\mathbb{R}^d) := \left\{\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^d),\, \nabla\psi \in H^{s-1}(\mathbb{R}^d)\right\}.$$

In this chapter, C is a constant and for a function f in a normed space $(X, |\cdot|)$ or a parameter γ, $C(|f|, \gamma)$ is a constant depending on $|f|$ and γ whose exact value has non importance. The norm $|\cdot|_2$ is the L^2-norm and $|\cdot|_\infty$ is the L^∞-norm in $\mathbb{R}^d$. Let $f \in \mathcal{C}^0(\mathbb{R}^d)$ and $m \in \mathbb{N}$ such that $\frac{f}{1+|x|^m} \in L^\infty(\mathbb{R}^d)$. We define the Fourier multiplier $f(D) : H^m(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ as

$$\forall u \in H^m(\mathbb{R}^d),\ \widehat{f(D)u}(\xi) = f(\xi)\widehat{u}(\xi).$$

In $\mathbb{R}^d$ we denote the gradient operator by ∇ and in Ω or $S = \mathbb{R}^d \times (-1, 0)$ the gradient operator is denoted $\nabla_{X,z}$. Finally, we denote by $\Lambda := \sqrt{1 + |D|^2}$ with $D = -i\nabla$.

In this chapter, $(,)$ is the standard $L^2(\mathbb{R}^d)$ scalar product.

2.2 Local existence of the water waves equations

This part is devoted to the wellposedness of the water waves equations (Theorems 2.2.3 and 2.2.4). We carefully study the dependence on the parameters ε, β, λ and μ on the existence time and on the size of the solution. Contrary to [80] and [71], we exhibit the nonlinearities of the water waves equations in order to obtain a better existence time in the presence of a source term.

2.2.1 The model

In this part, we present a nondimensionalized version of the water waves equations. In order to derive some asymptotic models to the water waves equations we introduce some dimensionless parameters linked to the physical scales of the system. The first one is the ratio between the typical free surface amplitude a and the water depth H. We define $\varepsilon := \frac{a}{H}$, called the nonlinearity parameter. The second one is the ratio between H and the characteristic horizontal scale L. We define $\mu := \frac{H^2}{L^2}$, called the shallowness parameter. The third one is the ratio between the order of bottom bathymetry amplitude a_{bott} and H. We define $\beta := \frac{a_{\text{bott}}}{H}$, called the bathymetric parameter. Finally, we denote by λ the ratio of the typical landslide amplitude $a_{\text{bott,m}}$ and a_{bott}. We also nondimensionalize the variables and the unknowns. We introduce (see also Figure 2.1)

$$\begin{cases} X' = \dfrac{X}{L},\ z' = \dfrac{z}{H},\ \zeta' = \dfrac{\zeta}{a},\ b' = \dfrac{b}{a_{\text{bott}}},\ b_0' = \dfrac{b_0}{a_{\text{bott}}},\ b_m' = \dfrac{b_m}{a_{\text{bott,m}}},\ t' = \dfrac{\sqrt{gH}}{L}t, \\ \left(\Phi^S\right)' = \dfrac{H}{aL\sqrt{gH}}\Phi^S,\ \left(\Phi^B\right)' = \dfrac{L}{Ha_{\text{bott,m}}\sqrt{gH}}\Phi^B,\ \psi' = \dfrac{H}{aL\sqrt{gH}}\psi,\ P' = \dfrac{P}{a\rho g}. \end{cases} \tag{2.12}$$

Then,

$$\Omega_t' = \{(X', z') \in \mathbb{R}^{d+1}\ ,\ -1 + \beta b'(t', X') < z' < \varepsilon\zeta'(t', X')\}.$$

Remark 2.2.1. *It is worth noting that the nondimensionalization of Φ^S, ψ and t comes from the linear wave theory (in shallow water regime, the characteristic speed is $\sqrt{gH}$). See paragraph **1.3.2** in [80]. Let us explain the nondimensionalization of Φ^B. Consider the linear case*

$$\begin{cases} \Delta_{X,z}\Phi^B = 0,\ -H < z < 0, \\ \Phi^B{}_{|z=0} = 0\ ,\ \partial_z\Phi^B{}_{|z=-H} = \partial_t b. \end{cases}$$

A straightforward computation gives $\Phi^B = \frac{\sinh(z|D|)}{|D|\cosh(H|D|)}\partial_t b$. If the typical wavelength is L, the typical wave number is $\frac{2\pi}{L}$. Furthermore, the typical order of magnitude of $\partial_t b$ is $\frac{a_{bott,m}\sqrt{gH}}{L}$. Then, the order of magnitude of Φ^B in the shallow water case is

$$\frac{L}{2\pi}\frac{\sqrt{gH}a_{bott,m}}{L}\frac{\sinh(2\pi\frac{H}{L})}{\cosh(2\pi\frac{H}{L})} \sim \frac{\sqrt{gH}a_{bott,m}H}{L}.$$

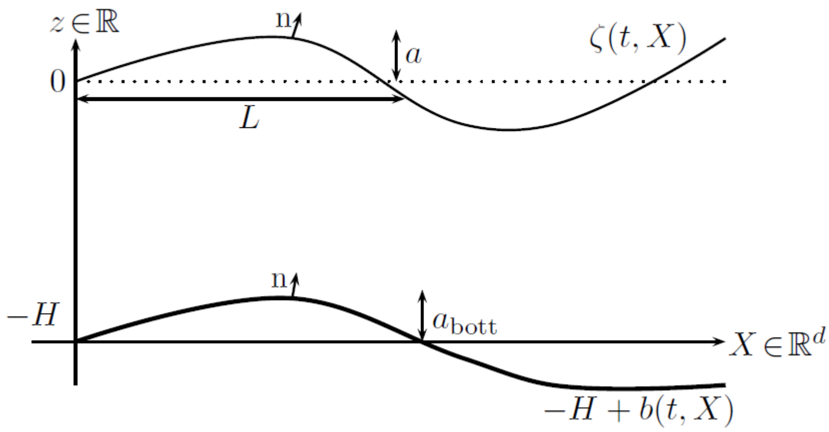

Figure 2.1: Typical scales

For the sake of clarity, we omit the primes. We can now nondimensionalize the water waves problem. Using the notation

$$\nabla^{\mu}_{X,z} := (\sqrt{\mu}\nabla_X \ , \partial_z \ )^t \text{ and } \Delta^{\mu}_{X,z} \ := \mu \Delta_X \ + \partial_z^2,$$

the water waves equations (2.11) become in dimensionless form

$$\begin{cases} \partial_t \zeta - \dfrac{1}{\mu} G_{\mu}[\varepsilon\zeta, \beta b](\psi) = \dfrac{\beta\lambda}{\varepsilon} G^{NN}_{\mu}[\varepsilon\zeta, \beta b](\partial_t b), \\ \partial_t \psi + \zeta + \dfrac{\varepsilon}{2} |\nabla \psi|^2 - \dfrac{\varepsilon}{2\mu} \dfrac{\left(G_{\mu}[\varepsilon\zeta, \beta b](\psi) + \frac{\lambda\beta\mu}{\varepsilon} G^{NN}_{\mu}[\varepsilon\zeta, \beta b](\partial_t b) + \mu\varepsilon \nabla\zeta \cdot \nabla\psi \right)^2}{(1 + \varepsilon^2 \mu |\nabla \zeta|^2)} = -P. \end{cases} \tag{2.13}$$

In the following $\partial_{\mathbf{n}}$ is the upward *conormal* derivative

$$\partial_{\mathbf{n}} \Phi^S = \mathbf{n} \cdot \begin{pmatrix} \sqrt{\mu} I_d & 0 \\ 0 & 1 \end{pmatrix} \nabla^{\mu}_{X,z} {\Phi^S}_{|\partial\Omega}.$$

Then, The Dirichlet-Neumann operator $G_{\mu}[\varepsilon\zeta, \beta b]$ is

$$G_{\mu}[\varepsilon\zeta, \beta b](\psi) := \sqrt{1 + \varepsilon^2 |\nabla\zeta|^2} \partial_{\mathbf{n}} {\Phi^S}_{|z=\varepsilon\zeta} = -\mu\varepsilon \nabla\zeta \cdot \nabla_X {\Phi^S}_{|z=\varepsilon\zeta} + \partial_z {\Phi^S}_{|z=\varepsilon\zeta} \ , \tag{2.14}$$

where Φ^S satisfies

$$\begin{cases} \Delta^{\mu}_{X,z} \Phi^S = 0 \text{ in } \Omega_t \ , \\ {\Phi^S}_{|z=\varepsilon\zeta} = \psi \ , \ \partial_{\mathbf{n}} {\Phi^S}_{|z=-1+\beta b} = 0, \end{cases} \tag{2.15}$$

while the Neumann-Neumann operator $G^{NN}_{\mu}[\varepsilon\zeta, \beta b]$ is

$$G^{NN}_{\mu}[\varepsilon\zeta, \beta b](\partial_t b) := \sqrt{1 + \varepsilon^2 |\nabla\zeta|^2} \partial_{\mathbf{n}} {\Phi^B}_{|z=\varepsilon\zeta} = -\mu \nabla(\varepsilon\zeta) \cdot \nabla_X {\Phi^B}_{|z=\varepsilon\zeta} + \partial_z {\Phi^B}_{|z=\varepsilon\zeta} \ , \tag{2.16}$$

where Φ^B satisfies

$$\begin{cases} \Delta^{\mu}_{X,z}\Phi^B = 0 \text{ in } \Omega_t \ , \\ \Phi^B{}_{|z=\varepsilon\zeta} = 0 \ , \ \sqrt{1+\beta^2|\nabla b|^2}\partial_{\mathbf{n}}\Phi^B{}_{|z=-1+\beta b} = \partial_t b. \end{cases} \tag{2.17}$$

Remark 2.2.2. *We adimensionalized the Dirichlet-Neumann and the Neumann-Neumann operators as follows*

$$G[\zeta,b](\psi) = \frac{aL\sqrt{gH}}{H^2} G_\mu[\varepsilon\zeta', \beta b'](\psi'), \ G^{NN}[\zeta,b](\partial_t b) = \frac{a_{bott,m}\sqrt{gH}}{L} G^{NN}_\mu[\varepsilon\zeta', \beta b'](\partial_{t'} b').$$

We add two classical assumptions. First, we assume some constraints on the nondimensionalized parameters and we suppose there exist $\rho_{\max} > 0$ and $\mu_{\max} > 0$, such that

$$0 < \varepsilon, \beta, \beta\lambda \leq 1 \ , \ \frac{\beta\lambda}{\varepsilon} \leq \rho_{\max} \text{ and } \mu \leq \mu_{\max}. \tag{2.18}$$

Furthermore, we assume that the water depth is bounded from below by a positive constant

$$\exists\, h_{\min} > 0 \ , \ \varepsilon\zeta + 1 - \beta b \geq h_{\min}. \tag{2.19}$$

In order to quasilinearize the water waves equations, we have to introduce the vertical speed at the surface $\underline{w}$ and horizontal speed at the surface $\underline{V}$. We define

$$\underline{w} := \underline{w}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right) = \frac{G_\mu[\varepsilon\zeta,\beta b](\psi) + \mu\frac{\beta\lambda}{\varepsilon}G^{NN}_\mu[\varepsilon\zeta,\beta b](\partial_t b) + \varepsilon\mu\nabla\zeta\cdot\nabla\psi}{1+\varepsilon^2\mu|\nabla\zeta|^2}, \tag{2.20}$$

and

$$\underline{V} := \underline{V}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right) = \nabla\psi - \varepsilon\underline{w}[\varepsilon\zeta,\beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right)\nabla\zeta. \tag{2.21}$$

2.2.2 Notations for this section and statement of the main results

In this chapter, $d = 1$ or 2, $t_0 > \frac{d}{2}$, $N \in \mathbb{N}$ and $s \geq 0$. The constant $T \geq 0$ represents a final time. The pressure P and the bottom b are given functions. We suppose that $b \in W^{3,\infty}(\mathbb{R}^+; H^N(\mathbb{R}^d))$ and $P \in W^{1,\infty}(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$. We denote by M_N a constant of the form

$$M_N = C\left(\frac{1}{h_{\min}}, \mu_{\max}, \varepsilon|\zeta|_{H^{\max(t_0+2,N)}}, \beta|b|_{L^\infty_t H_X^{\max(t_0+2,N+1)}}\right). \tag{2.22}$$

We denote by $U := (\zeta,\psi)^t$ the unknowns of our problem. We want to express (2.11) as a quasilinear system. It is well-known that the good energy for the water waves problem is

$$\mathcal{E}^N(U) = |\mathfrak{P}\psi|^2_{H^{\frac{3}{2}}} + \sum_{\alpha\in\mathbb{N}^d, |\alpha|\leq N} \left(|\zeta_{(\alpha)}|_2^2 + |\mathfrak{P}\psi_{(\alpha)}|_2^2\right), \tag{2.23}$$

where $\zeta_{(\alpha)} := \partial^\alpha\zeta$, $\psi_{(\alpha)} := \partial^\alpha\psi - \varepsilon\underline{w}\partial^\alpha\zeta$ and $\mathfrak{P} := \frac{|D|}{\sqrt{1+\sqrt{\mu}|D|}}$. This energy is motivated by the linearization of the system around the rest state (see Part **4.1** in [80]). $\mathfrak{P}$ acts as the square root of the Dirichlet-Neumann operator (see Proposition A.2.4). Here, $\zeta_{(\alpha)}$ and $\psi_{(\alpha)}$ are the *Alinhac's good unknowns of the system* (see [7] and [6] in the case of the standard water waves problem).

We define $U_{(\alpha)} := (\zeta_{(\alpha)}, \psi_{(\alpha)})^t$. We can introduce an associated energy space. Considering a $T \geq 0$, we define

$$E_T^N := \{U \in \mathcal{C}([0,T]; H^{t_0+2}(\mathbb{R}^d) \times \dot{H}^2(\mathbb{R}^d)) \ , \mathcal{E}^N(U) \in L^\infty([0,T])\}. \tag{2.24}$$

Our main results are the following theorems. We give two existence results. The first theorem extends the result of Iguchi (Theorem 2.4 in [71]) since we give a control of the dependence of the solution with respect to the parameters ε, β and μ and we add a non constant pressure at the surface and also extends the result of Lannes (Theorem 4.16 in [80]), since we improve the regularity of the initial data and we add a non constant pressure pressure at the surface and a moving bottom. Notice that we explain later what is Condition (2.29) (it corresponds to the positivity of the so called Rayleigh-Taylor coefficient).

Theorem 2.2.3. *Let $A > 0$, $t_0 > \frac{d}{2}$, $N \geq \max(1, t_0) + 3$, $U^0 \in E_0^N$, $b \in W^{3,\infty}(\mathbb{R}^+; H^{N+1}(\mathbb{R}^d))$ and $P \in W^{1,\infty}(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$ such that*

$$\mathcal{E}^N\left(U^0\right) + \frac{\beta\lambda}{\varepsilon} \left|\partial_t b\right|_{L_t^\infty H_X^N} + \left|\nabla P\right|_{L_t^\infty H_X^N} \leq A.$$

We suppose that the parameters $\varepsilon, \beta, \mu, \lambda$ satisfy (2.18) and that (2.19) and (2.29) are satisfied initially. Then, there exists $T > 0$ and a unique solution $U \in E_T^N$ to (2.13) with initial data U^0. Moreover, we have

$$T = \min\left(\frac{T_0}{\max(\varepsilon,\beta)}, \frac{T_0}{\frac{\beta\lambda}{\varepsilon}\left|\partial_t b\right|_{L_t^\infty H_X^N} + \left|\nabla P\right|_{L_t^\infty H_X^N}}\right) \ , \ \frac{1}{T_0} = c^1 \ \textit{and} \ \sup_{t\in[0,T]} \mathcal{E}^N(U) = c^2,$$

with $c^j = C\left(A, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, \mu_{\max}, \rho_{\max}, |b|_{W_t^{3,\infty} H_X^{N+1}}, |\nabla P|_{W_t^{1,\infty} H_X^N}\right)$.

Notice that if $\partial_t b$ and P are of size $\max(\varepsilon, \beta)$, we find the same existence time that in Theorem 4.16 in [80]. The second result shows that it is possible to go beyond the time scale of the previous theorem. Although the norm of the solution is not uniformly bounded in terms of ε and β, we are able to make this dependence precise. This theorem will be used to justify some of the asymptotic models derived in Section 2.3 over large time scales when the pressure at the surface and the moving bottom are not supposed small. We introduce $\delta := \max(\varepsilon, \beta^2)$.

Theorem 2.2.4. *Under the assumptions of the previous theorem, there exists $T_0 > 0$ such that $U \in E^N_{\frac{T_0}{\sqrt{\delta}}}$. Moreover, for all $\alpha \in \left[0, \frac{1}{2}\right]$, we have*

$$\frac{1}{T_0} = c^1 \ \textit{and} \ \sup_{t\in\left[0,\frac{T_0}{\delta^\alpha}\right]} \mathcal{E}^N(U) \leq \frac{c^3}{\delta^{2\alpha}}$$

where $c^j = C\left(A, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, \mu_{\max}, \rho_{\max}, |b|_{W_t^{3,\infty} H_X^{N+1}}, |\partial_t \nabla P|_{W_t^{1,\infty} H_X^N}\right)$.

Notice that when $\partial_t b$ and P are of size $\max(\varepsilon, \beta)$, the existence time of Theorem 2.2.3 is better than the one of Theorem 2.2.4. Theorem 2.2.4 is only useful when $\partial_t b$ and P are not small. Notice finally that Condition (2.29) is satisfied if ε is small enough. Hence, since in Section 2.3, we suppose that ε is small, it is reasonable to assume it.

2.2.3 Quasilinearization

Firstly, we give some controls of $|\mathfrak{P}\psi|_{H^s}$ and $|\mathfrak{P}\psi_{(\alpha)}|_{H^s}$ with respect to the energy $\mathcal{E}^N(U)$.

Proposition 2.2.5. *Let $T > 0$, $t_0 > \frac{d}{2}$ and $N \geq 2 + \max(1, t_0)$. Consider $U \in E_T^N$, $b \in W^{1,\infty}(\mathbb{R}^+; H^{N+1}(\mathbb{R}^d))$, such that ζ and b satisfy Condition* (2.19) *for all $0 \leq t \leq T$. We assume also that μ satisfies* (2.18). *Then, for $0 \leq t \leq T$, for $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq N-1$ and for $0 \leq s \leq N - \frac{1}{2}$,*

$$|\partial^\alpha \mathfrak{P}\psi|_2 + |\mathfrak{P}\psi_{(\alpha)}|_{H^1} + |\mathfrak{P}\psi|_{H^s} \leq M_N \mathcal{E}^N(U)^{\frac{1}{2}} + \frac{\beta\lambda}{\varepsilon} M_N \left|\partial_t b\right|_{L_t^\infty H_X^N}.$$

Proof. For the first inequality, we have thanks to Proposition B.1.1,

$$\begin{aligned}|\partial^\alpha \mathfrak{P}\psi|_2 &\leq |\mathfrak{P}\psi_{(\alpha)}|_2 + \varepsilon |\mathfrak{P}(\underline{w}\partial^\alpha\zeta)|_2,\\ &\leq |\mathfrak{P}\psi_{(\alpha)}|_2 + \frac{\varepsilon}{\mu^{\frac{1}{4}}} |\underline{w}\partial^\alpha\zeta|_{H^{\frac{1}{2}}}.\end{aligned}$$

Since $\psi \in \dot{H}^2(\mathbb{R}^d)$, by Proposition A.2.17, $\underline{w} \in H^1(\mathbb{R}^d)$, and thanks to Proposition B.2.1, we obtain

$$|\partial^\alpha \mathfrak{P}\psi|_2 \leq |\mathfrak{P}\psi_{(\alpha)}|_2 + C\varepsilon \left|\frac{w}{\mu^{\frac{1}{4}}}\right|_{H^1} |\zeta|_{H^N} \leq |\mathfrak{P}\psi_{(\alpha)}|_2 + M_N \left(|\mathfrak{P}\psi|_{H^{\frac{3}{2}}} + \frac{\beta\lambda}{\varepsilon} |\partial_t b|_{H^1} \right).$$

The other inequalities follow with the same arguments, see for instance Lemma 4.6 in [80]. □

The following statement is a first step to the quasilinearization of the water waves equations. It is essentially Proposition 4.5 in [80] and Lemma 6.2 in [71]. However, we improve the minimal regularity of U (we decrease the minimal value of N to 4 when $d = 1$) and we provide the dependence on $\partial_t b$ contrary to [71]. For those reasons, we give a proof of this proposition.

Proposition 2.2.6. *Let $t_0 > \frac{d}{2}$, $T > 0$, $N \geq \max(t_0, 1) + 3$, $b \in W^{1,\infty}(\mathbb{R}^+; H^{N+1}(\mathbb{R}^d))$ and $U \in E_T^N$, such that ζ and b satisfy Condition* (2.19) *for all $0 \leq t \leq T$. We assume also that μ satisfies* (2.18). *Then, for all $\alpha \in \mathbb{N}^d$, $1 \leq |\alpha| \leq N$, we have,*

$$\begin{aligned}\partial^\alpha \left(\frac{1}{\mu} G_\mu[\varepsilon\zeta, \beta b](\psi) + \frac{\lambda\beta}{\varepsilon} G_\mu^{NN}[\varepsilon\zeta, \beta b](\partial_t b) \right) = &\frac{1}{\mu} G_\mu[\varepsilon\zeta, \beta b](\psi_{(\alpha)}) + \frac{\beta\lambda}{\varepsilon} G_\mu^{NN}[\varepsilon\zeta, \beta b](\partial^\alpha \partial_t b)\\ &- \varepsilon \mathbf{1}_{\{|\alpha|=N\}} \nabla \cdot (\zeta_{(\alpha)} \underline{V}) + R_\alpha.\end{aligned}$$

Furthermore R_α is controlled

$$|R_\alpha|_2 \leq M_N |(\varepsilon\zeta, \beta b)|_{H^N} \mathcal{E}^N(U)^{\frac{1}{2}} + \frac{\beta\lambda}{\varepsilon} M_N \left|\partial_t b\right|_{L_t^\infty H_X^N}.$$

Proof. We adapt and follow the proof of Proposition 4.5 in [80]. See also Proposition 6.4 in [71]. Using Proposition A.2.20, we obtain

$$\begin{aligned}\partial^\alpha \left(\frac{1}{\mu} G_\mu[\varepsilon\zeta, \beta b](\psi) + \frac{\lambda\beta}{\varepsilon} G_\mu^{NN}[\varepsilon\zeta, \beta b](\partial_t b) \right) = &\frac{1}{\mu} G_\mu[\varepsilon\zeta, \beta b](\psi_{(\alpha)}) + \frac{\beta\lambda}{\varepsilon} G_\mu^{NN}[\varepsilon\zeta, \beta b](\partial^\alpha \partial_t b)\\ &- \varepsilon \mathbf{1}_{\{|\alpha|=N\}} \nabla \cdot (\zeta_{(\alpha)} \underline{V}) - \beta G_\mu^{NN}[\varepsilon\zeta, \beta b] \left(\nabla \cdot \left(\partial^\alpha b \, \underline{\widetilde{V}} \right) \right)\\ &+ R_\alpha,\end{aligned}$$

where $\underline{\widetilde{V}} = \underline{\widetilde{V}}[\varepsilon\zeta, \beta b](\psi, \partial_t b)$ is the horizontal velocity at the bottom and is defined in Equation (2.21) and R_α is a sum of terms of the form

$$A_{j,\iota,\nu} := d^j \left(\frac{1}{\mu} G_\mu(\partial^\nu \psi) + \frac{\beta\lambda}{\varepsilon} G_\mu^{NN}(\partial^\nu \partial_t b) \right) .(\partial^{\iota^1}\zeta, ..., \partial^{\iota^j}\zeta; \partial^{\iota^1} b, ..., \partial^{\iota^j} b),$$

where j is an integer and $\iota^1, ..., \iota^j$ and ν are multi-index, and

$$\sum_{1 \leq l \leq j} |\iota^l| + |\nu| = N,$$

with $(j, |\iota^{l_0}|, |\nu|) \neq (1, N, 0)$ and $(0, 0, N)$. Here ι^{l_0} is such that $\max_{1 \leq l \leq j} |\iota^l| = |\iota^{l_0}|$. In particular, $1 \leq |\iota^{l_0}| \leq N$. First, using Propositions A.2.13, A.2.18 and Product estimates B.2.1, we get

$$\left| \beta G_\mu^{NN}[\varepsilon\zeta, \beta b] \left(\nabla \cdot \left(\partial^\alpha b \, \underline{\widetilde{V}} \right) \right) \right|_2 \leq \beta M_N \left(|\mathfrak{P}\psi|_{H^2} + |\partial_t b|_{H^1} \right).$$

Then, we distinguish several cases.

a) $|\iota^{l_0}| + |\nu| \leq N - 2$ and $|\iota^{l_0}| \leq N - 3$ or $|\iota^{l_0}| + |\nu| \leq N$, $|\iota^{l_0}| \leq N - 3$ and $|\nu| \leq N - 2$:

Applying the second point of Proposition A.2.23 and the first point of Proposition A.2.24 with $s = \frac{1}{2}$ and $t_0 = \min(t_0, \frac{3}{2})$, we get that

$$|A_{j,\iota,\nu}|_2 \leq M_N \prod_l |(\varepsilon\partial^{\iota^l}\zeta, \beta\partial^{\iota^l} b)|_{H^3} \left[|\mathfrak{P}\partial^\nu \psi|_{H^1} + \frac{\beta\lambda}{\varepsilon} |\partial^\nu \partial_t b|_2 \right],$$

and the result follows by Proposition 2.2.5.

b) $|\iota^{l_0}| = N - 2$ and $|\nu| = 0$,1 or 2 :

We apply the fourth point of Proposition A.2.23 and the second point of Proposition A.2.24 with $s = \frac{1}{2}$ and $t_0 = \max(t_0, 1)$,

$$|A_{j,\iota,\nu}|_2 \leq M_N |(\varepsilon\partial^{\iota^{l_0}}\zeta, \beta\partial^{\iota^{l_0}} b)|_{H^{\frac{3}{2}}} \prod_{l \neq l_0} |(\varepsilon\partial^{\iota^l}\zeta, \beta\partial^{\iota^l} b)|_{H^{N-2}} \left[|\mathfrak{P}\partial^\nu \psi|_{H^{N-2}} + \frac{\beta\lambda}{\varepsilon} |\partial^\nu \partial_t b|_{H^{N-2}} \right].$$

Then, we get the result thanks to Proposition 2.2.5.

c) $\iota^1 = \iota$ with $|\iota| = N - 1$, $|\nu| = j = 1$:

We proceed as in Proposition 4.5 in [80], using Proposition A.2.12 and Propositions A.2.20, A.2.13, A.2.17.

d) $|\iota^{l_0}| = N - 1$ and $|\nu| = 0$:

Here $j = 2$. For instance we consider that $l_0 = 1$ and $|\iota^2| = 1$. Using the second inequality of Proposition A.2.24 we have

$$\left| d^2 G_\mu^{NN}(\partial_t b).(\partial^{\iota^1}\zeta, \partial^{\iota^2}\zeta; \partial^{\iota^1} b, \partial^{\iota^2} b) \right|_2 \leq M_N \left| \left(\varepsilon\partial^{\iota^1}\zeta, \beta\partial^{\iota^1} b \right) \right|_{H^1} \left| \left(\varepsilon\partial^{\iota^2}\zeta, \beta\partial^{\iota^2} b \right) \right|_{H^2} |\partial_t b|_{H^2}.$$

Furthermore, using two times Proposition A.2.20, we get

$$\frac{1}{\mu}d^2G_\mu(\psi).(\partial^{\iota^1}\zeta,\partial^{\iota^2}\zeta;\partial^{\iota^1}b,\partial^{\iota^2}b) = -\frac{\varepsilon}{\sqrt{\mu}}dG_\mu[\varepsilon\zeta,\beta b]\left(\partial^{\iota^1}\zeta\frac{1}{\sqrt{\mu}}\underline{w}(\psi,0)\right).(\partial^{\iota^2}\zeta,0)$$
$$-\frac{\varepsilon}{\sqrt{\mu}}G_\mu[\varepsilon\zeta,\beta b]\left(\partial^{\iota^1}\zeta\frac{1}{\sqrt{\mu}}d\underline{w}(\psi,0).(\partial^{\iota^2}\zeta,0)\right)$$
$$-\varepsilon\nabla\cdot\left(\partial^{\iota^1}\zeta d\underline{V}(\psi,0).(\partial^{\iota^2}\zeta,0)\right)$$
$$+\beta dG_\mu^{NN}[\varepsilon\zeta,\beta b]\left(\partial^{\iota^1}b\,\underline{\widetilde{V}}(\psi,0)\right).(0,\partial^{\iota^2}b)$$
$$+\beta G_\mu^{NN}[\varepsilon\zeta,\beta b]\left(\partial^{\iota^1}b\,d\underline{\widetilde{V}}(\psi,0).(0,\partial^{\iota^2}b)\right).$$

The control follows from the first inequality of Proposition A.2.12, Proposition A.2.17, and Propositions A.2.22, A.2.17 and A.2.18.

e) $|\nu| = N-1$ and $|\iota^{l_0}| = 1$:

Here, $j=1$. It is clear that

$$\left|\frac{\beta\lambda}{\varepsilon}dG_\mu^{NN}(\partial^\nu\partial_t b).(\partial^{\iota^1}\zeta;\partial^{\iota^1}b)\right|_2 \leq \frac{\beta\lambda}{\varepsilon}M_N\,|\partial_t b|_{H^N}.$$

Furthermore,

$$\frac{1}{\mu}dG_\mu(\partial^\nu\psi).(\partial^{\iota^1}\zeta;\partial^{\iota^1}b) = \frac{1}{\mu}dG_\mu(\psi_{(\nu)}).(\partial^{\iota^1}\zeta;\partial^{\iota^1}b) + \frac{1}{\sqrt{\mu}}dG_\mu\left(\frac{\varepsilon}{\sqrt{\mu}}\underline{w}\partial^\nu\zeta\right).(\partial^{\iota^1}\zeta;\partial^{\iota^1}b).$$

Then, using Proposition A.2.12 and Proposition 2.2.5, we get the result. □

This Proposition enables to quasilinearize the first equation of the water waves equations. For the second equation, it is the purpose of the following proposition.

Proposition 2.2.7. *Let $t_0 > \frac{d}{2}$, $T>0$, $N \geq \max(t_0,1)+3$, $b \in W^{1,\infty}(\mathbb{R}^+;H^{N+1}(\mathbb{R}^d))$ and $U \in E_T^N$, such that ζ and b satisfy* (2.19) *for all $0 \leq t \leq T$. We assume also that μ satisfies* (2.18)*. Then, for all $\alpha \in \mathbb{N}^d$, $1 \leq |\alpha| \leq N$, we have,*

$$\partial^\alpha\left[\frac{\varepsilon}{2}|\nabla\psi|^2 - \frac{\varepsilon}{2\mu}(1+\varepsilon^2\mu|\nabla\zeta|^2)\underline{w}^2\right] = \varepsilon\underline{V}\cdot(\nabla\psi_{(\alpha)} + \varepsilon\partial^\alpha\zeta\nabla\underline{w}) - \frac{\varepsilon}{\mu}\underline{w}\,\partial^\alpha G_\mu(\psi)$$
$$-\beta\lambda\underline{w}\,\partial^\alpha G_\mu^{NN}(\partial_t b) + S_\alpha.$$

Furthermore S_α is controlled

$$|\mathfrak{P}S_\alpha|_2 \leq \varepsilon M_N\mathcal{E}^N(U) + C\left(M_N, \frac{\beta\lambda}{\varepsilon}\,|\partial_t b|_{L_t^\infty H_X^N}\right)\varepsilon\mathcal{E}^N(U)^{\frac{1}{2}} + \varepsilon M_N\left(\frac{\beta\lambda}{\varepsilon}\,|\partial_t b|_{L_t^\infty H_X^N}\right)^2.$$

Proof. The proof of this Proposition is similar to the proof of Proposition 4.10 in [80] expect we use Propositions A.2.17 and A.2.20. See also Proposition 6.4 in [71]. □

Thanks to this linearization, we can "quasilinarize" equations (2.13). It is the purpose of the next proposition. Let us introduce, the Rayleigh-Taylor coefficient

$$\begin{aligned}\underline{\mathfrak{a}} := \underline{\mathfrak{a}}(U,\beta b) =& 1 + \varepsilon \partial_t \left(\underline{w}[\varepsilon\zeta,\beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right)\right) \\ &+ \varepsilon^2 \underline{V}[\varepsilon\zeta,\beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right) \cdot \nabla \left(\underline{w}[\varepsilon\zeta,\beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right)\right).\end{aligned} \tag{2.25}$$

This quantity plays an important role and is linked to the pressure in the fluid domain (see Part 2.2.5). We also introduce two new operators,

$$\mathcal{A}[U,\beta b] := \begin{pmatrix} 0 & -\frac{1}{\mu} G_\mu[\varepsilon\zeta,\beta b] \\ \underline{\mathfrak{a}}(U,\beta b) & 0 \end{pmatrix} \tag{2.26}$$

and

$$\mathcal{B}[U,\beta b] := \begin{pmatrix} \varepsilon\nabla\cdot(\cdot\underline{V}) & 0 \\ 0 & \varepsilon\underline{V}\cdot\nabla \end{pmatrix}. \tag{2.27}$$

We can now quasilinearize the water waves equations. We use the same arguments as in Proposition 4.10 in [80] and part **6** in [71]. Notice that we give here a precise estimate with respect to $\partial_t b$ and P of the residuals R_α and S_α and that the minimal value of N, regularity of U, is smaller than in Proposition 4.10 in [80].

Proposition 2.2.8. *Let $t_0 > \frac{d}{2}$, $T > 0$, $N \geq \max(t_0, 1) + 3$, $U \in E_T^N$ satisfies* (2.19) *for all $0 \leq t \leq T$ and solving* (2.13)*, $b \in W^{2,\infty}(\mathbb{R}^+; H^{N+1}(\mathbb{R}^d))$, $P \in L^\infty(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$. We assume also that μ satisfies* (2.18)*. Then, for all $\alpha \in \mathbb{N}^d$, $1 \leq |\alpha| \leq N$, we have,*

$$\begin{aligned}\partial_t U_{(\alpha)} + \mathcal{A}[U,\beta b](U_{(\alpha)}) + \mathbf{1}_{\{|\alpha|=N\}}\mathcal{B}[U,\beta b](U_{(\alpha)}) =& \left(\frac{\lambda\beta}{\varepsilon} G_\mu^{NN}[0,0](\partial^\alpha\partial_t b), -\partial^\alpha P\right)^t \\ &+ \left(\widetilde{R_\alpha}, S_\alpha\right)^t.\end{aligned} \tag{2.28}$$

Furthermore, $\widetilde{R_\alpha}$ and S_α satisfy

$$\begin{cases} |\widetilde{R_\alpha}|_2 \leq M_N |(\varepsilon\zeta,\beta b)|_{H^N} \mathcal{E}^N(U)^{\frac{1}{2}} + \dfrac{\beta\lambda}{\varepsilon} M_N \left|\partial_t b\right|_{L_t^\infty H_X^N}, \\ |\mathfrak{P} S_\alpha|_2 \leq \varepsilon M_N \mathcal{E}^N(U) + C\left(M_N, \dfrac{\beta\lambda}{\varepsilon}\left|\partial_t b\right|_{L_t^\infty H_X^N}\right) \varepsilon\mathcal{E}^N(U)^{\frac{1}{2}} + \varepsilon M_N \left(\dfrac{\beta\lambda}{\varepsilon}\left|\partial_t b\right|_{L_t^\infty H_X^N}\right)^2. \end{cases}$$

Proof. Thanks to Proposition A.2.24, we get

$$\begin{aligned}\left|G_\mu^{NN}[\varepsilon\zeta,\beta b](\partial^\alpha\partial_t b) - G_\mu^{NN}[0,0](\partial^\alpha\partial_t b)\right|_2 &\leq \int_0^1 \left|dG_\mu^{NN}[z\varepsilon\zeta, z\beta b](\partial^\alpha\partial_t b).(\zeta,b)\right|_2 dz \\ &\leq M_N |(\varepsilon\zeta,\beta b)|_{H^N} \left|\partial^\alpha\partial_t b\right|_{L_t^\infty H_X^N}.\end{aligned}$$

Then, denoting $\widetilde{R_\alpha} = R_\alpha + G_\mu^{NN}[\varepsilon\zeta,\beta b](\partial^\alpha\partial_t b) - G_\mu^{NN}[0,0](\partial^\alpha\partial_t b)$, we obtain the first equation thanks to Proposition 2.2.6. For the second equation, using Proposition 2.2.7 and the first equation of the water waves problem, we have

$$\begin{aligned}\partial_t\partial^\alpha\psi &= -\partial^\alpha\zeta - \varepsilon\underline{V}\cdot(\nabla\psi_{(\alpha)} + \varepsilon\partial^\alpha\zeta\nabla\underline{w}) + \frac{\varepsilon}{\mu}\underline{w}\,\partial^\alpha G_\mu(\psi) + \beta\underline{w}\,\partial^\alpha G_\mu^{NN}(\partial_t b) - \partial^\alpha P + S_\alpha\\ &= -\partial^\alpha\zeta - \varepsilon\underline{V}\cdot(\nabla\psi_{(\alpha)} + \varepsilon\partial^\alpha\zeta\nabla\underline{w}) + \varepsilon\underline{w}\partial_t\partial^\alpha\zeta - \partial^\alpha P + S_\alpha\\ &= -\partial^\alpha\zeta(1+\varepsilon\partial_t\underline{w} + \varepsilon^2\underline{V}\cdot\nabla\underline{w}) - \varepsilon\underline{V}\cdot\nabla\psi_{(\alpha)} + \varepsilon\partial_t(\underline{w}\partial^\alpha\zeta) - \partial^\alpha P + S_\alpha\\ &= -\underline{\mathfrak{a}}\partial^\alpha\zeta - \varepsilon\underline{V}\cdot\nabla\psi_{(\alpha)} + \varepsilon\partial_t(\underline{w}\partial^\alpha\zeta) - \partial^\alpha P + S_\alpha,\end{aligned}$$

and the result follows.

□

In the case of a constant pressure at the surface and a fixed bottom, it is well-known that system (2.28) is symmetrizable if

$$\exists\, \mathfrak{a}_{min} > 0 \ ,\ \underline{\mathfrak{a}}(U,\beta b) \geq \mathfrak{a}_{\min}. \tag{2.29}$$

Then, we introduce the symmetrizer

$$\mathcal{S}[U,\beta b] := \begin{pmatrix} \underline{\mathfrak{a}}(U,\beta b) & 0 \\ 0 & \frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b]\end{pmatrix}. \tag{2.30}$$

This symmetrization has an associated energy

$$\begin{aligned}\mathcal{F}^\alpha(U) &= \frac{1}{2}\left(\mathcal{S}[U,\beta b](U_{(\alpha)}), U_{(\alpha)}\right),\ \text{ if } \alpha\neq 0,\\ \mathcal{F}^0(U) &= \frac{1}{2}|\zeta|^2_{H^{\frac{3}{2}}} + \frac{1}{2}\left(\Lambda^{\frac{3}{2}}\psi, \frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\Lambda^{\frac{3}{2}}\psi)\right),\\ \mathcal{F}^{[N]}(U) &= \sum_{|\alpha|\leq N}\mathcal{F}^\alpha(U).\end{aligned} \tag{2.31}$$

As in Lemma 4.27 in [80], it can be shown that $\mathcal{F}^{[N]}$ and $\mathcal{E}^{[N]}$ are equivalent in the following sense.

Proposition 2.2.9. *Let $T>0$, $N\in\mathbb{N}$, $U\in E_T^N$ satisfying* (2.19) *and* (2.29) *for all $0\leq t\leq T$. Then, for all $0\leq k\leq N$, $\mathcal{F}^{[k]}$ is comparable to $\mathcal{E}^k$*

$$\frac{1}{|\underline{\mathfrak{a}}(U,\beta b)|_{L^\infty} + M_N}\mathcal{F}^{[k]}[U,b] \leq \mathcal{E}^k(U) \leq \left(M_N + \frac{1}{\mathfrak{a}_{min}}\right)\mathcal{F}^{[k]}[U,b]. \tag{2.32}$$

Remark 2.2.10. *We said that the water waves equations* (2.13) *are symmetrizable under the assumption that the Rayleigh-Taylor coefficient is positive. In fact, Ebin ([56]) shown that if this condition is not satisfied, the water waves equations are illposed.*

2.2.4 Local existence

The water water equations can be written as follow :

$$\partial_t U + \mathcal{N}(U) = (0,-P)^t, \tag{2.33}$$

with $\mathcal{N}(U) = (\mathcal{N}_1(U),\mathcal{N}_2(U))^t$ and

$$\begin{aligned} \mathcal{N}_1(U) &:= -\frac{1}{\mu} G_\mu[\varepsilon\zeta, \beta b](\psi) - \frac{\beta\lambda}{\varepsilon} G_\mu^{NN}[\varepsilon\zeta, \beta b](\partial_t b), \\ \mathcal{N}_2(U) &:= \zeta + \frac{\varepsilon}{2}|\nabla\psi|^2 - \frac{\varepsilon}{2\mu}\left(1+\varepsilon^2\mu|\nabla\zeta|^2\right)\left(\underline{w}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right)\right)^2. \end{aligned} \tag{2.34}$$

According to our quasilinearization, we need that $\underline{\mathfrak{a}}$ be a positive real number. Therefore, we have to express $\underline{\mathfrak{a}}$ without partial derivative with respect to t, particularly when $t = 0$. It is easy to check that

$$\begin{aligned} \underline{\mathfrak{a}}(U, \beta b) = 1 &+ \varepsilon^2 \underline{V}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right) \cdot \nabla\left[\underline{w}[\varepsilon\zeta, \beta b]\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right)\right] \\ &+ \varepsilon d\underline{w}\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right).(-\mathcal{N}_1(U), \partial_t b) + \varepsilon\underline{w}[\varepsilon\zeta, \beta b]\left(-P - \mathcal{N}_2(U), \frac{\beta\lambda}{\varepsilon}\partial_t^2 b\right). \end{aligned} \tag{2.35}$$

The following Proposition gives estimates for $\underline{\mathfrak{a}}(U, \beta b)$ and is adapted from Proposition 6.6 in [71].

Proposition 2.2.11. *Let $T > 0$, $t_0 > \frac{d}{2}$, $N \geq \max(t_0, 1) + 3$, $(\zeta, \psi) \in E_T^N$ is a solution of the water waves equations (2.13), $P \in L^\infty(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$ and $b \in W^{2,\infty}(\mathbb{R}^+; H^{N+1}(\mathbb{R}^d))$, such that Condition (2.19) is satisfied. We assume also that μ satisfies (2.18). Then, for all $0 \leq t \leq T$,*

$$\begin{aligned} |\underline{\mathfrak{a}}(U, \beta b) - 1|_{H^{t_0}} \leq & C\Big(M_N, \max(\beta\lambda, \beta)\,|\partial_t b|_{L_t^\infty H_X^N}, \varepsilon\mathcal{E}^N(U)^{\frac{1}{2}}\Big)\,\varepsilon\mathcal{E}^N(U)^{\frac{1}{2}} \\ &+ \varepsilon M_N\Big(|\nabla P|_{L_t^\infty H_X^N} + \frac{\beta\lambda}{\varepsilon}\,|\partial_t^2 b|_{L_t^\infty H_X^N}\Big). \end{aligned}$$

Furthermore, if $\partial_t^3 b \in L^\infty(\mathbb{R}^+; H^N(\mathbb{R}^d))$ and $\partial_t P \in L^\infty(\mathbb{R}^+; \dot{H}^N(\mathbb{R}^d))$, then,

$$\begin{aligned} |\partial_t(\underline{\mathfrak{a}}(U, \beta b))|_{H^{t_0}} \leq & C\Big(M_N, \max(\beta\lambda, \beta)\,|\partial_t b|_{W_t^{1,\infty} H_X^N}, |\nabla P|_{L_t^\infty H_X^N}, \varepsilon\mathcal{E}^N(U)^{\frac{1}{2}}\Big)\,\varepsilon\mathcal{E}^N(U)^{\frac{1}{2}} \\ &+ \varepsilon C\Big(M_N, \max(\beta\lambda, \beta)\,|\partial_t b|_{L_t^\infty H_X^N}\Big)\Big(|\nabla P|_{W_t^{1,\infty} H_X^N} + \frac{\beta\lambda}{\varepsilon}\,|\partial_t^2 b|_{W_t^{1,\infty} H_X^N}\Big). \end{aligned}$$

Proof. Using the first point of Proposition A.2.17 and Product estimate B.2.1 we have

$$|\underline{V}[\varepsilon\zeta, \beta b](\varepsilon\psi, \beta\lambda\partial_t b) \cdot \nabla\left[\underline{w}[\varepsilon\zeta, \beta b]\left(\varepsilon\psi, \beta\lambda\partial_t b\right)\right]|_{H^{t_0}} \leq M_N\left(|\mathfrak{P}\varepsilon\psi|_{H^{t_0+\frac{1}{2}}} + \beta\lambda\,|\partial_t b|_{L_t^\infty H_X^{t_0}}\right)^2.$$

Furthermore, thanks to the first point of Proposition A.2.24 and the first point of Proposition A.2.23 we obtain

$$\left|\varepsilon d\underline{w}\left(\psi, \frac{\beta\lambda}{\varepsilon}\partial_t b\right).(-\mathcal{N}_1(U), \partial_t b)\right|_{H^{t_0}} \leq M_N|(\varepsilon\mathcal{N}_1(U), \beta\partial_t b)|_{H^{t_0+1}}\left(|\mathfrak{P}\varepsilon\psi|_{H^{t_0+\frac{1}{2}}} + \beta\lambda\,|\partial_t b|_{L_t^\infty H_X^{t_0}}\right).$$

Then, the first inequality follows easily from Proposition A.2.17, Proposition 2.2.5 and Product estimate B.2.1. The second inequality can be proved similarly. □

Remark 2.2.12. *Notice that in the previous proposition, we only use the fact that $\partial_t^2 b$ belongs to $L^\infty(\mathbb{R}^+; H^{t_0+1}(\mathbb{R}^d))$ and that $\partial_t^3 b$ belongs to $L^\infty(\mathbb{R}^+; H^{t_0}(\mathbb{R}^d))$.*

We can now prove Theorems 2.2.3 and 2.2.4. We recall that $\delta := \max(\varepsilon, \beta^2)$.

Proof. We slice up this proof in three parts. First we regularize and symmetrize the equations, then we find some energy estimates and finally we conclude by convergence. We only give the energy estimates in this chapter and a carefully study of the nonlinearities of the water waves equations is done. We refer to the proof of Theorem 4.16 in [80] for the regularization, the convergence and the uniqueness (see also part 7 in [71]). For Theorem 2.2.3 (respectively Theorem 2.2.4), we assume that U solves (2.13) on $[0,T]$ $\left(\text{respectively on } \left[0, \frac{T}{\sqrt{\delta}}\right]\right)$ and that (2.19) and (2.29) are satisfied for $\frac{h_{min}}{2}$ and $\frac{\mathfrak{a}_{min}}{2}$ on $[0,T]$ $\left(\text{respectively on } \left[0, \frac{T}{\sqrt{\delta}}\right]\right)$ for some $T > 0$.

a) $|\alpha| = 0$, The 0 - energy

We proceed as in Subsection 4.3.4.3 in [80] and part 6 in [71]. We have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}^0(U) =& \frac{1}{2\mu}\left(dG_\mu[\varepsilon\zeta,\beta b](\Lambda^{\frac{3}{2}}\psi).(\partial_t\zeta,\partial_t b), \Lambda^{\frac{3}{2}}\psi\right) + \frac{\beta\lambda}{\varepsilon}\left(\Lambda^{\frac{3}{2}}G_\mu^{NN}[\varepsilon\zeta,\beta b](\partial_t b), \Lambda^{\frac{3}{2}}\zeta\right) \\ &- \left(\Lambda^{\frac{3}{2}}\left(\mathcal{N}_2(U)-\zeta\right), \frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\Lambda^{\frac{3}{2}}\psi)\right) - \left(\frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\Lambda^{\frac{3}{2}}\psi), \Lambda^{\frac{3}{2}}P\right).\end{aligned} \tag{2.36}$$

We have to control all the term in the r.h.s.

$\star$ Control of $\frac{\beta\lambda}{\varepsilon}\left(\Lambda^{\frac{3}{2}}G_\mu^{NN}[\varepsilon\zeta,\beta b](\partial_t b), \Lambda^{\frac{3}{2}}\zeta\right)$.

Using Proposition A.2.13, we get

$$\left|\frac{\beta\lambda}{\varepsilon}\left(\Lambda^{\frac{3}{2}}G_\mu^{NN}[\varepsilon\zeta,\beta b](\partial_t b), \Lambda^{\frac{3}{2}}\zeta\right)\right| \leq M_N\frac{\beta\lambda}{\varepsilon}\left|\partial_t b\right|_{L_t^\infty H_X^N}\mathcal{E}^N(U)^{\frac{1}{2}}.$$

$\star$ Control of $\left(\Lambda^{\frac{3}{2}}\left(\mathcal{N}_2(U)-\zeta\right), \frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\Lambda^{\frac{3}{2}}\psi)\right)$.

Using Proposition 2.2.5 and Proposition A.2.17, we get

$$\begin{aligned}\left|\left(\Lambda^{\frac{3}{2}}(\mathcal{N}_2(U)-\zeta), \frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\Lambda^{\frac{3}{2}}\psi)\right)\right| &\leq \left|\mathcal{N}_2(U)-\zeta\right|_{H^{\frac{3}{2}}}\left|\frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\Lambda^{\frac{3}{2}}\psi)\right|_2, \\ &\leq \varepsilon M_N\mathcal{E}^N(U)^{\frac{3}{2}} + M_N\left(\frac{\beta\lambda}{\varepsilon}\left|\partial_t b\right|_{L_t^\infty H_X^N}\right)^2\varepsilon\mathcal{E}^N(U).\end{aligned}$$

$\star$ Control of $\left(\frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\Lambda^{\frac{3}{2}}\psi), \Lambda^{\frac{3}{2}}P\right)$.

We get, using Propositions A.2.2 and A.2.4,

$$\left|\left(\frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\Lambda^{\frac{3}{2}}\psi), \Lambda^{\frac{3}{2}}P\right)\right| \leq M_N E^N(U)^{\frac{1}{2}}\left|\nabla P\right|_{L_t^\infty H_X^N}.$$

$\star$ Control of $\frac{1}{2\mu}\left(dG_\mu[\varepsilon\zeta,\beta b](\Lambda^{\frac{3}{2}}\psi).(\partial_t\zeta,\partial_t b),\Lambda^{\frac{3}{2}}\psi\right)$.

Using the second point of Proposition A.2.12, Proposition A.2.13 and Proposition 2.2.5, we get

$$\begin{aligned}\left|\frac{1}{\mu}\Big(dG_\mu[\varepsilon\zeta,\beta b](\Lambda^{\frac{3}{2}}\psi).(\partial_t\zeta,\partial_t b),\Lambda^{\frac{3}{2}}\psi\Big)\right| &\leq M_N\left|(\varepsilon\mathcal{N}_1(U),\beta\partial_t b)\right|_{H^{N-2}}\left|\mathfrak{P}\psi\right|^2_{H^{\frac{3}{2}}}\\ &\leq M_N\varepsilon\mathcal{E}^N(U)^{\frac{3}{2}}+\max(\beta,\beta\lambda)\left|\partial_t b\right|_{L^\infty_t H^N_X}\mathcal{E}^N(U).\end{aligned}$$

Finally, gathering all the previous estimates, we get that

$$\begin{aligned}\frac{d}{dt}\mathcal{F}^0(U)\leq \varepsilon M_N\mathcal{E}^N(U)^{\frac{3}{2}}&+M_N C\left(\rho_{\max},\left|\partial_t b\right|_{L^\infty_t H^N_X}\right)\max(\varepsilon,\beta)\mathcal{E}^N(U)\\ &+M_N\sqrt{E^N(U)}\left(\left|\nabla P\right|_{L^\infty_t H^N_X}+\frac{\beta\lambda}{\varepsilon}\left|\partial_t b\right|_{L^\infty_t H^N_X}\right).\end{aligned}\tag{2.37}$$

b) <u>$|\alpha|>0$, the higher orders energies</u>

We proceed as in Subsection 4.3.4.3 in [80] and part 6 in [71]. A simple computation gives

$$\begin{aligned}\frac{d}{dt}(\mathcal{F}^\alpha(U)) = &-\varepsilon\mathbf{1}_{\{|\alpha|=N\}}\left(\underline{\mathfrak{a}}\zeta_{(\alpha)},\nabla\cdot\left(\zeta_{(\alpha)}\underline{V}\right)\right)+\left(\underline{\mathfrak{a}}\zeta_{(\alpha)},\frac{\beta\lambda}{\varepsilon}G^{NN}_\mu[0,0](\partial_t\partial^\alpha b)+\widetilde{R_\alpha}\right)\\ &-\varepsilon\mathbf{1}_{\{|\alpha|=N\}}\left(\frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\psi_{(\alpha)}),\underline{V}\cdot\nabla\psi_{(\alpha)}\right)+\left(\frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\psi_{(\alpha)}),S_\alpha-\partial^\alpha P\right)\\ &+\frac{1}{2}\left(\partial_t\underline{\mathfrak{a}}\zeta_{(\alpha)},\zeta_{(\alpha)}\right)+\frac{1}{2}\left(\frac{1}{\mu}dG_\mu[\varepsilon\zeta,\beta b](\psi_{(\alpha)}).(\partial_t\zeta,\partial_t b),\psi_{(\alpha)}\right).\end{aligned}\tag{2.38}$$

We have to control all the term in the r.h.s.

$\star$ Control of $\left(\partial_t\underline{\mathfrak{a}}\zeta_{(\alpha)},\zeta_{(\alpha)}\right)$.

Using the second point of Proposition 2.2.11 we get

$$\begin{aligned}\left|\left(\partial_t\underline{\mathfrak{a}}\zeta_{(\alpha)},\zeta_{(\alpha)}\right)\right|\leq M_N C&\left(\rho_{\max},\left|\partial_t b\right|_{W^{1,\infty}_t H^N_X},\left|\nabla P\right|_{L^\infty_t H^N_X},\varepsilon\mathcal{E}^N(U)^{\frac{1}{2}}\right)\varepsilon\mathcal{E}^N(U)^{\frac{3}{2}}\\ &+C\Big(M_N,\beta\lambda\left|\partial_t b\right|_{L^\infty_t H^N_X}\Big)\Big(\left|\nabla P\right|_{W^{1,\infty}_t H^N_X}+\frac{\beta\lambda}{\varepsilon}\left|\partial_t^2 b\right|_{W^{1,\infty}_t H^N_X}\Big)\varepsilon\mathcal{E}^N(U).\end{aligned}$$

$\star$ Control of $\left(\underline{\mathfrak{a}}\zeta_{(\alpha)},\frac{\beta\lambda}{\varepsilon}G^{NN}_\mu[0,0](\partial_t\partial^\alpha b)\right)$.

We get, thanks to Proposition 2.2.11 and A.2.13,

$$\left|\left(\underline{\mathfrak{a}}\zeta_{(\alpha)},\frac{\beta\lambda}{\varepsilon}G^{NN}_\mu[0,0](\partial_t\partial^\alpha b)\right)\right|\leq C\Big(\rho_{\max},\mu_{\max},\left|b\right|_{W^{2,\infty}_t H^N_X},\left|\nabla P\right|_{L^\infty_t H^N_X},\varepsilon\mathcal{E}^N(U)^{\frac{1}{2}}\Big)\frac{\beta\lambda}{\varepsilon}\left|\partial_t b\right|_{L^\infty_t H^N_X}\mathcal{E}^N(U)^{\frac{1}{2}}.$$

$\star$ Controls of $\varepsilon\mathbf{1}_{\{|\alpha|=N\}}\left(\underline{\mathfrak{a}}\zeta_{(\alpha)},\nabla\cdot\left(\zeta_{(\alpha)}\underline{V}\right)\right)$.

A simple computation gives, using Proposition 2.2.11 and Proposition A.2.17,

$$\left|\varepsilon\left(\underline{\mathfrak{a}}\zeta_{(\alpha)},\nabla\cdot\left(\zeta_{(\alpha)}\underline{V}\right)\right)\right| = \left|\varepsilon\left(\underline{\mathfrak{a}}\zeta_{(\alpha)}\nabla\cdot\underline{V},\zeta_{(\alpha)}\right)\right|$$
$$\leq C\left(\rho_{\max},\mu_{\max},|b|_{W_t^{2,\infty}H_X^N},|\nabla P|_{L_t^{\infty}H_X^N},\delta\mathcal{E}^N(U)\right)\varepsilon\left[\mathcal{E}^N(U)^{\frac{3}{2}}+\mathcal{E}^N(U)\right].$$

⋆ Controls of the terms $\left(\frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\psi_{(\alpha)}),S_\alpha-\partial^\alpha P\right)$, $\left(\frac{1}{\mu}dG_\mu[\varepsilon\zeta,\beta b](\psi_{(\alpha)}).(\partial_t\zeta,\partial_t b),\psi_{(\alpha)}\right)$ and $\left(\underline{\mathfrak{a}}\zeta_{(\alpha)},\widetilde{R_\alpha}\right)$.

We can use the same arguments as in the third and the fourth point of part a) and Propositions 2.2.8 and 2.2.11.

⋆ Control of $\left(\frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](\psi_{(\alpha)}),\underline{V}\cdot\nabla\psi_{(\alpha)}\right)$.

Thanks to Proposition A.2.19, we get the control.

Gathering the previous estimates and using Proposition 2.2.9, we obtain that

$$\frac{d}{dt}\mathcal{F}^N(U) \leq C\left(\rho_{\max},\frac{1}{h_{\min}},\mu_{\max},\frac{1}{\mathfrak{a}_{\min}},|b|_{W_t^{3,\infty}H_X^N},|\nabla P|_{W_t^{1,\infty}H_X^N},\varepsilon\mathcal{F}^N(U)^{\frac{1}{2}}\right)\times$$
$$\left(\varepsilon\mathcal{F}^N(U)^{\frac{3}{2}}+\max(\varepsilon,\beta)\mathcal{F}^N(U)+\mathcal{F}^N(U)^{\frac{1}{2}}\left[\frac{\beta\lambda}{\varepsilon}|\partial_t b|_{L_t^\infty H_X^N}+|\nabla P|_{L_t^\infty H_X^N}\right]\right). \tag{2.39}$$

Then, we easily prove Theorem 2.2.3, using the same arguments as Subsection 4.3.4.4 in [80]. Furthermore, for $\alpha\in\left[0,\frac{1}{2}\right]$, defining $\widetilde{\mathcal{F}^N}(U)(\tau)=\delta^{2\alpha}\mathcal{F}^N(U)\left(\frac{\tau}{\delta^\alpha}\right)$, we get

$$\frac{d}{d\tau}\widetilde{\mathcal{F}^N}(U)\leq C\left(\rho_{\max},\mu_{\max},\frac{1}{\mathfrak{a}_{\min}},\frac{1}{h_{\min}},|b|_{W_t^{3,\infty}H_X^N},|\nabla P|_{W_t^{1,\infty}H_X^N},\widetilde{\mathcal{F}^N}(U)\right).$$

We can also apply the same arguments as Subsection 4.3.4.4 in [80] and Theorem 2.2.4 follows. □

2.2.5 The Rayleigh-Taylor coefficient

The classical Rayleigh-Taylor criterion ([138]) is the fact that

$$\inf_{\mathbb{R}^d}[(-\partial_z\mathcal{P})_{|z=\varepsilon\zeta}]>0,$$

where $\mathcal{P}=\mathcal{P}(x,z,t)$ is here the pressure in the domain Ω_t given by the Euler equation. In this part, we express the Rayleigh-Taylor coefficient $\underline{\mathfrak{a}}$ in terms of the pressure and we show that the positivity of $\underline{\mathfrak{a}}$ is linked to the Rayleigh-Taylor criterion. Recall that

$$\underline{\mathfrak{a}}:=1+\varepsilon\partial_t\underline{w}+\varepsilon^2\underline{V}\cdot\nabla\underline{w}.$$

We nondimensionalize $\mathcal{P}$ by $\mathcal{P}'=\frac{\mathcal{P}}{\rho H g}$ and we forget the prime in the following. Notice that $\mathcal{P}_{|z=\varepsilon\zeta}=\varepsilon P$.

Proposition 2.2.13. *The Rayleigh-Taylor coefficient* $\mathfrak{a}$ *defined in* (2.25) *can be expressed as*

$$\underline{\mathfrak{a}}=-\varepsilon(\partial_z\mathcal{P})_{z=\varepsilon\zeta}.$$

Proof. This proof is very similar to the one in Subsection 4.3.5 in [80]. We consider the adimensionalization of the velocity

$$\mathbf{U} = \frac{aL\sqrt{gH}}{H^2}\nabla^\mu\left(\Phi^S + \frac{\beta\mu\lambda}{\varepsilon}\Phi^B\right)$$

and $(V, w) := \mathbf{U}$. Then, (V, w) satisfy the nondimensionalized Euler equations

$$\begin{cases} \partial_t V + \varepsilon(V \cdot \nabla_X + \frac{1}{\mu}\partial_z)V = -\nabla_X \mathcal{P}, \\ \partial_t w + \varepsilon(V \cdot \nabla_X + \frac{1}{\mu}\partial_z)w = -(\partial_z \mathcal{P} + \frac{1}{\varepsilon}). \end{cases}$$

We take the trace at the surface of these equations denoted $\mathbf{U}_{|z=\varepsilon\zeta} := (\underline{V}, \underline{w})$. A straightforward computation gives

$$\begin{cases} \partial_t \underline{V} + \varepsilon\underline{V} \cdot \nabla_X \underline{V} + \varepsilon(\partial_t\zeta + \varepsilon\underline{V} \cdot \nabla\zeta - \frac{1}{\mu}\underline{w})(\partial_z \underline{V})_{|z=\varepsilon\zeta} = -(\nabla_X \mathcal{P})_{|z=\varepsilon\zeta}, \\ \partial_t \underline{w} + \varepsilon\underline{V} \cdot \nabla_X \underline{w} + \varepsilon(\partial_t\zeta + \varepsilon\underline{V} \cdot \nabla\zeta - \frac{1}{\mu}\underline{w})(\partial_z \underline{w})_{|z=\varepsilon\zeta} = -(\partial_z \mathcal{P})_{|z=\varepsilon\zeta} - \frac{1}{\varepsilon}. \end{cases}$$

Then, remarking that the first equation of the water waves equations (2.13) can be written as

$$\partial_t\zeta + \varepsilon\underline{V} \cdot \nabla\zeta - \frac{1}{\mu}\underline{w} = 0,$$

we obtain that

$$\partial_t \underline{w} + \varepsilon\underline{V} \cdot \nabla_X \underline{w} = -(\partial_z \mathcal{P})_{|z=\varepsilon\zeta} - \frac{1}{\varepsilon}.$$

□

2.2.6 Hamiltonian system

In this section we prove that the water waves problem (2.13) is a Hamiltonian system. This extends the classical result of Zakharov ([153]) to the case where the bottom is moving and the atmospheric pressure is not constant (see also [44]). In the case of a moving bottom, Guyenne and Nicholls already pointed out it in [66] [(1)]. We have to introduce the Dirichlet-Dirichlet and the Neumann-Dirichlet operators

$$\begin{cases} G_\mu^{DD}[\varepsilon\zeta, \beta b](\psi) = \left(\Phi^S\right)_{|z=-1+\beta b}, \\ G_\mu^{ND}[\varepsilon\zeta, \beta b](\partial_t b) = \left(\Phi^B\right)_{|z=-1+\beta b}, \end{cases} \tag{2.40}$$

where Φ^S is defined in (2.15) and Φ^B is defined in (2.17). We postpone the study of these operators to appendix A (Section A.2).

Remark 2.2.14. *If we denote* $\Phi := \Phi^S + \frac{\beta\lambda\mu}{\varepsilon}\Phi^B$, Φ *satisfies*

$$\begin{cases} \Delta^\mu_{X,z}\Phi = 0 \ \text{in } \Omega_t \ , \\ \Phi_{|z=\varepsilon\zeta} = \psi \ , \ \sqrt{1+\beta^2|\nabla b|^2}\partial_{\boldsymbol{n}}\Phi_{|z=-1+\beta b} = \dfrac{\beta\lambda\mu}{\varepsilon}\partial_t b. \end{cases}$$

Then

$$\sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_{\boldsymbol{n}}\Phi_{|z=\varepsilon\zeta} = G_\mu[\varepsilon\zeta, \beta b](\psi) + \frac{\beta\mu\lambda}{\varepsilon}G_\mu^{NN}[\varepsilon\zeta, \beta b](\partial_t b), \tag{2.41}$$

[1] It seems that there is a typo in their hamiltonian; "$-\zeta v$" should read "$+\zeta v$".

and

$$\Phi_{|z=-1+\beta b} = G_\mu^{DD}[\varepsilon\zeta,\beta b](\psi) + \frac{\beta\mu\lambda}{\varepsilon} G_\mu^{ND}[\varepsilon\zeta,\beta b](\partial_t b). \tag{2.42}$$

We also have to introduce the dual spaces of the Beppo Levi spaces $\dot{H}^{-s}(\mathbb{R}^d)$ which are the spaces

$$H_*^s(\mathbb{R}^d) := \left\{ u \in H^s(\mathbb{R}^d),\, \exists \underline{u} \in H^{s+1}(\mathbb{R}^d),\, u = |D|\underline{u} \right\},$$

endowed with norm $|\cdot|_{H_*^s} := \left| \frac{\cdot}{|D|} \right|_{H^{s+1}}$. In the end of Subsection A.1.1, we give different properties of these spaces. We can now show that the water waves equations (2.13) is a Hamiltonian system.

Theorem 2.2.15. *Let $T > 0$, $t_0 > \frac{d}{2}$. We assume that $\zeta, b \in \mathcal{C}^0([0,T]; H^{t_0+1}(\mathbb{R}^d))$, that $\psi \in \mathcal{C}^0([0,T]; \dot{H}^2(\mathbb{R}^d))$, that $\partial_t b \in \mathcal{C}^0([0,T]; H_*^1(\mathbb{R}^d))$ and that $P \in \mathcal{C}^0([0,T]; L^2(\mathbb{R}^d))$. We also suppose that (ζ,ψ) is a solution of* (2.13) *and verified Condition* (2.19)*. Then, if we define $H = H(\zeta,\psi) = \mathcal{T}(\zeta,\psi) + \mathcal{U}(\zeta,\psi)$, where $\mathcal{T}(\zeta,\psi) = \mathcal{T}$ is*

$$\mathcal{T} = \frac{1}{2\mu}\int_{\Omega_t} \left| \nabla_{X,z}^\mu \left(\Phi^S + \frac{\beta\lambda\mu}{\varepsilon}\Phi^B \right) \right|^2 + \int_{\mathbb{R}^d} \frac{\beta\lambda}{\varepsilon}\partial_t b \left(G_\mu^{DD}[\varepsilon\zeta,\beta b](\psi) + \frac{\beta\lambda\mu}{\varepsilon} G_\mu^{ND}[\varepsilon\zeta,\beta b](\partial_t b) \right), \tag{2.43}$$

and $\mathcal{U}(\zeta,\psi) = \mathcal{U}$ is

$$\mathcal{U} = \frac{1}{2}\int_{\mathbb{R}^d} \zeta^2 dX + \int_{\mathbb{R}^d} \zeta P dX, \tag{2.44}$$

the water waves equations (2.13) *take the form*

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \partial_\zeta H \\ \partial_\psi H \end{pmatrix}.$$

Remark 2.2.16. *$\mathcal{T}$ is the sum of the kinetic energy and the moving bottom contribution and $\mathcal{U}$ the sum of the potential energy and the pressure contribution. Using Green's formula and Remark 2.2.14 we obtain that*

$$\begin{aligned}
\mathcal{T} =& \frac{1}{2}\int_{\mathbb{R}^d} \psi \left(\frac{1}{\mu} G_\mu[\varepsilon\zeta,\beta b](\psi) + \frac{\beta\lambda}{\varepsilon} G_\mu^{NN}[\varepsilon\zeta,\beta b](\partial_t b) \right) dX \\
&+ \frac{1}{2}\int_{\mathbb{R}^d} \frac{\beta\lambda}{\varepsilon}\partial_t b \left(G_\mu^{DD}[\varepsilon\zeta,\beta b](\psi) + \frac{\beta\lambda\mu}{\varepsilon} G_\mu^{ND}[\varepsilon\zeta,\beta b](\partial_t b) \right) dX,
\end{aligned}$$

Remark 2.2.17. *The fact that $\partial_t b$ has to be in $H_*^1(\mathbb{R}^d)$ is useful to make sense to $\mathcal{T}$ (see Proposition A.2.8). Furthermore, notice that if $b \in \mathcal{C}^0([0,T]; L^1(\mathbb{R}^d))$ and $\partial_t b \in \mathcal{C}^0([0,T]; H_*^1(\mathbb{R}^d))$, Remark A.1.15 shows that the quantity $\int_{\mathbb{R}^d} b(t,\cdot) dX$ is constant. It means that the mass of the seabed stays constant during the movement which is a quite reasonable assumption for submarine landslides.*

Proof. Using the linearity of the Dirichlet-Neumann and the Dirichlet-Dirichlet operators with respect to ψ and the fact that the adjoint of $G_\mu^{NN}[\varepsilon\zeta,\beta b]$ is $G_\mu^{DD}[\varepsilon\zeta,\beta b]$ (see Proposition A.2.8), we get that

$$\partial_\psi H = \frac{1}{\mu} G_\mu[\varepsilon\zeta,\beta b](\psi) + \frac{\beta\lambda}{\varepsilon} G_\mu^{NN}[\varepsilon\zeta,\beta b](\partial_t b).$$

Applying Proposition A.2.20 (which provides explicit expressions for shape derivatives) and remark 2.2.16, we obtain that

$$\begin{aligned}
2\partial_\zeta H &= -\frac{\varepsilon}{\mu} G_\mu[\varepsilon\zeta, \beta b](\psi)\underline{w} + \varepsilon \nabla\psi \cdot \underline{V} - \varepsilon \frac{\beta\lambda}{\varepsilon} G_\mu^{NN}[\varepsilon\zeta, \beta b](\partial_t b)\underline{w} + 2P + 2\zeta, \\
&= -\frac{\varepsilon}{\mu} G_\mu[\varepsilon\zeta, \beta b](\psi)\underline{w} + \varepsilon \nabla\psi \cdot \nabla\psi - \varepsilon^2 \underline{w} \nabla\psi \cdot \nabla\zeta - \varepsilon \frac{\beta\lambda}{\varepsilon} G_\mu^{NN}[\varepsilon\zeta, \beta b](\partial_t b)\underline{w} + 2P + 2\zeta, \\
&= \varepsilon \left|\nabla\psi\right|^2 - \frac{\varepsilon}{\mu} \underline{w}^2 \left(1 + \varepsilon^2 \mu |\nabla\zeta|^2\right) + 2P + 2\zeta,
\end{aligned}$$

which ends the proof. □

2.3 Asymptotic models

In this part, we derive some asymptotic models in order to model two different types of tsunamis. The most important phenomenon that we want to catch is the Proudman resonance ([120], see also [109] or [144] for numerical simulations) and the submarine landslide tsunami phenomenon (see [86], [141] or [142] for numerical simulations). The resonance occurs in a linear case. The duration of the resonance depends on the phenomenon. For a meteotsunami, the duration of the resonance corresponds to the time the meteorological disturbance takes to reach the coast (see [109]). However, for a landslide tsunami, the duration of the resonance corresponds to the duration of the landslide (which depends on the size of the slope, see [86] or [141]). If the landslide is offshore, it is unreasonable to assume that the duration of the landslide is the time the water waves take to reach the coast. A variation of the pressure of 1 hPa creates a water wave of 1 cm whereas a moving bottom of 1 cm tends to create a water wave of 1 cm. Therefore we assume in the following that $a_{\text{bott,m}} = a$ (and hence $\beta\lambda = \varepsilon$). However, it is important to notice that even if for storms, a variation of the pressure of 100 hPa is very huge, it is quite ordinary that a submarine landslide have a thickness of 1 m. Typically, a storm makes a variation of few Hpa, and the thickness of a submarine landslide is few dm (we refer to [86]).

In this part, we only study the propagation of such phenomena. Therefore, we take $d = 1$. In the following, we give three linear asymptotic models of the water waves equations and we give examples of pressures and moving bottoms that create a resonance. The pressure at the surface P and the moving bottom b_m move from the left to the right. We consider that the system is initially at rest. We start this part by giving an asymptotic expansion with respect to μ and $\max(\varepsilon, \beta)$ of $G_\mu[\varepsilon\zeta, \beta b]$ and $G_\mu^{NN}[\varepsilon\zeta, \beta b]$.

Proposition 2.3.1. *Let $t_0 > \frac{d}{2}$, ζ and $b \in H^{t_0+2}(\mathbb{R}^d)$ such that Condition* (2.19) *is satisfied. We suppose that the parameters ε, β and μ satisfy* (2.18)*. Then, for all $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$ with $0 \leq s \leq t_0 + \frac{3}{2}$, we have*

$$\left|G_\mu^{NN}[\varepsilon\zeta, \beta b](B) - G_\mu^{NN}[0,0](B)\right|_{H^{s-\frac{1}{2}}} \leq M_0 \left|(\varepsilon\zeta, \beta b)\right|_{H^{t_0+2}} |B|_{H^{s-\frac{1}{2}}}$$

and

$$\left|G_\mu^{NN}[0,0](B) - B\right|_{H^{s-\frac{1}{2}}} \leq C\mu\, |B|_{H^{s+\frac{3}{2}}}.$$

Proof. The first inequality follows from Proposition A.2.24 and the second from Remark A.2.1. □

In the same way, using Proposition A.2.23 and Remark A.2.1, we have the following proposition.

Proposition 2.3.2. *Let $t_0 > \frac{d}{2}$, ζ and $b \in H^{t_0+2}(\mathbb{R}^d)$ such that Condition* (2.19) *is satisfied. We suppose that the parameters ε, β and μ satisfy* (2.18)*. Then, for all $\psi \in \dot{H}^{s+1}(\mathbb{R}^d)$ with $0 \leq s \leq t_0 + \frac{3}{2}$, we have*

$$\left|G_\mu[\varepsilon\zeta,\beta b](\psi) - G_\mu[0,0](\psi)\right|_{H^{s-\frac{1}{2}}} \leq \mu M_0 \left|(\varepsilon\zeta,\beta b)\right|_{H^{t_0+2}} \left|\mathfrak{P}\psi\right|_{H^{s+\frac{1}{2}}}$$

and

$$\left|\frac{1}{\mu} G_\mu[0,0](\psi) + \Delta\psi\right|_{H^{s-\frac{1}{2}}} \leq \mu C \left|\nabla\psi\right|_{H^{s+\frac{5}{2}}}.$$

We denote by $\overline{V}$ the vertically averaged horizontal component,

$$\overline{V} = \overline{V}[\varepsilon\zeta,\beta b]\left(\psi,\partial_t b\right) = \frac{1}{1+\varepsilon\zeta-\beta b}\int_{-1+\beta b}^{\varepsilon\zeta} \nabla_X\left(\Phi[\varepsilon\zeta,\beta b]\left(\psi,\partial_t b\right)(\cdot,z)\right)dz, \tag{2.45}$$

where $\Phi = \Phi[\varepsilon\zeta,\beta b]\left(\psi,\partial_t b\right)$ satisfies

$$\begin{cases} \Delta^\mu_{X,z}\Phi = 0, \ -1+\beta b \leq z \leq \varepsilon\zeta, \\ \Phi_{|z=\varepsilon\zeta} = \psi \ , \ \sqrt{1+\beta^2|\nabla b|^2}\partial_{\mathbf{n}}\Phi_{|z=-1+\beta b} = \mu\partial_t b. \end{cases}$$

The following Proposition is Remark 3.36 and a small adaptation of Proposition 3.37 and Lemma 5.4 in [80] (see also Subsection A.5.5 in [80]).

Proposition 2.3.3. *Let $T > 0$, $t_0 > \frac{d}{2}$, $0 \leq s \leq t_0$ and $\zeta, b \in W^{1,\infty}\left([0,T]; H^{t_0+2}(\mathbb{R}^d)\right)$ such that Condition* (2.19) *is satisfied on $[0,T]$. We suppose that the parameters ε, β and μ satisfy* (2.18)*. We also assume that $\psi \in W^{1,\infty}\left([0,T]; \dot{H}^{s+3}(\mathbb{R}^d)\right)$. Then,*

$$G_\mu[\varepsilon\zeta,\beta b](\psi) + \mu G^{NN}_\mu[\varepsilon\zeta,\beta b](\partial_t b) = -\mu\nabla\cdot\left((1+\varepsilon\zeta-\beta b)\overline{V}\right) + \mu\partial_t b,$$

and

$$\begin{cases} \left|\overline{V} - \nabla\psi\right|_{H^s} \leq \mu C\left(\frac{1}{h_{\min}}, \mu_{\max}, \varepsilon|\zeta|_{H^{t_0+2}}, \beta|b|_{L^\infty_t H^{t_0+2}_X}\right) \max\left(|\nabla\psi|_{H^{s+2}}, |\partial_t b|_{L^\infty_t H^{s+1}_X}\right), \\ \left|\partial_t\overline{V} - \nabla\partial_t\psi\right|_{H^s} \leq \mu C\left(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{t_0+2}}, |\partial_t\zeta|_{H^{t_0+2}}, |b|_{W^{2,\infty}_t H^{t_0+2}_X}, |\nabla\psi|_{H^{s+2}}, |\partial_t\nabla\psi|_{H^{s+2}}\right). \end{cases}$$

In this part, we will consider symmetrizable linear hyperbolic systems of the first order. We refer to [16] for more details about the wellposedness. In the following, we will only give the energy associated to the symmetrization.

2.3.1 A shallow water model for very small topography variation

2.3.1.1 Linear asymptotic

We consider the case that ε, β, μ are small. Physically, this means that we consider small amplitudes for the surface and the bottom (compared to the mean depth) and waves with large wavelengths (compared to the mean depth). The asymptotic regime (in the sense of Definition 4.19 in [80]) is

$$\mathcal{A}_{LW} = \{(\varepsilon, \beta, \lambda, \mu),\, 0 < \mu,\, \varepsilon,\, \beta \leq \delta_0,\, \beta\lambda = \varepsilon\}, \tag{2.46}$$

with $\delta_0 \ll 1$.

Proposition 2.3.4. *Let $t_0 > \frac{d}{2}$, $N \geq \max(1, t_0) + 3$, $U^0 \in E_0^N$, $P \in W^{1,\infty}(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$ and $b \in W^{3,\infty}(\mathbb{R}^+; H^{N+1}(\mathbb{R}^d))$. We suppose (2.19) and (2.29) are satisfied initially. Then, there exists $T > 0$, such that for all $(\varepsilon, \beta, \lambda, \mu) \in \mathcal{A}_{LW}$, there exists a solution $U = (\zeta, \psi) \in E^N_{\frac{T}{\sqrt{\delta_0}}}$ to the water waves equations with initial data U^0 and this solution is unique. Furthermore, for all $\alpha \in \left[0, \frac{1}{3}\right)$,*

$$\left|\zeta - \widetilde{\zeta}\right|_{L^\infty\left(\left[0, \frac{T}{\delta_0^\alpha}\right]; H^{N-4}(\mathbb{R}^d)\right)} + \left|\nabla\psi - \nabla\widetilde{\psi}\right|_{L^\infty\left(\left[0, \frac{T}{\delta_0^\alpha}\right]; H^{N-2}(\mathbb{R}^d)\right)} \leq T\delta_0^{1-3\alpha}\widetilde{C},$$

where

$$\widetilde{C} = C\left(\mathcal{E}^N\left(U^0\right), \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{W_t^{3,\infty}H_X^N}, |\nabla P|_{W_t^{1,\infty}H_X^N}\right),$$

and with, $(\widetilde{\zeta}, \widetilde{\psi})$ solution of the waves equation

$$\begin{cases} \partial_t\widetilde{\zeta} + \Delta_X\widetilde{\psi} = \partial_t b, \\ \partial_t\widetilde{\psi} + \widetilde{\zeta} = -P, \end{cases} \tag{2.47}$$

with initial data U^0.

Proof. First, the system (2.47) is wellposed since it can be symmetrized thanks to the energy

$$\mathcal{E}(t) = \left|\widetilde{\zeta}\right|_2^2 + \left|\nabla\widetilde{\psi}\right|_2^2.$$

Using Theorem 2.2.4 we get a uniform time of existence $\frac{T}{\sqrt{\delta_0}} > 0$ for the water waves equation and for all parameters in $\mathcal{A}_{LW}$. Then, using Proposition 2.3.1, Proposition 2.3.2, Proposition A.2.17 and B.1.1 and standard controls we get that

$$\begin{cases} \partial_t\zeta + \Delta_X\psi = \partial_t b + R_1, \\ \partial_t\psi + \zeta = -P + R_2, \end{cases} \tag{2.48}$$

with

$$\begin{cases} |R_1|_{H^{N-4}} \leq C\Big(\varepsilon|\zeta|_{H^N}, |b|_{L_t^\infty H_X^N}\Big)(|(\varepsilon\zeta, \beta b)|_{H^N} + \mu)\max\left(|\mathfrak{P}\psi|_{H^{N-\frac{1}{2}}}, |\partial_t b|_{H^N}\right), \\ |R_2|_{H^{N-1}} \leq \varepsilon C\Big(\varepsilon|\zeta|_{H^N}, |b|_{L_t^\infty H_X^N}\Big)\max\left(|\mathfrak{P}\psi|^2_{H^{N-\frac{1}{2}}}, |\partial_t b|^2_{H^N}\right). \end{cases}$$

If we denote $\zeta_1 = \zeta - \widetilde{\zeta}$ and $\psi_1 = \psi - \widetilde{\psi}$, we see that (ζ_1, ψ_1) satisfies

$$\begin{cases} \partial_t\zeta_1 + \Delta_X\psi_1 = R_1, \\ \partial_t\psi_1 + \zeta_1 = R_2. \end{cases}$$

Differentiating the energy

$$\mathcal{E}^N(t) = \frac{1}{2}|\zeta_1|^2_{H^{N-4}} + \frac{1}{2}|\nabla\psi_1|^2_{H^{N-2}},$$

we get the estimate thanks to Proposition 2.2.5 and energy estimate in Theorem 2.2.4. □

This model is well-known in the physics literature ([120] when $\partial_t b \equiv 0$, [141] when $P \equiv 0$ and [86]).

2.3.1.2 Resonance in shallow waters for very small topography variation

We consider the equation (2.47) for $d = 1$. We transform it in order to have a unique equation for $h := \widetilde{\zeta} - b$,

$$\begin{cases} \partial_t^2 h - \partial_X^2 h = \partial_X^2 \left(P + b\right), \\ h_{|t=0} = -b(0,.), \\ \partial_t h_{|t=0} = 0. \end{cases} \tag{2.49}$$

We denote $f(t,X) := (P+b)\,(t,X)$, which represents a disturbance. We want to understand the resonance for landslide and meteo tsunamis. In both cases, it is a linear respond, in the shallow water case, of a body of water due to a moving pressure or a moving bottom, when the speed of the storm or the landslide is close to the typical wave celerity (here 1). We can compute h thanks to the d'Alembert's formula

$$\begin{aligned} h(t,X) = & \underbrace{-\frac{1}{2}\left(b(0,X-t) + b(0,X+t)\right)}_{h_T(t,X)} + \underbrace{\frac{1}{2}\int_0^t \partial_X f(\tau, X+t-\tau) d\tau}_{:=h_L(t,X)} \\ & \underbrace{-\frac{1}{2}\int_0^t \partial_X f(\tau, X-t+\tau) d\tau}_{:=h_R(t,X)}. \end{aligned}$$

We are interesting in disturbances f moving from the left to the right (propagation to a coast). Therefore, we study only h_R. The following Proposition shows that a disturbance moving with a speed equal to 1 makes appear a resonance.

Proposition 2.3.5. *Let $f \in L^\infty(\mathbb{R}^+; H^1(\mathbb{R}^d))$ and $\partial_X f \in L^\infty_{t\times X}(\mathbb{R}\times\mathbb{R}^d)$. Then, for all $X \in \mathbb{R}$, $t > 0$,*

$$|h_R(t,X)| \leq \frac{t}{2} \left|\partial_X f\right|_\infty.$$

Furthermore, if $f(t,X) = f_0(X-t)$, $f_0 \in H^1(\mathbb{R}^d)$ and $|f_0'(X_0 - t_0)| = |f'|_\infty$ the equality holds for (t_0, X_0). If $f(t,X) = f_0(X - Ut)$ with $f_0 \in H^1(\mathbb{R}^d)$ and $U \neq 1$,

$$|h_R|_\infty \leq \min\left(\frac{|f_0|_\infty}{|1-U|}, \frac{t}{2}\left|f_0'\right|_\infty\right).$$

Proof. If $f(t,X) = f_0(X - Ut)$,

$$h_R(t,X) = -\frac{1}{2}\int_0^t f_0'(X - t + (1-U)\tau) d\tau,$$

and the result follows. □

This Proposition corresponds to the historical work of Proudman ([120]). We rediscover the fact that the resonance occurs if the speed of the disturbance is 1. For a disturbance with a speed different from 1, we notice a saturation effect (also pointed out in [141]). The graph in Figure

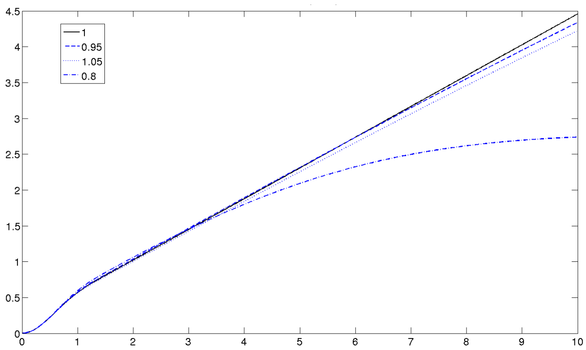

Figure 2.2: Evolution of the maximum of h, solution of equation (2.49), with different values of the speed U.

2.2, gives the typical evolution of $|h(t,\cdot)|_\infty$ with respect to the time t for different values of the speed. We can see the saturation effect. We compute h with a finite difference method and we take $f(t,X) = e^{-\frac{1}{2}(X-Ut)^2}$. We also see that the landslide resonance and the Proudman resonance have the same effects. There are however two important differences between these two phenomena. The first one is the duration of the resonance. A landslide is quicker than a meteorological effect. The second one, is the fact that the typical size of the landslide (few dm) is bigger than the size of a storm (few hPa). For instance, for a moving storm which creates a variation of the pressure of 3 hPa during $15t_0$, the final wave can reach a amplitude of 13 cm (it is for example the case of the meteotsunami in Nagasaki in 1979, see [109]). Conversely, an offshore landslide with a thickness of 1 m that lasts t_0, can create a wave of 50 cm (which corresponds to the results in [141]). Therefore, we see that the principal difference between an offshore landslide and a moving storm is the size.

2.3.2 A shallow water model when the topography is not small

2.3.2.1 Linear asymptotic

In this case, we suppose only that ε and μ are small. We recall that $\beta b(t,X) = \beta b_0(X) + \beta\lambda b_m(t,X)$. Then, we assume also that $1 - b_0 \geq h_{\min} > 0$. In the following, we denote $h_0 := 1 - \beta b_0$. The asymptotic regime is

$$\mathcal{A}_{LVW} = \{(\varepsilon,\beta,\lambda,\mu),\, 0<\varepsilon,\, \mu \leq \delta_0,\, 0<\beta\leq 1,\, \beta\lambda=\varepsilon\}\,, \tag{2.50}$$

with $\delta_0 \ll 1$. We can now give a asymptotic model.

Proposition 2.3.6. *Let* $t_0 > \frac{d}{2}$*,* $N \geq \max(1,t_0)+4$*,* $b \in W^{3,\infty}(\mathbb{R}^+; H^{N+1}(\mathbb{R}^d))$*,* $U^0 = (\zeta_0,\psi_0) \in E_0^N$*, and* $P \in W^{1,\infty}(\mathbb{R}^+;\dot{H}^{N+1}(\mathbb{R}^d))$*. We suppose that* (2.19) *and* (2.29) *are satisfied initially. We suppose also that* $b_0 \in H^N(\mathbb{R}^d)$ *and that* $h_0 = 1-\beta b_0 \geq h_{\min}$*. Then, there exists* $T>0$*, such that for all* $(\varepsilon,\beta,\lambda,\mu) \in \mathcal{A}_{LVW}$*, there exists a unique solution* $U = (\zeta,\psi) \in E_T^N$ *to the water waves equations with initial data* U^0*. Furthermore, for* $\overline{V}$ *as in* (2.45)*,*

$$|\zeta-\zeta_1|_{L^\infty([0,T];H^{N-4}(\mathbb{R}^d))} + \left|\overline{V}-\overline{V}_1\right|_{L^\infty([0,T];H^{N-4}(\mathbb{R}^d))} \leq T\delta_0\widetilde{C},$$

where

$$\widetilde{C} = C\left(\mathcal{E}^N\left(U^0\right), \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{W_t^{3,\infty} H_X^N}, |\nabla P|_{W_t^{1,\infty} H_X^N}\right),$$

and $(\zeta_1, \overline{V}_1)$ *solution of the waves equation*

$$\begin{cases} \partial_t \zeta_1 + \nabla \cdot \left(h_0 \overline{V}_1\right) = \partial_t b_m, \\ \partial_t \overline{V}_1 + \nabla \zeta_1 = -\nabla P, \\ (\zeta_1)_{|t=0} = \zeta_0,\ (V_1)_{|t=0} = \overline{V}\left[\varepsilon\zeta_0, \beta b_{|t=0}\right]\left(\psi_0, (\partial_t b)_{|t=0}\right). \end{cases} \tag{2.51}$$

Proof. The system (2.51) is wellposed since it can be symmetrized thanks to the energy

$$\mathcal{E}(t) = \frac{1}{2}|\zeta_1|_2^2 + \frac{1}{2}\left(h_0 \overline{V}_1, \overline{V}_1\right).$$

For the inequality, we proceed as in Proposition 2.3.4, differentiating the energy

$$\mathcal{E}^N(t) = \frac{1}{2}|\zeta_2|_{H^{N-4}}^2 + \frac{1}{2}\left(h_0 \Lambda^{N-4}\overline{V}_2, \Lambda^{N-4}\overline{V}_2\right),$$

with $\zeta_2 = \zeta - \zeta_1$ and $\overline{V}_2 = \overline{V} - \overline{V}_1$. Using Gronwall's Lemma, Proposition 2.3.3 and standard controls, we get result. □

This model is well-known in the physics literature to investigate the landslide tsunami phenomenon (see [141]).

2.3.2.2 Amplification in shallow waters when β is large

In this part, $d = 1$ and we suppose that $P = 0$. The same study can be done for a non constant pressure. For the sake of simplicity, we assume also that initially the velocity of the landslide is zero and hence that $(\partial_t b_m)_{|t=0} = 0$ (the bottom does not move at the beginning). We transform the system (2.51) in order to get an equation for ζ_1 only. We obtain that ζ_1 satisfies

$$\partial_t^2 \zeta_1 - \partial_X \left(h_0 \partial_X \zeta_1\right) = \partial_t^2 b_m, \tag{2.52}$$

with $(\zeta_1)_{|t=0} = 0$ and $(\partial_t \zeta_1)_{|t=0} = 0$. Notice that this equation can not be transformed into a system of two transport equation with opposite speeds. Hence, there are no analytical solutions. We wonder now if we can catch an elevation of the sea level with this asymptotic model. Therefore, we are looking for solutions of the form

$$\zeta_2(t, X) = t\zeta_3(t, X). \tag{2.53}$$

The following proposition gives example of such solutions for bounded moving bottoms (with finite energy).

Proposition 2.3.7. *Suppose that* $h_0 \geq h_{\min} > 0$ *with* $h_0 \in H^1(\mathbb{R})$*. Let* $\left(\zeta_3, \overline{V}_3\right)$ *be a solution of*

$$\begin{cases} \partial_t \zeta_3 + \partial_X \left(h_0 \overline{V}_3\right) = 0, \\ \partial_t \overline{V}_3 + \partial_X \zeta_3 = 0, \end{cases}$$

with $\left(\zeta_3, \overline{V}_3\right)_{|t=0} = (0, f')$ *with* $f \in H^1(\mathbb{R})$*. Then,* $\zeta_1(t, X) = t\zeta_3(t, X)$ *is a non trivial solution of* (2.52) *with*

$$b_m(t,X) = 2\int_0^t \zeta_3(s,X)ds, \tag{2.54}$$

and $b_m(t,\cdot)$ is bounded in $L^2(\mathbb{R}^d)$ and in $L^\infty(\mathbb{R}^d)$ uniformly with respect to t

$$|b_m(t,\cdot)|_2 + |b_m(t,\cdot)|_{L^\infty} \leq C,$$

where C is independent on t.

Proof. Plugging the expression of ζ and b_m in (2.52), we get the first result. We have to show that $\zeta_3 \in L^1(\mathbb{R}^+; L^2(\mathbb{R}^d))$. Consider the linear hyperbolic equation

$$\begin{cases} \partial_t \eta + \partial_X (h_0 W) = 0, \\ \partial_t W + \partial_X \eta = 0, \end{cases}$$

with $(\eta, W)_{|t=0} = (-f, 0)$. This system has a unique solution $(\eta, W) \in \mathcal{C}^0(\mathbb{R}; H^1(\mathbb{R}))$. Furthermore, $(\partial_t \eta, \partial_t W) \in \mathcal{C}^0(\mathbb{R}; L^2(\mathbb{R}))$, and $(\partial_t \eta, \partial_t W)$ satisfies the same linear hyperbolic system as $(\zeta_3, \overline{V}_3)$. By uniqueness, $\zeta_3 = \partial_t \eta$ and

$$b_m(t,X) = 2\eta(t,X) + 2f(X).$$

Since, for all t,

$$\int_{\mathbb{R}} \eta(t,X)^2 + h_0(X) W(t,X)^2 dX = \int_{\mathbb{R}} f(X)^2 dX,$$

and $h_0 \geq h_{\min} > 0$, we get the control of $|b_m(t,\cdot)|_2$. Finally, η satisfies the waves equation

$$\partial_t^2 \eta - \partial_X (h_0 \partial_X \eta) = 0,$$

with $(\eta, \partial_t \eta)_{|t=0} = (-f, 0) \in H^1(\mathbb{R}^d)$. Then, for all t,

$$\int_{\mathbb{R}} |\partial_t \eta(t,X)|^2 + h_0(X) |\partial_X \eta(t,X)|^2 dX = \int_{\mathbb{R}} h_0(X) f'(X)^2 dX.$$

Therefore, $|\eta|_{H^1}$ (and $|\eta|_{L^\infty}$ by Sobolev embedding) is controlled uniformly with respect to t. □

In the following, we compute numerically some solutions of Equations (2.52) of the form (2.53) with a finite difference method. We take $b_0(X) = -\tanh(X)$, $\beta = \frac{1}{2}$ and $(\partial_t \zeta_3)_{|t=0} = (4X^2 - 2)e^{-X^2}$. The figure 2.3 is the evolution of the maximum of ζ_1. The figure 2.4 is the graph at different times of the waves and the landslide. The dashed curves are the landslide, the solid curves are the waves and the dotted curve is the slope. Therefore, we see that an important elevation of the sea level is possible even if we do not consider that the seabed is flat. This is what happened during the meteotsunami in Nagasaki bay in 1979. The shelf in the East China sea gradually decreases from 20 meters to 200 meters over 500 kilometers (see [67], [144]) and hence is favourable for an amplification.

Remark 2.3.8. *In order to simplify, we consider that the system is initially at rest. But our study can easily be extended to waves with non trivial initial data. In particular, we can study a wave amplified by a landslide. This is what seemed to happen during the 2011 Tohoku tsunami, responsible of the Fukushima nuclear disaster. Indeed, no models and numerical simulations validated the run-up heights of up to 40 meters measured along the coast of the north east part of Honshu Island. Hence, in [136], they proposed that a landslide amplified the tsunami wave*

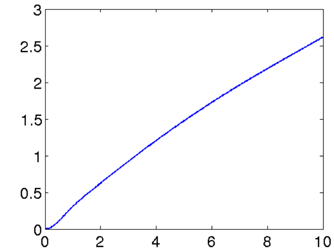

Figure 2.3: Evolution of the maximum of ζ_1, solution of (2.52), for a non flat bottom b_0.

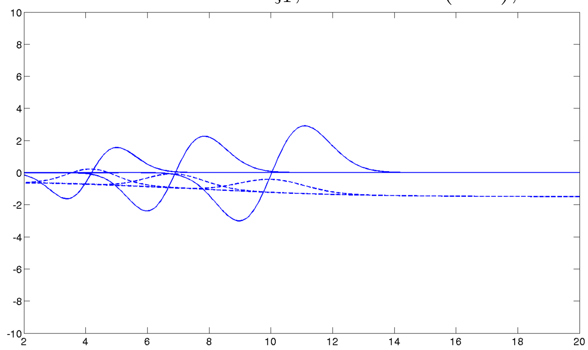

Figure 2.4: Evolution of the surface ζ_1 (solid line), solution of (2.52), and the landslide b_m (dashed line).

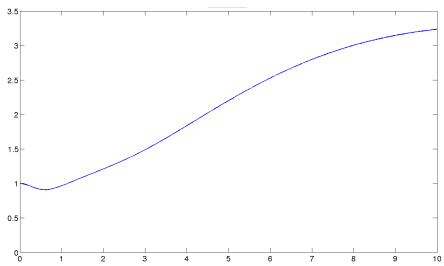

Figure 2.5: Evolution of the maximum of h, solution of (2.52), with non trivial initial data and with b_m like in Figure 2.4.

and they validated this assumption thanks to numerical simulations and physical considerations. We compute numerically this amplification. We consider a wave moving with a speed equal to 1 *(typical speed in the sea after nondimensionalization) that is amplified by a landslide. Figure 2.5 represents the evolution of the maximum of this wave. We can see that an amplification occurs relatively quickly.*

Remark 2.3.9. *We saw in Subsection 2.3.1 that the main mathematical argument that explains the Proudman resonance is the non L^∞-dispersion of the* 1*D wave equation. It allows us to show that a resonance could occur with a factor of amplification of t (linear amplification). When we add a bottom, our model is a variable coefficient* 1*D wave equation. In our knowledge, there are no mathematical results about non L^∞-dispersion for the variable* 1*D wave equation.*

We wonder now if a landslide with a speed equal to 1 can also create a resonance. The following results show that it is true and that the factor of amplification is of order t.

Lemma 2.3.10. *Let $f \in H^3(\mathbb{R}^d)$ such that, there exists $\alpha > 1$, for $k \in \{0,1,2,3\}$,*

$$\left| f^{(k)}(X) \right| \leq \frac{C}{(1+|X|)^{\alpha}}.$$

Let $h_0 \in W^{1,\infty}(\mathbb{R})$, such that $h_0 \geq h_{\min} > 0$ and

$$|1 - h_0(X)| \leq \frac{C}{(1+|X|)^{\alpha+1}} \text{ and } |h_0'(X)| \leq \frac{C}{(1+|X|)^{\alpha+1}}.$$

Then, if we denote $\zeta(t,X) = tf'(X-t) - f(X-t)$ and $v(t,X) = tf'(X-t)$, (ζ, v) is solution of

$$\begin{cases} \partial_t \zeta + \partial_X (h_0 v) = g + 2f'(X-t), \\ \partial_t v + \partial_X \zeta = 0, \end{cases}$$

where g satisfies

$$|g(t,\cdot)|_{L^2_X} \leq \frac{C}{(1+|t|)^{\alpha-1}} \text{ and } |\partial_t g(t,\cdot)|_{L^2_X} \leq \frac{C}{(1+|t|)^{\alpha-1}}.$$

Proof. we compute g

$$g(t,X) = t\left((h_0(X)-1)f''(X-t) + h_0'(X)f'(X-t)\right).$$

The result follows from Peetre's inequality. □

We also have an integrability result for the variable wave equation.

Lemma 2.3.11. *Let $h_0 \in W^{1,\infty}(\mathbb{R})$, such that $h_0 \geq h_{\min} > 0$ and let $u = (\zeta, v)$ be a solution of*

$$\begin{cases} \partial_t \zeta + \partial_X (h_0 v) = g, \\ \partial_t v + \partial_X \zeta = 0. \end{cases}$$

with initial data equal to 0 and where $g, \partial_t g \in L^1_t\left(\mathbb{R}^+; L^2_X(\mathbb{R})\right)$ and $g \in L^\infty_t\left(\mathbb{R}^+; L^2_X(\mathbb{R})\right)$. Then, $u \in L^\infty_{(t,X)}(\mathbb{R}^2)$.

Proof. We denote $E(t) = (\zeta,\zeta)_2 + (h_0 v, v)_2 + (\partial_t \zeta, \partial_t \zeta)_2 + (h_0 \partial_t v, \partial_t v)_2$. Using the equation we get

$$\frac{d}{dt} E(t) \leq C.\left(|g(t,\cdot)|_{L^2_X} + |\partial_t g(t,\cdot)|_{L^2_X} \right) \sqrt{E(t)}.$$

Since $|g(t,\cdot)|_{L^2_X}$ and $|\partial_t g(t,\cdot)|_{L^2_X}$ are integrable, the energy E is bounded. Finally, using the equations, we can control the H^1-norm of u and we get the result. □

Gathering these two results we obtain the following proposition.

Proposition 2.3.12. *Let $(\zeta_0, v_0) \in H^1(\mathbb{R})$, $f \in H^3(\mathbb{R}^d)$ such that, there exists $\alpha > 2$, for $k \in \{0,1,2,3\}$,*

$$\left| f^{(k)}(X) \right| \leq \frac{C}{(1+|X|)^{\alpha}}.$$

Let $h_0 \in W^{1,\infty}(\mathbb{R})$, such that $h_0 \geq h_{\min} > 0$ and

$$|1-h_0(X)| \leq \frac{C}{(1+|X|)^{\alpha+1}} \text{ and } |h_0'(X)| \leq \frac{C}{(1+|X|)^{\alpha+1}}.$$

Consider the solution (ζ, v) of

$$\begin{cases} \partial_t \zeta + \partial_X (h_0 v) = 2f'(X-t), \\ \partial_t v + \partial_X \zeta = 0, \\ (\zeta, v)_{|t=0} = (\zeta_0, v_0). \end{cases}$$

Then, there exists a constant $C > 0$, such that $|(\zeta, v)(t, \cdot)|_{L^\infty_X} \geq C(t-1)$.

Hence, for slopes which converge quickly enough to a flat bottom, we get a resonance for a landslide with a speed equal to 1. This fact was conjectured by physicists thanks to various numerical simulations ([57]).

Remark 2.3.13. *We saw that we can not expect analytical solution for Equation* (2.52)*. In order to get some, physicists commonly assume that the slope varies slowly. They consider a slope with h_0 of the form $h_0(x) = h_0(\alpha x)$ where $\alpha\beta$ is small. By neglecting all the terms of order $\mathcal{O}(\alpha\beta)$, we obtain*

$$\partial_t^2 \zeta - (1-h_0)\partial_X^2 \zeta = \partial_t^2 b_m,$$

which can be viewed of the composition of two transport equations with opposite speeds thanks to an appropriate change of variables. Then, it was shown in this setting that a resonance is possible for landslides which adapt their speeds with the slope (see for instance [54] and [55]).

2.3.3 Linear asymptotic and resonance in intermediate depths

In this case, we consider only that ε, β are small. Physically, this means that we consider small amplitudes for the surface and the bottom (compared to the mean depth) and that the depth is comparable to wavelength of the waves. In this part, we generalize the Proudman resonance in deeper waters. The asymptotic regime is

$$\mathcal{A}_{LWW} = \{(\varepsilon, \beta, \lambda, \mu),\ 0 < \varepsilon,\ \beta \leq \delta_0,\ \beta\lambda = \varepsilon \text{ and } 0 < \mu \leq \mu_{\max}\}, \tag{2.55}$$

with $\delta_0 \ll 1$ and $0 < \mu_{\max}$. Using the energy

$$\mathcal{E}(t) = \frac{1}{2}|\zeta|_2^2 + \frac{1}{2}\left(\frac{1}{\mu}G_\mu[0,0](\psi), \psi\right),$$

and proceeding as in Proposition 2.3.4 (we need also Proposition A.2.4), we get a new asymptotic model.

Proposition 2.3.14. *Let $t_0 > \frac{d}{2}$, $N \geq \max(1, t_0) + 3$, $b \in W^{3,\infty}(\mathbb{R}^+; H^{N+1}(\mathbb{R}^d))$, $U^0 = (\zeta_0, \psi_0) \in E_0^N$ and $P \in W^{1,\infty}(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$. We suppose that* (2.19) *and* (2.29) *are satisfied initially. Then, there exists $T > 0$, such that for all $(\varepsilon, \beta, \lambda, \mu) \in \mathcal{A}_{LWW}$, there exists a unique solution $U = (\zeta, \psi) \in E^N_{\frac{T}{\sqrt{\delta_0}}}$ to the water waves equations with initial data U^0. Furthermore, for all $\alpha \in \left[0, \frac{1}{3}\right)$,*

$$\left|\zeta - \widetilde{\zeta}\right|_{L^\infty\left(\left[0, \frac{T}{\delta_0^\alpha}\right]; H^{N-2}(\mathbb{R}^d)\right)} + \left|\frac{|D|}{\sqrt{1+|D|}}\left(\psi - \widetilde{\psi}\right)\right|_{L^\infty\left(\left[0, \frac{T}{\delta_0^\alpha}\right]; H^{N-2}(\mathbb{R}^d)\right)} \leq T\delta_0^{1-3\alpha}\widetilde{C},$$

where

$$\widetilde{C} = C\left(\mathcal{E}^N\left(U^0\right), \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, \mu_{\max}, |b|_{W_t^{3,\infty}H_X^N}, |\nabla P|_{W_t^{1,\infty}H_X^N}\right),$$

where $\left(\widetilde{\zeta}, \widetilde{\psi}\right)$ *is a solution of the waves equation*

$$\begin{cases} \partial_t \widetilde{\zeta} - \dfrac{1}{\mu} G_\mu[0,0](\widetilde{\psi}) = G_\mu^{NN}[0,0](\partial_t b), \\ \partial_t \widetilde{\psi} + \widetilde{\zeta} = -P, \end{cases} \tag{2.56}$$

with initial data U^0.

The Proudman resonance is a phenomenon which occurs in shallow water regime. We wonder if there is also a resonance in deeper waters. In this part, we only work with a non constant pressure and hence $\partial_t b = 0$. The same study can be done for a moving bottom. We consider the equation (2.56) for $d = 1$. Since, the initial data does not affect the possible resonance, we suppose in the following that $U^0 = 0$. We transform the system (2.56) in order to have a unique equation for $\widetilde{\zeta}$ (in the following we denote $\widetilde{\zeta}$ by ζ to simplify the notation)

$$\begin{cases} \partial_t^2 \zeta + \dfrac{1}{\mu} G_\mu[0,0](\zeta) = -\dfrac{1}{\mu} G_\mu[0,0](P), \\ \zeta_{|t=0} = 0,\ \partial_t \zeta_{|t=0} = 0. \end{cases}$$

We can solve explicitly the previous equation, we get that

$$\begin{aligned} \widehat{\zeta}(t,\xi) = & \underbrace{\frac{i}{2}\int_0^t \xi \sqrt{\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}}\, \widehat{P}(\tau,\xi) e^{i(t-\tau)\xi\sqrt{\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}}} d\tau}_{:=\widehat{\zeta}_L(t,\xi)} \\ & \underbrace{-\frac{i}{2}\int_0^t \xi \sqrt{\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}}\, \widehat{P}(\tau,\xi) e^{i(\tau-t)\xi\sqrt{\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}}} d\tau}_{:=\widehat{\zeta}_R(t,\xi)}. \end{aligned}$$

In order to find a resonant pressure, we suppose that P has the form $e^{-ita(D)}P_0$, where a is a real smooth odd function which is sublinear, there exists $C > 0$ such that $|a(\xi)| \leq C|\xi|$. We also suppose that the phase velocity of the disturbance is positive, $\frac{a(\xi)}{\xi} \geq 0$. P_0 is a smooth function in a Sobolev space. We denote $\omega(\xi) = \sqrt{\frac{\tanh(\xi)}{\xi}}$. A simple computation gives that

$$|\zeta_L(t,\cdot)| \leq |\widehat{\zeta_L}(t,\cdot)|_{L^1} \leq \left|\widehat{P_0}\right|_{L^1}.$$

Furthermore, we have

$$\begin{aligned} |\widehat{\zeta_R}(t,\xi)| &= \frac{1}{2}\left|\int_0^t \xi\omega(\sqrt{\mu}\xi)\widehat{P}_0(\xi) e^{i\tau\left(\xi\omega(\sqrt{\mu}\xi) - a(\xi)\right)} d\tau\right| \\ &\leq \frac{t}{2}\left|\xi\omega(\sqrt{\mu}\xi)\widehat{P}_0(\xi)\right|, \end{aligned}$$

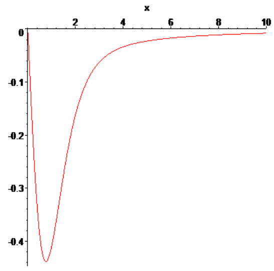

Figure 2.6: Profile of ϕ'' .

with an equality if and only if $a(\xi) = \xi\omega(\sqrt{\mu}\xi)$. Hence, it is natural to consider that

$$\widehat{P}(t,\xi) = e^{-it\xi\omega(\sqrt{\mu}\xi)} P_0(\xi). \tag{2.57}$$

A simple computation gives

$$\zeta_R(t,X) = -\frac{it}{2}\int_{\mathbb{R}} \xi\omega(\sqrt{\mu}\xi)\widehat{P_0}(\xi)e^{-it\xi\omega(\sqrt{\mu}\xi)}e^{iX\xi}d\xi. \tag{2.58}$$

We wonder now if a resonance occurs. We need a dispersion estimate for the linear water waves equation.

Proposition 2.3.15. *Let $f \in W^{1,1}(\mathbb{R})$ such that $\widehat{f}(0) = 0$. Then,*

$$\left|\int_{\mathbb{R}} e^{-it\xi\omega(\sqrt{\mu}\xi)}e^{iX\xi}\widehat{f}(\xi)d\xi\right| \leq \frac{C}{\sqrt{t}}\left(\frac{1}{\sqrt{\mu}}\left|\frac{1}{\sqrt{|\xi|}}\left(\widehat{f}\right)'\right|_{L^1(\mathbb{R})} + \mu^{\frac{1}{8}}\left||\xi|^{\frac{3}{4}}\left(\widehat{f}\right)'\right|_{L^1(\mathbb{R})}\right).$$

Remark 2.3.16. *In [106], Mésognon-Gireau improved this result by only assuming f and xf in $H^1(\mathbb{R})$ (no condition on $\widehat{f}(0)$) and he got a dispersion of order $\frac{1}{t^{\frac{1}{3}}}$. Noticed that the result of Mésognon-Gireau is more convenient if one wants to apply this dispersion property to the water waves equations (2.13) (see [107] for instance). We also refer to [31] for the case of an infinite depth.*

Proof. We denote $I(t)$,

$$\begin{aligned} I(t) &:= \int_{\mathbb{R}} e^{-it\xi\omega(\sqrt{\mu}\xi)}e^{iX\xi}\widehat{f}(\xi)d\xi \\ &= \frac{1}{\sqrt{\mu}}\int_{\mathbb{R}} e^{-i\frac{t}{\sqrt{\mu}}\left(y\omega(y)-\frac{X}{t}y\right)}\widehat{f}\left(\frac{y}{\sqrt{\mu}}\right)dy. \end{aligned}$$

We denote ϕ,

$$\phi(y) = y\omega(y) - \frac{X}{t}y,$$

and y_0 the unique minimum of ϕ''. Figure 2.6 represents ϕ'' on $[0,+\infty[$.
To estimate $I(t)$ we decompose $I(t)$ into four parts.

$$
\begin{aligned}
I_1(t) &= \frac{1}{\sqrt{\mu}} \int_0^{y_0} e^{-i\frac{t}{\sqrt{\mu}}\phi(y)} \widehat{f}\left(\frac{y}{\sqrt{\mu}}\right) dy \\
&= \frac{1}{\sqrt{\mu}} \int_0^{y_0} -\frac{d}{dy}\left(\int_y^{y_0} e^{-i\frac{t}{\sqrt{\mu}}\phi(z)} dz\right) \widehat{f}\left(\frac{y}{\sqrt{\mu}}\right) dy \\
&= \frac{1}{\mu} \int_0^{y_0} \int_y^{y_0} e^{-i\frac{t}{\sqrt{\mu}}\phi(z)} dz \left(\widehat{f}\right)'\left(\frac{y}{\sqrt{\mu}}\right) dy.
\end{aligned}
$$

Then, using Van der Corput's Lemma (see [133]) and the fact that for $z \in [y, y_0]$, $|\phi''(z)| \geq |\phi''(y)|$ and $|\phi''(z)| \geq Cz$,

$$
\begin{aligned}
|I_1(t)| &\leq \frac{C}{\mu^{\frac{3}{4}}\sqrt{t}} \int_0^{y_0} \left|\frac{1}{\sqrt{y}} \left(\widehat{f}\right)'\left(\frac{y}{\sqrt{\mu}}\right)\right| dy \\
&\leq \frac{C}{\sqrt{\mu}\sqrt{t}} \int_0^{+\infty} \left|\frac{1}{\sqrt{\xi}} \left(\widehat{f}\right)'(\xi)\right| d\xi.
\end{aligned}
$$

Furthermore, for $M > y_0$ large enough,

$$
\begin{aligned}
I_2(t) &= \frac{1}{\sqrt{\mu}} \int_{y_0}^{M} e^{-i\frac{t}{\sqrt{\mu}}\phi(y)} \widehat{f}\left(\frac{y}{\sqrt{\mu}}\right) dy \\
&= \frac{1}{\sqrt{\mu}} \int_{y_0}^{M} \frac{d}{dy}\left(\int_{y_0}^{y} e^{-i\frac{t}{\sqrt{\mu}}\phi(z)} dz\right) \widehat{f}\left(\frac{y}{\sqrt{\mu}}\right) dy \\
&= \int_{y_0}^{M} e^{-i\frac{t}{\sqrt{\mu}}\phi(z)} \frac{dz}{\sqrt{\mu}} \widehat{f}\left(\frac{M}{\sqrt{\mu}}\right) - \frac{1}{\mu} \int_{y_0}^{M} \int_{y_0}^{y} e^{-i\frac{t}{\sqrt{\mu}}\phi(z)} dz \left(\widehat{f}\right)'\left(\frac{y}{\sqrt{\mu}}\right) dy.
\end{aligned}
$$

Then, using Van der Corput's Lemma and the fact that for $z \in [y_0, y]$, $|\phi''(z)| \geq |\phi''(y)|$ and $|\phi''(z)| \geq Cz^{-\frac{3}{2}}$,

$$
\begin{aligned}
|I_2(t)| &\leq \left|\frac{M}{\sqrt{\mu}} \widehat{f}\left(\frac{M}{\sqrt{\mu}}\right)\right| + \frac{C}{\mu^{\frac{3}{4}}\sqrt{t}} \int_{y_0}^{M} \left|y^{\frac{3}{4}} \left(\widehat{f}\right)'\left(\frac{y}{\sqrt{\mu}}\right)\right| dy \\
&\leq \left|\widehat{f}'\left(\frac{M}{\sqrt{\mu}}\right)\right| + \frac{C\mu^{\frac{1}{8}}}{\sqrt{t}} \int_0^{+\infty} \left|\xi^{\frac{3}{4}} \left(\widehat{f}\right)'(\xi)\right|.
\end{aligned}
$$

Tending M to $+\infty$ we get the result. The control for $\xi < 0$ is similar. □

Therefore, in the linear case, we have also a resonance.

Corollary 2.3.17. *Let $P_0 \in H^3(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$ such that $XP_0 \in H^3(\mathbb{R})$ and let $0 < \mu \leq \mu_{\max}$. We consider*

$$
\zeta_R(t, X) = -\frac{it}{2} \int_{\mathbb{R}} \xi \omega(\sqrt{\mu}\xi) \widehat{P_0}(\xi) e^{-it\xi\omega(\sqrt{\mu}\xi)} e^{iX\xi} d\xi.
$$

Then,

$$|\zeta_R(t,\cdot)|_\infty \leq C(\mu_{\max})\sqrt{\frac{t}{\mu}}\left(|P_0|_{H^3} + |P_0|_{L^1} + |XP_0|_{H^3}\right),$$

and

$$\underline{\lim}_{t\to+\infty} \left|\frac{1}{\sqrt{t}}\zeta_R(t,\cdot)\right|_\infty \geq C(P_0) > 0.$$

Proof. We take $\widehat{f}(\xi) = \xi\omega(\sqrt{\mu}\xi)\widehat{P}_0(\xi)$. Then,

$$\left|\left(\widehat{f}\right)'(\xi)\right| \leq (1+\sqrt{\mu}|\xi|)\left|\widehat{P}_0(\xi)\right| + |\xi|\left|\left(\widehat{P}_0\right)'(\xi)\right|,$$

and the first inequality follows from the previous Proposition. For the second inequality, we use a stationary phase approximation. We denote $\phi(\xi) = \xi\omega(\xi)$. Let $\xi_0 > 0$, such that $\left|\xi_0\widehat{P_0}(\xi_0)\right| = \left|\xi\widehat{P_0}\right|_{L^\infty}$, and $X_\mu < 0$, such that $\phi'(\sqrt{\mu}\xi_0) = X_\mu$. Then, we have,

$$\begin{aligned}\underline{\lim}_{t\to+\infty}\left|\frac{1}{\sqrt{t}}\zeta_R(t,tX_\mu)\right| &= \underline{\lim}_{t\to+\infty}\frac{\sqrt{t}}{2\mu}\left|\int_{\mathbb{R}}\xi\omega(\xi)\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)e^{-i\frac{t}{\sqrt{\mu}}\xi(\omega(\xi)-X_\mu)}d\xi\right| \\ &= \frac{\sqrt{2\pi}}{2\mu^{\frac{1}{4}}}\left|\frac{\omega(\xi_0\sqrt{\mu})\xi_0\widehat{P_0}(\xi_0)}{\sqrt{|\phi''(\xi_0\sqrt{\mu})|}}\right|.\end{aligned}$$

Since $|\phi''(\xi)| \leq C|\xi|$ and $\omega(\xi_0\sqrt{\mu}) \geq C(\xi_0)\sqrt{\mu}$, we get the result.

□

Remark 2.3.18. *Notice that for all $s \in \mathbb{R}$,*

$$\left|\zeta_R(t,\cdot) + \frac{t}{2}P_0'(\cdot - t)\right|_{H^s} \leq \sqrt{\mu}t^2\left|\nabla P_0\right|_{H^{s+2}}.$$

Hence, by tending formally μ to 0, we rediscover the result we get in the shallow water case (section 2.3.1).

Remark 2.3.19. *Notice that for a general pressure term $P(t,X)$ we can show that the amplitude ζ satisfying*

$$\begin{aligned}\widehat{\zeta}(t,\xi) = &\frac{i}{2}\int_0^t \xi\sqrt{\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}}\widehat{P}(\tau,\xi)e^{i(t-\tau)\xi\sqrt{\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}}}d\tau \\ &-\frac{i}{2}\int_0^t \xi\sqrt{\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}}\widehat{P}(\tau,\xi)e^{i(\tau-t)\xi\sqrt{\frac{\tanh(\sqrt{\mu}|\xi|)}{\sqrt{\mu}|\xi|}}}d\tau,\end{aligned}$$

satisfies also

$$|\zeta(t,\cdot)|_\infty \leq C(\mu_{\max})\sqrt{\frac{t}{\mu}}\left(|P|_{L^\infty(\mathbb{R}^+;L^1(\mathbb{R}^d))} + |P|_{L^\infty(\mathbb{R}^+;H^3(\mathbb{R}^d)} + |XP|_{L^\infty(\mathbb{R}^+;H^3(\mathbb{R}^d)}\right).$$

Hence, contrary to the shallow water case, we can not hope a linear amplification with respect to the time t. Corollary 2.3.17 also shows that the factor of amplification of $\sqrt{t}$ is optimal.

Hence, we observe that in intermediate water depths, a resonance can occur but with a factor of amplification of $\sqrt{t}$ and not t. But we saw that in the shallow water case, the resonance occurs for a moving pressure with a speed equal to 1, $P(t,X) = P_0(X-t)$. We wonder if this pressure can create a resonance. The following Proposition shows that the previous pressure can create a resonance with a factor of amplification of $t^{\frac{1}{3}}$.

Proposition 2.3.20. *Let $0 < \mu \leq \mu_{\max}$. Let $P_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ such that $\widehat{P_0}(0) \neq 0$. We consider the amplitude ζ_R created by $P(t,X) = P_0(X-t)$*

$$\widehat{\zeta_R}(t,\xi) = -\frac{i}{2}\xi\omega(\sqrt{\mu}\xi)\widehat{P_0}(\xi)e^{-it\xi}\int_{-t}^{0} e^{is\xi\left(\omega(\sqrt{\mu}\xi)-1\right)}ds. \tag{2.59}$$

Then,

$$|\zeta_R(t,\cdot)|_\infty \leq C(\mu_{\max})\left(\frac{t^{\frac{1}{3}}}{\mu}|P_0|_{L^1} + \mu^{\frac{1}{4}}|P_0|_{H^1}\right).$$

Furthermore,

$$\varliminf_{t\to+\infty}\left|\frac{1}{t^{\frac{1}{3}}}\zeta_R(t,\cdot)\right|_\infty \geq \frac{C}{\mu^{\frac{2}{3}}}\left|\widehat{P_0}(0)\right|.$$

Proof. We have

$$\begin{aligned}\zeta_R(t,X) &= -\frac{i}{2}\int_{\mathbb{R}}\xi\omega(\sqrt{\mu}\xi)\widehat{P_0}(\xi)e^{-it\xi}\int_{-t}^{0}e^{is\xi\left(\omega(\sqrt{\mu}\xi)-1\right)}e^{iX\xi}dsd\xi \\ &= -\frac{i}{2}\frac{1}{\mu}\int_{\mathbb{R}}\xi\omega(\xi)\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)e^{-i\frac{t}{\sqrt{\mu}}\xi}\int_{-t}^{0}e^{i\frac{s}{\sqrt{\mu}}\xi(\omega(\xi)-1)}e^{i\frac{X}{\sqrt{\mu}}\xi}dsd\xi.\end{aligned}$$

We decompose this integral into 3 parts.

$$\begin{aligned}|I_1(t)| &= \left|\frac{1}{\mu}\int_{|\xi|\leq t^{-\frac{1}{3}}}\xi\omega(\xi)\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)e^{-i\frac{t}{\sqrt{\mu}}\xi}\int_{-t}^{0}e^{i\frac{s}{\sqrt{\mu}}\xi(\omega(\xi)-1)}e^{i\frac{X}{\sqrt{\mu}}\xi}d\xi ds\right| \\ &\leq \frac{t^{\frac{1}{3}}}{\mu}\left|\widehat{P_0}\right|_\infty.\end{aligned}$$

Furthermore, since $|\omega(\xi)-1| \geq C\xi^2$ for $0 \leq |\xi| \leq 1$, we have

$$\begin{aligned}|I_2(t)| &= \left|\frac{1}{\mu}\int_{t^{-\frac{1}{3}}\leq|\xi|\leq 1}\xi\omega(\xi)\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)e^{-i\frac{t}{\sqrt{\mu}}\xi}\int_{-t}^{0}e^{i\frac{s}{\sqrt{\mu}}\xi(\omega(\xi)-1)}e^{i\frac{X}{\sqrt{\mu}}\xi}d\xi ds\right| \\ &= \left|\frac{1}{\sqrt{\mu}}\int_{t^{-\frac{1}{3}}\leq|\xi|\leq 1}e^{i\frac{X}{\sqrt{\mu}}\xi}\frac{\omega(\xi)}{\omega(\xi)-1}\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right)\left(e^{-i\frac{t}{\sqrt{\mu}}\xi} - e^{-i\frac{t}{\sqrt{\mu}}\xi\omega(\xi)}\right)d\xi\right| \\ &\leq C\frac{t^{\frac{1}{3}}}{\sqrt{\mu}}\left|\widehat{P_0}\right|_\infty.\end{aligned}$$

Finally,

$$
\begin{aligned}
|I_3(t)| &= \left| \frac{1}{\mu} \int_{|\xi|\geq 1} \xi\omega(\xi)\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right) e^{-i\frac{t}{\sqrt{\mu}}\xi} \int_{-t}^{0} e^{i\frac{s}{\sqrt{\mu}}\xi\left(\omega(\sqrt{\mu}\xi)-1\right)} e^{i\frac{X}{\sqrt{\mu}}\xi} d\xi ds \right| \\
&= \left| \frac{1}{\sqrt{\mu}} \int_{|\xi|\geq 1} e^{i\frac{X}{\sqrt{\mu}}\xi} \frac{\omega(\xi)}{\omega(\xi)-1} \widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right) \left(e^{-i\frac{t}{\sqrt{\mu}}\xi} - e^{-i\frac{t}{\sqrt{\mu}}\xi\omega(\xi)}\right) d\xi \right| \\
&\leq C \int_{|\xi|\geq \frac{1}{\sqrt{\mu}}} \left|\widehat{P_0}(\xi)\right| d\xi, \\
&\leq C\mu^{\frac{1}{4}} \left|P_0\right|_{H^1},
\end{aligned}
$$

and the first inequality follows. For the second inequality, we use a stationary phase approximation. We denote $\phi(\xi) := \xi(\omega(\xi)-1)$. We recall that $\phi(\xi) = -\frac{1}{6}\xi^3 + o(\xi^3)$. Using a generalization of Morse Lemma at the order 3, there exists $a > 0$ and $\psi \in \mathcal{C}^\infty([-a,a])$, such that for all $|y| \leq a$,

$$
\phi(\psi(y)) = \frac{1}{6}\phi'''(0)y^3, \psi(0) = 0 \text{ and } \psi'(0) = 1.
$$

Then,

$$
\begin{aligned}
I(s) &:= \int_{\mathbb{R}} \omega(\xi)\xi\widehat{P_0}\left(\frac{\xi}{\sqrt{\mu}}\right) e^{i\frac{s}{\sqrt{\mu}}\xi(\omega(\xi)-1)} d\xi \\
&= \int_{-a}^{a} \psi'(y)\omega(\psi(y))\psi(y)\widehat{P_0}\left(\frac{\psi(y)}{\sqrt{\mu}}\right) e^{i\frac{s}{6\sqrt{\mu}}y^3} dy + o(s^{-\frac{2}{3}}) \\
&= \left(\frac{6\sqrt{\mu}}{s}\right)^{\frac{2}{3}} \widehat{P_0}(0) \int_{z\in\mathbb{R}} z e^{iz^3} dz + o(s^{-\frac{2}{3}}).
\end{aligned}
$$

Therefore,

$$
\lim_{t\to+\infty} \left| \frac{1}{t^{\frac{1}{3}}} \zeta_R(t,t) \right| = \frac{C}{\mu^{\frac{2}{3}}} \left|\widehat{P_0}(0)\right|.
$$

□

Then, in intermediate water depths, a traveling pressure with a constant speed equal to 1 is also resonant. It takes more time to obtain a significant elevation of the level of the sea compare to the shallow water case. In the following, we compute numerically some solutions. We take $P_0(X) = -e^{-X^2}$ and $\mu = 1$. The figure 2.7 displays the evolution of a water wave because of a pressure of the form (2.57). The solid curve is the wave and the dashed curve is the moving pressure. The figure 2.8 displays the evolution is a water wave when the pressure moves with a speed 1. The figure 2.9 compares the evolution of the maximum of the resonant case and the case when the speed is equal to 1.

Remark 2.3.21. *In our work, we neglect the Coriolis effect. However, in view of the duration of the meteotsunami phenomenon, it would be more realistic to consider it. It will be studied in Chapter 4.*

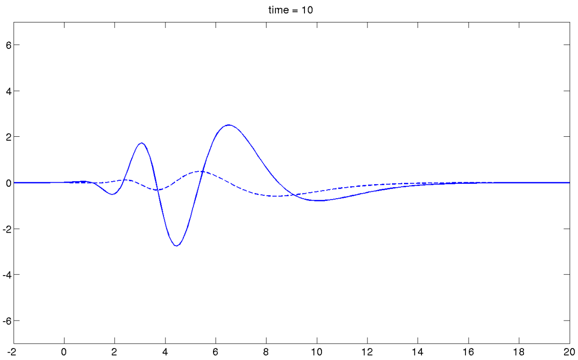

Figure 2.7: Evolution of the surface elevation ζ_R in (2.58) (solid line) because of a resonant moving pressure P in (2.57) (dashed line).

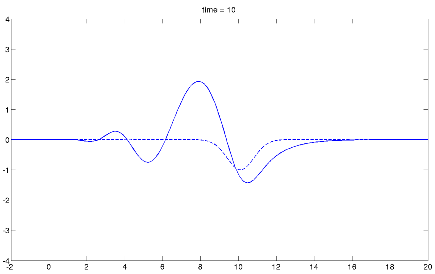

Figure 2.8: Evolution of the surface elevation ζ_R in (2.59) (solid line) because of a moving pressure P with a speed of 1 (dashed line).

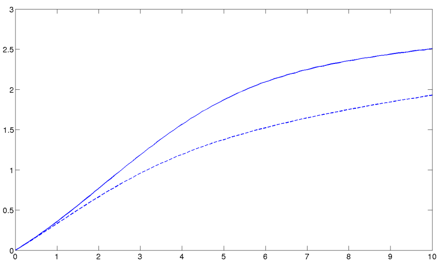

Figure 2.9: Evolution of the maximum of ζ_R in the resonant case (solid line) and the moving pressure with a speed of 1 (dashed line).

2.3.4 Toward nonlinear asymptotic models

Previously, we gave linear asymptotic models to understand the Proudman resonance. However, sometime, the nonlinear effects can not be neglected. We propose in this part numerical simulations to understand how the nonlinear effects perturb the resonance. The first model is the nonlinear shallow water equation and the second one is the Saut-Xu equation. Since our goal is to catch the nonlinear effects, we consider that the bottom is flat and we only work with a non constant pressure. We show in both cases that an amplification due to the source term is possible but some shocks can occur and create a saturation of the L^∞-norm.

2.3.4.1 The nonlinear shallow water equation

The linear asymptotic model that we studied in Part 2.3.1 is a shallow water model and we saw that a Proudman resonance is possible for pressure with a speed equal to 1 and the factor of amplification of t. In order to study the impact of nonlinear effects, we can use the so-called nonlinear shallow water equations

$$\begin{cases} \partial_t \zeta + \partial_x((1+\varepsilon\zeta)v) = 0 \\ \partial_t v + \partial_x \zeta + \varepsilon v \partial_x v = -P. \end{cases} \tag{2.60}$$

To simulate numerically this system, we use a well balanced finite volume scheme with a rusanov flux. We refer to [11] and [23] for more details about this scheme. We take P of the form $P(t,x) = P_0(X-t)$ with $P_0(X) = \exp(-\frac{1}{3}X^2)$ and we take $\varepsilon = 1$. Figure 2.10 shows the evolution of the surface in the linear and nonlinear case. Figure 2.11 compares the maximum of the two cases. In Figure 2.10, we see that in the nonlinear case, the water wave is about to break. It creates a saturation of the L^∞-norm. This effect is well-known for water waves without a forcing term (see for instance the numerical simulations in [23]).

In [144], Vilibic also studied the Proudman resonance thanks to the nonlinear shallow water equations. He shown that an amplification due to the source term of the amplitude of the surface is possible (see Figure 2 and figure 5 in [144]).

Remark 2.3.22. *In [118] (see also [63]), Pelinovsky et al. use the forced KdV equation to model water waves generated by atmospheric disturbances moving with long wave speeds. It could be interesting to study the Proudman resonance in this framework.*

2.3.4.2 The Saut-Xu equations

We studied in Part 2.3.3 the Proudman resonance in deep water, i.e typically for $\mu = 1$. The idea to get a weakly nonlinear model in deep water is to forget all the terms of order $\mathcal{O}(\varepsilon^2)$ in the water waves equations (2.13). Then, we get the following system

$$\begin{cases} \partial_t \zeta - \mathcal{H} v + \varepsilon \left(\mathcal{H}\left(\zeta \partial_x \mathcal{H} v\right) + \partial_x\left(\zeta v\right)\right) = 0 \\ \partial_t v + \partial_x \zeta + \varepsilon v \partial_x v - \varepsilon\sqrt{\mu}\partial_x \zeta \mathcal{H} \partial_x \zeta = -P, \end{cases} \tag{2.61}$$

where $\zeta = \zeta(t,x)$ is the free surface, $v = v(t,x)$ is the horizontal velocity at the surface and $\mathcal{H}$ is the Fourier multiplier, $\mathcal{H} = -\frac{\tanh(D)}{D}\partial_x$. This model was derived by Matsuno ([97]) and is now called the Matsuno equations. It is important to notice that this model is only derived. In our knowledge, the wellposedness of the Matsuno equations is still an open problem (see the paper of Ambrose, Bona and Nicholls [10]). To avoid this difficulty, Saut and Xu ([125]) developed an equivalent problem to the Matsuno system which is consistent to the water waves problem and with the same accuracy ($\mathcal{O}(\varepsilon^2)$). Then, they proved that this new system is wellposed. The Saut-Xu equations are

$$\begin{cases} \partial_t \zeta - \mathcal{H}_\mu v + \varepsilon\sqrt{\mu}\left(\frac{1}{2} v \partial_x \zeta + \frac{1}{2}\mathcal{H}_\mu\left(v \partial_x \mathcal{H}_\mu \zeta\right) + \mathcal{H}_\mu\left(\zeta \partial_x \mathcal{H}_\mu v\right) + \zeta \partial_x v\right) = 0, \\ \partial_t v + \partial_x \zeta + \frac{3\varepsilon\sqrt{\mu}}{2} v \partial_x v - \frac{\varepsilon\sqrt{\mu}}{2}\partial_x \zeta \mathcal{H}_\mu \partial_x \zeta - \frac{\varepsilon\sqrt{\mu}}{2} v \mathcal{H}_\mu^2 \partial_x v = -P. \end{cases} \tag{2.62}$$

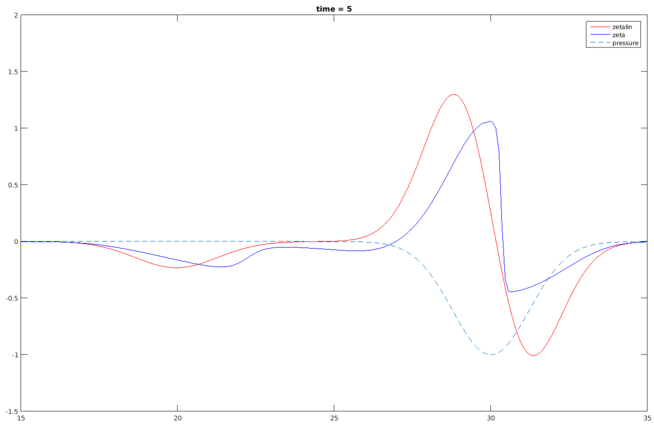

Figure 2.10: Evolution of the surface in the nonlinear case (blue line), the linear case (red line). The dashed in the corresponding pressure.

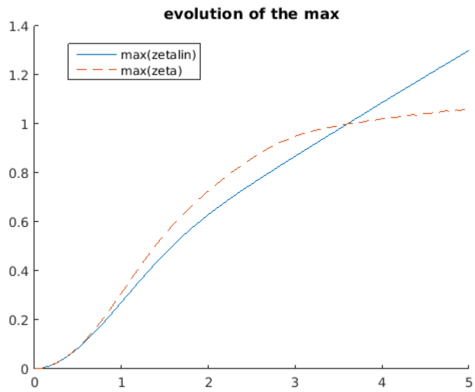

Figure 2.11: Comparison of the maximum of the surface between the linear case (blue line) and the nonlinear case (red line).

In Section 3.3, we introduce a splitting scheme to solve numerically the Saut-Xu system and we show that this scheme converges (Theorem 3.4.6). We also refer to Section 3.2 for more details about the Saut-Xu equations. In the following we give different numerical simulations to understand how the nonlinear effect can perturb the Proudman resonance. We recall that if the pressure P satisfies

$$\widehat{P}(t,\xi) = e^{-it\xi\sqrt{\frac{\tanh(\xi)}{\xi}}} P_0(\xi), \tag{2.63}$$

we have a resonance with a factor of amplification of $\sqrt{t}$. In the first simulation, we compare the linear case and the nonlinear case. We take P as in (2.63) with $P_0(X) = \exp(-\frac{1}{3}X^2)$ and we take $\varepsilon = 0.1$. Figure 2.12 displays the evolution of the surface in the linear and nonlinear cases. Figure 2.13 compares the maximum of the two cases. We notice that the nonlinear effects can increase the maximum at the beginning (compared to the linear situation) and an amplification

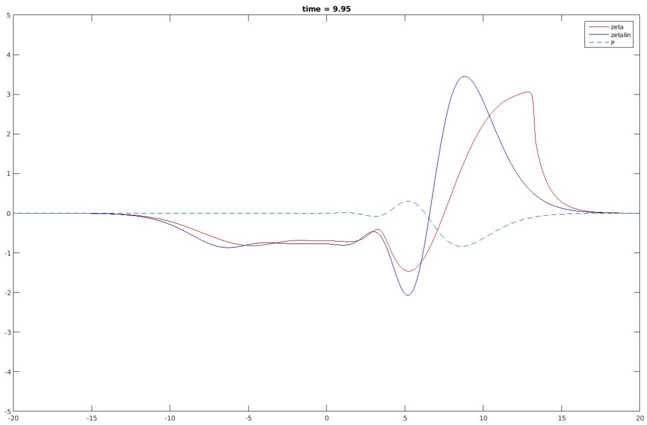

Figure 2.12: Evolution of the surface in the nonlinear case (red line), the linear case (blue line). The dashed line is the corresponding pressure.

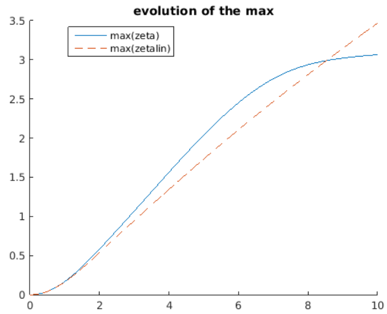

Figure 2.13: Comparison of the maximum of the surface between the linear case (red line) and the nonlinear case (blue line).

occurs, but, after a while, a saturation of the L^∞-norm is possible since the wave breaks. This situation is similar to the one in the shallow water case. The only difference, is the fact that the Saut-Xu system is dispersive and hence the wave breaking is delayed.

In the following simulation, we take $\varepsilon = 0.1$ and

$$P(t, X) = P_0(X - t) \text{ and } \zeta_0(X) = v_0(X) = -P_0(X) = \left(\operatorname{sech}\left(\frac{\sqrt{3}}{2}x\right)\right)^2. \tag{2.64}$$

Figure 2.14 displays the evolution of the surface and Figure 2.15 displays the evolution the maximum of the surface. We see that an amplification due to the source term is possible even in the nonlinear case.

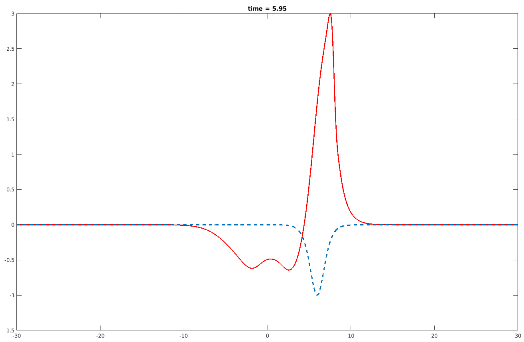

Figure 2.14: Evolution of the surface with initial data (2.64). The dashed line is the corresponding pressure in (2.64).

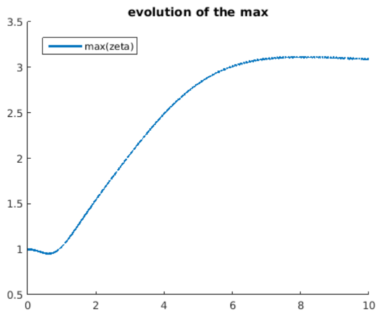

Figure 2.15: Evolution of the maximum of the the surface with initial data (2.64).

2.3.4.3 Conclusion

We studied numerically nonlinear effects on the Proudman resonance thanks to two different systems. We saw that, in both cases, the source term can amplify the amplitude of the water wave. However, if the water wave is too big, wave breaking can occur. This leads to a saturation of the L^∞-norm. It could be interesting to study mathematically if we actually get a shock.

Chapter 3

A splitting method for deep water with bathymetry

Sommaire

Ce chapitre est l'article [24]. Il a été écrit en collaboration avec Afaf Bouharguane (IMB).

3.1 Introduction

3.1.1 Presentation of the problem

In this chapter we derive and prove the wellposedness of a deep water model that generalizes the Saut-Xu system for nonflat bottoms. Then, we present a new numerical method based on a splitting approach for studying this system. The advantage of this method is that it does not require any low pass filter to avoid spurious oscillations. We prove a local error estimate and we show that our scheme represents a good approximation of order one in time. Then, we perform some numerical experiments which confirm our theoretical result and we study two physical phenomena : the behaviour of a KdV soliton when the shallowness parameter increases; the homogenization effect of rapidly varying topographies on water waves.

The study of the influence of the topography on water waves is an important issue in oceanography. Many phenomena are linked to the variation of the topography : shoaling, rip currents, diffraction, Bragg reflection. Since the direct study on the Euler equations is quite involved, several authors derived and justified asymptotic models according to different small parameters. A usual way to derive asymptotic models is to start from the Zakharov/Craig-Sulem-Sulem formulation [153, 47, 48], which is a good formulation for irrotational water waves, and to expand the Dirichlet-Neumann operator. Then, in the shallow water regime for example, several models were obtained like the Saint-Venant equations or the Green-Naghdi or Boussinesq equations, see [9, 80], [70] for instance. The present chapter addresses the influence of the bathymetry in deep water, in the sense explained below.

In this chapter, a denotes the typical amplitude of the water waves, L the typical length, H the typical height and a_{bott} the typical amplitude of the bathymetry. Then, we introduce three parameters : $\varepsilon = \frac{a}{H}$ the nonlinearity parameter, $\mu = \frac{H^2}{L^2}$ the shallowness parameter and $\beta = \frac{a_{bott}}{H}$ the bathymetric parameter. We denote by d the horizontal dimension, which is equal to 1 or 2. We recall that assuming μ small leads to shallow water models. In deep water, which is typically the case when μ is of order 1, it is quite common to assume that the steepness parameter $\varepsilon\sqrt{\mu} = \frac{a}{L}$ is small. The first nonlinear asymptotic model with a small steepness assumption was derived by Matsuno, when $d = 1$ for a flat bottom ([97]) and a slowly varying bottom ([98]), and when $d = 2$ for weakly transverse water waves (γ is of order $\mathcal{O}(\varepsilon)$) and a flat bottom ([99]). Then, Choi ([38]) extended [99] for general water waves (see also the work of Smith [132]). Finally, Bonneton and Lannes ([22]) gave a formulation of the Matsuno equations when $d = 1$ and $d = 2$ in the case of a small non-flat bottom (β of order $\mathcal{O}(\varepsilon)$). It is important to notice that these models are only formally derived. It is proven in [9] that smooth enough solutions to theses models are close to the solutions of the water waves equations but, to our knowledge, the wellposedness of the Matsuno equations, even in the case of a flat bottom, is still an open problem. This system could be illposed (see Ambrose, Bona and Nicholls [10]). To avoid this difficulty, Saut and Xu ([125]) developed an equivalent problem to the Matsuno system which is consistent with the water waves problem and with the same accuracy. Then, they proved that this new system is wellposed. However, this model is for a flat bottom. In this chapter, we shall derive, use, and prove the wellposedness of a generalization of the Saut-Xu system with a non-flat bottom.

Many authors developed numerical approaches to study the impact of the bottom on water waves, see for instance [100], [91], [132], [65], [35], [20], [21]). However to our knowledge, when one works with deep water models, there is no convergence result in the literature. After the original work of Craig and Sulem ([47]) and the paper of Craig et al. ([45]), Guyenne and Nicholls ([66]) developed a numerical method based on a pseudospectral method and a fourth-order Runge-Kutta scheme for the time integration. The linear terms are solved exactly whereas the nonlinear terms are viewed as source terms. Their approach has been developed for the whole water waves equations but we could easily adapt it to our system. However with their scheme, we observe spurious oscillations in the wave profile that lead to instabilities. These errors seem to appear when the nonlinear part is evaluated via the Fourier transform. This is the aliasing phenomenon. Guyenne and Nicholls also observe these oscillations and, to fix it, they apply at every time step a low-pass filter. The scheme that we propose in this chapter avoids this low-pass filter.

We present a new numerical method based on a splitting approach for studying nonlinear water waves in the presence of a bottom. We remark that the Saut-Xu system contains a dispersive part and a nonlinear transport part. Thus, the splitting method becomes an interesting alternative to solve the system since this approach is commonly used to split different physical terms, see for instance [123]. We also motivate our decomposition by the fact that, due to the pseudodifferential operator, some terms in the dispersive part may be computed efficiently using the fast Fourier transform. The transport part is computed thanks a Lax-wendroff method. Various versions of the splitting method have been developed for instance for the nonlinear Schrodinger, the viscous Burgers equation, Korteweg-de-Vries equations [32, 68, 92, 124, 135]. Thanks to this splitting, we only use a pseudospectral method for the nonlocal terms (contrary to [47, 66]), which limits the aliasing phenomenon and allows us to avoid a low-pass filter.

We denote by Φ^t the nonlinear flow associated to the Saut-Xu system (3.6), $\Phi^t_{\mathcal{A}}$ and $\Phi^t_{\mathcal{D}}$, respectively, the evolution operator associated with the transport part (see equation (3.11)) and with the dispersive part (see equation (3.12)). We consider the Lie formula defined by

$$\mathcal{Y}^t = \Phi^t_{\mathcal{A}} \circ \Phi^t_{\mathcal{D}}. \tag{3.1}$$

Since the Saut-Xu system (3.6) is a quasilinear system, we have derivatives losses in the proof of the convergence of our splitting scheme. In Theorem 3.4.6, we show that the numerical solution converges to the solution of the Saut-Xu system (3.6) in the $H^{N+\frac{1}{2}} \times H^N$-norm for initial data in $H^{N+\frac{1}{2}-4} \times H^{N-4}(\mathbb{R})$.

Notice that it is not painful to generalize the present work to the Lie formula $\Phi^t_{\mathcal{D}} \circ \Phi^t_{\mathcal{A}}$. We also make the choice to prove a convergence result for a Lie splitting but our proof can be adapted to a Strang splitting or a more complex one. Finally, notice that our scheme can be used for others equations (see Remark 3.4.7).

The chapter is organised as follows. In the next section, we extend the Saut-Xu system by adding a topography effect and we prove a local wellposedness result. We also show that the flow map Φ^t is uniformly Lipschitzean. In section 3.3, we split the problem and we give some estimates on $\Phi^t_{\mathcal{A}}$ and $\Phi^t_{\mathcal{D}}$. In Section 3.4, we prove a local error estimate and we show that the Lie method represents a good approximation of order one in time (Theorem 3.4.6). Finally, in Section 3.5, we perform some numerical experiments which confirm our theoretical result and we illustrate two physical phenomena : the behaviour of a KdV soliton when the shallowness parameter increases and the homogenization effect of rapidly varying topographies on water waves.

3.1.2 Notations and assumptions

- x denotes the horizontal variable and z the vertical variable. In this chapter, we only study the case $d = 1$ ($x \in \mathbb{R}$).
- We assume that

$$0 \leq \varepsilon, \beta \leq 1,\ \exists \mu_{\max} > \mu_{\min} > 0,\ \mu_{\max} \geq \mu \geq \mu_{\min}. \tag{3.2}$$

  We explain in Remark 3.2.1 our assumption on μ.

- We denote $\delta = \max(\varepsilon, \beta)$.
- Let $f \in \mathcal{C}^0(\mathbb{R})$ and $m \in \mathbb{N}$ such that $\frac{f}{1+|x|^m} \in L^\infty(\mathbb{R})$. We define the Fourier multiplier $f(D) : H^m(\mathbb{R}) \to L^2(\mathbb{R})$ as

$$\forall u \in H^m(\mathbb{R}),\ \widehat{f(D)u}(\xi) = f(\xi)\widehat{u}(\xi).$$

- D denotes the Fourier multiplier corresponding to $\frac{\partial_x}{i}$.
- We denote by $C(c_1, c_2, ...)$ a generic positive constant, strictly positive, which depends on parameters $c_1, c_2, \cdots$.

3.2 The Saut-Xu system

In this part, we extend the Saut-Xu system ([125]) for a non-flat bottom. Then, we give a wellposedness result that generalizes the one of Saut-Xu.

The Matsuno system, which is a full dispersion model for deep water, is an asymptotic model of the water waves equations with an accuracy of order $\mathcal{O}\left(\delta^2\right)$. Bonneton and Lannes [22] formulated it in the following way in the presence of a non flat topography

$$\begin{cases} \partial_t \zeta - \frac{1}{\sqrt{\mu}\nu} \mathcal{H}_\mu v + \frac{\varepsilon}{\nu}\left(\mathcal{H}_\mu\left(\zeta \partial_x \mathcal{H}_\mu v\right) + \partial_x\left(\zeta v\right)\right) = \frac{\beta}{\nu} \partial_x\left(B_\mu v\right) \\ \partial_t v + \partial_x \zeta + \frac{\varepsilon}{\nu} v \partial_x v - \varepsilon\sqrt{\mu}\partial_x \zeta \mathcal{H}_\mu \partial_x \zeta = 0, \end{cases} \tag{3.3}$$

where $\zeta = \zeta(t,x)$ is the free surface, $v = v(t,x)$ is the horizontal velocity at the surface, $\nu = \frac{\tanh(\sqrt{\mu})}{\sqrt{\mu}}$ and $\mathcal{H}_\mu$ and B_μ are Fourier multipliers,

$$\mathcal{H}_\mu = -\frac{\tanh(\sqrt{\mu}D)}{D}\partial_x \text{ and } B_\mu = \operatorname{sech}(\sqrt{\mu}D)\left(b \operatorname{sech}(\sqrt{\mu}D)\ \cdot\ \right),$$

and b is the topography. It is important to notice that in this context, the fluid domain is

$$\Omega_t := \{(x,z) \in \mathbb{R}^2\ ,\ -1 + \beta b(x) < z < \varepsilon\zeta(t,x)\}.$$

In [9], Alvarez-Samaniego and Lannes show that the Matsuno system (3.3) is consistent with the Zakharov/Craig-Sulem-Sulem formulation when $\beta = 0$ and it is not painful to generalize their result to the case where $\beta \neq 0$. In [125], Saut and Xu obtained a new model with the same accuracy than the Matsuno system thanks to a nonlinear change of variables in the case of a flat bottom. The advantage of this model is that they proved a local wellposedness on large time for this new model. We follow their approach. We define new variables

$$\widetilde{v} = v + \frac{\varepsilon\sqrt{\mu}}{2} v\mathcal{H}_\mu \partial_x \zeta \text{ and } \widetilde{\zeta} = \zeta - \frac{\varepsilon\sqrt{\mu}}{4} v^2. \tag{3.4}$$

Then, up to terms of order $\mathcal{O}\left(\delta^2\right)$, $\widetilde{\zeta}$ and $\widetilde{v}$ satisfy (we omit the tildes for the sake of simplicity)

$$\begin{cases} \partial_t\zeta + \left(\frac{\varepsilon}{\nu} - \frac{\varepsilon\sqrt{\mu}}{2}\right) v\partial_x\zeta - \frac{1}{\sqrt{\mu}\nu}\mathcal{H}_\mu v + \frac{\varepsilon}{\nu}\left(\frac{1}{2}\mathcal{H}_\mu\left(v\partial_x\mathcal{H}_\mu\zeta\right) + \mathcal{H}_\mu(\zeta\partial_x\mathcal{H}_\mu v) + \zeta\partial_x v\right) = \frac{\beta}{\nu}\partial_x\left(B_\mu v\right) \\ \partial_t v + \left(\frac{\varepsilon}{\nu} + \frac{\varepsilon\sqrt{\mu}}{2}\right) v\partial_x v + \partial_x\zeta - \frac{\varepsilon\sqrt{\mu}}{2}\partial_x\zeta\mathcal{H}_\mu\partial_x\zeta - \frac{\varepsilon}{2\nu} v\mathcal{H}_\mu^2\partial_x v = 0. \end{cases} \tag{3.5}$$

Since our motivation is the study of water waves in deep water (μ of order $\mathcal{O}(1)$), we assume that $\nu = \frac{1}{\mu}$. Hence, we study the following system, which is the system studied by Saut and Xu,

$$\begin{cases} \partial_t\zeta - \mathcal{H}_\mu v + \varepsilon\sqrt{\mu}\left(\frac{1}{2}v\partial_x\zeta + \frac{1}{2}\mathcal{H}_\mu(v\partial_x\mathcal{H}_\mu\zeta) + \mathcal{H}_\mu(\zeta\partial_x\mathcal{H}_\mu v) + \zeta\partial_x v\right) = \beta\sqrt{\mu}\partial_x(B_\mu v) \\ \partial_t v + \partial_x\zeta + \frac{3\varepsilon\sqrt{\mu}}{2} v\partial_x v - \frac{\varepsilon\sqrt{\mu}}{2}\partial_x\zeta\mathcal{H}_\mu\partial_x\zeta - \frac{\varepsilon\sqrt{\mu}}{2} v\mathcal{H}_\mu^2\partial_x v = 0, \end{cases} \tag{3.6}$$

In the following, we denote $\mathbf{U} = (\zeta, v)^t$ and we define the energy of the system for $N \in \mathbb{N}$ by

$$\mathcal{E}^N(\mathbf{U}) = \frac{1}{\sqrt{\mu}}\left|\Lambda^N\zeta\right|_2^2 + \left||D|^{\frac{1}{2}}\Lambda^N\zeta\right|_2^2 + |v|_{H^N}^2. \tag{3.7}$$

We also denote by E_μ^N the energy space related to this norm.

Remark 3.2.1. *Notice that if μ satisfies condition* (3.2)*, the energy $\mathcal{E}^N$ is equivalent to the $H^{N+\frac{1}{2}} \times H^N$-norm. Furthermore, the assumption $\mu_{\min} \leq \mu$ is essential for the Theorem 3.2.2.*

The main result of this section is the following local wellposedness result. We recall that $\delta = \max(\varepsilon, \beta)$.

Theorem 3.2.2. *Let $N \geq 2$, $\mathbf{U}_0 \in H^{N+\frac{1}{2}}(\mathbb{R}) \times H^N(\mathbb{R})$ and $b \in L^\infty(\mathbb{R})$. We assume that ε, β, μ satisfy Condition* (3.2) *and*

$$|\mathbf{U}_0|_{H^{N+\frac{1}{2}}\times H^N} + |b|_{L^\infty} \leq M.$$

Then, there exists a time $T_0 = T_0\left(M, \frac{1}{\mu_{\min}}, \mu_{\max}\right)$ independent of ε, μ and β and a unique solution $\mathbf{U} \in \mathcal{C}\left(\left[0, \frac{T_0}{\delta}\right], E_\mu^N\right)$ of the system (3.6) *with initial data $\mathbf{U}_0$. Furthermore, we have the following energy estimate, for all $t \in \left[0, \frac{T_0}{\delta}\right]$,*

$$\mathcal{E}^N(\mathbf{U}(t,\cdot)) \leq e^{\delta C_0 t}\mathcal{E}^N(\mathbf{U}_0),$$

where $C_0 = C\left(M, \frac{1}{\mu_{\min}}, \mu_{\max}\right)$.

Proof. We refer to Paragraph IV in [125] for a complete proof and we focus only on the bottom contribution. We quasilinearize System (3.6). For $0 \leq \alpha \leq N$, we denote $\mathbf{U}^{(\alpha)} = (\partial^\alpha\zeta, \partial^\alpha v)$. Then, applying ∂^α to System (3.6), we get

$$\partial_t\mathbf{U}^{(\alpha)} + \mathcal{L}\mathbf{U}^{(\alpha)} + \frac{\varepsilon\sqrt{\mu}}{2}\mathbb{1}_{\{\alpha\neq 0\}}\mathcal{B}[\mathbf{U}]\mathbf{U}^{(\alpha)} = \beta\sqrt{\mu}\left(\partial_x\partial^\alpha\left(B_\mu v\right), 0\right)^t + \varepsilon\sqrt{\mu}\mathcal{G}^\alpha,$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{\mu\nu}}\mathcal{H}_\mu \\ \partial_x & 0 \end{pmatrix}$$

$$\mathcal{B}[\mathbf{U}] = \begin{pmatrix} \mathcal{H}_\mu\left(v\mathcal{H}_\mu\partial_x\,\cdot\,\right) + v\partial_x & \mathcal{H}_\mu\left(\,\cdot\,\mathcal{H}_\mu\partial_x\zeta\right) - \partial_x\zeta\mathcal{H}_\mu^2 \\ -\partial_x\zeta\mathcal{H}_\mu\partial_x - \mathcal{H}_\mu\partial_x\zeta\partial_x & 3v\partial_x - v\mathcal{H}_\mu^2\partial_x \end{pmatrix},$$

and $\mathcal{G}^\alpha = (\mathcal{G}_1^\alpha, \mathcal{G}_2^\alpha)^t$ with

$$\mathcal{G}_1^\alpha = \partial_x^\alpha g(\zeta, v) - \frac{1}{2}\sum_{1\leq\gamma\leq\alpha-1} C_\alpha^\gamma\left(\mathcal{H}_\mu(\partial_x^\gamma v\mathcal{H}_\mu\partial_x^{1+\alpha-\gamma}\zeta) + \partial_x^\gamma v\partial_x^{1+\alpha-\gamma}\zeta\right) - \frac{1}{2}\partial_x\zeta(\mathcal{H}_\mu^2+1)\partial_x^\alpha v$$

$$\mathcal{G}_2^\alpha = \frac{1}{2}\sum_{1\leq\gamma\leq\alpha-1} C_\alpha^\gamma\partial_x^{1+\gamma}\zeta\mathcal{H}_\mu\partial_x^{1+\alpha-\gamma}\zeta + \sum_{1\leq\gamma\leq\alpha} C_\alpha^\gamma\left(-\frac{3}{2}\partial_x^\gamma v\partial_x^{1+\alpha-\gamma}v + \frac{1}{2}\partial_x^\gamma v\mathcal{H}_\mu^2\partial_x^{1+\alpha-\gamma}v\right)$$

where

$$g(\zeta, v) = -[\mathcal{H}_\mu, \zeta]\mathcal{H}_\mu\partial_x v - \zeta(\mathcal{H}_\mu^2+1)\partial_x v.$$

Then we can show, as in Paragraph IV. B in [125], that

$$|\mathcal{G}^\alpha|_2 + \left||D|^{\frac{1}{2}}\mathcal{G}^\alpha\right|_2 \leq C\left(\frac{1}{\mu_{\min}}\right)\mathcal{E}^N(\mathbf{U}).$$

As Saut and Xu, we define a symmetrizer for $\mathcal{L}$

$$\mathcal{S} = \begin{pmatrix} \frac{D}{\tanh(\sqrt{\mu}D)} & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that $\sqrt{(\mathcal{S}\,\cdot, \cdot)}$ is a norm equivalent to $\sqrt{\mathcal{E}^0}$. For the bottom contribution, we easily get

$$\left|\left(\frac{D}{\tanh(\sqrt{\mu}D)}\partial^\alpha\zeta, \partial_x\partial^\alpha\mathrm{sech}\,(\sqrt{\mu}D)\,(b\,\mathrm{sech}\,(\sqrt{\mu}D)\,v)\right)\right| \leq C\left(\frac{1}{\mu_{\min}}\right)|b|_\infty\,\mathcal{E}^N(U).$$

Then, we obtain,

$$\mathcal{E}^N(\mathbf{U}) \leq \mathcal{E}^N(\mathbf{U}_0) + \delta C\left(\frac{1}{\mu_{\min}}, \mu_{\max}\right)\int_0^t\left(\mathcal{E}^N(U)^{\frac{3}{2}} + \mathcal{E}^N(U)\right)(s)ds,$$

and there exists a time $T > 0$, such that, for all $t \in \left[0, \frac{T}{\delta}\right]$,

$$\mathcal{E}^N(\mathbf{U}(t,\cdot)) \leq C\left(\frac{1}{\mu_{\min}}, \mu_{\max}, \mathcal{E}^N(\mathbf{U}_0)\right).$$

The energy estimate follows from the Gronwall Lemma.

□

In order to use a Lady Windermere's fan argument to prove the convergence of the numerical scheme, we need a Lipschitz property for the flow of the Saut-Xu system (3.6). We first give a control of the differential of the flow with respect to the initial datum.

Proposition 3.2.3. *Let $N \geq 2$, $\mathbf{V}_0 \in H^{N+\frac{1}{2}}(\mathbb{R}) \times H^N(\mathbb{R})$, $\mathbf{U}_0 \in H^{N+1+\frac{1}{2}}(\mathbb{R}) \times H^{N+1}(\mathbb{R})$, and $b \in L^\infty(\mathbb{R})$. We assume that ε, β, μ satisfy Condition (3.2) and*

$$|\mathbf{V}_0|_{H^{N+\frac{1}{2}} \times H^N} + |\mathbf{U}_0|_{H^{N+1+\frac{1}{2}} \times H^{N+1}} + |b|_{L^\infty} \leq M.$$

Then, there exists a time $T = T\left(M, \frac{1}{\mu_{\min}}, \mu_{\max}\right)$ independent of ε, μ and β such that $(\Phi^t)'(\mathbf{U}_0) \cdot (\mathbf{V}_0)$ exists on $\left[0, \frac{T}{\delta}\right]$. Furthermore, we have, for all $0 \leq t \leq \frac{T}{\delta}$,

$$\left|(\Phi^t)'(\mathbf{U}_0) \cdot (\mathbf{V}_0)\right|_{H^{N+\frac{1}{2}} \times H^N} \leq C\left(\frac{1}{\mu_{\min}}, \mu_{\max}, M\right) |\mathbf{V}_0|_{H^{N+\frac{1}{2}} \times H^N}.$$

Proof. This proof is similar to the one in Paragraph IV in [125]. We denote $\mathbf{U}(t) = (\zeta(t), v(t))$ the solution of the Saut-Xu system (3.6) with initial data $\mathbf{U}_0$. We denote also $(\eta(t), w(t)) = (\Phi^t)'(\mathbf{U}_0) \cdot (\mathbf{V}_0)$. Then, (η, w) satisfy the following system

$$\partial_t \begin{pmatrix} \eta \\ w \end{pmatrix} = \mathcal{L} \begin{pmatrix} \eta \\ w \end{pmatrix} + \varepsilon\sqrt{\mu}\mathcal{N}[(\zeta, v)]\partial_x \begin{pmatrix} \eta \\ w \end{pmatrix} + \varepsilon\sqrt{\mu}\mathcal{N}[(\eta, w)]\partial_x \begin{pmatrix} \zeta \\ v \end{pmatrix} = \beta\sqrt{\mu}\left(\partial_x\left(B_\mu w\right), 0\right)^t, \quad (3.8)$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{\mu}\nu}\mathcal{H}_\mu \cdot \\ \partial_x & 0 \end{pmatrix}$$

$$\mathcal{N}[(\zeta, v)] = \begin{pmatrix} \frac{1}{2}\mathcal{H}_\mu\left(v\mathcal{H}_\mu \cdot\right) + \frac{1}{2}v & \mathcal{H}_\mu\left(\zeta\mathcal{H}_\mu \cdot\right) + \zeta \\ -\frac{1}{2}\partial_x\zeta\mathcal{H}_\mu & \frac{3}{2}v - \frac{1}{2}v\mathcal{H}_\mu^2 \end{pmatrix}.$$

We quasilinearize System (3.8). For $0 \leq \alpha \leq N$, we denote $\mathbf{V}^{(\alpha)} = (\partial^\alpha\eta, \partial^\alpha w)$. Then, applying ∂^α to System (3.6), we get

$$\partial_t \mathbf{V}^{(\alpha)} + \mathcal{L}\mathbf{V}^{(\alpha)} + \frac{\varepsilon\sqrt{\mu}}{2}\mathbb{1}_{\{\alpha \neq 0\}}\left(\mathcal{B}[\mathbf{U}]\mathbf{V}^{(\alpha)} + \mathcal{B}[\mathbf{V}]\partial^\alpha\mathbf{U}\right) = \sqrt{\mu}\beta\begin{pmatrix} \partial_x\partial^\alpha\left(B_\mu w\right) \\ 0 \end{pmatrix} + \varepsilon\sqrt{\mu}\mathcal{G}^\alpha,$$

where

$$\mathcal{B}[\mathbf{U}] = \begin{pmatrix} \mathcal{H}_\mu\left(v\mathcal{H}_\mu\partial_x \,\cdot\,\right) + v\partial_x & \mathcal{H}_\mu\left(\,\cdot\,\mathcal{H}_\mu\partial_x\zeta\right) - \partial_x\zeta\mathcal{H}_\mu^2 \\ -\partial_x\zeta\mathcal{H}_\mu\partial_x - \mathcal{H}_\mu\partial_x\zeta\partial_x & 3v\partial_x - v\mathcal{H}_\mu^2\partial_x \end{pmatrix}.$$

Then, we can show, as in Paragraph IV. B in [125], that

$$|\mathcal{G}^\alpha|_2 + \left||D|^{\frac{1}{2}}\mathcal{G}^\alpha\right|_2 \leq \varepsilon\sqrt{\mu}C\left(\frac{1}{\mu_{\min}}\right)\mathcal{E}^N(\mathbf{U}). \quad (3.9)$$

We recall that we can symmetrize $\mathcal{L}$ thanks to

$$\mathcal{S} = \begin{pmatrix} \frac{D}{\tanh(\sqrt{\mu}D)} & 0 \\ 0 & 1 \end{pmatrix}.$$

We define the energy associated to this symmetrizer

$$F^\alpha(\mathbf{V}) = \left|\sqrt{\frac{D}{\tanh(\sqrt{\mu}D)}}\partial^\alpha\eta\right|_2^2 + |\partial^\alpha w|_2^2,$$

and

$$F^N(\mathbf{V}) = \sum_{0\leq\alpha\leq N} F^\alpha(\mathbf{V}).$$

We have, for $\alpha \neq 0$,

$$\begin{aligned}\frac{d}{dt}F^\alpha(\mathbf{V}) &= \left(\mathcal{G}^\alpha, \mathcal{S}\mathbf{V}^{(\alpha)}\right) - \frac{\varepsilon\sqrt{\mu}}{2}\left(\mathcal{B}[\mathbf{U}]\mathbf{V}^{(\alpha)}, \mathcal{S}\mathbf{V}^{(\alpha)}\right) - \frac{\varepsilon\sqrt{\mu}}{2}\left(\mathcal{B}[\mathbf{V}]\partial^\alpha\mathbf{U}, \mathcal{S}\mathbf{V}^{(\alpha)}\right)\\ &\qquad + \beta\sqrt{\mu}\left(\partial_x\partial^\alpha\left(B_\mu v\right), \mathcal{S}\mathbf{V}^{(\alpha)}\right)\\ &= I + II + III + IIII.\end{aligned}$$

We can estimate I and II as in Paragraph IV. B in [125] thanks to estimate (3.9). For *IIII*, we can proceed as in the previous theorem. For the term *III*, we get, thanks to Proposition B.4.1,

$$|III| \leq \varepsilon\sqrt{\mu}\left|(\zeta,v)\right|_{H^{N+1+\frac{1}{2}}\times H^{N+1}}\left|(\eta,w)\right|_{H^{N+\frac{1}{2}}\times H^N}$$

Then, we get

$$\frac{d}{dt}F^N(\mathbf{V}) \leq \delta\sqrt{\mu}C(M)\left(F^N(\mathbf{V}) + \sqrt{F^N(\mathbf{V})}\right),$$

and the result follows.

□

We give now a Lipschitz property for the flow of the Saut-Xu system (3.6).

Proposition 3.2.4. *Let $N \geq 2$, $\mathbf{U}_0, \mathbf{V}_0 \in H^{N+1+\frac{1}{2}} \times H^{N+1}(\mathbb{R})$ and $b \in L^\infty(\mathbb{R})$. We assume that ε, β, μ satisfy Condition (3.2) and*

$$\left|\mathbf{V}_0\right|_{H^{N+1+\frac{1}{2}}\times H^{N+1}} + \left|\mathbf{U}_0\right|_{H^{N+1+\frac{1}{2}}\times H^{N+1}} + |b|_{L^\infty} \leq M.$$

Then, there exists a time T independent of ε, μ and β and two unique solutions $\mathbf{U}, \mathbf{V}$ of the system (3.6) on $\left[0, \frac{T}{\delta}\right]$ with initial data $\mathbf{U}_0$ and $\mathbf{V}_0$. Furthermore, we have the following lipschitz estimate, for all $0 \leq t \leq \frac{T}{\delta}$,

$$\left|\mathbf{U}(t,\cdot) - \mathbf{V}(t,\cdot)\right|_{H^{N+\frac{1}{2}}\times H^N} \leq K\left|\mathbf{U}_0 - \mathbf{V}_0\right|_{H^{N+\frac{1}{2}}\times H^N}, \tag{3.10}$$

where $K = C\left(\frac{1}{\mu_{\min}}, \mu_{\max}, M\right)$.

Proof. The existence of $\mathbf{U}, \mathbf{V}$ and T follow from the previous theorem. Furthermore, we have

$$\mathbf{U}(t) - \mathbf{V}(t) = \int_{s=0}^1 \left(\Phi^t\right)'\left(\mathbf{V}_0 + s\left(\mathbf{U}_0 - \mathbf{V}_0\right)\right)\cdot\left(\mathbf{U}_0 - \mathbf{V}_0\right).$$

The result follows from Proposition 3.2.3. □

3.3 A splitting scheme

In this section, we split the Saut-Xu system (3.6) and we give some estimates for the sub-problems. We consider, separately, the transport part

$$\begin{cases} \partial_t \zeta + \frac{\varepsilon\sqrt{\mu}}{2}\left(\left(\mathcal{H}_\mu^2+1\right)v\right)\partial_x\zeta = 0 \\ \partial_t v + \frac{3\varepsilon\sqrt{\mu}}{2} v\partial_x v = 0, \end{cases} \tag{3.11}$$

and the dispersive part

$$\begin{cases} \partial_t\zeta - \mathcal{H}_\mu v + \varepsilon\sqrt{\mu}\left(\frac{1}{2}\mathcal{H}_\mu\left(v\partial_x\mathcal{H}_\mu\zeta\right) + \mathcal{H}_\mu\left(\zeta\partial_x\mathcal{H}_\mu v\right) + \zeta\partial_x v - \frac{1}{2}\partial_x\zeta\mathcal{H}_\mu^2 v\right) = \beta\sqrt{\mu}\partial_x\left(B_\mu v\right) \\ \partial_t v + \partial_x\zeta - \frac{\varepsilon\sqrt{\mu}}{2}\partial_x\zeta\mathcal{H}_\mu\partial_x\zeta - \frac{\varepsilon\sqrt{\mu}}{2} v\mathcal{H}_\mu^2\partial_x v = 0. \end{cases} \tag{3.12}$$

We denote by $\Phi_{\mathcal{A}}^t$ the flow of System (3.11) and by $\Phi_{\mathcal{D}}^t$ the flow of System (3.12).

Remark 3.3.1. *Notice that we keep the term $\zeta\partial_x v$ in the first equation and we decompose $v\partial_x\zeta$ as $v\partial_x\zeta = \partial_x\zeta\left(\mathcal{H}_\mu^2+1\right)v - \partial_x\zeta\mathcal{H}_\mu^2 v$. This will be useful for the local wellposedness of the dispersive part.*

In the following, we prove the local existence on large time for Systems (3.11) and (3.12).

3.3.1 The transport equation

The system (3.11) is a transport equation. Then, it is easy to get the following result.

Proposition 3.3.2. *Let $s_1 \geq 0$, $s_2 > \frac{3}{2}$ and $M > 0$. We assume that ε, μ satisfies Condition* (3.2). *Then, there exists a time $T_1 = T_1(M, \mu_{\max}) > 0$, such that if*

$$|\zeta_0|_{H^{s_1}} + |v_0|_{H^{s_2}} \leq M,$$

we have a unique solution $(\zeta, v) \in \mathcal{C}\left(\left[0, \frac{T_1}{\varepsilon}\right], H^{s_1}(\mathbb{R}) \times H^{s_2}(\mathbb{R})\right)$, to System (3.11) *with initial data (ζ_0, v_0). Furthermore, we have, for all, $t \leq \frac{T_1}{\varepsilon}$,*

$$|\zeta(t,\cdot)|_{H^{s_1}} + |v(t,\cdot)|_{H^{s_2}} \leq e^{C_1 t}|\boldsymbol{U}_0|_{H^{s_1}\times H^{s_2}}, \tag{3.13}$$

where $C_1 > 0$ depends on M and $\mu_{\max}$.

Proof. The proof follows from the fact that the quasilinear system (3.11) is symmetric. Thanks to the Coifman-Meyer estimate (see Lemma B.3.1), we get

$$\frac{d}{dt}\left(|\zeta(t,\cdot)|_{H^{s_1}}^2 + |v(t,\cdot)|_{H^{s_2}}^2\right) \leq \varepsilon\sqrt{\mu}\left(|\zeta(t,\cdot)|_{H^{s_1}}^2 + |v(t,\cdot)|_{H^{s_2}}^2\right)^{\frac{3}{2}}.$$

Then, we see that the energy is bounded uniformly with respect to ε and μ and applying the Gronwall lemma, we get the result. □

3.3.2 The dispersive equation

The system (3.12) contains all the dispersive terms of the Saut-Xu system. We have the following estimate for the flow.

Proposition 3.3.3. *Let $N \geq 2$, and $b \in L^\infty(\mathbb{R})$. We assume that ε, β, μ satisfy Condition* (3.2)*. Then, there exists a time $T_2 = T_2\left(M, \frac{1}{\mu_{\min}}, \mu_{\max}\right)$ such that if*

$$|\zeta_0|_{H^{N+\frac{1}{2}}} + |v_0|_{H^N} + |b|_{L^\infty} \leq M,$$

we have a unique solution $(\zeta, v) \in \mathcal{C}\left(\left[0, \frac{T_2}{\delta}\right], H^{N+\frac{1}{2}}(\mathbb{R}) \times H^N(\mathbb{R})\right)$ to the system (3.12) *with initial data (ζ_0, v_0). Furthermore, we have, for all $t \leq \frac{T_2}{\delta}$,*

$$|\zeta(t,\cdot)|_{H^{N+\frac{1}{2}}} + |v(t,\cdot)|_{H^N} \leq e^{C_2 t} |\mathbf{U}_0|_{H^{N+1/2} \times H^N}, \tag{3.14}$$

where C_2 is a positive constant which depends on $\mu_{\max}, \frac{1}{\mu_{\min}}, M$.

Proof. The proof is a small adaptation of the proof of Theorem 3.2.2 and part IV in [125]. We notice that in the proof of Saut and Xu, the transport part can be treated separately and does not influence the control of the others terms. Hence, we can use the same estimate and we get that

$$\frac{d}{dt}\mathcal{E}^N(\zeta, v) \leq C\left(\frac{1}{\mu_{\min}}\right)\left(\frac{\varepsilon}{\nu}\mathcal{E}^N(\zeta, v)^{\frac{3}{2}} + \frac{\beta}{\nu}\mathcal{E}^N(\zeta, v)\right).$$

Since μ is bounded from above and from below, there exists a constant $C = C\left(\mu_{\max}, \frac{1}{\mu_{\min}}\right)$ such that

$$\frac{1}{C}\mathcal{E}^N(\zeta, v) \leq |\zeta|^2_{H^{N+\frac{1}{2}}} + |v|^2_{H^N} \leq C\mathcal{E}^N(\zeta, v),$$

and we see that the energy is bounded uniformly with respect to ε and μ. Applying the Gronwall lemma, we get the result. ☐

Remark 3.3.4. *Under the assumption of Proposition 3.3.3, we get from relations* (3.13) *and* (3.14) *that, there exists a time $T_3 > 0$, such that for all $t \in \left[0, \frac{T_3}{\delta}\right]$,*

$$|\mathcal{Y}^t \mathbf{U}_0|_{H^{N+1/2} \times H^N} \leq e^{C_3 \delta t} |\mathbf{U}_0|_{H^{N+1/2} \times H^N},$$

where T_3 and C_3 depend only on M, $\mu_{\max}$ and $\frac{1}{\mu_{\min}}$.

3.4 Error estimates

The goal of this part is to prove the main result of this chapter (Theorem 3.4.6). Our analysis is based on energy estimates.

3.4.1 The local error estimate

The local error is the following quantity

$$e(t, \mathbf{U}_0) = \Phi^t(\mathbf{U}_0) - \mathcal{Y}^t(\mathbf{U}_0). \tag{3.15}$$

Our approach is similar to the one developed in [36] (see Lemma 3.4 in [36]). We use the fact that $\Phi^t(\mathbf{U}_0)$ satisfies a symmetrizable system. Therefore, e satisfies this system up to a remainder and then, we can control e thanks to energy estimates. In the following we give different technical lemmas in order to control the local error. The main result of this part is Proposition 3.4.5. We recall that the transport operator is the operator $\mathcal{A}$

$$\mathcal{A}(\zeta,v) = -\frac{\varepsilon\sqrt{\mu}}{2}\begin{pmatrix}\left((\mathcal{H}_\mu^2+1)v\right)\partial_x\zeta\\ 3v\partial_x v\end{pmatrix}.$$

The following proposition gives an estimate of the differential of the transport operator.

Lemma 3.4.1. *Let $s_1, s_2 \geq 0$ and ε, μ satisfying Condition* (3.2)*. Then,*

$$|\mathcal{A}'(\zeta,v).(\eta,w)|_{H^{s_1}\times H^{s_2}} \leq \varepsilon C(\mu_{\max})\,|(\zeta,v)|_{H^{s_1+1}\times H^{s_2+1}}\,|(\eta,w)|_{H^{s_1+1}\times H^{s_2+1}}\,.$$

Proof. We have

$$\mathcal{A}'(\zeta,v).(\eta,w) = -\frac{\varepsilon\sqrt{\mu}}{2}\begin{pmatrix}\left((\mathcal{H}_\mu^2+1)v\right)\partial_x\eta + \left((\mathcal{H}_\mu^2+1)w\right)\partial_x\zeta\\ 3w\partial_x v + 3v\partial_x w\end{pmatrix},$$

and the estimate follows from the product estimate B.2.1. □

We can do the same for the dispersive part (using also Proposition B.4.1). We recall that the dispersive operator is the operator $\mathcal{D}$

$$\mathcal{D}(\zeta,v) = \begin{pmatrix}\mathcal{H}_\mu v + \varepsilon\sqrt{\mu}\left(\frac{1}{2}\mathcal{H}_\mu\left(v\partial_x\mathcal{H}_\mu\zeta\right) + \mathcal{H}_\mu\left(\zeta\partial_x\mathcal{H}_\mu v\right) + \zeta\partial_x v - \frac{1}{2}\partial_x\zeta\mathcal{H}_\mu^2 v\right) - \beta\sqrt{\mu}\partial_x\left(B_\mu v\right)\\ -\partial_x\zeta + \frac{\varepsilon\sqrt{\mu}}{2}\partial_x\zeta\mathcal{H}_\mu\partial_x\zeta + \frac{\varepsilon\sqrt{\mu}}{2}v\mathcal{H}_\mu^2\partial_x v\end{pmatrix}$$

Lemma 3.4.2. *Let $s > 0$, ε, β, μ satisfying Condition* (3.2) *and $b \in L^\infty(\mathbb{R})$. Then,*

$$|\mathcal{D}'(\zeta,v).(\eta,w)|_{H^s\times H^s} \leq C(\mu_{\max})\left(1+\beta\,|b|_{L^\infty}+\varepsilon\,|(\zeta,v)|_{H^{s+1}\times H^{s+1}}\right)|(\eta,w)|_{H^{s+1}\times H^{s+1}}.$$

Furthermore, we have to control the derivative of the flow $\Phi^t_{\mathcal{A}}$ with respect to the initial data. We denote it by $(\Phi^t_{\mathcal{A}})'$. We recall that $\delta = \max(\varepsilon,\beta)$.

Lemma 3.4.3. *Let $s_1, s_2 \geq 0$, $M > 0$, ε, β, μ satisfying Condition* (3.2) *and $b \in L^\infty(\mathbb{R})$. Let $(\zeta_0, v_0) \in H^{s_1+1}\times H^{s_2+1}(\mathbb{R}^d)$ such that,*

$$|(\zeta_0,v_0)|_{H^{s_1+1}\times H^{s_2+1}} \leq M.$$

Then, there exists a time $T = T(M,\mu_{\max})$, such that $(\Phi^t_{\mathcal{A}})'(\zeta_0,v_0)\cdot(\eta_0,w_0)$ exists for all $t \in \left[0,\frac{T}{\delta}\right]$ and if we denote

$$\begin{pmatrix}\eta\\ w\end{pmatrix} = \left(\Phi^t_{\mathcal{A}}\right)'(\zeta_0,v_0)\cdot(\eta_0,w_0)\,,$$

for all $0 \leq t \leq \frac{T}{\delta}$,

$$|(\eta,w)(t,\cdot)|_{H^{s_1}\times H^{s_2}} \leq |(\eta_0,w_0)|_{H^{s_1}\times H^{s_2}}\,C\left(\mu_{\max},M\right).$$

Proof. The quantity (η,w) satisfies the following linear system

$$\begin{cases}\partial_t\eta + \frac{\varepsilon\sqrt{\mu}}{2}\left(\mathcal{H}_\mu^2+1\right)v\partial_x\eta + \frac{\varepsilon\sqrt{\mu}}{2}\left(\mathcal{H}_\mu^2+1\right)w\partial_x\zeta = 0,\\ \partial_t w + \frac{3\varepsilon\sqrt{\mu}}{2}v\partial_x w + \frac{3\varepsilon\sqrt{\mu}}{2}w\partial_x v = 0,\end{cases}$$

where $(\zeta,v) = \Phi^t_{\mathcal{A}}(\zeta_0,v_0)$. The result follows from energy estimates, the Gronwall lemma and Proposition 3.3.2. □

In the following, we use the fact $\Phi^t_{\mathcal{A}} \circ \Phi^t_{\mathcal{D}}$ satisfies the Saut-Xu system (3.6) up to a remainder. The following lemma is the key point for the control of this remainder.

Lemma 3.4.4. *Let $N \geq 2$, $M > 0$, ε, β, μ satisfying Condition* (3.2) *and $b \in L^\infty(\mathbb{R})$. Let $\mathbf{U} = (\zeta, v) \in H^{N+\frac{1}{2}} \times H^N(\mathbb{R}^d)$ such that,*

$$|b|_{L^\infty} + |\mathbf{U}|_{H^{N+\frac{1}{2}} \times H^N(\mathbb{R})} \leq M.$$

Then, there exists a time $T = T\left(M, \mu_{\max}, \frac{1}{\mu_{\min}}\right) > 0$, such that $\Phi^t_{\mathcal{A}}(\mathbf{U})$ exists for all $0 \leq t \leq \frac{T}{\delta}$, and furthermore,

$$\left|\left(\Phi^t_{\mathcal{A}}\right)'(\mathbf{U}) \cdot \mathcal{D}(\mathbf{U}) - \mathcal{D}\left(\Phi^t_{\mathcal{A}}(\mathbf{U})\right)\right|_{H^{N-2} \times H^{N-2}} \leq \varepsilon C\left(M, \mu_{\max}, \frac{1}{\mu_{\min}}\right) t.$$

Proof. The existence of T follows from Proposition 3.3.2. Then, we notice that

$$\left(\Phi^t_{\mathcal{A}}\right)'(\mathbf{U}) \cdot \mathcal{D}(\mathbf{U}) - \mathcal{D}\left(\Phi^t_{\mathcal{A}}(\mathbf{U})\right) = \int_0^t \mathcal{A}'(\Phi^s_{\mathcal{A}}(\mathbf{U})) \cdot \left((\Phi^s_{\mathcal{A}})'(\mathbf{U}) \cdot \mathcal{D}(\mathbf{U})\right) - \mathcal{D}'(\Phi^s_{\mathcal{A}}(\mathbf{U})) \cdot \mathcal{A}(\Phi^s_{\mathcal{A}}(\mathbf{U})).$$

Using Lemmas 3.4.1, 3.4.2 and Proposition 3.3.2, we get,

$$\begin{aligned}\left|\left(\Phi^t_{\mathcal{A}}\right)'(\mathbf{U}) \cdot \mathcal{D}(\mathbf{U}) - \mathcal{D}\left(\Phi^t_{\mathcal{A}}(\mathbf{U})\right)\right|_{H^{N-2} \times H^{N-2}} \leq C(\mu_{\max}, M) \int_0^t & \varepsilon \left|(\Phi^s_{\mathcal{A}})'(\mathbf{U}) \cdot \mathcal{D}(\mathbf{U})\right|_{H^{N-1} \times H^{N-1}} \\ & + |\mathcal{A}(\Phi^s_{\mathcal{A}}(\mathbf{U}))|_{H^{N-1} \times H^{N-1}}.\end{aligned}$$

Then, using Lemma 3.4.3, the product estimate B.2.1 and the expression of $\mathcal{A}$, we obtain

$$\left|\left(\Phi^t_{\mathcal{A}}\right)'(\mathbf{U}) \cdot \mathcal{D}(\mathbf{U}) - \mathcal{D}\left(\Phi^t_{\mathcal{A}}(\mathbf{U})\right)\right|_{H^{N-2} \times H^{N-2}} \leq \varepsilon C(\mu_{\max}, M) \int_0^t |\mathcal{D}(\mathbf{U})|_{H^{N-1} \times H^{N-1}} + |\Phi^s_{\mathcal{A}}(\mathbf{U})|^2_{H^N \times H^N}.$$

Finally, the result follows from the expression of $\mathcal{D}$, the product estimate B.2.1 and Proposition B.4.1. □

We can now give the main result of this part, the local error estimate. We recall that $\delta = \max(\varepsilon, \beta)$.

Proposition 3.4.5. *Let $N \geq 4$, $M > 0$, ε, β, μ satisfying Condition* (3.2) *and $b \in L^\infty(\mathbb{R})$. Let $\mathbf{U}_0 = (\zeta_0, v_0)$ such that,*

$$|b|_{L^\infty} + |\mathbf{U}_0|_{H^{N+\frac{1}{2}} \times H^N} \leq M.$$

Then, there exists a time $T_4 = T_4\left(M, \frac{1}{\mu_{\min}}, \mu_{\max}\right) > 0$, such that the local error $e(t, \mathbf{U})$ defined in (3.15) *exists for all $0 \leq t \leq \frac{T_4}{\delta}$, and furthermore,*

$$|e(t, \mathbf{U}_0)|_{H^{N-4+\frac{1}{2}} \times H^{N-4}} \leq \delta C_4 t^2,$$

where $C_4 = C\left(\frac{1}{\mu_{\min}}, \mu_{\max}, M\right)$.

Proof. From Propositions 3.3.2 and 3.3.3, we obtain the existence of T. We denote

$$\mathbf{U}(t) = \begin{pmatrix} \zeta(t) \\ v(t) \end{pmatrix} = \Phi^t\left(\mathbf{U}_0\right) \text{ and } \mathbf{V}(t) = \begin{pmatrix} \eta(t) \\ w(t) \end{pmatrix} = \Phi^t_{\mathcal{A}}\left(\Phi^t_{\mathcal{D}}\left(\mathbf{U}_0\right)\right).$$

Then, from Theorem 3.2.2 and Propositions 3.3.2 and 3.3.3 we also have, for all $0 \leq t \leq \frac{T}{\delta}$,

$$\left|\mathbf{U}(t,\cdot)\right|_{H^{N+\frac{1}{2}} \times H^N} + \left|\mathbf{V}(t,\cdot)\right|_{H^{N+\frac{1}{2}} \times H^N} \leq C\left(\frac{1}{\mu_{\min}}, \mu_{\max}, M\right). \tag{3.16}$$

We know that (ζ, v) satisfy the Saut-Xu system (3.6). Furthermore, (η, w) satisfy also the Saut-Xu system (3.6) up to a remainder

$$\partial_t \begin{pmatrix} \eta \\ w \end{pmatrix} = \mathcal{A}\left(\eta, w\right) + \mathcal{D}\left(\eta, w\right) + \mathcal{R}(t),$$

where $\mathcal{R}(t) = \left(\Phi^t_{\mathcal{A}}\right)'\left(\Phi^t_{\mathcal{D}}\left(\mathbf{U}_0\right)\right) \cdot \mathcal{D}\left(\Phi^t_{\mathcal{D}}\left(\mathbf{U}_0\right)\right) - \mathcal{D}\left(\Phi^t_{\mathcal{A}}\left(\Phi^t_{\mathcal{D}}\left(\mathbf{U}_0\right)\right)\right)$. Therefore, the local error e satisfies the following system

$$\partial_t e = \begin{pmatrix} 0 & H_\mu \\ -\partial_x & 0 \end{pmatrix} e + \begin{pmatrix} 0 & \beta\sqrt{\mu}B_\mu \\ 0 & 0 \end{pmatrix} e + \mathcal{T}_\mu\left(\left(\zeta, v\right), \left(\eta, w\right)\right) - \mathcal{R}(t), \tag{3.17}$$

where the operator $\mathcal{T}_\mu\left(\mathbf{U}, \mathbf{V}\right)$ is quadratic and satisfies the following estimate, for $0 \leq s \leq N-1$,

$$\left|\mathcal{T}_\mu\left(\left(\zeta, v\right), \left(\eta, w\right)\right)\right|_{H^s \times H^s} \leq \varepsilon C\left(\frac{1}{\mu_{\min}}, \mu_{\max}, M\right) |e|_{H^{s+1} \times H^{s+1}}. \tag{3.18}$$

Then, since $e_{|t=0} = 0$,

$$e(t,\cdot) = \int_0^t \partial_t e(s,\cdot) ds,$$

and since e satisfies (3.17), we obtain, thanks to Estimates (3.16), (3.18) and Lemma 3.4.4,

$$\left|e\left(t,\cdot\right)\right|_{H^{N-2} \times H^{N-2}} \leq C\left(\frac{1}{\mu_{\min}}, \mu_{\max}, M\right) t. \tag{3.19}$$

Furthermore, we recall that the Saut-Xu system (3.6) is symmetrizable thanks to the symmetrizer (see Theorem 3.2.2)

$$\mathcal{S} = \begin{pmatrix} \frac{D}{\tanh(\sqrt{\mu}D)} & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, applying $\mathcal{S}$ to the system (3.17), and using the fact that $\sqrt{(\mathcal{S}\cdot,\cdot)}$ is a norm equivalent to the $H^{\frac{1}{2}} \times L^2$-norm, we obtain, thanks to estimates (3.16), (3.18) and (3.19) and Lemma 3.4.4,

$$\frac{d}{dt}\mathcal{F}(e) \leq C\left(\frac{1}{\mu_{\min}}, \mu_{\max}, M\right)\left(\beta\mathcal{F}(e) + \varepsilon t\sqrt{\mathcal{F}(e)}\right),$$

where $\mathcal{F}(e) = \sum\limits_{|\alpha| \leq N-4} (S\partial^\alpha e, \partial^\alpha e)$. Then, we get

$$\mathcal{F}(e)(t) \leq \delta C\left(\frac{1}{\mu_{\min}}, \mu_{\max}, M\right) \int_0^t \mathcal{F}(e)(s) + s\sqrt{\mathcal{F}(e)(s)} ds.$$

Denoting $\mathcal{M}(t) = \max\limits_{[0,t]}\sqrt{\mathcal{F}(e)(t)}$, we have

$$\mathcal{M}(t) \leq \delta C\left(\frac{1}{\mu_{\min}}, \mu_{\max}, M\right) \int_0^t \mathcal{M}(s) + s ds,$$

and the result follows from the Grönwall's lemma. □

3.4.2 Global error estimate

In this part, we prove our main result. We denote by

$$\mathbf{U}_k = \left(\mathcal{Y}^{\Delta t}\right)^k (\mathbf{U}_0)$$

the numerical solution and by $\mathbf{U}(t_k) := \Phi^{k\Delta t}(\mathbf{U}_0)$ the exact solution at the time $t_k = k\Delta t$. We recall that $\delta = \max(\varepsilon, \beta)$.

Theorem 3.4.6. *Let $N \geq 4$, $M > 0$, ε, β, μ satisfying Condition* (3.2) *and $b \in L^\infty(\mathbb{R})$. Let $\mathbf{U}_0 = (\zeta_0, v_0)$ such that,*

$$|b|_{L^\infty} + |\mathbf{U}_0|_{H^{N+\frac{1}{2}} \times H^N} \leq M.$$

Then, there exist a time $T = T\left(M, \mu_{\max}, \frac{1}{\mu_{\min}}\right) > 0$ and constants $\gamma, \nu, \Delta t_0, C_0 > 0$ such that for all $\Delta t \in]0, \Delta t_0]$ and for all $n \in \mathbb{N}$ such that $0 \leq n\Delta t \leq \frac{T}{\delta}$,

$$|\mathbf{U}_n|_{H^{N+\frac{1}{2}} \times H^N} \leq \nu \text{ and } \left|\Phi^{n\Delta t}(\mathbf{U}_0) - \left(\mathcal{Y}^{\Delta t}\right)^n (\mathbf{U}_0)\right|_{H^{N-4+\frac{1}{2}} \times H^{N-4}} \leq \gamma \Delta t.$$

Proof. The proof is based on a Lady's Windermere's fan argument and is similar to the one in [32]. In order to simplify the notations, we forget the dependence on $\frac{1}{\mu_{\min}}$ and $\mu_{\max}$ in all the constants. We denote by X^N the following space

$$X^N = H^{N+\frac{1}{2}} \times H^N.$$

Thanks to Theorem 3.2.2, we know that there exists a time $T_0(M) > 0$, such that $\Phi^t \mathbf{U}_0$ exists for all $t \in \left[0, \frac{T_0(M)}{\delta}\right]$ and there exists ρ such that, for all $t \in \left[0, \frac{T_0(M)}{\delta}\right]$,

$$|\mathbf{U}(t_k)|_{X^N} = \left|\Phi^t(\mathbf{U}_0)\right|_{X^N} \leq \rho.$$

We prove by induction that there exists $\Delta t_0, \gamma, \nu$ such that if $0 < \Delta t \leq \Delta t_0$, for all $n \in \mathbb{N}$ with $n\Delta t \leq \frac{T}{\delta}$,

$$\begin{aligned}
&(i)\ |\mathbf{U}_n|_{X^N} \leq 2M,\\
&(ii)\ |\mathbf{U}_n|_{X^N} \leq e^{C_3(2M)\delta n\Delta t} |\mathbf{U}_0|_{X^N},\\
&(iii)\ |\mathbf{U}_n - \mathbf{U}(t_n)|_{X^{N-4}} \leq \gamma \Delta t,
\end{aligned}$$

with

$$\begin{aligned}
&T = \min\left(T_0(M), \frac{\ln(2)}{C_3(2M)}\right), \ \Delta t_0 = \min\left(T_0(2M), T_3(2M), \frac{\ln(2)}{C_0(2M)}\right),\\
&\gamma = T \max(K, 1) C_4(\max(2M, \rho)),
\end{aligned}$$

and where $K = K(4M)$ is a constant from Inequality (3.10) and C_0, T_0, C_3, T_3, C_4 are constants from Theorem 3.2.2, Remark 3.3.4 and Proposition 3.4.5. The above properties are satisfied for $n = 0$. Let $n \geq 1$, and suppose that the induction assumption is true for $0 \leq k \leq n-1$. First, using Remark 3.3.4 and the induction assumption, we have, since $\Delta t \leq T_3(2M)$,

$$|\mathbf{U}_n|_{X^N} = \left|\mathcal{Y}^{\Delta t}\left(\mathbf{U}_{n-1}\right)\right|_{X^N} \leq e^{\delta C_3(2M)\Delta t} |\mathbf{U}_{n-1}|_{X^N} \leq e^{C_3(2M)\delta n \Delta t} |\mathbf{U}_0|_{X^N}.$$

Inequality (i) follows from the fact that $\delta n \Delta t \leq T \leq \frac{\ln(2)}{C_3(2M)}$. Furthermore, we have the following telescopic series (see [68] or [32])

$$\mathbf{U}_n - \mathbf{U}(t_n) = \sum_{0 \leq k \leq n-1} \Phi^{(n-k-1)\Delta t} \circ \mathcal{Y}^{\Delta t}\left(\mathbf{U}_k\right) - \Phi^{(n-k-1)\Delta t} \circ \Phi^{\Delta t}\left(\mathbf{U}_k\right). \tag{3.20}$$

For $k \leq n-2$, since $\mathcal{Y}^{\Delta t}\mathbf{U}_k = \mathbf{U}_{k+1}$, using the induction assumption, we have

$$\left|\mathcal{Y}^{\Delta t}\left(\mathbf{U}_k\right)\right|_{X^{N-3}} \leq 2M,$$

and from Theorem 3.2.2, since $\Delta t \leq \min\left(T_0(2M), \frac{\ln(2)}{C_0(2M)}\right)$, we get

$$\left|\Phi^{\Delta t}\left(\mathbf{U}_k\right)\right|_{X^{N-3}} \leq e^{C_0(2M)\Delta t}|\mathbf{U}_k|_{X^{N-3}} \leq 4M,$$

Therefore, from Proposition 3.2.4 and up to replacing $K = K(4M)$ with $\max(K, 1)$, we obtain, for $k \leq n-1$ and $n\Delta t \leq \frac{T}{\delta}$,

$$\left|\Phi^{(n-k-1)\Delta t} \circ \mathcal{Y}^{\Delta t}\left(\mathbf{U}_k\right) - \Phi^{(n-k-1)\Delta t} \circ \Phi^{\Delta t}\left(\mathbf{U}_k\right)\right|_{X^{N-4}} \leq K\left|\mathcal{Y}^{\Delta t}\left(\mathbf{U}_k\right) - \Phi^{\Delta t}\left(\mathbf{U}(t_k)\right)\right|_{X^{N-4}}.$$

Then, using Proposition 3.4.5 and Inequality (i), we infer

$$\left|\Phi^{(n-k-1)\Delta t} \circ \mathcal{Y}^{\Delta t}\left(\mathbf{U}_k\right) - \Phi^{(n-k-1)\Delta t} \circ \Phi^{\Delta t}\left(\mathbf{U}_k\right)\right|_{X^{N-4}} \leq \delta C_4(\max(2M,\rho))K(\Delta t)^2.$$

Therefore, using the telescopic series (3.20), we get

$$|\mathbf{U}_n - \mathbf{U}(t_n)|_{X^{N-4}} \leq nC_4(\max(2M,\rho))K\delta(\Delta t)^2 \leq C_4(\max(2M,\rho))KT\Delta t.$$

□

Remark 3.4.7. *The method proposed in this chapter can be used for others equations. For instance, we can extend the work [68] by considering equations of the form*

$$\partial_t u = iP(D)u + \varepsilon u\partial_x u + \varepsilon uig(D)u, \tag{3.21}$$

where u is a real function, P is a real polynomial and g a smooth real function satisfying, for all $\xi \in \mathbb{R}$,

$$P(\xi) = -P(-\xi),\ g(\xi) = -g(-\xi),\ \exists C > 0,\ |g'(\xi)| \leq C. \tag{3.22}$$

We split (3.21) *in two parts : the transport part,*

$$\partial_t u = \varepsilon u \partial_x u, \tag{3.23}$$

that we can compute using a Lax Wendroff scheme, and the non local part,

$$\partial_t u = iP(D)u + \varepsilon uig(D)u, \tag{3.24}$$

that we can compute using a pseudo-spectral method. Thanks to Conditions (3.22)*, energy estimates and a commutator estimate for $g(D)$ similar than Proposition B.4.2 (see for instance [79] for such a result), we can show the wellposedness of Equations* (3.21) *and* (3.24) *over a time $\frac{T}{\varepsilon}$. Then, proceeding as in this section, we can obtain a similar result than Theorem 3.4.6.*

3.5 Numerical experiments

The aim of this section is to numerically verify the Lie method convergence rate in $\mathcal{O}(\Delta t)$ for the Saut-Xu system (3.6) and to illustrate some physical phenomena. To solve the dispersive equation (3.12), discrete Fourier transform is used and for the transport equation (3.11), we consider a Lax-Wendroff scheme. In the both cases, we use a Euler method for the time integration. For the latter problem, we have to be careful of the numerical instability, and that why the time and the space steps are chosen in a way that the classical CFL condition is satisfied.

In others works and particularly on the whole water waves problem (see for example [47], [66], [111] and references therein), several authors use a discrete Fourier transform even for the transport part. They observe spurious oscillations in the wave profile that lead to instabilities. These errors seem to appear when they evaluate the nonlinear part via Fourier transform because additional terms appear in the approximation, this is the aliasing phenomenon. To fix this problem, they apply at every time step a low-pass filter. The main interest of our scheme is that we do not need one.

3.5.1 Example 1: Convergence curve

In this example, we consider the following initial data:

$$\zeta_0(x) = \operatorname{sech}\left(\frac{\sqrt{3}}{2}x\right), \quad v_0 = \zeta_0. \tag{3.25}$$

with two different bathymetries: a bump and a ripple bottom. Note that in order to avoid numerical reflections due to the boundaries and justify of the use of the Fast Fourier Transform, we decide to take rapidly decreasing initial data. Figures 3.1 and 3.2 display the evolution for different times of the free surface ζ for these two test cases. We take $\varepsilon = 0.1, \mu = 1, \beta = \frac{1}{2}$ and the final time $T = 10$.

Figures 3.3 displays the convergence curve for this example. We plot the logarithm of the error (in norm $H^1 \times L^2$) in function of the logarithm of the time step Δt. The convergence numerical order is then given by the slope of this curve. For reference, a small line (the dashed line) of slope one is added in this figure. We see that the numerical rate of convergence is greater than 1.

3.5.2 Example 2: Linear versus nonlinear

In this example, we compare the effect of the nonlinear terms on the evolution of the free surface. The linear case corresponds to $\varepsilon = 0$ and for the nonlinear, we take $\varepsilon = 0.1$. We take $\mu = 1$ and the initial data (3.25). Figure 3.4 displays the evolution of the free surface for $T = 15$. We notice that the leading wave in the nonlinear solution is higher than in the linear one. This fact is also noticed in [66] (Section 4.5) when the authors study the propagation of a tsunami.

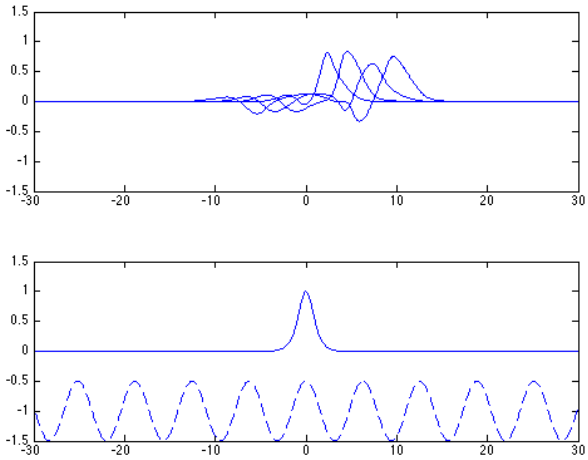

Figure 3.1: Upper: Evolution of the free surface for different times. Lower: bottom topography and initial condition.

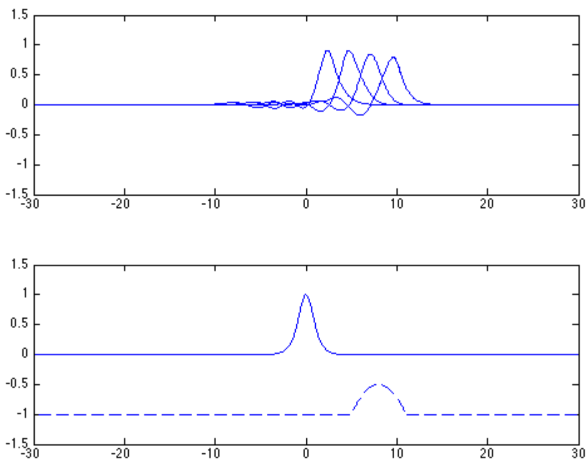

Figure 3.2: Upper: Evolution of the free surface for different times. Lower: bottom topography and initial condition.

3.5.3 Example 3: Boussinesq regime

In the shallow water regime (μ small), there is a huge literature for asymptotic models (see for instance [80]). Among all these asymptotic models, we have the KdV equation. It is a model obtained under the Boussinesq regime, i.e. when $\varepsilon = \mu$, $\beta = 0$ and μ small. In the following, we formally derive a KdV equation from the Saut-Xu equations and we give numerical simulations in this setting.

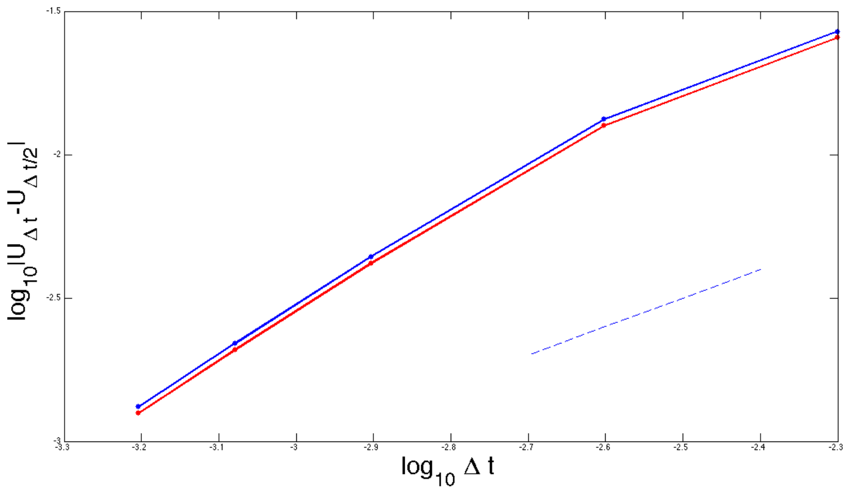

Figure 3.3: Convergence curve for the Lie method for two bottoms: bump (red line) and ripple bottom (blue line).

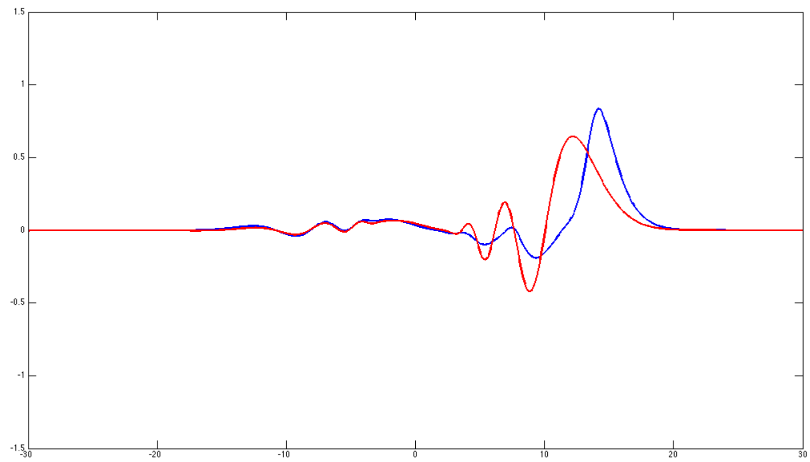

Figure 3.4: Evolution of the free surface for $\varepsilon = 0$ (red) and $\varepsilon = 0.1$ (blue).

We recall that, without the assumption $\nu = \frac{1}{\mu}$, the Saut-Xu equations are given by the system (3.5). Notice also that

$$\mathcal{H}_\mu = -\sqrt{\mu}\partial_x - \frac{1}{3}\mu^{\frac{3}{2}}\partial_x^3 + \mathcal{O}(\mu^2). \tag{3.26}$$

Then if we assume that $\mu = \varepsilon$, $\nu = 1$ (since $\nu \sim 1$ if μ is small) and we drop all the terms of order $\mathcal{O}(\mu^2)$ in System (3.5), we obtain the following equations

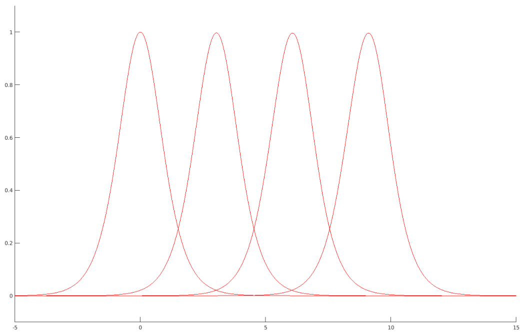

Figure 3.5: Evolution of the soliton at different times $t = 0, 3, 6, 9$ ($\varepsilon = 0.01$).

$$\begin{cases} \partial_t \zeta + \partial_x v + \mu v \partial_x \zeta - \frac{1}{2}\mu^{\frac{3}{2}} v \partial_x \zeta + \frac{1}{3}\mu \partial_x^3 v + \mu \zeta \partial_x v = 0, \\ \partial_t v + \partial_x \zeta + \mu v \partial_x v + \mu^{\frac{3}{2}} \frac{1}{2} v \partial_x v = 0. \end{cases} \tag{3.27}$$

Formally, the solutions of this system are close to the solutions of (3.5) with an accuracy of order $\mathcal{O}(\mu^2)$. Notice that this system is not a standard Boussinesq system (in the sense of [17] or [80]) because of our nonlinear change of variables (3.4). Using the approach developed in [129], [19], [9] (see also Part 7.1.1 in [80]) we can check that, formally, the following KdV equation is an asymptotic model of the system (3.27)

$$\partial_\tau f + \frac{3}{2} f \partial_\xi f + \frac{1}{6} \partial_\xi^3 f = 0. \tag{3.28}$$

This means that if we solve (3.27) with the initial data (f_0, f_0) and (3.28) with the initial datum f_0, the solution $(\zeta, v)(t, x)$ of (3.27) is close to $(f, f)(\mu t, x - t)$. Furthermore, if we take $f_0(x) = \alpha \text{sech}^2 \left(\sqrt{\frac{3}{4}\alpha} x \right)$, the solution f of the KdV equation with this initial datum is the soliton $f(\tau, \xi) = f_0(\xi - c\tau)$ with $c = \frac{\alpha}{2}$. Hence, in this case, the solution of (3.27) and (3.5) are close to a soliton.

In the following we check that the solution to (3.6) is indeed close to the KdV solution when μ is small. We simulate one soliton. We take $\varepsilon = \mu = 0.01$, $\alpha = 1$ and the final time is $T = 10$. Figure 3.5 represents the evolution of this soliton at different times.

In deep water (μ not small), the KdV approximation ceases to be a good approximation. In order to get some insight on the range of validity of the KdV approximation, we compare in Figure 3.6 the solution of (3.6) to the exact soliton after a time $T = 10$ for various values of μ. We notice that even for $\mu = 0.1$ and a final time $T = \frac{1}{\mu}$, the KdV approximation remains a good approximation of the Saut-Xu system.

Remark 3.5.1. *Notice that in Section 3.3, we crucially use the fact that μ is bounded from below. Therefore, we do not have a proof of the convergence of our scheme in the shallow water regime. However, we see that our scheme also works in this context.*

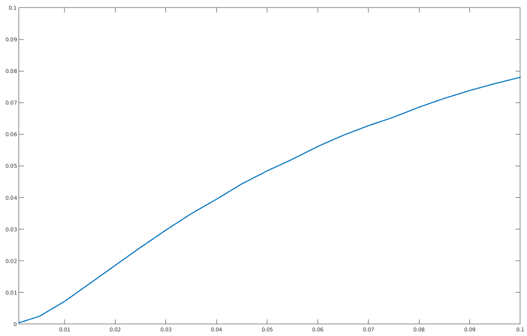

Figure 3.6: Difference after a time $T = 10$ between a real soliton and a soliton generated by our scheme with the same initial data for different values of $\varepsilon = \mu$. Abscissa : value of ε; Ordinate : difference after a final time $T = 10$.

3.5.4 Example 4: Rapidly varying topographies

In this example we study the evolution of water waves over a rapidly varying periodic bottom. We assume that $\mu = 1$. This problem is linked to the Bragg reflection phenomenon (see for instance [100], [91], [66]). We take

$$\zeta_0 = v_0 = \left(\operatorname{sech}\left(\frac{\sqrt{3}}{2}x\right)\right)^2 \quad \text{and } b(x) = \cos(\alpha x). \tag{3.29}$$

Figure 3.7 compares the evolution of water waves when we take $\alpha = 10$ (blue line) and when we take $b(x) = 0$ (blue line). Figure 3.8 displays the difference between the case of a flat bottom and the case of a bottom of the form $b(x) = \cos(\alpha x)$ for different values of α. We observe an homogenization effect when α is large. It seems that when α goes to infinity, the solution of the Saut-Xu equations converges to the solution of the Saut-Xu equations with a flat bottom (corresponding to the mean of b). Notice that this result is different from what we could see in the literature (for instance [40] or [46]), since we take a bottom of the form $b(x) = \cos(\alpha x)$ and not of the form $b(x) = \frac{1}{\alpha}\cos(\alpha x)$. Our numerical simulations suggest therefore a homogenization effect for large amplitude bottom variations that has not been investigated so far.

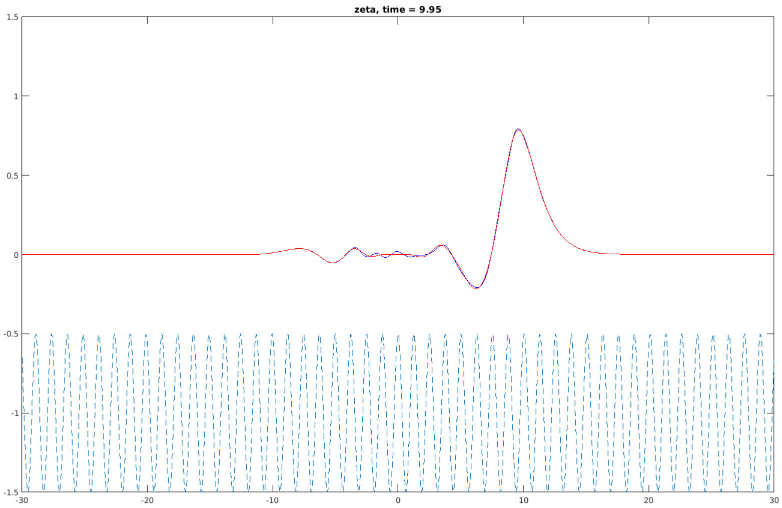

Figure 3.7: Comparison between the evolution of a water wave (blue line) over a bottom of the form $b(x) = \cos(10x)$ (dashed line) and the evolution of a water wave over a flat bottom (red line). $\varepsilon = 0.05$, $\beta = 0.5$.

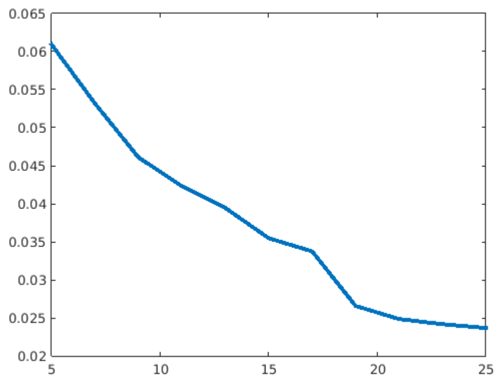

Figure 3.8: Difference between a water wave over a rapidly varying topography $b(x) = \cos(\alpha x)$ and a water wave over a flat bottom. Abscissa : value of α; Ordinate : difference after a final time $T = 10$.

Chapter 4

Coriolis effect on water waves

Sommaire

Ce chapitre est basé sur l'article [101]. Nous avons ajouté la section 4.4 où nous montrons qu'à partir d'une solution régulière de la formulation de Castro-Lannes nous pouvons reconstruire la pression à l'intérieur du fluide et retrouver la solution du système d'Euler à surface libre correspondante. Nous avons aussi ajouté la sous-section 4.5.4 où nous montrons que, lorsque $d = 2$ et en présence de la force de Coriolis, la résonance de Proudman n'est pas possible à cause des effets dispersives.

4.1 Introduction

4.1.1 Presentation of the problem

This chapter is devoted to the study of water waves under the influence of the gravity and the Coriolis force. It is quite common in the physical literature that the rotating shallow water equations are used to study such water waves. We prove a local wellposedness theorem for the water waves equations with vorticity and Coriolis force, taking into account the dependence on various physical parameters and we justify rigorously the shallow water model. We also consider a possible non constant pressure at the surface that can be used to describe meteorological disturbances such as storms or pressure jumps for instance.

There has been a lot of interest on the Cauchy problem for the irrotational water waves problem since the work of Wu ([150] and [151]). More relevant for our present work is the Eulerian approach developed by Lannes ([78]) in the presence of a bottom. Another program initiated by Craig ([43]) consists in justifying the use of the many asymptotic models that exist in the physical literature to describe the motion of water waves. This requires a local wellposedness result that is uniform with respect to the small parameters involved (typically, the shallow water parameter). This was achieved by Alvarez-Samaniego and Lannes ([9]) for many regimes; other references in this direction are ([128], [129], [70]). The irrotational framework is however not always the relevant one. When dealing with wave-current interactions or, at larger scales, if one wants to take into account the Coriolis force. The latter configuration motivates the present study. Several authors considered the local wellposedness theory for the water waves equations in the presence of vorticity ([42], [90], [155], [156]). Recently, Castro and Lannes proposed a generalization of the Zakharov-Craig-Sulem formulation (see [153], [47], [48], [80], [3] for an explanation of this formulation), and gave a system of three equations that allow for the presence of vorticity. Then, they used it to derive new shallow water models that describe wave current interactions and more generally the coupling between waves and vorticity effects ([34] and [33]). In this chapter, we base our study on their formulation.

This chapter is organized in four parts : firstly we derive a generalization of the Castro-Lannes formulation (4.20) that takes into account the Coriolis forcing as well as non flat bottoms and a non constant pressure at the surface; secondly, we prove a local wellposedness result taking account the dependence of small parameters; thirdly, we show how we can reconstruct the pressure in the fluid domain from the Castro-Lannes formulation; Finally, we justify that the rotational shallow water model is a good asymptotic model of the rotational water waves equations under a Coriolis forcing.

We model the sea by an incompressible ideal fluid bounded from below by the seabed and from above by a free surface. We suppose that the seabed and the surface are graphs above the still water level. The pressure at the surface is of the form $P + P_{\text{ref}}$ where $P(t, \cdot)$ models a meteorological disturbance and P_{ref} is a constant which represents the pressure far from the meteorological disturbance. We denote by d the horizontal dimension, which is equal to 1 or 2. The horizontal variable is $X \in \mathbb{R}^d$ and $z \in \mathbb{R}$ is the vertical variable. H is the typical water depth. The water occupies the domain $\Omega_t := \{(X,z) \in \mathbb{R}^{d+1}\ ,\ -H + b(X) < z < \zeta(t,X)\}$. The water is homogeneous (constant density ρ), inviscid with no surface tension. We denote by $\mathbf{U}$ the velocity of the fluid, $\mathbf{V}$ is the horizontal component of the velocity and $\mathbf{w}$ its vertical component. The water is under the influence of the gravity $\boldsymbol{g} = -g\boldsymbol{e_z}$ and the rotation of the Earth with a rotation vector $\mathbf{f} = \frac{f}{2}\mathbf{e_z}$. Finally, we define the pressure in the fluid domain by $\mathcal{P}$. The equations governing the motion of the surface of an ideal fluid under the influence of gravity and Coriolis force are the free surface Euler Coriolis equations [(1)]

$$\begin{cases} \partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_{X,z})\,\mathbf{U} + \mathbf{f} \times \mathbf{U} = -\frac{1}{\rho}\nabla_{X,z}\mathcal{P} - g\mathbf{e_z} \text{ in } \Omega_t, \\ \operatorname{div} \mathbf{U} = 0 \text{ in } \Omega_t, \end{cases} \tag{4.1}$$

with the boundary conditions

$$\begin{cases} \partial_t \zeta - \underline{\mathrm{U}} \cdot N = 0, \\ \mathrm{U}_b \cdot N_b = 0, \end{cases} \tag{4.2}$$

where $N = \begin{pmatrix} -\nabla\zeta \\ 1 \end{pmatrix}$, $N_b = \begin{pmatrix} -\nabla b \\ 1 \end{pmatrix}$, $\underline{\mathrm{U}} = \begin{pmatrix} \underline{\mathrm{V}} \\ \underline{\mathrm{w}} \end{pmatrix} = \mathbf{U}_{|z=\zeta}$ and $\mathrm{U}_b = \begin{pmatrix} \mathrm{V}_b \\ \mathrm{w}_b \end{pmatrix} = \mathbf{U}_{|z=-H+b}$.

The pressure $\mathcal{P}$ can be decomposed as the surface contribution and the internal pressure

$$\mathcal{P}(t,X,z) = P(t,X) + P_{\mathrm{ref}} + \widetilde{\mathcal{P}}(t,X,z),$$

with $\widetilde{\mathcal{P}}_{|z=\zeta} = 0$.

Remark 4.1.1. *In this chapter, we identify functions on $\mathbb{R}^2$ as function on $\mathbb{R}^3$. Then, the gradient, the curl and the divergence operators become in the two dimensional case*

$$\nabla_{X,z} f = \begin{pmatrix} \partial_x f \\ 0 \\ \partial_z f \end{pmatrix},\ \operatorname{curl} \mathbf{A} = \begin{pmatrix} -\partial_z \mathbf{A}_2 \\ \partial_z \mathbf{A}_1 - \partial_x \mathbf{A}_3 \\ -\partial_x \mathbf{A}_2 \end{pmatrix},\ \operatorname{div} \mathbf{A} = \partial_x \mathbf{A}_1 + \partial_z \mathbf{A}_3.$$

In order to obtain some asymptotic models we nondimensionalize the previous equations. There are five important physical parameters : the typical amplitude of the surface a, the typical amplitude of the bathymetry a_{bott}, the typical horizontal scale L, the characteristic water depth H and the typical Coriolis frequency f. Then we can introduce four dimensionless parameters

$$\varepsilon = \frac{a}{H},\ \beta = \frac{a_{\mathrm{bott}}}{H},\ \mu = \frac{H^2}{L^2} \text{ and } \mathrm{Ro} = \frac{a}{fL}\sqrt{\frac{g}{H}}, \tag{4.3}$$

where ε is called the nonlinearity parameter, β the bathymetric parameter, μ the shallowness parameter and Ro the Rossby number. We also nondimensionalize the variables and the unknowns. We introduce (see Section 2.2.1 and [34] for an explanation of this nondimensionalization)

$$\begin{cases} X' = \frac{X}{L},\ z' = \frac{z}{H},\ \zeta' = \frac{\zeta}{a},\ b' = \frac{b}{a_{\mathrm{bott}}},\ t' = \frac{\sqrt{gH}}{L}t, \\ \mathbf{V}' = \sqrt{\frac{H}{g}}\frac{\mathbf{V}}{a},\ \mathbf{w}' = H\sqrt{\frac{H}{g}}\frac{\mathbf{w}}{aL},\ P' = \frac{P}{\rho g a} \text{ and } \widetilde{\mathcal{P}}' = \frac{\widetilde{\mathcal{P}}}{\rho g H}. \end{cases} \tag{4.4}$$

[1] We consider that the centrifugal potential is constant and included in the pressure term.

In this chapter, we use the following notations

$$\nabla^\mu_{X',z'} = \begin{pmatrix} \sqrt{\mu}\nabla_{X'} \\ \partial_{z'} \end{pmatrix}, \ \mathrm{curl}^\mu = \nabla^\mu_{X',z'} \times , \ \mathrm{div}^\mu = \nabla^\mu_{X',z'} \cdot . \tag{4.5}$$

We also define

$$\mathbf{U}^\mu = \begin{pmatrix} \sqrt{\mu}\mathbf{V}' \\ \mathbf{w}' \end{pmatrix}, \ \boldsymbol{\omega}' = \frac{1}{\mu}\mathrm{curl}^\mu \mathbf{U}^\mu, \ \underline{\mathrm{U}}^\mu = \mathbf{U}^\mu_{|z'=\varepsilon\zeta'}, \ \mathrm{U}^\mu_b = \mathbf{U}^\mu_{|z'=-1+\beta b'}, \tag{4.6}$$

and

$$N^\mu = \begin{pmatrix} -\varepsilon\sqrt{\mu}\nabla\zeta' \\ 1 \end{pmatrix}, \ N^\mu_b = \begin{pmatrix} -\beta\sqrt{\mu}\nabla b' \\ 1 \end{pmatrix}. \tag{4.7}$$

Notice that our nondimensionalization of the vorticity allows us to consider only weakly sheared flows (see [33], [139], [121]). The nondimensionalized fluid domain is

$$\Omega'_{t'} := \{(X', z') \in \mathbb{R}^{d+1} \ , \ -1 + \beta b'(X') < z' < \varepsilon\zeta'(t', X')\}. \tag{4.8}$$

Finally, if $\mathbf{V} = \begin{pmatrix} \mathrm{V}_1 \\ \mathrm{V}_2 \end{pmatrix} \in \mathbb{R}^2$, we define $\mathbf{V}$ by $\mathbf{V}^\perp = \begin{pmatrix} -\mathrm{V}_2 \\ \mathrm{V}_1 \end{pmatrix}$. Then, the Euler Coriolis equations (4.1) become

$$\begin{cases} \partial_{t'}\mathbf{U}^\mu + \dfrac{\varepsilon}{\mu}\left(\mathbf{U}^\mu \cdot \nabla^\mu_{X',z'}\right)\mathbf{U}^\mu + \dfrac{\varepsilon\sqrt{\mu}}{\mathrm{Ro}}\begin{pmatrix} \mathbf{V}'^\perp \\ 0 \end{pmatrix} = -\sqrt{\mu}\begin{pmatrix} \nabla P' \\ 0 \end{pmatrix} - \dfrac{1}{\varepsilon}\nabla^\mu_{X',z'}\widetilde{\mathcal{P}}' - \dfrac{1}{\varepsilon}\mathbf{e_z} \text{ in } \Omega'_t, \\ \mathrm{div}^\mu_{X',z'} \ \mathbf{U}^\mu = 0 \text{ in } \Omega'_t, \end{cases} \tag{4.9}$$

with the boundary conditions

$$\begin{cases} \partial_{t'}\zeta' - \dfrac{1}{\mu}\underline{\mathrm{U}}^\mu \cdot N^\mu = 0, \\ \mathrm{U}^\mu_b \cdot N^\mu_b = 0. \end{cases} \tag{4.10}$$

In the following we omit the primes. In [34], Castro and Lannes derived a formulation of the water waves equations with vorticity. We outline the main ideas of this formulation and extend it to take into account the Coriolis force. Even in absence of Coriolis forcing, our results extend the result of [34] by allowing non flat bottoms. First, applying the curl^μ operator to the first equation of (4.9) we obtain an equation on $\boldsymbol{\omega}$

$$\partial_t\boldsymbol{\omega} + \frac{\varepsilon}{\mu}\left(\mathbf{U}^\mu \cdot \nabla^\mu_{X,z}\right)\boldsymbol{\omega} = \frac{\varepsilon}{\mu}\boldsymbol{\omega} \cdot {\nabla^\mu}_{X,z}\mathbf{U}^\mu + \frac{\varepsilon}{\mu\mathrm{Ro}}\partial_z\mathbf{U}^\mu. \tag{4.11}$$

Furthermore, taking the trace at the surface of the first equation of (4.9) we get

$$\partial_t\underline{\mathrm{U}}^\mu + \varepsilon\left(\underline{\mathrm{V}} \cdot \nabla_X\right)\underline{\mathrm{U}}^\mu + \frac{\varepsilon\sqrt{\mu}}{\mathrm{Ro}}\begin{pmatrix} \underline{\mathrm{V}}^\perp \\ 0 \end{pmatrix} = -\sqrt{\mu}\begin{pmatrix} \nabla P \\ 0 \end{pmatrix} - \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{\varepsilon}\left(\partial_z\widetilde{\mathcal{P}}\right)_{|z=\varepsilon\zeta} N^\mu. \tag{4.12}$$

Then, in order to eliminate the term $\left(\partial_z\widetilde{\mathcal{P}}\right)_{|z=\varepsilon\zeta} N^\mu$, we have to introduce the following quantity. If $\mathbf{A}$ is a vector field on Ω_t, we define $\mathrm{A}_{/\!/}$ as

$$\mathrm{A}_{/\!/} = \frac{1}{\sqrt{\mu}}\underline{\mathrm{A}}_h + \varepsilon\underline{\mathrm{A}}_v\nabla\zeta, \tag{4.13}$$

where A_h is the horizontal component of A, A_v its vertical component, $\underline{\mathrm{A}} = \mathbf{A}_{|z=\varepsilon\zeta}$ and $\mathrm{A}_b = \mathbf{A}_{|z=-1+\beta b}$. Notice that,

$$\underline{\mathrm{A}} \times N^\mu = \sqrt{\mu} \begin{pmatrix} -\mathrm{A}_{/\!/}^\perp \\ -\varepsilon\sqrt{\mu}\mathrm{A}_{/\!/}^\perp \cdot \nabla\zeta \end{pmatrix}. \tag{4.14}$$

Therefore, taking the orthogonal of the horizontal component of the vectorial product of (4.12) with N^μ we obtain

$$\partial_t \mathrm{U}_{/\!/}^\mu + \nabla\zeta + \frac{\varepsilon}{2}\nabla\left|\mathrm{U}_{/\!/}^\mu\right|^2 - \frac{\varepsilon}{2\mu}\nabla\left[\left(1+\varepsilon^2\mu\left|\nabla\zeta\right|^2\right)\underline{\mathrm{w}}^2\right] + \varepsilon\left(\underline{\omega}\cdot N^\mu + \frac{1}{\mathrm{Ro}}\right)\underline{\mathrm{V}}^\perp = -\nabla P. \tag{4.15}$$

Since $\mathrm{U}_{/\!/}^\mu$ is a vector field on $\mathbb{R}^2$, we have the classical Hodge-Weyl decomposition

$$\mathrm{U}_{/\!/}^\mu = \nabla\frac{\nabla}{\Delta}\cdot\mathrm{U}_{/\!/}^\mu + \nabla^\perp\frac{\nabla^\perp}{\Delta}\cdot\mathrm{U}_{/\!/}^\mu. \tag{4.16}$$

In the following we denote by $\psi := \frac{\nabla}{\Delta}\cdot\mathrm{U}_{/\!/}$ and $\widetilde{\psi} := \frac{\nabla^\perp}{\Delta}\cdot\mathrm{U}_{/\!/}$ [2]. Applying the operator $\frac{\nabla}{\Delta}\cdot$ to (4.15), we obtain

$$\partial_t\psi + \zeta + \frac{\varepsilon}{2}\left|\mathrm{U}_{/\!/}^\mu\right|^2 - \frac{\varepsilon}{2\mu}\left(1+\varepsilon^2\mu\left|\nabla\zeta\right|^2\right)\underline{\mathrm{w}}^2 + \varepsilon\frac{\nabla}{\Delta}\cdot\left[\left(\underline{\omega}\cdot N^\mu + \frac{1}{\mathrm{Ro}}\right)\underline{\mathrm{V}}^\perp\right] = -P. \tag{4.17}$$

Moreover, using the following vectorial identity

$$\left(\nabla_{X,z}^\mu \times \mathbf{U}^\mu\right)_{|z=\varepsilon\zeta}\cdot N^\mu = \mu\nabla^\perp\cdot\mathrm{U}_{/\!/}^\mu, \tag{4.18}$$

we have

$$\Delta\widetilde{\psi} = \left(\underline{\omega}\cdot N^\mu\right). \tag{4.19}$$

We can now give the nondimensionalized Castro-Lannes formulation of the water waves equations with vorticity in the presence of Coriolis forcing. It is a system of three equations for the unknowns $(\zeta, \psi, \boldsymbol{\omega})$

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu}\underline{\mathrm{U}}^\mu\cdot N^\mu = 0, \\ \partial_t\psi+\zeta+\frac{\varepsilon}{2}\left|\mathrm{U}_{/\!/}^\mu\right|^2-\frac{\varepsilon}{2\mu}\left(1+\varepsilon^2\mu\left|\nabla\zeta\right|^2\right)\underline{\mathrm{w}}^2+\varepsilon\frac{\nabla}{\Delta}\cdot\left[\left(\underline{\omega}\cdot N^\mu + \frac{1}{\mathrm{Ro}}\right)\underline{\mathrm{V}}^\perp\right] = -P, \\ \partial_t\boldsymbol{\omega}+\frac{\varepsilon}{\mu}\left(\mathbf{U}^\mu\cdot\nabla_{X,z}^\mu\right)\boldsymbol{\omega} = \frac{\varepsilon}{\mu}\left(\boldsymbol{\omega}\cdot\nabla_{X,z}^\mu\right)\mathbf{U}^\mu + \frac{\varepsilon}{\mu\mathrm{Ro}}\partial_z\mathbf{U}^\mu, \end{cases} \tag{4.20}$$

where $\mathbf{U}^\mu := \mathbf{U}^\mu[\varepsilon\zeta, \beta b](\psi, \boldsymbol{\omega})$ is the unique solution in $H^1(\Omega_t)$ of

$$\begin{cases} \mathrm{curl}^\mu\ \mathbf{U}^\mu = \mu\boldsymbol{\omega} \text{ in } \Omega_t, \\ \mathrm{div}^\mu\ \mathbf{U}^\mu = 0 \text{ in } \Omega_t, \\ \underline{\mathrm{U}}_{/\!/}^\mu = \nabla\psi + \frac{\nabla^\perp}{\Delta}\left(\underline{\omega}\cdot N^\mu\right), \\ \mathrm{U}_b^\mu\cdot N_b^\mu = 0. \end{cases} \tag{4.21}$$

[2] We define rigorously these operators in Section 4.2.1.

We add a technical assumption. We assume that the water depth is bounded from below by a positive constant

$$\exists\, h_{\min} > 0 \;,\; \varepsilon\zeta + 1 - \beta b \geq h_{\min}. \tag{4.22}$$

We also suppose that the dimensionless parameters satisfy

$$\exists \mu_{\max},\; 0 < \mu \leq \mu_{\max},\; 0 < \varepsilon \leq 1,\; 0 < \beta \leq 1 \text{ and } \frac{\varepsilon}{\mathrm{Ro}} \leq 1. \tag{4.23}$$

Remark 4.1.2. *The assumption $\varepsilon \leq Ro$ is equivalent to $fL \leq \sqrt{gH}$. This means that the typical rotation speed due to the Coriolis force is less than the typical water wave celerity. For water waves, this assumption is common (see for instance [117], [61]). Typically for offshore long water waves at mid-latitudes, we have $L = 100km$ and $H = 1km$ and $f = 10^{-4}Hz$. Then, $\frac{\varepsilon}{Ro} = 10^{-1}$.*

4.1.2 Existence result

In this part, we give our main result. It is a wellposedness result for the system (4.46) which is a straightened system of the Castro-Lannes formulation. This result extends Theorem 4.7 and Theorem 5.1 in [34] by adding a non flat bottom and a Coriolis forcing. We define the energy $\mathcal{E}^N$ in Subsection 4.3.1.

Theorem 4.1.3. *Assume that the initial data, b and P are smooth enough and the initial vorticity is divergent free. Assume also that Conditions* (4.22) *and* (4.54) *are satisfied initially. Then, there exists $T > 0$, and a unique solution to the water waves equations* (4.46) *on* $[0, T]$. *Moreover,*

$$T = \min\left(\frac{T_0}{\max(\varepsilon, \beta, \frac{\varepsilon}{\mathrm{Ro}})}, \frac{T_0}{|\nabla P|_{L^\infty_t H^N_X}}\right) \;,\; \frac{1}{T_0} = c^1 \text{ and } \sup_{t\in[0,T]} \mathcal{E}^N(\zeta(t), \psi(t), \omega(t)) = c^2,$$

where c^j is a constant which depends on the initial conditions, P and b.

A full version is given in Subsection 4.3.4. This theorem allows us to investigate the justification of asymptotic models in the presence of Coriolis forcing. In the case of a constant pressure at the surface and without a Coriolis forcing, our existence time is similar to Theorem 3.16 in [80] (see also [9]); without a Coriolis forcing, it is as Theorem 2.2.3 in the second chapter.

4.1.3 Notations for this chapter

- If $\mathbf{A} \in \mathbb{R}^3$, we denote by $\mathbf{A}_h$ its horizontal component and by $\mathbf{A}_v$ its vertical component.

- If $\mathbf{V} = \begin{pmatrix} \mathrm{V}_1 \\ \mathrm{V}_2 \end{pmatrix} \in \mathbb{R}^2$, we define the orthogonal of $\mathbf{V}$ by $\mathbf{V}^\perp = \begin{pmatrix} -\mathrm{V}_2 \\ \mathrm{V}_1 \end{pmatrix}$.

- In this chapter, $C(\cdot)$ is a nondecreasing and positive function whose exact value has no importance.

- Consider a vector field $\mathbf{A}$ or a function $\mathbf{w}$ defined on Ω. Then, we denote $\mathrm{A} = \mathbf{A} \circ \Sigma$ and $\mathrm{w} = \mathbf{w} \circ \Sigma$, where Σ is defined in (4.40). Furthermore, we denote $\underline{\mathrm{A}} = \mathbf{A}_{|z=\varepsilon\zeta} = \mathrm{A}_{|z=0}$, $\underline{\mathrm{w}} = \mathbf{w}_{|z=\varepsilon\zeta} = \mathrm{w}_{|z=0}$ and $\mathrm{A}_b = \mathbf{A}_{|z=-1+\beta b} = \mathrm{A}_{|z=-1}$, $\mathrm{w}_b = \mathbf{w}_{|z=-1+b} = \mathrm{w}_{|z=-1}$.

- If $s \in \mathbb{R}$ and f is a function on $\mathbb{R}^d$, $|f|_{H^s}$ is its H^s-norm and $|f|_2$ is its L^2-norm. The quantity $|f|_{W^{k,\infty}}$ is $W^{k,\infty}(\mathbb{R}^d)$-norm of f, where $k \in \mathbb{N}^*$, and $|f|_{L^\infty}$ its $L^\infty(\mathbb{R}^d)$-norm.

- The operator (,) is the L^2-scalar product in $\mathbb{R}^d$.

- If $N \in \mathbb{N}^*$, $\mathbf{A}$ is defined on Ω and $\mathrm{A} = \mathbf{A} \circ \Sigma$, $||\mathrm{A}||_{H^N}$ and $||\mathbf{A}||_{H^N}$ are respectively the $H^N(\mathcal{S})$-norm of A and the $H^N(\Omega)$-norm of $\mathbf{A}$. The L^p-norm are denoted $||\cdot||_p$.

- The norm $||\cdot||_{H^{s,k}}$ is defined in Definition 4.2.10.

- The space $H^s_*(\mathbb{R}^d)$, $\dot{H}^s(\mathbb{R}^d)$ and $H_b(\mathrm{div}_0^\mu, \Omega)$ are defined in Subsection 4.2.1.

- If f is a function defined on $\mathbb{R}^d$, we denote ∇f the gradient of f.

- If $\mathbf{w}$ is a function defined on Ω, $\nabla_{X,z}\mathbf{w}$ is the gradient of $\mathbf{w}$ and $\nabla_X \mathbf{w}$ its horizontal component. We have the same definition for functions defined on $\mathcal{S}$.

- $\mathfrak{P}$, Λ and M_N are defined in Subsection 4.2.1.

4.2 The div-curl problem

In [34], A. Castro and D. Lannes study the system (4.21) in the case of a flat bottom ($b = 0$). The purpose of this part is to extend their results in the case of a non flat bottom.

4.2.1 Notations for this part

In this chapter, we use the Beppo-Levi spaces (see [53])

$$\forall s \geq 0,\, \dot{H}^s(\mathbb{R}^d) = \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}^d),\, \nabla f \in H^{s-1}(\mathbb{R}^d) \right\} \text{ and } |\cdot|_{\dot{H}^s} = |\nabla \cdot|_{H^{s-1}} .$$

The dual space of $\dot{H}^s(\mathbb{R}^d)/_{\mathbb{R}}$ is the space (see [29])

$$H^{-s}_*(\mathbb{R}^d) = \left\{ u \in H^{-s}(\mathbb{R}^d),\, \exists v \in H^{-s+1}(\mathbb{R}^d),\, u = |D|\, v \right\} \text{ and } |\cdot|_{H^{-s}_*} = \left| \frac{\cdot}{|D|} \right|_{H^{-s+1}} .$$

Notice that $\dot{H}^1(\mathbb{R}^d)/_{\mathbb{R}}$ is a Hilbert space (see Proposition A.1.1). Then, we can rigorously define the Hodge-Weyl decomposition and the operators $\frac{\nabla}{\Delta}\cdot$ and $\frac{\nabla^\perp}{\Delta}\cdot$. For $f \in L^2(\mathbb{R}^d)^d$, $u = \frac{\nabla}{\Delta} \cdot f$ is defined as the unique solution, up to a constant, in $\dot{H}^1(\mathbb{R}^d)$ of the variational problem

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla \phi = \int_{\mathbb{R}^d} f \cdot \nabla \phi \ ,\, \forall \phi \in \dot{H}^1(\mathbb{R}^d).$$

The operator $\frac{\nabla^\perp}{\Delta}\cdot$ can be defined similarly. Then, it is easy to check that the operators $\frac{\nabla^\perp}{\Delta}\cdot$ and $\frac{\nabla^\perp}{\Delta}\cdot$ belong to $\mathcal{L}\left(H^s(\mathbb{R}^d)^d, \dot{H}^{s+1}(\mathbb{R}^d)\right)$, for all $s \geq 0$.

The subspace of $L^2(\Omega)^3$ of functions whose rotationnal is in $L^2(\Omega)^3$ is the space

$$H(\mathrm{curl}^\mu, \Omega) = \left\{ \mathbf{A} \in L^2(\Omega)^3,\, \mathrm{curl}^\mu \mathbf{A} \in L^2(\Omega)^3 \right\} .$$

The subspace of $L^2(\Omega)^3$ of divergence free vector fields is the space

$$H(\mathrm{div}_0^\mu, \Omega) = \left\{ \mathbf{A} \in L^2(\Omega)^3,\, \mathrm{div}^\mu \mathbf{A} = 0 \right\} .$$

Remark 4.2.1. *Notice that $\mathbf{A} \in H(\mathrm{div}_0^\mu, \Omega)$ implies that $\left(\mathbf{A}_{|\partial\Omega} \cdot n\right)$ belongs to $H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathbf{A} \in H(\mathrm{curl}^\mu, \Omega)$ implies that $\left(\mathbf{A}_{|\partial\Omega} \times n\right)$ belongs to $H^{-\frac{1}{2}}(\partial\Omega)$ (see [51]).*

Finally, we define $H_b(\mathrm{div}_0^\mu, \Omega)$ as

$$H_b(\mathrm{div}_0^\mu, \Omega) = \left\{ \mathbf{A} \in H(\mathrm{div}_0^\mu, \Omega) \, , \, A_b \cdot N_b^\mu \in H_*^{-\frac{1}{2}}(\mathbb{R}^d) \right\} .$$

Remark 4.2.2. *We have a similar equation to* (4.18) *at the bottom*

$$\frac{1}{\mu} \left(\nabla_{X,z}^\mu \times \mathbf{U}^\mu \right)_{|z=-1+\beta b} \cdot N_b^\mu = \nabla^\perp \cdot (\mathbf{V}_b + \beta \mathbf{w}_b \nabla b) \, ,$$

hence, in the following, we suppose that $\boldsymbol{\omega} \in H_b(\mathrm{div}_0^\mu, \Omega)$.

We define $\mathfrak{P}$ and Λ as the Fourier multiplier in $\mathcal{S}'(\mathbb{R}^d)$,

$$\mathfrak{P} = \frac{|D|}{\sqrt{1+\sqrt{\mu}\,|D|}} \text{ and } \Lambda = \sqrt{1+|D|^2}.$$

Then it is important to notice that, if $\boldsymbol{\omega} \in H_b(\mathrm{div}_0^\mu, \Omega)$, the quantity $\frac{1}{\mathfrak{P}}(\omega_b \cdot N_b^\mu)$ makes sense and belongs to $L^2(\mathbb{R}^d)$.

In the following M_N is a constant of the form

$$M_N = C \left(\mu_{\max}, \frac{1}{h_{\min}}, \varepsilon \left| \zeta \right|_{H^N}, \beta \left| \nabla b \right|_{H^N}, \beta \left| b \right|_{L^\infty} \right). \tag{4.24}$$

4.2.2 Existence and uniqueness

In this part, we forget the dependence on t. First, notice that we can split the problem into two parts. Let $\Phi \in \dot{H}^2(\Omega)$ the unique solution of the Laplace problem (see [80])

$$\begin{cases} \Delta_{X,z}^\mu \Phi = 0 \text{ in } \Omega, \\ \Phi_{|z=\varepsilon\zeta} = \psi, \; \left(N_b^\mu \cdot \nabla_{X,z}^\mu \Phi \right)_{|z=-1+\beta b} = 0. \end{cases} \tag{4.25}$$

Using the vectorial identity

$$\left(\nabla_{X,z}^\mu \Phi \right)_{/\!/} = \nabla \psi,$$

it is easy to check that if $\mathbf{U}^\mu$ satisfies (4.21), $\widetilde{\mathbf{U}}^\mu := \mathbf{U}^\mu - \nabla_{X,z}^\mu \Phi$ satisfies

$$\begin{cases} \mathrm{curl}^\mu \, \widetilde{\mathbf{U}}^\mu = \mu \, \boldsymbol{\omega} \text{ in } \Omega_t, \\ \mathrm{div}^\mu \, \widetilde{\mathbf{U}}^\mu = 0 \text{ in } \Omega_t, \\ \widetilde{\mathrm{U}}_{/\!/}^\mu = \dfrac{\nabla^\perp}{\Delta} \left(\underline{\boldsymbol{\omega}} \cdot N^\mu \right) \text{at the surface}, \\ \widetilde{\mathbf{U}}_b^\mu \cdot N_b^\mu = 0 \text{ at the bottom}. \end{cases} \tag{4.26}$$

In the following we focus on the system (4.26). We give 4 intermediate results in order to get the existence and uniqueness. The first Proposition shows how to control the norm of the gradient of a function with boundary conditions as in (4.26).

Proposition 4.2.3. *Let ζ, $b \in W^{2,\infty}(\mathbb{R}^d)$, $\mathbf{A} \in H(\mathrm{div}_0^{\mu}, \Omega) \cap H(\mathrm{curl}^{\mu}, \Omega)$. Then, for all $C \in H^1(\mathbb{R}^d)^3$, we have*

$$\int_{\Omega} \nabla^{\mu}_{X,z} \mathbf{A} : \nabla^{\mu}_{X,z} \mathbf{C} = \int_{\Omega} \mathrm{curl}^{\,\mu} \mathbf{A} : \mathrm{curl}^{\,\mu} \mathbf{C} + \left\langle l^{\mu}[\varepsilon\zeta](\underline{\mathrm{A}}), \underline{C} \right\rangle_{H^{-\frac{1}{2}}-H^{\frac{1}{2}}} - \left\langle l^{\mu}[\beta b](\mathrm{A}_b), C_b \right\rangle_{H^{-\frac{1}{2}}-H^{\frac{1}{2}}}, \tag{4.27}$$

where for $\mathrm{B} \in H^{\frac{1}{2}}\left(\mathbb{R}^2\right)^3$ *and for* $\eta \in W^{2,\infty}(\mathbb{R}^d)$,

$$l^{\mu}[\eta](\mathrm{B}) = \begin{pmatrix} \sqrt{\mu} \nabla \mathrm{B}_v - \mu \left(\nabla^{\perp} \eta \cdot \nabla\right) \mathrm{B}_h^{\perp} \\ -\sqrt{\mu} \nabla \cdot \mathrm{B}_h \end{pmatrix}. \tag{4.28}$$

Furthermore, if $\widetilde{\psi} \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$ *and*

$$A_b \cdot N_b^{\mu} = 0 \text{ and } \mathrm{A}_{/\!/} = \nabla^{\perp} \widetilde{\psi},$$

we have the following estimate

$$\begin{aligned} \left|\left|\nabla^{\mu}_{X,z} \mathbf{A}\right|\right|_2^2 \leq & \left|\left|\mathrm{curl}^{\,\mu} \mathbf{A}\right|\right|_2^2 + \mu C\left(\varepsilon \left|\zeta\right|_{W^{2,\infty}}, \beta \left|b\right|_{W^{2,\infty}}\right) \left(\left|\underline{\mathrm{A}}\right|_2^2 + \left|\mathrm{A}_{bh}\right|_2^2\right) \\ & + \mu C\left(\mu_{\max}, \varepsilon \left|\zeta\right|_{W^{2,\infty}}, \beta \left|b\right|_{W^{2,\infty}}\right) \left|\sqrt{1 + \sqrt{\mu}|D|} \nabla \widetilde{\psi}\right|_2 \left|\sqrt{1 + \sqrt{\mu}\,|D|} \underline{\mathrm{A}}_h\right|_2. \end{aligned} \tag{4.29}$$

Proof. Using the Einstein summation convention and denoting $\nabla^{\mu}_{X,z} = \left(\partial_1^{\mu}, \partial_2^{\mu}, \partial_3^{\mu}\right)^T$, a simple computation gives (see Lemma 3.2 in [34] or Chapter 9 in [51]),

$$\left|\left|\nabla^{\mu} \mathbf{A}\right|\right|_2^2 = \left|\left|\mathrm{curl}^{\mu} \mathbf{A}\right|\right|_2^2 + \left|\left|\mathrm{div}^{\mu} \mathbf{A}\right|\right|_2^2 + \int_{\partial\Omega} n_i^{\mu} \mathbf{A}_j \partial_j^{\mu} \mathbf{A}_i - n_j^{\mu} \mathbf{A}_j \partial_i^{\mu} \mathbf{A}_i. \tag{4.30}$$

In this case, $\partial\Omega$ is the union of two surfaces and $\overrightarrow{n^{\mu}} = \pm \begin{pmatrix} -\sqrt{\mu}\nabla\eta \\ 1 \end{pmatrix}$, where η is the corresponding surface. Then, one can check that (see also Lemma 3.8 in [34]),

$$\int_{\{z=\eta\}} n_i^{\mu} \mathbf{A}_j \partial_j^{\mu} \mathbf{A}_i - n_j^{\mu} \mathbf{A}_j \partial_i^{\mu} \mathbf{A}_i = \pm \int_{\mathbb{R}^d} \mathbf{A}_{\eta,h} \cdot \left(2\sqrt{\mu} \nabla_X \mathrm{A}_{\eta,v} - \mu \left(\nabla \eta^{\perp} \cdot \nabla\right) \mathbf{A}_{\eta,h}^{\perp}\right), \tag{4.31}$$

where $\mathbf{A}_{\eta} := \mathbf{A}_{|z=\eta}$. The first part of the Proposition follows by polarization of Equations (4.30) and (4.31) (as quadratic forms). For the second estimate, since $\mathrm{A}_b \cdot N_b^{\mu} = 0$, we get at the bottom that

$$\begin{aligned} \int_{\{z=-1+\beta b\}} n_i^{\mu} \mathbf{A}_j \partial_j^{\mu} \mathbf{A}_i - n_j^{\mu} \mathbf{A}_j \partial_i^{\mu} \mathbf{A}_i &= -2 \int_{\mathbb{R}^d} \mu\beta \left(\partial_x b \mathrm{A}_{bx} \partial_y \mathrm{A}_{by} + \partial_y b \mathrm{A}_{by} \partial_x \mathrm{A}_{bx} + \partial_{xy}^2 b \mathrm{A}_{bx} \mathrm{A}_{by}\right) - \mathrm{A}_z \sqrt{\mu} \mathrm{div}_X \mathrm{A}_{bh} \\ &= -\mu\beta \int_{\mathbb{R}^d} \partial_x^2 b \, \mathrm{A}_{bx}^2 + \partial_y^2 b \, \mathrm{A}_{by}^2 + 2\partial_{xy}^2 b \, \mathrm{A}_{bx} \mathrm{A}_{by}. \end{aligned}$$

At the surface, since $\underline{\mathrm{A}}_h = \sqrt{\mu} \nabla^{\perp} \widetilde{\psi} - \varepsilon \sqrt{\mu} \underline{\mathrm{A}}_v \nabla\zeta$ (we use equality (4.14)), we have

$$\int_{\{z=\varepsilon\zeta\}} n_i^\mu \mathbf{A}_j \partial_j^\mu \mathbf{A}_i - n_j^\mu \mathbf{A}_j \partial_i^\mu \mathbf{A}_i = -2\int_{\mathbb{R}^d} \varepsilon\mu \left(\partial_x \zeta \underline{\mathrm{A}}_y \partial_y \underline{\mathrm{A}}_x + \partial_y \zeta \underline{\mathrm{A}}_x \partial_x \underline{\mathrm{A}}_y + \partial_{xy}^2 \zeta \underline{\mathrm{A}}_x \underline{\mathrm{A}}_y\right) + \sqrt{\mu}\left(\underline{\mathrm{A}}_h \cdot \nabla_X\right) \underline{\mathrm{A}}_z$$

$$= \varepsilon\mu \int_{\mathbb{R}^d} \underline{\mathrm{A}}_x^2 \partial_y^2 \zeta + \underline{\mathrm{A}}_y^2 \partial_x^2 \zeta - 2\underline{\mathrm{A}}_x \underline{\mathrm{A}}_y \partial_{xy}^2 \zeta + \underline{\mathrm{A}}_z^2 \left[\partial_x^2 \zeta + \partial_y^2 \zeta\right]$$

$$- 2\varepsilon\mu^{\frac{3}{2}} \int_{\mathbb{R}^d} \underline{\mathrm{A}}_h \cdot \nabla^\perp \Big(\nabla\widetilde{\psi} \cdot \nabla\zeta\Big).$$

Then,

$$\left| 2\varepsilon\mu^{\frac{3}{2}} \int_{\mathbb{R}^d} \underline{\mathrm{A}}_h \cdot \nabla^\perp \Big(\nabla\widetilde{\psi} \cdot \nabla\zeta\Big) \right| \leq \varepsilon\mu \left| \sqrt{1 + \sqrt{\mu}|D|}\, \underline{\mathrm{A}}_h \right|_2 \left| \sqrt{\mu}\mathfrak{P}\left(\nabla\widetilde{\psi} \cdot \nabla\zeta\right) \right|_2.$$

and estimate (4.29) follows easily from Lemma B.1.2.

□

The second Proposition gives a control of the L^2-norm of the trace.

Proposition 4.2.4. *Let ζ, $b \in W^{1,\infty}\left(\mathbb{R}^d\right)$, $\mathbf{A} \in H(\mathrm{div}_0^\mu, \Omega) \cap H(\mathrm{curl}^\mu, \Omega)$ and $\widetilde{\psi} \in \dot{H}^1\left(\mathbb{R}^d\right)$ such that*

$$\mathrm{A}_b \cdot N_b^\mu = 0 \text{ and } \mathrm{A}_{/\!/} = \nabla^\perp \widetilde{\psi}.$$

Then,

$$|\underline{\mathrm{A}}|_2^2 + |\mathrm{A}_b|_2^2 \leq C. \left(\mu \left|\nabla\widetilde{\psi}\right|_2^2 + ||\mathrm{curl}^{\,\mu} \mathbf{A}||_2 \, ||\mathbf{A}||_2\right). \tag{4.32}$$

Proof. Using the fact that $\partial_z \mathbf{A}_h = -\left(\mathrm{curl}^{\,\mu} \mathbf{A}\right)_h^\perp + \sqrt{\mu}\nabla_X \mathbf{A}_v$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |\underline{\mathrm{A}}_h|^2 &= \int_{\mathbb{R}^d} |\mathrm{A}_{bh}|^2 + 2\int_\Omega \partial_z \mathbf{A}_h \cdot \mathbf{A}_h \\
&= \int_{\mathbb{R}^d} |\mathrm{A}_{bh}|^2 - 2\int_\Omega \left(\mathrm{curl}^{\,\mu} \mathbf{A}\right)_h^\perp \cdot \mathbf{A}_h + 2\sqrt{\mu}\int_\Omega \nabla_X \mathbf{A}_v \cdot \mathbf{A}_h \\
&= \int_{\mathbb{R}^d} |\mathrm{A}_{bh}|^2 - 2\int_\Omega \left(\mathrm{curl}^{\,\mu} \mathbf{A}\right)_h^\perp \cdot \mathbf{A}_h + 2\int_\Omega \partial_z \mathbf{A}_v \mathbf{A}_v + 2\sqrt{\mu}\Bigg(\int_{\mathbb{R}^d} \beta\left(\nabla b \cdot \mathrm{A}_{bh}\right)\mathrm{A}_{bv} \\
&\qquad - \int_{\mathbb{R}^d} \left(\varepsilon\nabla\zeta \cdot \underline{\mathrm{A}}_h\right)\underline{\mathrm{A}}_v\Bigg),
\end{aligned}$$

where the third equality is obtained by integrating by parts the third integral and by using the fact that $\mathrm{div}^\mu \mathbf{A} = 0$. Furthermore, thanks to the boundary conditions and Equality (4.14), we have

$$\varepsilon\sqrt{\mu}\left(\nabla\zeta \cdot \underline{\mathrm{A}}_h\right)\underline{\mathrm{A}}_v = \sqrt{\mu}\nabla^\perp\widetilde{\psi} \cdot \underline{\mathrm{A}}_h - |\underline{\mathrm{A}}_h|^2 \text{ and } \beta\sqrt{\mu}\left(\nabla b \cdot \mathrm{A}_{bh}\right)\mathrm{A}_{bv} = \mathrm{A}_{bv}^2.$$

Then, we get

$$|\underline{\mathrm{A}}|_2^2 + |\mathrm{A}_b|_2^2 = 2\sqrt{\mu}\int_{\mathbb{R}^d} \nabla^\perp\widetilde{\psi} \cdot \underline{\mathrm{A}}_h + 2\int_{\mathbb{R}^d} \left(\mathrm{curl}^{\,\mu} \mathbf{A}\right)_h^\perp \cdot \mathbf{A}_h, \tag{4.33}$$

and the inequality follows. □

The third Proposition is a Poincaré inequality.

Proposition 4.2.5. *Let ζ, $b \in W^{1,\infty}(\mathbb{R}^d)$ and $\mathbf{A} \in H(\mathrm{div}_0^\mu, \Omega) \cap H(\mathrm{curl}^\mu, \Omega)$ such that*

$$\mathrm{A}_b \times N_b^\mu = 0 \text{ and } \underline{\mathrm{A}} \cdot N^\mu = 0.$$

Then,

$$||\mathbf{A}||_2 \leq C \left|\varepsilon\zeta - \beta b + 1\right|_{L^\infty} \left(||\mathrm{curl}^\mu \mathbf{A}||_2 + ||\partial_z \mathbf{A}||_2\right). \tag{4.34}$$

Proof. We have

$$|\mathbf{A}(X,z)|^2 = |\mathrm{A}_b(X)|^2 + 2\int_{s=-1+\beta b(X)}^{z} \partial_z \mathbf{A}(X,s) \cdot \mathbf{A}(X,s) dX ds.$$

Then, the result follows from the following lemma, which is a similar computation to the one in Proposition 4.2.4 (by switching the boundary conditions).

Lemma 4.2.6. *Let ζ, $b \in W^{1,\infty}(\mathbb{R}^d)$, $\mathbf{A} \in H(\mathrm{div}_0^\mu, \Omega) \cap H(\mathrm{curl}^\mu, \Omega)$ such that*

$$\mathrm{A}_b \times N_b^\mu = 0 \text{ and } \underline{\mathrm{A}} \cdot N^\mu = 0.$$

Then,

$$|\underline{\mathrm{A}}|_2^2 + |\mathrm{A}_b|_2^2 \leq C \,||\mathrm{curl}^{\,\mu} \mathbf{A}||_2 \,||\mathbf{A}||_2 \,. \tag{4.35}$$

□

Finally, the fourth Proposition links the regularity of $\widetilde{\psi}$ to the regularity of $\omega_b \cdot N_b^\mu$.

Proposition 4.2.7. *Let $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ be such that Condition (4.22) is satisfied and let $\boldsymbol{\omega} \in H_b(\mathrm{div}_0^\mu, \Omega)$. Then, there exists a unique solution $\widetilde{\psi} \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$ to the equation $\Delta\widetilde{\psi} = \underline{\boldsymbol{\omega}} \cdot N^\mu$ and we have*

$$\left|\nabla\widetilde{\psi}\right|_2 \leq \sqrt{\mu} C\left(\frac{1}{\mathrm{h}_{\min}}, \varepsilon\left|\zeta\right|_{W^{1,\infty}}, \beta\left|b\right|_{W^{1,\infty}}\right)\left(||\boldsymbol{\omega}||_2 + \frac{1}{\sqrt{\mu}}\left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N_b^\mu\right)\right|_2\right),$$

and

$$\left|\sqrt{1+\sqrt{\mu}|D|}\nabla\widetilde{\psi}\right|_2 \leq \sqrt{\mu} C\left(\frac{1}{\mathrm{h}_{\min}}, \varepsilon\left|\zeta\right|_{W^{1,\infty}}, \beta\left|b\right|_{W^{1,\infty}}\right)\left(||\boldsymbol{\omega}||_2 + \frac{1}{\sqrt{\mu}}\left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N_b^\mu\right)\right|_2\right).$$

Proof. The proof is a small adaptation of Lemma 3.7 and Lemma 5.5 in [34]. □

We can now prove an existence and uniqueness result for the system (4.21) and (4.26).

Theorem 4.2.8. *Let ζ, $b \in W^{2,\infty}(\mathbb{R}^d)$ such that Condition (4.22) is satisfied, $\psi \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$ and $\boldsymbol{\omega} \in H_b(\mathrm{div}_0^\mu, \Omega)$. There exists a unique solution $\mathbf{U}^\mu = \mathbf{U}^\mu[\varepsilon\zeta, \beta b](\psi, \boldsymbol{\omega}) \in H^1(\Omega)$ to (4.21). Furthermore, $\mathbf{U}^\mu = \nabla_{X,z}^\mu \Phi + \mathrm{curl}^\mu \mathbf{A}$, where Φ satisfies (4.25) and $\mathbf{A}$ satisfies*

$$\begin{cases} \mathrm{curl}^{\mu}\mathrm{curl}^{\mu}\mathbf{A} = \mu\,\boldsymbol{\omega} \text{ in } \Omega_t, \\ \mathrm{div}^{\mu}\mathbf{A} = 0 \text{ in } \Omega_t, \\ N_b^{\mu} \times \mathrm{A}_b = 0, \\ N^{\mu} \cdot \underline{\mathrm{A}} = 0, \\ \left(\mathrm{curl}^{\mu}\mathbf{A}\right)_{/\!/} = \dfrac{\nabla^{\perp}}{\Delta}\left(\underline{\omega} \cdot N^{\mu}\right), \\ N_b^{\mu} \cdot \left(\mathrm{curl}^{\,\mu}\mathbf{A}\right)_{|z=-1+\beta b} = 0. \end{cases} \tag{4.36}$$

Finally, one has

$$||\mathbf{U}^{\mu}||_2 \leq \sqrt{\mu} C\left(\mu_{\max}, \frac{1}{\mathrm{h}_{\min}}, \varepsilon\,|\zeta|_{W^{2,\infty}}, \beta\,|b|_{W^{2,\infty}}\right)\left(\sqrt{\mu}\,||\boldsymbol{\omega}||_2 + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N_b^{\mu}\right)\right|_2 + |\mathfrak{P}\psi|_2\right), \tag{4.37}$$

and

$$\left|\left|\nabla_{X,z}^{\mu}\mathbf{U}^{\mu}\right|\right|_2 \leq \mu C\left(\mu_{\max}, \frac{1}{\mathrm{h}_{\min}}, \varepsilon\,|\zeta|_{W^{2,\infty}}, \beta\,|b|_{W^{2,\infty}}\right)\left(||\boldsymbol{\omega}||_2 + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N_b^{\mu}\right)\right|_2 + |\mathfrak{P}\psi|_{H^1}\right). \tag{4.38}$$

Proof. The uniqueness follows easily from the last Propositions. The existence of Φ and the control of its norm are proved in Section A.1.1. We focus on the existence of a solution of (4.36). The main idea is the following variational formulation for the system (4.36) (we refer to Lemma 3.5 and Proposition 5.3 in [34] for the details). We denote by

$$\mathcal{X} = \left\{\mathbf{C} \in H^1(\Omega), \mathrm{div}^{\,\mu}\mathbf{C} = 0, \underline{\mathrm{A}} \cdot N^{\mu} = 0 \text{ and } \mathrm{A}_b \times N_b^{\mu} = 0\right\},$$

and $\widetilde{\psi}$ the unique solution in $\dot{H}^1(\mathbb{R}^d)$ of $\Delta\widetilde{\psi} = \underline{\boldsymbol{\omega}} \cdot N^{\mu}$. Then, $\mathrm{A} \in \mathcal{X}$ is a variational solution of System (4.36) if

$$\forall \mathbf{C} \in \mathcal{X}, \int_{\Omega} \mathrm{curl}^{\mu}\,\mathbf{A} \cdot \mathrm{curl}^{\mu}\,\mathbf{C} = \mu\int_{\Omega} \boldsymbol{\omega} \cdot \mathbf{C} + \mu\int_{\mathbb{R}^d} \nabla\widetilde{\psi} \cdot \mathrm{C}_{/\!/}, \tag{4.39}$$

The existence of such a $\mathbf{A}$ follows Lax-Milgram's theorem. In the following we only explain how we get the coercivity. Thanks to a similar computation that we used to prove Estimate (4.29) (by switching the boundary conditions), we get

$$\left|\left|\nabla_{X,z}^{\mu}\mathbf{A}\right|\right|_2^2 \leq ||\mathrm{curl}^{\,\mu}\mathbf{A}||_2^2 + \mu C\left(\varepsilon\,|\nabla\zeta|_{W^{2,\infty}}, \beta\,|\nabla b|_{W^{2,\infty}}\right)\left(|\underline{\mathrm{A}}|_2^2 + |\mathrm{A}_{bh}|_2^2\right).$$

Then, thanks to a similar computation that in Proposition 4.2.4 and Proposition 4.2.5, we obtain the coercivity

$$||\mathbf{A}||_2 + \left|\left|\nabla_{X,z}^{\mu}\mathbf{A}\right|\right|_2 \leq C\left(\mu_{\max}, \frac{1}{h_{\min}}, \varepsilon\,|\zeta|_{W^{2,\infty}}, \beta\,|b|_{W^{2,\infty}}\right)||\mathrm{curl}^{\mu}\mathbf{A}||_2.$$

Then, we can easily extend this for all $\mathbf{C}$ in $\left\{\mathbf{C} \in H^1(\Omega), \underline{\mathrm{C}} \cdot N^{\mu} = 0 \text{ and } \mathrm{C}_b \times N_b^{\mu} = 0\right\}$ (see Lemma 3.5 in [34]) and thanks to the variational formulation of $\mathbf{A}$ we get

$$||\mathrm{curl}^{\mu}\mathbf{A}||_2 \leq C\left(\mu_{\max}, \frac{1}{h_{\min}}, \varepsilon\,|\zeta|_{W^{2,\infty}}, \beta\,|b|_{W^{2,\infty}}\right)\left(\mu\,||\boldsymbol{\omega}||_2 + \sqrt{\mu}\left|\nabla\widetilde{\psi}\right|_2\right).$$

Using Proposition 4.2.7, we get the first estimate. The second estimate follows from the first estimate, the inequality (4.29), Proposition 2.4, Proposition 2.6 and the following Lemma.

Lemma 4.2.9. *Let ζ, $b \in W^{1,\infty}\left(\mathbb{R}^d\right)$ be such that Condition* (4.22) *is satisfied. Then, for all $u \in H^1(\Omega)$,*

$$\left|\sqrt{1+\sqrt{\mu}|D|}\underline{u}\right|_2 + \left|\sqrt{1+\sqrt{\mu}|D|}u_b\right|_2 \leq C\left(\frac{1}{h_{\min}}, \varepsilon\left|\zeta\right|_{W^{1,\infty}}, \beta\left|b\right|_{W^{1,\infty}}\right)\left(\left|\left|\nabla^{\mu}_{X,z}u\right|\right|_2 + ||u||_2\right).$$

Proof. The proof is a small adaptation of Lemma 5.4 in [34]. □

□

4.2.3 The transformed div-curl problem

In this section, we transform the div-curl problem in the domain Ω into a variable coefficients problem in the flat strip $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$. We introduce the diffeomorphism Σ,

$$\Sigma := \begin{array}{ccc} S & \to & \Omega \\ (X,z) & \mapsto & (X, z+\sigma(X,z)), \end{array} \tag{4.40}$$

where

$$\sigma(X,z) := z\left(\varepsilon\zeta(X) - \beta b(X)\right) + \varepsilon\zeta(X).$$

We keep the notations of [34]. We define

$$\mathrm{U}^{\sigma,\mu}\left[\varepsilon\zeta, \beta b\right](\psi, \omega) = \mathrm{U}^{\mu} = \begin{pmatrix} \sqrt{\mu}\mathrm{V} \\ \mathrm{w} \end{pmatrix} = \mathbf{U}^{\mu} \circ \Sigma,\ \omega = \boldsymbol{\omega} \circ \Sigma,$$

and

$$\nabla^{\sigma,\mu}_{X,z} = \left(J_{\Sigma}^{-1}\right)^t \nabla^{\mu}_{X,z},\ \text{where}\ \left(J_{\Sigma}^{-1}\right)^t = \begin{pmatrix} Id_{d\times d} & \frac{-\sqrt{\mu}\nabla\sigma}{1+\partial_z\sigma} \\ 0 & \frac{1}{1+\partial_z\sigma} \end{pmatrix}.$$

Furthermore, for $\mathrm{A} = \mathbf{A} \circ \Sigma$,

$$\mathrm{curl}^{\sigma,\mu}\mathrm{A} = \left(\mathrm{curl}^{\mu}\ \mathbf{A}\right) \circ \Sigma = \nabla^{\sigma,\mu}_{X,z} \times \mathrm{A},\ \mathrm{div}^{\sigma,\mu}\mathrm{A} = \left(\mathrm{div}^{\mu}\ \mathbf{A}\right) \circ \Sigma = \nabla^{\sigma,\mu}_{X,z} \cdot \mathrm{A}.$$

Finally, if A is vector field on $\mathcal{S}$,

$$\underline{\mathrm{A}} = \mathrm{A}_{|z=0},\ \mathrm{A}_b = \mathrm{A}_{|z=-1}\ \text{and}\ \mathrm{A}_{/\!/} = \frac{1}{\sqrt{\mu}}\underline{\mathrm{A}}_h + \varepsilon\underline{\mathrm{A}}_v\nabla\zeta.$$

Then, $\mathrm{U}^{\mu} = \mathrm{U}^{\mu}[\varepsilon\zeta, \beta b](\psi, \omega)$ is the unique solution in $H^1(\mathcal{S})$ of

$$\begin{cases} \mathrm{curl}^{\sigma,\mu}\ \mathrm{U}^{\mu} = \mu\,\omega\ \text{in}\ \mathcal{S}, \\ \mathrm{div}^{\sigma,\mu}\ \mathrm{U}^{\mu} = 0\ \text{in}\ \mathcal{S}, \\ \mathrm{U}^{\mu}_{/\!/} = \nabla\psi + \dfrac{\nabla^{\perp}}{\Delta}\left(\underline{\omega}\cdot N^{\mu}\right)\ \text{on}\ \{z=0\}, \\ \mathrm{U}^{\mu}_b \cdot N^{\mu}_b = 0\ \text{on}\ \{z=-1\}. \end{cases} \tag{4.41}$$

We also keep the notations in [96]. If $\mathrm{A} = \mathbf{A} \circ \Sigma$, we define

$$\partial_i^{\sigma}\mathrm{A} = \partial_i\mathbf{A} \circ \Sigma,\ i \in \{t,x,y,z\},\ \partial_i^{\sigma} = \partial_i - \frac{\partial_i\sigma}{1+\partial_z\sigma}\partial_z,\ i \in \{x,y,t\}\ \text{and}\ \partial_z^{\sigma} = \frac{1}{1+\partial_z\sigma}\partial_z.$$

Then, by a change of variables and Proposition 4.2.3 we get the following variational formulation for U^μ. For all $\mathrm{C} \in H^1(\mathcal{S})$,

$$\int_{\mathcal{S}} \nabla^\mu_{X,z}\mathrm{U}^\mu \cdot P(\Sigma)\, \nabla^\mu_{X,z}\mathrm{C} = \mu \int_{\mathcal{S}} (1+\partial_z\sigma)\, \omega \cdot \mathrm{curl}^{\sigma,\mu}\mathrm{C} + \int_{\mathbb{R}^d} l^\mu[\varepsilon\zeta](\underline{\mathrm{U}}^\mu)\cdot \underline{C} - \int_{\mathbb{R}^d} l^\mu[\beta b](\mathrm{U}^\mu_b)\cdot C_b, \tag{4.42}$$

where $P(\Sigma) = (1+\partial_z\sigma)\, J_\Sigma^{-1}\left(J_\Sigma^{-1}\right)^t$ and

$$l^\mu[\eta]\left(\mathrm{U}^\mu_{|z=\eta}\right) = \begin{pmatrix} \sqrt{\mu}\nabla \mathrm{w}_{|z=\eta} - \mu^{\frac{3}{2}}\left(\nabla^\perp\eta\cdot\nabla\right)\mathrm{V}^\perp_{|z=\eta} \\ -\mu\nabla\cdot \mathrm{V}_{|z=\eta} \end{pmatrix}.$$

In order to obtain higher order estimates on U^μ, we have to separate the regularity on z and the regularity on X. We use the following spaces.

Definition 4.2.10. *We define the spaces* $H^{s,k}$

$$H^{s,k}(\mathcal{S}) = \bigcap_{0\le l\le k} H^l_z\left(-1,0\,;H^{s-l}_X\left(\mathbb{R}^d\right)\right) \text{ and } |u|_{H^{s,k}} = \sum_{0\le l\le k}\left|\Lambda^{s-j}\partial_z^j u\right|_2.$$

Furthermore, if $\alpha \in \mathbb{N}^d \backslash\{0\}$, we define the *Alinhac's good unknown*

$$\psi_{(\alpha)} = \partial^\alpha\psi - \varepsilon\underline{\mathrm{w}}\partial^\alpha\zeta \text{ and } \psi_{(0)} = \psi. \tag{4.43}$$

This quantities play an important role in the wellposedness of the water waves equations (see [6] or [80]). In fact, more generally, if A is vector field on $\mathcal{S}$, we denote by

$$\mathrm{A}_{(\alpha)} = \partial^\alpha\mathrm{A} - \partial^\alpha\sigma\partial_z^\sigma\mathrm{A}\ ,\ \mathrm{A}_{(0)} = \mathrm{A}\ ,\ \underline{\mathrm{A}}_{(\alpha)} = \partial^\alpha\underline{\mathrm{A}} - \varepsilon\partial^\alpha\zeta\underline{\partial_z^\sigma\mathrm{A}} \text{ and } \underline{\mathrm{A}}_{(0)} = \underline{\mathrm{A}}. \tag{4.44}$$

We can now give high order estimates on U^μ. We recall that M_N is defined in (4.24).

Theorem 4.2.11. *Let $N \in \mathbb{N}$, $N \ge 5$. Then, under the assumptions of Theorem 4.2.8, for all $0 \le l \le 1$ and $0 \le l \le k \le N-1$, the straightened velocity U^μ, satisfies*

$$\left|\left|\nabla^\mu_{X,z}\mathrm{U}^\mu\right|\right|_{H^{k,l}} \le \mu M_N\left(|\mathfrak{P}\psi|_{H^1} + \sum_{1<|\alpha|\le k+1}\left|\mathfrak{P}\psi_{(\alpha)}\right|_2 + ||\omega||_{H^{k,l}} + \left|\frac{\Lambda^k}{\mathfrak{P}}\left(\omega_b\cdot N^\mu_b\right)\right|_2\right).$$

Proof. We start with $l = 0$. We follow the proof of Proposition 3.12 and Proposition 5.8 in [34]. Let $k \in [1, N-1]$, $\alpha \in \mathbb{N}^d$ with $|\alpha| \le k$. We take $C = \partial^{2\alpha}\mathrm{U}^\mu$ in (4.42)[3] and we get

$$\begin{aligned}\int_{\mathcal{S}}\nabla^\mu_{X,z}\mathrm{U}^\mu\cdot P(\Sigma)\,\nabla^\mu_{X,z}\partial^{2\alpha}\mathrm{U}^\mu = \mu\int_{\mathcal{S}}(1+\partial_z\sigma)\,\omega\cdot\mathrm{curl}^{\sigma,\mu}\partial^{2\alpha}\mathrm{U}^\mu + \int_{\mathbb{R}^d} l^\mu[\varepsilon\zeta](\underline{\mathrm{U}}^\mu)\cdot\partial^{2\alpha}\underline{\mathrm{U}}^\mu \\ - \int_{\mathbb{R}^d} l^\mu[\beta b](\mathrm{U}^\mu_b)\cdot\partial^{2\alpha}\mathrm{U}^\mu_b.\end{aligned}$$

We focus on the bottom contribution, which is the last term of the previous equation (see [34] for the other terms). Using the fact that $\mathrm{w}_b = \mu\beta\nabla b\cdot\mathrm{V}_b$, we have

[3] A. Castro and D. Lannes explain why we can take such a C in the variational formulation.

$$
\begin{aligned}
(-1)^{|\alpha|}\int_{\mathbb{R}^d} l^{\mu}[\beta b]\left(\mathrm{U}_b\right)\cdot\partial^{2\alpha}\mathrm{U}_b &= \int_{\mathbb{R}^d} 2\mu\partial^{\alpha}\nabla \mathrm{w}_b\cdot\partial^{\alpha}\mathrm{V}_b - \mu^2\beta\partial^{\alpha}\left[\left(\nabla^{\perp}b\cdot\nabla\right)\mathrm{V}_b^{\perp}\right]\cdot\partial^{\alpha}\mathrm{V}_b \\
&= \int_{\mathbb{R}^d} 2\mu^2\beta\partial^{\alpha}\nabla\left(\nabla b\cdot\mathrm{V}_b\right)\cdot\partial^{\alpha}\mathrm{V}_b - \mu^2\beta\partial^{\alpha}\left[\left(\nabla^{\perp}b\cdot\nabla\right)\mathrm{V}_b^{\perp}\right]\cdot\partial^{\alpha}\mathrm{V}_b \\
&= \underbrace{\int_{\mathbb{R}^d} 2\mu^2\beta\left(\nabla b\right)^t\cdot\partial^{\alpha}\nabla\mathrm{V}_b\cdot\partial^{\alpha}\mathrm{V}_b - \beta\mu^2\left[\left(\nabla^{\perp}b\cdot\nabla\right)\partial^{\alpha}\mathrm{V}_b^{\perp}\right]\cdot\partial^{\alpha}\mathrm{V}_b}_{I_1} \\
&\quad + \underbrace{\int_{\mathbb{R}^d} 2\mu^2\beta\left[\partial^{\alpha}\nabla,\nabla b\right]\mathrm{V}_b\cdot\partial^{\alpha}\mathrm{V}_b - \beta\mu^2\left[\partial^{\alpha},\left(\nabla^{\perp}b\cdot\nabla\right)\right]\mathrm{V}_b^{\perp}\cdot\partial^{\alpha}\mathrm{V}_b}_{I_2}.
\end{aligned}
$$

Then, a careful computation gives

$$
\begin{aligned}
|I_1| &= \left|\mu^2\beta\int_{\mathbb{R}^d}\partial_x^2 b\left(\partial^{\alpha}\mathrm{V}_{bx}\right)^2 + \partial_y^2 b\left(\partial^{\alpha}\mathrm{V}_{by}\right)^2 + 2\mu^2\beta\int_{\mathbb{R}^d}\partial_{xy}^2 b\,\partial^{\alpha}\mathrm{V}_{bx}\,\partial^{\alpha}\mathrm{V}_{by}\right| \\
&\leq \mu C\left(\delta,\frac{1}{h_{\min}},\varepsilon\left|\zeta\right|_{W^{1,\infty}},\beta\left|b\right|_{W^{2,\infty}}\right)\left|\left|\partial^{\alpha}\mathrm{U}^{\mu}\right|\right|_2^2 + \delta\left|\left|\nabla_{X,z}^{\mu}\partial^{\alpha}\mathrm{U}^{\mu}\right|\right|_2^2 \\
&\leq C\left(\delta,\frac{1}{h_{\min}},\varepsilon\left|\zeta\right|_{W^{1,\infty}},\beta\left|b\right|_{W^{2,\infty}}\right)\left|\left|\nabla_{X,z}^{\mu}\mathrm{U}^{\mu}\right|\right|_{H^{k-1}}^2 + \delta\left|\left|\nabla_{X,z}^{\mu}\partial^{\alpha}\mathrm{U}^{\mu}\right|\right|_2^2,
\end{aligned}
$$

where $\delta > 0$ is small enough and where we use the following Lemma.

Lemma 4.2.12. *Let ζ, $b \in W^{1,\infty}\left(\mathbb{R}^d\right)$, such that Condition* (4.22) *is satisfied. Then, for all $u \in H^1\left(\mathcal{S}\right)$* and $\delta > 0$,

$$
\left|\underline{u}\right|_2^2 + \left|u_b\right|_2^2 \leq C\left(\delta,\frac{1}{h_{\min}},\varepsilon\left|\zeta\right|_{W^{1,\infty}},\beta\left|b\right|_{W^{1,\infty}}\right)\left|\left|u\right|\right|_2^2 + \delta\left|\left|\partial_z u\right|\right|_2^2.
$$

Furthermore, using Lemma B.3.1 and the previous Lemma, we get

$$
\begin{aligned}
|I_2| &\leq C\mu\beta\left|\nabla b\right|_{H^{k+1}}\left|\mathrm{U}_b^{\mu}\right|_{H^k}\left|\partial^{\alpha}\mathrm{U}_b^{\mu}\right|_2 \\
&\leq \mu C\left(\delta,\frac{1}{h_{\min}},\varepsilon\left|\zeta\right|_{W^{1,\infty}},\beta\left|b\right|_{W^{1,\infty}},\beta\left|\nabla b\right|_{H^{k+1}}\right)\left|\left|\nabla_{X,z}^{\mu}\mathrm{U}^{\mu}\right|\right|_{H^{k-1}}^2 + \delta\left|\left|\nabla_{X,z}^{\mu}\partial^{\alpha}\mathrm{U}^{\mu}\right|\right|_2^2.
\end{aligned}
$$

For the surface contribution, we can do the same thing as in Proposition 3.12 and Proposition 5.8 in [34], using the previous Lemma to control $\partial^{\alpha}\mathrm{w}$. Finally, for the other terms, the main idea is the following Lemma (which is a small adaptation of Lemma 3.13 and Lemma 5.6 in [34]).

Lemma 4.2.13. *Let $\widetilde{\psi}$ the unique solution of $\Delta\widetilde{\psi} = \underline{\omega}\cdot N^{\mu}$ in $\dot{H}^1(\mathbb{R}^d)$. Under the assumptions of the Theorem, we have the following estimate*

$$
\left|\mathfrak{P}\nabla^{\perp}\widetilde{\psi}\right|_{H^k} \leq M_N\left(\left|\left|\omega\right|\right|_{H^{k,0}} + \left|\frac{\Lambda^k}{\mathfrak{P}}\left(\omega_b\cdot N_b^{\mu}\right)\right|_2\right).
$$

Gathering the previous estimates with the estimate without the bottom contribution in Proposition 5.8 in [34], we get

$$||\partial^{\alpha}\nabla^{\mu}\mathrm{U}^{\mu}||_2 \leq \mu M_N\left(|\mathfrak{P}\psi|_{H^1}+\sum_{1<|\alpha|\leq k+1}\left|\mathfrak{P}\psi_{(\alpha)}\right|+||\omega||_{H^{k,0}}+\left|\frac{\Lambda^k}{\mathfrak{P}}(\omega_b\cdot N_b^{\mu})\right|_2\right)+M_N\left|\left|\Lambda^{k-1}\nabla^{\mu}_{X,z}\mathrm{U}^{\mu}\right|\right|_2,$$

and the inequality follows by a finite induction on k. If $l=1$, we can adapt the proof of Corollary 3.14 in [34] easily. □

Remark 4.2.14. *Notice that for $k\geq 2$, we have*

$$\left|\frac{\Lambda^k}{\mathfrak{P}}\left(\omega_b\cdot N_b^{\mu}\right)\right|_2 \leq C\left(\frac{1}{h_{\min}},\mu_{\max},\beta\left|\nabla b\right|_{H^{k+1}}\right)\left(||\omega||_{H^{k,1}}+\left|\frac{1}{\mathfrak{P}}\left(\omega_b\cdot N_b^{\mu}\right)\right|_2\right),$$

thanks to Lemma B.1.4, Lemma 4.2.9 and Lemma B.2.1.

4.2.4 Time derivatives and few remarks about the good unknown

This part is devoted to recall and adapt some results in [34]. Unlike the previous Propositions, adding a non flat bottom is not painful. That is why we do not give proofs. We refer to section 3.5 and 3.6 in [34] for the details. Firstly, in order to obtain an energy estimate of the Castro-Lannes water waves formulation, we need to control $\partial_t\mathrm{U}^{\mu}$. This is the purpose of the following result.

Proposition 4.2.15. *Let $T>0$, $\zeta\in\mathcal{C}^1\left([0,T],W^{2,\infty}\left(\mathbb{R}^d\right)\right)$, $b\in W^{2,\infty}\left(\mathbb{R}^d\right)$ such that (4.22) is satisfied for $0\leq t\leq T$, $\psi\in\mathcal{C}^1\left([0,T],\dot{H}^{\frac{3}{2}}\left(\mathbb{R}^d\right)\right)$ and $\omega\in\mathcal{C}^1\left([0,T],L^2\left(\mathcal{S}\right)^{d+1}\right)$ such that $\nabla^{\mu,\sigma}_{X,z}\cdot\omega=0$ for $0\leq t\leq T$. Then,*

$$\partial_t\left(\mathrm{U}^{\sigma,\mu}[\varepsilon\zeta,\beta b]\left(\psi,\omega\right)\right)=\mathrm{U}^{\sigma,\mu}[\varepsilon\zeta,\beta b]\Big(\partial_t\psi-\varepsilon\underline{\mathrm{w}}\partial_t\zeta+\varepsilon\sqrt{\mu}\frac{\nabla}{\Delta}\cdot\left(\underline{\omega_h}^{\perp}\partial_t\zeta\right),\partial_t^{\sigma}\omega\Big)$$
$$+\partial_t\sigma\partial_z^{\sigma}\left(\mathrm{U}^{\mu,\sigma}[\varepsilon\zeta,\beta b]\left(\psi,\omega\right)\right).$$

Furthermore, for $N\geq 5$, $\mathrm{U}^{\mu}=\mathrm{U}^{\sigma,\mu}[\varepsilon\zeta,\beta b](\psi,\omega)$ satisfies

$$\sqrt{\mu}\,||\partial_t\mathrm{U}^{\mu}||_2+\left|\left|\partial_t\nabla^{\mu}_{X,z}\mathrm{U}^{\mu}\right|\right|_{H^{N-2,0}}\leq\mu\max\left(M_N,\varepsilon\left|\partial_t\zeta\right|_{H^{N-1}}\right)\times$$
$$\left(|\mathfrak{P}\partial_t\psi|_{H^1}+\sum_{1<|\alpha|\leq N-1}\left|\mathfrak{P}\partial_t\psi_{(\alpha)}\right|_2+||\partial_t\omega||_{H^{N-2,0}}+\left|\frac{\Lambda^{N-2}}{\mathfrak{P}}\left(\partial_t\omega_b\cdot N_b^{\mu}\right)\right|_2\right.$$
$$\left.+\,|\mathfrak{P}\psi|_{H^1}+\sum_{1<|\alpha|\leq N}\left|\mathfrak{P}\psi_{(\alpha)}\right|_2+||\omega||_{H^{N-1,1}}+\left|\frac{1}{\mathfrak{P}}\left(\omega_b\cdot N_b^{\mu}\right)\right|_2\right).$$

Secondly, in the context of water waves, the *Alinhac's good unknown* plays a crucial role. Masmoudi and Rousset remarked in [96] that the *Alinhac's good unknown* $\mathrm{U}^{\mu}_{(\alpha)}$ is almost incompressible and Castro and Lannes showed that the $\mathrm{curl}^{\sigma,\mu}\mathrm{U}^{\mu}_{(\alpha)}$ is also well controlled. This is the purpose of the following Proposition. We recall that $\mathrm{U}^{\mu}_{(\alpha)}$ is defined in (4.44).

Proposition 4.2.16. *Let $N \geq 5$, $\zeta \in H^N(\mathbb{R}^d)$, $b \in L^\infty \cap \dot{H}^{N+1}(\mathbb{R}^d)$ such that Condition (4.22) is satisfied and $\omega \in H^{N-1}(\mathcal{S})$ such that $\nabla^{\sigma,\mu} \cdot \omega = 0$. Then if we denote by $\mathrm{U}^\mu = \mathrm{U}^{\mu,\sigma}[\varepsilon\zeta, \beta b]$, we have for $1 \leq |\alpha| \leq N$,*

$$\left|\left|\nabla^{\sigma,\mu}_{X,z} \cdot \mathrm{U}^\mu_{(\alpha)}\right|\right|_2 + \left|\left|\nabla^{\sigma,\mu}_{X,z} \times \mathrm{U}^\mu_{(\alpha)} - \mu\partial^\alpha\omega\right|\right|_2$$
$$\leq \mu \left|(\varepsilon\zeta, \beta b)\right|_{H^N} M_N \left(|\mathfrak{P}\psi|_{H^1} + \sum_{1<|\alpha'|\leq|\alpha|} \left|\mathfrak{P}\psi_{(\alpha')}\right|_2 + ||\omega||_{H^{\max(|\alpha|-1,1)}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N^\mu_b\right)\right|_2 \right),$$

and

$$\left|\mathfrak{P}\psi_{(\alpha)}\right|_2 \leq M_N \left(|\mathfrak{P}\psi|_{H^3} + \frac{1}{\sqrt{\mu}} \sum_{1<|\alpha'|\leq|\alpha|-1} \left|\left|\nabla_X \mathrm{U}^\mu_{(\alpha')}\right|\right|_2 + ||\omega||_{H^{N-1}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N^\mu_b\right)\right|_2 \right).$$

Furthermore, we can control $|\mathfrak{P}\psi|_{H^3}$ by $\mathrm{U}^{\sigma,\mu}[\varepsilon\zeta, \beta b](\psi, \omega)$ and ω.

Proposition 4.2.17. *Let $N \geq 5$, $\zeta \in H^N(\mathbb{R}^d)$, $b \in L^\infty \cap \dot{H}^{N+1}(\mathbb{R}^d)$ such that Condition (4.22) is satisfied and $\omega \in H^{2,1}(\mathcal{S})$ such that $\nabla^{\sigma,\mu} \cdot \omega = 0$. Then,*

$$|\mathfrak{P}\psi|_{H^3} \leq M_N \left(\frac{1}{\sqrt{\mu}} \left|\left|\Lambda^3 \mathrm{U}^{\sigma,\mu}[\varepsilon\zeta, \beta b]\,(\psi, \omega)\right|\right|_2 + ||\omega||_{H^{2,1}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N^\mu_b\right)\right|_2 \right).$$

Proof. The proof is a small adaptation of Lemma 3.23 in [34]. □

Finally, we give a result that is useful for the energy estimate. Since the proof is a little different from Corollary 3.21 in [34], we give it. Notice that the main difference with Corollary 3.21 in [34] is the fact that we do not have a flat bottom.

Proposition 4.2.18. *Let $N \geq 5$, $\zeta \in H^N(\mathbb{R}^d)$, $b \in L^\infty \cap \dot{H}^{N+1}(\mathbb{R}^d)$ and $\omega \in H^{N-1}(\mathcal{S})$ such that $\nabla^{\sigma,\mu} \cdot \omega = 0$. Then, for $k = x, y$, $|\gamma| \leq N-1$, α such that $\partial^\alpha = \partial_k \partial^\gamma$* and $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^d)$, we have

$$\left(\varphi, \frac{1}{\mu}\partial_k \underline{\mathrm{U}}^\mu_{(\gamma)} \cdot N^\mu\right) \leq M_N \left(|\mathfrak{P}\psi|_{H^1} + \sum_{1<|\alpha'|\leq|\alpha|} \left|\mathfrak{P}\psi_{(\alpha')}\right|_2 + ||\omega||_{H^{|\alpha|-1}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N^\mu\right)\right|_2 \right) \times$$
$$\left[|\mathfrak{P}\varphi|_2 + \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\varphi\right|_2 \right],$$

where we denote by $\mathrm{U}^\mu = \mathrm{U}^{\sigma,\mu}[\varepsilon\zeta, \beta b]$.

Proof. Notice that when $\gamma \neq 0$,

$$\partial_k \mathrm{U}^\mu_{(\gamma)} = \mathrm{U}^\mu_{(\alpha)} - \partial^\gamma \sigma \partial_k \partial^\sigma_z \mathrm{U}^\mu.$$

Then, using Lemma 4.2.9, it is easy to check that

$$\left(\varphi, \underline{\partial^\gamma \sigma \partial_k \partial^\sigma_z \mathrm{U}^\mu} \cdot N^\mu\right) \leq M_N \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\varphi\right|_2 \left|\left|\nabla^\mu_{X,z} \mathrm{U}^\mu\right|\right|_{H^2}.$$

Furthermore, using the Green identity we get

$$\left(\varphi, \mathrm{U}^{\mu}_{(\alpha)} \cdot N^{\mu}\right) = \int_{\mathcal{S}} (1+\partial_z \sigma)\, \varphi^{\dagger} \nabla^{\sigma,\mu}_{X,z} \cdot \mathrm{U}^{\mu}_{(\alpha)} + \int_{\mathcal{S}} (1+\partial_z\sigma)\, \mathrm{U}^{\mu}_{(\alpha)} \cdot \nabla^{\sigma,\mu}_{X,z} \varphi^{\dagger} + \left(\varphi^{\dagger}_b, \left(\mathrm{U}^{\mu}_{(\alpha)}\right)_b \cdot N^{\mu}_b\right),$$

where $\varphi^{\dagger} = \chi\left(z\sqrt{\mu}|D|\right)\varphi$ and χ is an even positive compactly supported function equal to 1 near 0. Then, using the fact that $\mathrm{U}^{\mu}_b \cdot N^{\mu}_b = 0$ and the trace Lemma, we get

$$\begin{aligned}
\left(\varphi^{\dagger}_b \,,\, \left(\mathrm{U}^{\mu}_{(\alpha)}\right)_b \cdot N^{\mu}_b\right) &= \left(\chi(\sqrt{\mu}|D|)\varphi \,,\, \partial^{\alpha}\mathrm{U}^{\mu}_b \cdot N^{\mu}_b - \beta\partial^{\alpha} b \left(\partial^{\sigma}_z \mathrm{U}^{\mu}\right)_b \cdot N^{\mu}_b\right) \\
&= \left(\chi(\sqrt{\mu}|D|)\varphi \,,\, \mu\beta \left[\nabla b, \partial^{\alpha}\right] \cdot \mathrm{V}_b - \beta\partial^{\alpha} b \left(\partial^{\sigma}_z \mathrm{U}^{\mu}\right)_b \cdot N^{\mu}_b\right) \\
&\leq M_N \left(\sqrt{\mu}\, ||\mathrm{U}^{\mu}||_{H^N} + ||\mathrm{U}^{\mu}||_{H^{2,2}}\right) |\chi(\sqrt{\mu}|D|)\varphi|_2 \,.
\end{aligned}$$

Therefore, using Proposition 4.2.16, Theorem 4.2.11 and the following Lemma (Lemma 2.20 and Lemma 2.34 in [80]) we get the control.

Lemma 4.2.19. *Let $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^d)$ and χ an even positive compactly supported function equal to 1 near 0. Then,*

$$||\chi\left(z\sqrt{\mu}|D|\right)\varphi||_2 \leq C \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\varphi\right|_2 \text{ and } \left|\left|\nabla^{\mu}_{X,z}\left(\chi\left(z\sqrt{\mu}|D|\right)\varphi\right)\right|\right|_2 \leq C\sqrt{\mu}\,|\mathfrak{P}\varphi|_2 \,.$$

□

4.3 Well-posedness of the water waves equations

4.3.1 Framework

In this section, we prove a local well-posedness result of the water waves equations. We improve the result of [34] by adding a non flat bottom, a non constant pressure at the surface and a Coriolis forcing. In order to work on a fixed domain, we seek a system of 3 equations on ζ, ψ and $\omega = \boldsymbol{\omega} \circ \Sigma$. We keep the first and the second equations of the Castro-Lannes formulation (4.20). It is easy to check that ω satisfies

$$\partial^{\sigma}_t \omega + \frac{\varepsilon}{\mu}\left(\mathrm{U}^{\mu} \cdot \nabla^{\sigma,\mu}_{X,z}\right)\omega = \frac{\varepsilon}{\mu}\left(\omega \cdot \nabla^{\sigma,\mu}_{X,z}\right)\mathrm{U}^{\mu} + \frac{\varepsilon}{\mu \mathrm{Ro}}\partial^{\sigma}_z \mathrm{U}^{\mu}, \tag{4.45}$$

where $\mathrm{U}^{\mu} = \mathrm{U}^{\sigma,\mu}[\varepsilon\zeta, \beta b](\psi, \omega)$. Then, in the following the water waves equations will be the system

$$\begin{cases}
\partial_t \zeta - \frac{1}{\mu}\underline{\mathrm{U}}^{\mu} \cdot N^{\mu} = 0, \\
\partial_t \psi + \zeta + \frac{\varepsilon}{2}\left|\mathrm{U}^{\mu}_{/\!/}\right|^2 - \frac{\varepsilon}{2\mu}\left(1+\varepsilon^2\mu\,|\nabla\zeta|^2\right)\underline{\mathrm{w}}^2 + \varepsilon\frac{\nabla}{\Delta}\cdot\left[\left(\underline{\omega}\cdot N^{\mu} + \frac{1}{\mathrm{Ro}}\right)\underline{\mathrm{V}}^{\perp}\right] = -P, \\
\partial^{\sigma}_t \omega + \frac{\varepsilon}{\mu}\left(\mathrm{U}^{\mu} \cdot \nabla^{\sigma,\mu}_{X,z}\right)\omega = \frac{\varepsilon}{\mu}\left(\omega \cdot \nabla^{\sigma,\mu}_{X,z}\right)\mathrm{U}^{\mu} + \frac{\varepsilon}{\mu\mathrm{Ro}}\partial^{\sigma}_z \mathrm{U}^{\mu}.
\end{cases} \tag{4.46}$$

The following quantity is the energy that we will use to get the local wellposedness

$$\mathcal{E}^N(\zeta,\psi,\omega) = \frac{1}{2}|\zeta|^2_{H^N} + \frac{1}{2}|\mathfrak{P}\psi|^2_{H^3} + \frac{1}{2}\sum_{1\leq|\alpha|\leq N}\left|\mathfrak{P}\psi_{(\alpha)}\right|^2_2 + \frac{1}{2}||\omega||^2_{H^{N-1}} + \frac{1}{2}\left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N^{\mu}_b\right)\right|^2_2,$$

where we recall that $\psi_{(\alpha)}$ is defined in (4.43). For $T \geq 0$, we also introduce the energy space

$$\mathrm{E}_T^N = \left\{(\zeta, \psi, \omega) \in \mathcal{C}\left([0,T], H^2(\mathbb{R}^d) \times \dot{H}^2(\mathbb{R}^d) \times H^2(\mathcal{S})\right), \mathcal{E}^N(\zeta,\psi,\omega) \in L^\infty([0,T])\right\}.$$

We also recall that M_N is defined in (4.24). We organize this section in three parts. First, we give an a priori estimate for the vorticity. Then, we explain briefly how we can quasilinearize the system and how we obtain a priori estimates for the full system. The last part of this section is devoted to the proof of the main result.

4.3.2 A priori estimate for the vorticity

In this part, we give a priori estimate for the straightened equation of the vorticity.

Proposition 4.3.1. *Let $N \geq 5$, $T > 0$, $b \in L^\infty \cap \dot{H}^{N+1}(\mathbb{R}^d)$ and $(\zeta, \psi, \omega) \in \mathrm{E}_T^N$ such that* (4.45) *and Condition* (4.22) *hold on $[0,T]$. We also assume that on $[0,T]$*

$$\partial_t \zeta - \frac{1}{\mu} \mathrm{U}^{\sigma,\mu}[\varepsilon\zeta, \beta b] \cdot N^\mu = 0.$$

Then,

$$\frac{d}{dt}\left(||\omega||_{H^{N-1}}^2 + \left| \frac{1}{\mathfrak{P}} \left(\omega_b \cdot N_b^\mu\right)\right|_2^2 \right) \leq M_N \left(\varepsilon \mathcal{E}^N(\zeta,\psi,\omega)^{\frac{3}{2}} + \max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right) \mathcal{E}^N(\zeta,\psi,\omega)\right).$$

Proof. We denote $\mathrm{U}^{\sigma,\mu}[\varepsilon\zeta, \beta b] = \mathrm{U}^\mu = \begin{pmatrix} \sqrt{\mu}\mathrm{V} \\ \mathrm{w} \end{pmatrix}$. We can reformulate Equation (4.45) as

$$\partial_t \omega + \varepsilon \left(\mathrm{V} \cdot \nabla_X\right) \omega + \frac{\varepsilon}{\mu} a \partial_z \omega = \frac{\varepsilon}{\mu}\left(\omega \cdot \nabla_{X,z}^{\sigma,\mu}\right) \mathrm{U}^\mu + \frac{\varepsilon}{\mu \mathrm{Ro}} \partial_z^\sigma \mathrm{U}^\mu,$$

where

$$a = \frac{1}{1 + \partial_z \sigma} \left(\mathrm{U}^\mu \cdot \begin{pmatrix} -\sqrt{\mu}\nabla_X \sigma \\ 1 \end{pmatrix} - (z+1)\underline{\mathrm{U}}^\mu \cdot N^\mu \right).$$

Notice that $\underline{a} = a_b = 0$. Then, we get

$$\partial_t \left|\left|\omega\right|\right|_2^2 = \varepsilon \int_S \left(\nabla_X \cdot \mathrm{V} + \frac{1}{\mu} \partial_z a \right) \omega^2 + \frac{2}{\mu}\left(\omega \cdot \nabla_{X,z}^{\sigma,\mu}\right) \mathrm{U}^\mu \cdot \omega + \frac{1}{\mathrm{Ro}} \partial_z^\sigma \mathrm{U}^\mu \cdot \omega,$$

and

$$\partial_t \left|\left|\omega\right|\right|_2^2 \leq \frac{\varepsilon}{\mu} C\left(\frac{1}{h_{\min}}, \varepsilon \left|\zeta\right|_{W^{1,\infty}}, \beta \left|b\right|_{W^{1,\infty}}\right)\Big(\left[\left|\left|\nabla_{X,z}^\mu \mathrm{U}^\mu\right|\right|_\infty + \sqrt{\mu}\left|\left|\mathrm{U}^\mu\right|\right|_\infty\right] \left|\left|\omega\right|\right|_2^2 + \frac{1}{\mathrm{Ro}} \left|\left|\nabla_{X,z}^\mu \mathrm{U}^\mu\right|\right|_\infty \left|\left|\omega\right|\right|_2 \Big),$$

where we use the fact that

$$\left|\underline{\mathrm{U}}^\mu \cdot N^\mu\right|_{L^\infty} \leq C\left(\varepsilon \left|\zeta\right|_{W^{1,\infty}}, \beta \left|b\right|_{W^{1,\infty}}\right) \left(\left|\left|\partial_z \mathrm{U}^\mu\right|\right|_\infty + \sqrt{\mu} \left|\left|\mathrm{U}^\mu\right|\right|_\infty \right).$$

The estimate for the L^2-norm of ω follows thanks to Theorem, 4.2.8, Theorem 4.2.11 and Remark 4.2.14. For the high order estimates, we differentiate Equation (4.45) and we easily obtain the control thanks to Theorem 4.2.11 and Remark 4.2.14 (see the proof of Proposition 4.1 in [34]). Finally, taking the trace at the bottom of the vorticity equation in System (4.20), we get the following equation for $\omega_b \cdot N_b^\mu$,

$$\partial_t \left(\omega_b \cdot N_b^\mu\right) + \varepsilon \nabla \cdot \left(\left[\omega_b \cdot N_b^\mu + \frac{1}{\mathrm{Ro}} \right] \mathrm{V}_b \right) = 0, \tag{4.47}$$

and then,

$$\partial_t \left| \frac{1}{\mathfrak{P}} \left(\omega_b \cdot N_b^\mu\right) \right|_2^2 \leq 2\varepsilon \left| \sqrt{1+\sqrt{\mu}|D|} \left(\left[\omega_b \cdot N_b^\mu + \frac{1}{\mathrm{Ro}} \right] \mathrm{V}_b \right) \right|_2 \left| \frac{1}{\mathfrak{P}} \left(\omega_b \cdot N_b^\mu\right) \right|_2 .$$

The control follows easily thanks to and Lemma 4.2.9, Theorem 4.2.8, Theorem 4.2.11 and Remark 4.2.14. □

Remark 4.3.2. *Notice that we can also take the trace at the surface of the vorticity equation and we obtain a transport equation for $\underline{\omega} \cdot N^\mu$,*

$$\partial_t \left(\underline{\omega} \cdot N^\mu\right) + \varepsilon \nabla \cdot \left(\left[\underline{\omega} \cdot N^\mu + \frac{1}{\mathrm{Ro}} \right] \underline{\mathrm{V}} \right) = 0. \tag{4.48}$$

4.3.3 Quasilinearization and a priori estimates

In this part, we quasilinearize the system (4.20). We introduce the Rayleigh-Taylor coefficient

$$\mathfrak{a} := \mathfrak{a}[\varepsilon\zeta, \beta b](\psi, \omega) = 1 + \varepsilon \left(\partial_t + \varepsilon \underline{\mathrm{V}}[\varepsilon\zeta, \beta b](\psi, \omega) \cdot \nabla\right) \underline{\mathrm{w}}[\varepsilon\zeta, \beta b](\psi, \omega). \tag{4.49}$$

It is well-known that the positivity of this quantity is essential for the wellposedness of the water waves equations (see for instance Remark 4.17 in [80] or [56]). Thanks to Equation (4.12), we can easily adapt Part 2.2.5 and check that the positivity of $\mathfrak{a}$ is equivalent to the classical Rayleigh-Taylor criterion ([138])

$$\inf_{\mathbb{R}^d} \left(-\partial_z \mathcal{P}_{|z=\varepsilon\zeta} \right) > 0,$$

where we recall that $\mathcal{P}$ is the pressure in the fluid domain. We can now give a quasilinearization of (4.46). We recall that the notation $\underline{\mathrm{U}}^\mu_{(\alpha)}$ is defined in (4.44) and $\psi_{(\alpha)}$ is defined in (4.43).

Proposition 4.3.3. *Let $N \geq 5$, $T > 0$, $b \in L^\infty \cap \dot{H}^{N+1}(\mathbb{R}^d)$, $P \in L^\infty_t\left(\mathbb{R}^+; \dot{H}^{N+1}_X\left(\mathbb{R}^d\right)\right)$ and $(\zeta, \psi, \omega) \in E^N_T$ solution of the system (4.46) such that (ζ, b) satisfy Condition (4.22) on $[0, T]$. Then, for $\alpha, \gamma \in \mathbb{N}^d$ and for $k \in \{x, y\}$ such that $\partial^\alpha = \partial_k \partial^\gamma$* and *$|\gamma| \leq N-1$, we have the following quasilinearization*

$$\begin{aligned} &\left(\partial_t + \varepsilon \underline{\mathrm{V}} \cdot \nabla\right) \partial^\alpha \zeta - \frac{1}{\mu} \partial_k \underline{\mathrm{U}}^\mu_{(\gamma)} \cdot N^\mu = R^1_\alpha, \\ &\left(\partial_t + \varepsilon \underline{\mathrm{V}} \cdot \nabla\right) \left(\underline{\mathrm{U}}^\mu_{(\gamma)/\!/} \cdot e_{\mathbf{k}} \right) + \mathfrak{a} \partial^\alpha \zeta = -\partial^\alpha P + R^2_\alpha, \end{aligned} \tag{4.50}$$

where

$$\left|R^1_\alpha\right|_2 + \left|R^2_\alpha\right|_2 + \left|\mathfrak{P} R^2_\alpha\right|_2 \leq M_N \left(\max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right) \mathcal{E}^N(\zeta, \psi, \omega) + \frac{\varepsilon}{\mathrm{Ro}} \sqrt{\mathcal{E}^N(\zeta, \psi, \omega)} \right). \tag{4.51}$$

Before proving this result, we introduce the following notation. For $\alpha \in \mathbb{N}^d$, $f, g \in H^{|\alpha|-1}(\mathbb{R}^d)$, we define the symmetric commutator

$$[\partial^\alpha, f, g] = \partial^\alpha (fg) - g\partial^\alpha f - f\partial^\alpha g.$$

Proof. Firstly, we apply ∂^α to the first equation of (4.46)

$$\partial_t \partial^\alpha \zeta + \varepsilon \underline{\mathrm{V}} \cdot \nabla \partial^\alpha \zeta + \varepsilon \partial^\alpha \underline{\mathrm{V}} \cdot \nabla \zeta - \frac{1}{\mu} \partial^\alpha \underline{\mathrm{w}} + \varepsilon \left[\partial^\alpha, \underline{\mathrm{V}}, \nabla \zeta\right] = 0.$$

Using Theorem 4.2.11 and the trace Lemma 4.2.12, we get the first equality. For the second equality we get, after applying ∂_k to the second equation of (4.20),

$$\begin{aligned}\partial_t \partial_k \psi + \partial_k \zeta + \varepsilon \underline{\mathrm{V}} \cdot \left((\partial_k \nabla \psi - \varepsilon \underline{\mathrm{w}} \nabla \partial_k \zeta) + \partial_k \nabla^\perp \widetilde{\psi} \right) - \frac{\varepsilon}{\mu} \underline{\mathrm{w}} \partial_k \left(\underline{\mathrm{U}}^\mu \cdot N^\mu \right) \\ - \varepsilon \partial_k \frac{\nabla^\perp}{\Delta} \cdot \left(\left(\underline{\omega} \cdot N^\mu + \frac{1}{\mathrm{Ro}} \right) \underline{\mathrm{V}} \right) = -\partial_k P.\end{aligned}$$

Then, applying ∂^γ and using Lemma 4.3 in [34] (we can easily adapt it thanks to Theorem 4.2.11 and Lemma 4.2.13) we get

$$\begin{aligned}\partial_t \partial^\alpha \psi + \partial^\alpha \zeta + \varepsilon \underline{\mathrm{V}} \cdot \left((\partial^\alpha \nabla \psi - \varepsilon \underline{\mathrm{w}} \nabla \partial^\alpha \zeta) + \partial^\alpha \nabla^\perp \widetilde{\psi} \right) \\ - \frac{\varepsilon}{\mu} \underline{\mathrm{w}} \partial^\alpha \left(\underline{\mathrm{U}}^\mu \cdot N^\mu \right) - \varepsilon \partial^\alpha \frac{\nabla^\perp}{\Delta} \cdot \left(\left(\underline{\omega} \cdot N^\mu + \frac{1}{\mathrm{Ro}} \right) \underline{\mathrm{V}} \right) = -\partial^\alpha P + \widetilde{R_\alpha^2},\end{aligned}$$

where $\widetilde{R_\alpha^2}$ is controlled

$$\left| \widetilde{R_\alpha^2} \right|_2 + \left| \mathfrak{P} \widetilde{R_\alpha^2} \right|_2 \leq \varepsilon M_N \mathcal{E}^N(\zeta, \psi, \omega) . \tag{4.52}$$

Using the first equation of (4.20) and the fact that $\Delta \widetilde{\psi} = \underline{\omega} \cdot N^\mu$, we obtain

$$\begin{aligned}\partial_t \psi_{(\alpha)} + \mathfrak{a} \partial^\alpha \zeta + \varepsilon \underline{\mathrm{V}} \cdot \nabla \psi_{(\alpha)} + \frac{\varepsilon}{\mathrm{Ro}} \partial^\alpha \frac{\nabla^\perp}{\Delta} \cdot \underline{\mathrm{V}} + \partial^\alpha P &= \varepsilon \partial^\alpha \frac{\nabla^\perp}{\Delta} \cdot \left(\underline{\omega} \cdot N^\mu \underline{\mathrm{V}} \right) \\ &\quad - \varepsilon \underline{\mathrm{V}} \cdot \nabla^\perp \partial^\alpha \widetilde{\psi} + \widetilde{R_\alpha^2} \\ &= \varepsilon \sum_{k \in \{1,2\}} (-1)^{k+1} \left[\partial^\alpha \frac{\partial_k}{\Delta}, \underline{\mathrm{V}}_{3-k} \right] \left(\underline{\omega} \cdot N^\mu \right) \\ &\quad + \widetilde{R_\alpha^2} \\ &:= \widetilde{R_\alpha^3} + \widetilde{R_\alpha^2},\end{aligned}$$

where $\partial_1 = \partial_x$ and $\partial_2 = \partial_y$. Then, using Theorem 3 in [79], Lemma B.1.2 and Lemma 4.2.9 we get

$$\left| \widetilde{R_\alpha^3} \right|_2 + \left| \mathfrak{P} \widetilde{R_\alpha^3} \right|_2 \leq \varepsilon M_N \left|\left| \mathrm{V} \right|\right|_{H^{N,1}} \left|\left| \omega \right|\right|_{H^{N-1,1}} + \varepsilon \left| \mathfrak{P} \frac{\nabla^\perp}{\Delta} \cdot \left(\underline{\omega} \cdot N^\mu \partial^\alpha \underline{\mathrm{V}} \right) \right|_2 .$$

Furthermore,

$$\left|\mathfrak{P}\frac{\nabla^\perp}{\Delta}\cdot\left(\underline{\omega}\cdot N^\mu\partial^\alpha\underline{\mathrm{V}}\right)\right|_2 \leq \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\left(\underline{\omega}\cdot N^\mu\partial^\alpha\underline{\mathrm{V}}\right)\right|_2 ,$$

$$\leq \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\left(\partial_k\left(\underline{\omega}\cdot N^\mu\right)\partial^\gamma\underline{\mathrm{V}}\right)\right|_2 + \left|\mathfrak{P}\left(\underline{\omega}\cdot N^\mu\partial^\gamma\underline{\mathrm{V}}\right)\right|_2 ,$$

$$\leq C\left(\varepsilon\left|\zeta\right|_{H^N}\right)\left|\underline{\omega}\right|_{H^{N-2}}\left(\left|\underline{\mathrm{V}}\right|_{H^{N-1}}+\left|\mathfrak{P}\partial^\gamma\underline{\mathrm{V}}\right|_2\right),$$

where we use Lemma B.1.3. The first term is controlled thanks to the trace Lemma 4.2.12 and Theorem 4.2.11. For the second term, we have

$$\partial^\gamma\underline{\mathrm{V}} = \nabla\partial^\gamma\psi - \varepsilon\underline{\mathrm{w}}\nabla\partial^\gamma\zeta - \varepsilon\partial^\gamma\underline{\mathrm{w}}\nabla\zeta + \nabla^\perp\partial^\gamma\widetilde{\psi} - \varepsilon\left[\partial^\gamma,\underline{\mathrm{w}},\nabla\zeta\right],$$

and the control follows from Lemma B.1.2, Lemma 4.2.9, Theorem 4.2.11 and Lemma 4.2.13. Then, we obtain

$$\partial_t\psi_{(\alpha)} + \mathfrak{a}\partial^\alpha\zeta + \varepsilon\underline{\mathrm{V}}\cdot\nabla\psi_{(\alpha)} + \frac{\varepsilon}{\mathrm{Ro}}\partial^\alpha\frac{\nabla^\perp}{\Delta}\cdot\underline{\mathrm{V}} + \partial^\alpha P = \widetilde{\widetilde{R_\alpha^2}},$$

where $\widetilde{\widetilde{R_\alpha^2}}$ satisfied also the estimate (4.52). Finally, we can adapt Lemma 4.4 in [34] thanks to Remark 4.3.2, Theorem 4.2.11 and Proposition 4.2.15 and we get

$$\partial_t\psi_{(\alpha)} = \partial_t\left(\mathrm{U}^\mu_{(\gamma)/\!/}\cdot\mathbf{e}_k\right) + \widetilde{R_\alpha},$$

where $\widetilde{R_\alpha}$ satisfies the same estimate as R_2 in (4.51). The third equality is a direct consequence of Proposition 4.3.1. □

In order to establish an a priori estimate we need to control the Rayleigh-Taylor coefficient $\mathfrak{a}$. The following Proposition is adapted from Proposition 2.2.11.

Proposition 4.3.4. *Let $T>0$, $t_0>\frac{d}{2}$, $N\geq 5$, $(\zeta,\psi,\omega)\in E_T^N$ is a solution of the water waves equations* (4.46)*, $P\in L^\infty(\mathbb{R}^+;\dot{H}^{N+1}(\mathbb{R}^d))$ and $b\in L^\infty\cap\dot{H}^{N+1}(\mathbb{R}^d)$, such that Condition* (4.22) *is satisfied. We assume also that ε,β,Ro,μ satisfy* (4.23)*. Then, for all $0\leq t\leq T$,*

$$\left|\mathfrak{a}-1\right|_{W^{1,\infty}} \leq C\left(M_N,\varepsilon\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)}\right)\varepsilon\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)} + \varepsilon M_N\left|\nabla P\right|_{L^\infty_t H^N_X}.$$

Furthermore, if $\partial_t P\in L^\infty(\mathbb{R}^+;\dot{H}^N(\mathbb{R}^d))$, then,

$$\left|\partial_t\mathfrak{a}\right|_{L^\infty} \leq C\Big(M_N,\left|\nabla P\right|_{L^\infty_t H^N_X},\varepsilon\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)}\Big)\varepsilon\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)} + \varepsilon M_N\left|\nabla P\right|_{W^{1,\infty}_t H^N_X}.$$

Proof. Using Proposition 4.2.15 we get that

$$\begin{aligned}\mathfrak{a}[\varepsilon\zeta,\beta b](\psi,\omega) = {} & 1+\varepsilon^2\underline{\mathrm{V}}\cdot\nabla\underline{\mathrm{w}} + \varepsilon\partial_t\zeta\underline{\partial_z^\sigma\mathrm{w}} \\ & + \varepsilon\underline{\mathrm{w}}[\varepsilon\zeta,\beta b]\left(\partial_t\psi - \varepsilon\underline{\mathrm{w}}[\varepsilon\zeta,\beta b](\psi,\omega)\partial_t\zeta + \varepsilon\sqrt{\mu}\frac{\nabla}{\Delta}\cdot\left(\underline{\omega_h}^\perp\partial_t\zeta\right),\partial_t^\sigma\omega\right).\end{aligned} \tag{4.53}$$

Then, using the equations satisfied by (ζ,ψ,ω), Theorems 4.2.8 and 4.2.11, Remark 4.2.14 and standard controls, we easily get the first inequality. The second inequality can be proved similarly. □

We can now establish an a priori estimate for the Castro-Lannes System with a Coriolis forcing under the positivity on the Rayleigh-Taylor coefficient

$$\exists\, \mathfrak{a}_{\min} > 0 \ , \ \mathfrak{a} \geq \mathfrak{a}_{\min}. \tag{4.54}$$

Theorem 4.3.5. *Let $N \geq 5$, $T > 0$, $b \in L^{\infty} \cap \dot{H}^{N+2}(\mathbb{R}^d)$, $P \in L^{\infty}_t\left(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d)\right)$ and $(\zeta, \psi, \omega) \in E^N_T$ solution of the water waves equations (4.46) such that (ζ, b) satisfy Condition (4.22) and $\mathfrak{a}[\varepsilon\zeta, \beta b]\,(\psi, \omega)$ satisfies (4.54) on $[0, T]$. We assume also that $\varepsilon, \beta, Ro, \mu$ satisfy (4.23). Then, for all $t \in [0, T]$,*

$$\begin{aligned}\frac{d}{dt}\mathcal{E}^N(\zeta,\psi,\omega) \leq C\Big(\mu_{\max}, \frac{1}{h_{\min}}, \varepsilon\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)}, \beta\left|\nabla b\right|_{H^{N+1}}, \beta\left|b\right|_{L^\infty}, \left|\nabla P\right|_{W^{1,\infty}_t H^N_X}\Big) \times \\ \left(\varepsilon\mathcal{E}^N(\zeta,\psi,\omega)^{\frac{3}{2}} + \max\left(\varepsilon, \beta, \frac{\varepsilon}{\mathrm{Ro}}\right)\mathcal{E}^N(\zeta,\psi,\omega) + \left|\nabla P\right|_{L^\infty_t H^N_X}\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)}\right).\end{aligned} \tag{4.55}$$

Proof. Compared to [34], we have here a non flat bottom, a Coriolis forcing and a non constant pressure. We focus on these terms. Inspired by [34] we can symmetrize the Castro-Lannes system. We define a modified energy

$$\begin{aligned}\mathcal{F}^N(\psi,\zeta,\omega) = \frac{1}{2}\Big(\|\boldsymbol{\omega}\|^2_{H^{N-1}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N^\mu_b\right)\right|_2^2 + \sum_{|\alpha|\leq 3}\left|\partial^\alpha\zeta\right|_2^2 + \frac{1}{\mu}\int_{\mathcal{S}}(1+\partial_z\sigma)\left|\partial^\alpha \mathrm{U}^\mu\right|^2 \\ + \sum_{k=x,y,1\leq|\gamma|\leq N-1}(\mathfrak{a}\partial_k\partial^\gamma\zeta, \partial_k\partial^\gamma\zeta) + \frac{1}{\mu}\int_{\mathcal{S}}(1+\partial_z\sigma)\left|\partial_k \mathrm{U}^\mu_{(\gamma)}\right|^2\Big).\end{aligned} \tag{4.56}$$

From Proposition 4.2.16 and Proposition 4.2.17 we get

$$\mathcal{E}^N(\psi,\zeta,\omega) \leq C\left(\frac{1}{\mathfrak{a}_{\min}}, M_N\right)\mathcal{F}^N(\psi,\zeta,\omega)\,,$$

and from Theorem 4.2.8, Theorem 4.2.11, Remark 4.2.14 and Proposition 4.3.4 we obtain that

$$\mathcal{F}^N(\psi,\zeta,\omega) \leq C\left(\frac{1}{h_{\min}}, \beta\left|b\right|_{L^\infty}, \beta\left|\nabla b\right|_{H^N}, \left|\nabla P\right|_{L^\infty_t H^N_X}, \varepsilon\sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}\right)\mathcal{E}^N(\psi,\zeta,\omega)\,.$$

Hence, in the following we estimate $\frac{d}{dt}\mathcal{F}^N(\psi,\zeta,\omega)$. We already did the work for the vorticity in Proposition 4.3.1. In the following R will be a remainder whose exact value has no importance and satisfying

$$\left|R\right|_2 \leq C\left(\frac{1}{h_{\min}}, \beta\left|b\right|_{L^\infty}, \beta\left|\nabla b\right|_{H^{N+1}}, \left|\nabla P\right|_{W^{1,\infty}_t H^N_X}, \varepsilon\sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}\right)\mathcal{E}^N(\psi,\zeta,\omega). \tag{4.57}$$

We start by the low order terms. Let $\alpha \in \mathbb{N}^d$, $|\alpha| \leq 3$. We apply ∂^α to the first equation of System (4.46) and we multiply it by ζ. Then, we apply ∂^α to the second equation and we multiply it by $\frac{1}{\mu}\underline{\mathrm{U}}^\mu \cdot N^\mu$. By summing these two equations, we obtain, thanks to Theorem 4.2.8, Theorem 4.2.11, Remark 4.2.14 and the trace Lemma,

$$\begin{aligned}\frac{1}{2}\partial_t\left(\partial^\alpha\zeta, \partial^\alpha\zeta\right) + \left(\partial_t\partial^\alpha\psi, \frac{1}{\mu}\partial^\alpha\underline{\mathrm{U}}^\mu \cdot N^\mu\right) + \frac{\varepsilon}{\mathrm{Ro}}\left(\frac{\nabla}{\Delta}\cdot\partial^\alpha\underline{\mathrm{V}}^\perp, \frac{1}{\mu}\partial^\alpha\underline{\mathrm{U}}^\mu \cdot N^\mu\right) \\ + \left(\partial^\alpha P, \frac{1}{\mu}\partial^\alpha\underline{\mathrm{U}}^\mu \cdot N^\mu\right) \leq \varepsilon\left|R\right|_2.\end{aligned} \tag{4.58}$$

Furthermore, using again the same Propositions as before, we get

$$\frac{\varepsilon}{\mathrm{Ro}}\left(\frac{\nabla}{\Delta}\cdot\partial^\alpha \underline{\mathrm{V}}^\perp, \frac{1}{\mu}\partial^\alpha \underline{\mathrm{U}}^\mu\cdot N^\mu\right)+\left(\partial^\alpha P, \frac{1}{\mu}\partial^\alpha \underline{\mathrm{U}}^\mu\cdot N^\mu\right) \leq \frac{\varepsilon}{\mathrm{Ro}}\,|R|_2 + M_N|\nabla P|_{L^\infty_t H^N_X}\sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}.$$

Then, we have to link $(\partial_t\partial^\alpha\psi, \partial^\alpha\underline{\mathrm{U}}^\mu\cdot N^\mu)$ to $\partial_t \int_{\mathcal{S}} (1+\partial_z\sigma)\,|\partial^\alpha \mathrm{U}^\mu|^2$. Remarking that $\psi = \underline{\phi}$, where ϕ satisfies

$$\begin{cases} \nabla^\mu_{X,z}\cdot P(\Sigma)\nabla^\mu_{X,z}\phi = 0 \text{ in } \mathcal{S}, \\ \phi_{|z=0} = \psi,\ \mathrm{e}_z\cdot P(\Sigma)\nabla^\mu\phi_{|z=-1} = 0, \end{cases} \tag{4.59}$$

we get thanks to Green's identity

$$\begin{aligned}\left(\partial_t\partial^\alpha\psi, \frac{1}{\mu}\partial^\alpha\underline{\mathrm{U}}^\mu\cdot N^\mu\right) &= \frac{1}{\mu}\int_{\mathcal{S}}(1+\partial_z\sigma)\,\nabla^{\sigma,\mu}_{X,z}\,(\partial_t\partial^\alpha\phi)\cdot\partial^\alpha\mathrm{U}^\mu \\ &+\frac{1}{\mu}\int_{\mathcal{S}}(1+\partial_z\sigma)\,\partial^\alpha\partial_t\phi\nabla^{\sigma,\mu}_{X,z}\cdot\partial^\alpha\mathrm{U}^\mu + \left(\partial_t\partial^\alpha\phi_b, \frac{1}{\mu}\partial^\alpha\mathrm{U}^\mu_b\cdot N^\mu_b\right).\end{aligned}$$

Then, notice that $\partial_k = \partial^\sigma_k + \partial_k\sigma\partial^\sigma_z$ for $k\in\{t,x,y\}$ and ∂^σ_k and $\nabla^{\sigma,\mu}_{X,z}$ commute. We differentiate Equation (4.59) with respect to t and we obtain thanks to Theorems 4.2.8, 4.2.11, Proposition 4.2.15 and Lemma 2.38 in [80] (irrotational theory),

$$\begin{aligned}\left(\partial_t\partial^\alpha\psi, \frac{1}{\mu}\partial^\alpha\underline{\mathrm{U}}^\mu\cdot N^\mu\right) &= \frac{1}{\mu}\int_{\mathcal{S}}(1+\partial_z\sigma)\,\partial^\sigma_t\partial^{\sigma,\alpha}\nabla^{\sigma,\mu}_{X,z}\phi\cdot\partial^\alpha\mathrm{U}^\mu \\ &+\left(\partial_t\partial^\alpha\phi_b, \frac{1}{\mu}\partial^\alpha\mathrm{U}^\mu_b\cdot N^\mu_b\right) + \max(\varepsilon,\beta)R.\end{aligned}$$

Using the fact that $\mathrm{w}_b = \mu\beta\nabla b\cdot\mathrm{V}_b$, we get

$$\left(\partial_t\partial^\alpha\phi_b, \frac{1}{\mu}\partial^\alpha\mathrm{U}^\mu_b\cdot N^\mu_b\right) \leq \beta M_N\,|\partial_t\partial^\alpha\phi_b|\,\sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}.$$

Then, by the trace Lemma, we finally obtain

$$\left(\partial_t\partial^\alpha\phi_b, \frac{1}{\mu}\partial^\alpha\mathrm{U}^\mu_b\cdot N^\mu_b\right) \leq \beta\,|R|_2\,.$$

Furthermore, remarking that $\mathrm{U}^\mu = \nabla^{\sigma,\mu}_{X,z}\phi + \mathrm{U}^{\sigma,\mu}[\varepsilon\zeta,\beta b]\,(0,\omega)$, we obtain, thanks to Proposition 4.2.15, Theorem 4.2.8 and Theorem 4.2.11,

$$\left(\partial_t\partial^\alpha\psi, \frac{1}{\mu}\partial^\alpha\underline{\mathrm{U}}^\mu\cdot N^\mu\right) = \frac{1}{\mu}\int_{\mathcal{S}}(1+\partial_z\sigma)\,\partial_t\partial^\alpha\mathrm{U}^\mu\cdot\partial^\alpha\mathrm{U}^\mu + \max\left(\varepsilon,\beta,\frac{\varepsilon}{\mathrm{Ro}}\right)R.$$

Using the following identity

$$\partial_t\int_{\mathcal{S}}(1+\partial_z\sigma)fg = \int_{\mathcal{S}}(1+\partial_z\sigma)\partial^\sigma_t fg + \int_{\mathcal{S}}(1+\partial_z\sigma)f\partial^\sigma_t g + \int_{\mathbb{R}^d}\varepsilon\partial_t\zeta\underline{fg}, \tag{4.60}$$

we obtain that

$$\frac{1}{\mu}\partial_t\int_{\mathcal{S}}(1+\partial_z\sigma)|\partial^\alpha\mathrm{U}^\mu|^2 \leq \max\left(\varepsilon,\beta,\frac{\varepsilon}{\mathrm{Ro}}\right)|R|_2 + M_N|\nabla P|_{L^\infty_t H^N_X}\sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}.$$

To control the high order terms of $\mathcal{F}^N(\psi,\zeta,\omega)$ we adapt Step 2 in Proposition 4.5 in [80]. Thanks to Proposition 4.3.3, we have

$$\begin{aligned}
&(\partial_t + \varepsilon \underline{V}\cdot\nabla)\,\partial^\alpha\zeta - \frac{1}{\mu}\partial_k \underline{\mathrm{U}}^\mu_{(\gamma)}\cdot N^\mu = R^1_\alpha,\\
&(\partial_t + \varepsilon \underline{V}\cdot\nabla)\left(\underline{\mathrm{U}}^\mu_{(\gamma)/\!/}\cdot e_{\mathbf{k}}\right) + \mathfrak{a}\partial^\alpha\zeta = -\partial^\alpha P + R^2_\alpha.
\end{aligned}$$

Then, we multiply the first equation by $\mathfrak{a}\partial^\alpha\zeta$ and the second by $\frac{1}{\mu}\partial_k\underline{\mathrm{U}}^\mu_{(\gamma)}\cdot N^\mu$ and we integrate over $\mathbb{R}^d$. Then, using Propositions 4.2.8, 4.2.18 and 4.3.4,

$$\begin{aligned}
\frac{1}{2}\partial_t\left(\mathfrak{a}\partial^\alpha\zeta,\partial\zeta\right) + \left((\partial_t + \varepsilon\underline{V}\cdot\nabla)\left(\underline{\mathrm{U}}^\mu_{(\gamma)/\!/}\cdot e_{\mathbf{k}}\right), \frac{1}{\mu}\partial_k\underline{\mathrm{U}}^\mu_{(\gamma)}\cdot N^\mu\right) &\leq \varepsilon\mathcal{E}^N(\psi,\zeta,\omega)^{\frac{3}{2}}\\
&+ \max\left(\varepsilon,\frac{\varepsilon}{\mathrm{Ro}}\right)|R|_2 + M_N|\nabla P|_{L^\infty_t H^N_X}\sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}.
\end{aligned}$$

We remark that

$$(\partial_t + \varepsilon\underline{V}\cdot\nabla)\left(\underline{\mathrm{U}}^\mu_{(\gamma)/\!/}\cdot e_{\mathbf{k}}\right) = \underline{\left(\partial^\sigma_t + \frac{\varepsilon}{\mu}\mathrm{U}^\mu\cdot\nabla^{\sigma,\mu}\right)\left(\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!/}\cdot e_{\mathbf{k}}\right)},$$

where $\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!/} = \mathrm{V}_{(\gamma)} + \mathrm{w}_{(\gamma)}\nabla\sigma$. Then, we have

$$\begin{aligned}
&\left((\partial_t + \varepsilon\underline{V}\cdot\nabla)\left(\underline{\mathrm{U}}^\mu_{(\gamma)/\!/}\cdot e_{\mathbf{k}}\right), \frac{1}{\mu}\partial_k\underline{\mathrm{U}}^\mu_{(\gamma)}\cdot N^\mu\right)\\
&\qquad = \frac{1}{\mu}\int_{\mathcal{S}}(1+\partial_z\sigma)\left(\partial^\sigma_t + \frac{\varepsilon}{\mu}\mathrm{U}^\mu\cdot\nabla^{\sigma,\mu}\right)\left(\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!/}\cdot e_{\mathbf{k}}\right)\nabla^{\sigma,\mu}_{X,z}\cdot\left(\partial_k\mathrm{U}^\mu_{(\gamma)}\right)\\
&\qquad + \frac{1}{\mu}\int_{\mathcal{S}}(1+\partial_z\sigma)\nabla^{\sigma,\mu}_{X,z}\left(\partial^\sigma_t + \frac{\varepsilon}{\mu}\mathrm{U}^\mu\cdot\nabla^{\sigma,\mu}\right)\left(\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!/}\cdot e_{\mathbf{k}}\right)\left(\partial_k\mathrm{U}^\mu_{(\gamma)}\right)\\
&\qquad + \left((\partial_t + \varepsilon\mathrm{V}_b\cdot\nabla)\left(\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!/}\cdot e_{\mathbf{k}}\right)_b, \frac{1}{\mu}\partial_k\left(\mathrm{U}^\mu_{(\gamma)}\right)_b\cdot N^\mu_b\right).
\end{aligned}$$

We focus on the last term (bottom contribution). The two other terms can be controlled as in Step 2 in Proposition 4.5 in [34]. Using the same computations as in Proposition 4.2.18, we have

$$\frac{1}{\mu}\partial_k\left(\mathrm{U}^\mu_{(\gamma)}\right)_b\cdot N^\mu_b = -\mu\beta\nabla\partial^\alpha b\cdot\mathrm{V}_b + \mathrm{l.o.t},$$

where l.o.t stands for lower order terms that can be controlled by the energy. Then, since $b\in\dot{H}^{N+2}(\mathbb{R}^d)$, we have by standard controls,

$$\left|\frac{1}{\mu}\partial_k\left(\mathrm{U}^\mu_{(\gamma)}\right)_b\cdot N^\mu_b\right|_{H^{\frac{1}{2}}} \leq \beta\,|\nabla b|_{H^{N+1}}\sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}.$$

Furthermore, using Propositions 4.2.8, 4.2.11 and 4.2.15 and standard controls, we have

$$\left|(\partial_t + \varepsilon\mathrm{V}_b\cdot\nabla)\left(\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!/}\cdot e_{\mathbf{k}}\right)_b\right|_{H^{-\frac{1}{2}}} \leq \varepsilon\,|R|_2 + M_N\sqrt{\mathcal{E}^N(\psi,\zeta,\omega)},$$

and the control follows easily. □

4.3.4 Existence result

We can now establish our existence theorem. Notice that thanks to Equation (4.53), we can define the Rayleigh-Taylor coefficient at time $t = 0$.

Theorem 4.3.6. *Let $A > 0$, $N \geq 5$, $b \in L^\infty \cap \dot{H}^{N+2}(\mathbb{R}^d)$, $P \in W^{1,\infty}(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$, $(\zeta_0, \psi_0, \omega_0) \in \mathrm{E}_0^N$ such that $\nabla_{X,z}^{\sigma,\mu} \cdot \omega_0 = 0$. We suppose that $(\varepsilon, \beta, \mu, \mathrm{Ro})$ satisfy (4.23). We also assume that*

$$\exists\, h_{\min},\, \mathfrak{a}_{\min} > 0 \ ,\ \varepsilon\zeta_0 + 1 - \beta b \geq h_{\min} \text{ and } \mathfrak{a}[\varepsilon\zeta, \beta b]\,(\psi, \omega)_{|t=0} \geq \mathfrak{a}_{\min}$$

and

$$\mathcal{E}^N(\zeta_0, \psi_0, \omega_0) + |\nabla P|_{L_t^\infty H_X^N} \leq A.$$

Then, there exists $T > 0$, and a unique solution $(\zeta, \psi, \omega) \in \mathrm{E}_T^N$ to the water waves equations (4.46) with initial data $(\zeta_0, \psi_0, \omega_0)$. Moreover,

$$T = \min\left(\frac{T_0}{\max(\varepsilon, \beta, \frac{\varepsilon}{\mathrm{Ro}})}, \frac{T_0}{|\nabla P|_{L_t^\infty H_X^N}}\right) \ ,\ \frac{1}{T_0} = c^1 \text{ and } \sup_{t \in [0,T]} \mathcal{E}^N\left(\zeta(t), \psi(t), \omega(t)\right) = c^2,$$

with $c^j = C\left(A, \mu_{\max}, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{L^\infty}, |\nabla b|_{H^{N+1}}, |\nabla P|_{W_t^{1,\infty} H_X^N}\right)$.

Proof. We do not give the proof. It is very similar to Theorem 4.7 in [34]. We can regularize the system (4.46) (see Step 2 of the proof of Theorem 4.7 in [34]) and thanks to the energy estimate of Theorem 4.3.5 we get the existence. The uniqueness mainly follows from a similar proposition to Corollary 3.19 in [34] which shows that the operator $\mathrm{U}^{\sigma,\mu}$ has a Lipschitz dependence on its coefficients. □

4.4 From the Castro-Lannes system to the Euler equations, reconstruction of the pressure.

In this part, we show how we can reconstruct the pressure in the fluid domain from the Castro-Lannes formulation. This part is similar to the work of Alazard-Burq-Zuily in the irrotational case (see [3]). We work with the dimensionalized version of the Castro-Lannes formulation and we assume that the pressure at the surface is equal to zero and that we do not have a Coriolis forcing. It is not painful to generalize this part when we add them. We recall that the Castro-Lannes formulation is the following system

$$\begin{cases} \partial_t \zeta - \underline{\mathrm{U}} \cdot N = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}\left|\mathrm{U}_{/\!/}\right|^2 - \frac{1}{2}\left(1 + |\nabla\zeta|^2\right)\underline{\mathrm{w}}^2 + \frac{\nabla}{\Delta} \cdot \left(\underline{\omega} \cdot N \underline{\mathrm{V}}^\perp\right) = 0, \\ \partial_t \boldsymbol{\omega} + \left(\mathbf{U} \cdot \nabla_{X,z}\right)\boldsymbol{\omega} = \left(\boldsymbol{\omega} \cdot \nabla_{X,z}\right)\mathbf{U}, \end{cases} \tag{4.61}$$

where $\mathbf{U} := \mathbf{U}[\zeta, b](\psi, \boldsymbol{\omega})$ is the unique solution in $H^1(\Omega_t)$ of

$$\begin{cases} \operatorname{curl} \mathbf{U} = \boldsymbol{\omega} \text{ in } \Omega_t, \\ \operatorname{div} \mathbf{U} = 0 \text{ in } \Omega_t, \\ \mathrm{U}_{/\!/} = \nabla\psi + \frac{\nabla^\perp}{\Delta}\left(\underline{\omega} \cdot N\right), \\ \mathrm{U}_b \cdot N_b = 0. \end{cases} \tag{4.62}$$

Our purpose is the reconstruction of the pressure in the fluid domain. The following proposition gives a necessary condition for the pressure $\mathcal{P}$.

Proposition 4.4.1. *Consider a regular solution* $(\zeta, \mathbf{U})$ *of the free surface Euler equations* (4.1) *with the boundary conditions* (4.2)*. Then the pressure* $\mathcal{P}$ *satisfies the Laplace problem*

$$\begin{cases} -\Delta_{X,z}\mathcal{P} = \nabla_{X,z} \cdot [(\mathbf{U} \cdot \nabla_{X,z})\, \mathbf{U}] \ \text{ in } \Omega_t, \\ \mathcal{P}_{|z=\zeta} = 0, \ \sqrt{1+|\nabla b|^2}\partial_n \mathcal{P}_{|z=-H+b} = -\left[(\mathrm{V}_b \cdot \nabla_X)\, \mathrm{U}_b\right] \cdot N_b. \end{cases}$$

Proof. We apply the divergence operator to the first equation of the free surface Euler equations (4.1) and we get the first equation. For the bottom contribution, we take the trace at the bottom of the first equation of the free surface Euler equations (4.1) and since $\mathbf{U}_b \cdot N_b = 0$, we get

$$[(\mathbf{U} \cdot \nabla_{X,z})\, \mathbf{U}]_{|z=-H+b} = (\mathrm{V}_b \cdot \nabla_X)\, \mathrm{U}_b.$$

□

Thanks to Section A.1, we get that the previous Laplace problem has a unique solution in $\dot{H}^2(\Omega_t)$ if $\zeta, b \in H^3(\mathbb{R}^d)$ and $\mathbf{U} \in H^2(\Omega_t)$. Then we denote by $\mathcal{P} = \mathcal{P}[\zeta, b](\psi, \boldsymbol{\omega})$ the solution of the following system

$$\begin{cases} -\Delta_{X,z}\mathcal{P} = \nabla_{X,z} \cdot [(\mathbf{U} \cdot \nabla_{X,z})\, \mathbf{U}] \ \text{ in } \Omega_t, \\ \mathcal{P}_{|z=\zeta} = 0, \ \sqrt{1+|\nabla b|^2}\partial_n \mathcal{P}_{|z=-H+b} = -\left[(\mathrm{V}_b \cdot \nabla_X)\, \mathrm{U}_b\right] \cdot N_b, \end{cases} \tag{4.63}$$

where $\mathbf{U} = \mathbf{U}[\zeta, b](\psi, \boldsymbol{\omega})$. In the following, we show that if $\mathcal{P}$ satisfies (4.63), $\mathbf{U}$ satisfies the first equation of (4.1). We denote by $\mathcal{K}$ the quantity

$$\mathcal{K}[\zeta, b](\psi, \boldsymbol{\omega}) = \partial_t \mathbf{U} + \nabla \mathcal{P} + (\mathbf{U} \cdot \nabla_{X,z})\, \mathbf{U} + g\mathbf{e_z}.$$

where $\mathcal{P}$ satisfies (4.63) and $\mathbf{U}$ verifies (4.62).

Proposition 4.4.2. *Let* $b \in H^3(\mathbb{R}^d)$*. If* $(\zeta, \psi, \boldsymbol{\omega})$ *is a smooth solution of the Castro-Lannes equations* (4.61)*, the quantity* $\mathcal{K} = \mathcal{K}[\zeta, b](\psi, \boldsymbol{\omega})$ *satisfies the following system*

$$\begin{cases} \operatorname{curl} \mathcal{K} = 0, \\ \operatorname{div} \mathcal{K} = 0, \\ \mathcal{K}_{/\!/} = 0, \\ \mathcal{K} \cdot N_b = 0, \end{cases} \tag{4.64}$$

and K *is equal to zero.*

Proof. Using Proposition 4.2.15, and the fact that $\partial_t \mathbf{U} \circ \Sigma = \partial_t^\sigma \mathrm{U}$ (see Subsection 4.2.3) we have

$$\partial_t \left(\mathbf{U}[\varepsilon\zeta, \beta b]\, (\psi, \boldsymbol{\omega})\right) = \mathbf{U}[\varepsilon\zeta, \beta b] \left(\partial_t \psi - \underline{\mathrm{w}} \partial_t \zeta + \frac{\nabla}{\Delta} \cdot \left(\underline{\omega_h}^\perp \partial_t \zeta\right), \partial_t \boldsymbol{\omega}\right).$$

It is clear that div $\mathcal{K} = 0$ by definition of $\mathcal{P}$ and curl $\mathcal{K} = 0$ since $\boldsymbol{\omega}$ satisfies the third equation of the Castro-Lannes formulation. The boundary condition at the bottom follows by definition of the pressure $\mathcal{P}$. For the boundary condition at the surface, since $(\nabla \mathcal{P})_{/\!/} = \nabla \underline{\mathcal{P}} = 0$, we get

$$\mathcal{K}_{/\!/} = \partial_t \nabla \psi - \nabla \left(\underline{\mathrm{w}} \partial_t \zeta\right) + \nabla \frac{\nabla}{\Delta} \cdot \left(\underline{\omega_h}^\perp \partial_t \zeta\right) + \frac{\nabla^\perp}{\Delta} \left(\underline{\partial_t \boldsymbol{\omega}} \cdot N\right) + [(\mathbf{U} \cdot \nabla_{X,z})\, \mathbf{U}]_{/\!/} + g\nabla\zeta$$

where we denote $\mathbf{U} = (\mathbf{V}, \mathbf{w})^t = \mathbf{U}[\varepsilon, b](\psi, \boldsymbol{\omega})$. Then, using the fact that

$$\nabla \frac{\nabla}{\Delta} \cdot + \nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot = Id,$$

and taking the trace at the surface of the equation div $\boldsymbol{\omega} = 0$, we get that

$$\begin{aligned}\nabla \frac{\nabla}{\Delta} \cdot \left(\underline{\omega_h}^{\perp} \partial_t \zeta\right) + \frac{\nabla^{\perp}}{\Delta} \left(\underline{\partial_t \boldsymbol{\omega}} \cdot N\right) &= \underline{\omega_h}^{\perp} \partial_t \zeta - \nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot \left(\underline{\omega_h}^{\perp} \partial_t \zeta\right) + \frac{\nabla^{\perp}}{\Delta} \left(\underline{\partial_t \boldsymbol{\omega}} \cdot N\right), \\ &= \underline{\omega_h}^{\perp} \partial_t \zeta + \frac{\nabla^{\perp}}{\Delta} \left(\underline{\omega} \cdot \partial_t N\right) + \frac{\nabla^{\perp}}{\Delta} \left(\partial_t \underline{\omega} \cdot N\right), \\ &= \underline{\omega_h}^{\perp} \partial_t \zeta + \frac{\nabla^{\perp}}{\Delta} \partial_t \left(\underline{\omega} \cdot N\right).\end{aligned}$$

Furthermore, using the fact that $\omega_h = \nabla \mathrm{w} - \partial_z \mathbf{V}$, we have

$$\underline{\omega_h}^{\perp} \partial_t \zeta - \nabla \left(\underline{\mathrm{w}} \partial_t \zeta\right) = -\underline{\partial_z \mathrm{V}} \partial_t \zeta - \underline{\partial_z \mathrm{w}} \partial_t \zeta \nabla \zeta - \underline{\mathrm{w}} \nabla \partial_t \zeta$$

and from the chain rule we get

$$\partial_t \nabla \psi + \partial_t \nabla \frac{\nabla^{\perp}}{\Delta} \left(\underline{\omega} \cdot N\right) = \partial_t \mathrm{U}_{/\!/} = \left(\partial_t \mathrm{U}\right)_{/\!/} + \underline{\partial_z \mathrm{V}} \partial_t \zeta + \underline{\partial_z \mathrm{w}} \partial_t \zeta \nabla \zeta + \underline{\mathrm{w}} \nabla \partial_t \zeta.$$

Therefore, we have

$$\mathcal{K}_{/\!/} = \left(\partial_t \mathrm{U}\right)_{/\!/} + \left[\left(\mathbf{U} \cdot \nabla_{X,z}\right) \mathbf{U}\right]_{/\!/} + g \nabla \zeta,$$

and using the same computations that Subsection 4.1.1, we obtain

$$\mathcal{K}_{/\!/} = \partial_t \mathrm{U}_{/\!/} + g \nabla \zeta + \frac{1}{2} \nabla \left|\mathrm{U}_{/\!/}\right|^2 - \frac{1}{2} \nabla \left[\left(1 + |\nabla \zeta|^2\right) \underline{\mathrm{w}}^2\right] + \underline{\omega} \cdot N \underline{\mathrm{V}}^{\perp}.$$

Then we decompose $\mathcal{K}_{/\!/}$ as $\mathcal{K}_{/\!/} = \nabla \frac{\nabla}{\Delta} \cdot \mathcal{K}_{/\!/} + \nabla^{\perp} \frac{\nabla^{\perp}}{\Delta} \cdot \mathcal{K}_{/\!/}$ and the first term is equal to zero because ψ satisfies the second equation of the Castro-Lannes system (4.61) and the second term is equal to zero since, by taking the trace of the third equation of the Castro-Lannes system (4.61), we have the following equation for $\underline{\omega} \cdot N$

$$\partial_t \left(\underline{\omega} \cdot N\right) + \nabla \cdot \left[\left(\underline{\omega} \cdot N\right) \underline{V}\right] = 0.$$

The fact that $\mathcal{K}_{/\!/}$ is equal to zero follows from Theorem 4.2.8. □

4.5 The nonlinear shallow water equations

4.5.1 The context

In this part we justify rigorously the derivation of the nonlinear rotating shallow water equations from the water waves equations. We recall that, in this chapter, we do not consider fast Coriolis forcing, i.e Ro $\leq \varepsilon$ (see Remark 4.1.2). The nonlinear shallow water equations (or Saint Venant equations) is a model used by the mathematical and physical communities to study the water waves in shallow waters. Coupled with a Coriolis term, it is usually uses to describe shallow waters under the influence of the Coriolis force (see for instance [27], [93] or [144]). But to our knowledge, there is no mathematical justification of this fact. Without the Coriolis term, many authors mathematically justify the Saint Venant equations; for the irrotationnal case, there are, for instance the works of Iguchi [70] and Alvarez-Samaniego and Lannes ([9]). It is also done in [80]. More recently, Castro and Lannes proposed a way to justify the Saint-Venant equations without the irrotational condition and with a flat bottom ([33] and [34]). We address here the case in which the Coriolis force and a non flat bottom are present. We denote the depth

$$h(t,X) = 1 + \varepsilon\zeta(t,X) - \beta b(X), \tag{4.65}$$

and the averaged horizontal velocity

$$\overline{\mathbf{V}} = \overline{\mathbf{V}}[\varepsilon\zeta, \beta b](\psi, \boldsymbol{\omega})(t,X) = \frac{1}{h(t,X)} \int_{z=-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \mathbf{V}[\varepsilon\zeta, \beta b](\psi, \boldsymbol{\omega})(t,X,z)dz. \tag{4.66}$$

The Saint-Venant equations (in the nondimensionalized form) are

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\overline{\mathbf{V}}) = 0, \\ \partial_t \overline{\mathbf{V}} + \varepsilon \left(\overline{\mathbf{V}} \cdot \nabla\right) \overline{\mathbf{V}} + \nabla\zeta + \frac{\varepsilon}{\mathrm{Ro}} \overline{\mathbf{V}}^{\perp} = -\nabla P. \end{cases} \tag{4.67}$$

It is well-known that the shallow water equations are wellposed (see Chapter 6 in [80] or [9] without the pressure term and the Coriolis forcing and [27]). We have the following Proposition.

Proposition 4.5.1. *Let $t_0 > \frac{d}{2}$, $s \geq t_0 + 1$ and $b \in H^{s+1}(\mathbb{R}^d)$, $\zeta_0 \in H^s(\mathbb{R}^d)$, $\overline{V}_0 \in H^s(\mathbb{R}^d)^d$. We assume that Condition* (4.22) *is satisfied by (ζ_0, b). Assume also that ε, β and Ro satisfy Condition* (4.23)*. Then, there exists $T > 0$ and $\left(\zeta, \overline{V}\right) \in \mathcal{C}^0\left(\left[0, \frac{T}{\max(\varepsilon,\beta)}\right], H^s(\mathbb{R}^d)^{d+1}\right)$ a unique solution to the Saint-Venant equations* (4.67) *with initial data $\left(\zeta_0, \overline{V}_0\right)$. Furthermore, for all $t \leq \frac{T}{\max(\varepsilon,\beta)}$,*

$$\frac{1}{T} = c^1 \text{ and } \left|\zeta(t,\cdot)\right|_{H^s} + \left|\overline{V}(t,\cdot)\right|_{H^{s+1}} \leq c^2,$$

with $c^j = C\left(\frac{1}{\mathrm{h_{min}}}, \left|\zeta_0\right|_{H^s}, \left|b\right|_{H^{s+1}}, \left|\overline{V}_0\right|_{H^s}\right)$.

4.5.2 WKB expansion with respect to μ

In this part, we study the dependence of U^μ with respect to μ. The first Proposition shows that $\overline{\mathbf{V}}$ is linked to $\underline{\mathrm{U}}^\mu \cdot N^\mu$.

Proposition 4.5.2. *Under the assumptions of Theorem 4.2.8, we have*

$$\underline{\mathrm{U}}^\mu \cdot N^\mu = -\mu\nabla \cdot \left(h\overline{\mathbf{V}}\right).$$

Proof. This proof is similar to Proposition 3.35 in [80]. Consider φ smooth and compactly supported in $\mathbb{R}^d$. Then, a simple computation gives

$$\begin{aligned} \int_{\mathbb{R}^d} \varphi \mathbf{U}^\mu \cdot N^\mu dX &= \int_\Omega \nabla^\mu_{X,z} \cdot (\varphi \mathbf{U}^\mu)\, dXdz, \\ &= \int_\Omega \mu \nabla\varphi \cdot \mathbf{V} dXdz, \\ &= -\mu \int_{\mathbb{R}^d} \varphi \nabla \cdot \left(\int_{z=-1+\beta b}^{\varepsilon\zeta} \mathbf{V} \right) dX. \end{aligned}$$

□

Then we need a WKB expansion with respect to μ of $\mathbf{U}^\mu$.

Proposition 4.5.3. *Let $t_0 > \frac{d}{2}$, $0 \leq s \leq t_0$, $\zeta \in H^{t_0+2}(\mathbb{R}^d)$, $b \in L^\infty \cap \dot{H}^{t_0+2}(\mathbb{R}^d)$. Under the assumptions of Theorem 4.2.8, we have*

$$\mathbf{U}^\mu = \begin{pmatrix} \sqrt{\mu}\overline{\mathbf{V}} + \mu \left(\int_z^{\varepsilon\zeta} \boldsymbol{\omega}_h^\perp - \mathbf{Q} \right) + \mu^{\frac{3}{2}} \widetilde{\mathbf{V}} \\ \mu \widetilde{\mathbf{w}} \end{pmatrix},$$

with

$$\mathbf{Q}(X) = \frac{1}{h(X)} \int_{z'=-1+\beta b(X)}^{\varepsilon\zeta(X)} \int_{s=z'}^{\varepsilon\zeta(X)} \boldsymbol{\omega}_h^\perp(X,s),$$

and

$$\left|\left| \widetilde{\mathbf{V}} \circ \Sigma \right|\right|_{H^{s,1}} + ||\widetilde{\mathbf{w}} \circ \Sigma||_{H^{s,1}} \leq C \left(\frac{1}{\mathrm{h}_{\min}}, \varepsilon \left|\zeta\right|_{H^{t_0+2}}, \beta \left|b\right|_{L^\infty}, \beta \left|\nabla b\right|_{H^{t_0+1}} \right) ||\mathbf{V} \circ \Sigma||_{H^{t_0+2,1}}.$$

Proof. This proof is inspired from the computations of Part 2.2 in [33] and Part 5.7.1 [34]. First, using the previous proposition, we get that

$$\underline{\mathrm{w}} = \varepsilon\mu\nabla\zeta \cdot \underline{\mathrm{V}} - \mu\nabla \cdot \left(h\overline{\mathbf{V}}\right).$$

Furthermore, using the fact that $\mathbf{U}^\mu$ is divergent free we have

$$\partial_z \mathbf{w} = -\mu\nabla_X \cdot \mathbf{V}.$$

Then, we obtain

$$\begin{aligned} \mathbf{w} &= \varepsilon\mu\nabla\zeta \cdot \underline{\mathrm{V}} - \mu\nabla \cdot \left(h\overline{\mathbf{V}}\right) + \mu \int_z^{\varepsilon\zeta} \nabla_X \mathbf{V} \\ &= -\mu\nabla_X \cdot \left(\int_{-1+\beta b}^z \mathbf{V} \right). \end{aligned}$$

The control of $\widetilde{\mathbf{w}}$ follows easily. Furthermore, using the ansatz

$$\mathbf{V} = \overline{\mathbf{V}} + \sqrt{\mu}\mathbf{V}_1, \tag{4.68}$$

and plugging it into the orthogonal of the horizontal part of $\mathrm{curl}^\mu \mathbf{U}^\mu = \mu\boldsymbol{\omega}$, we get that

$$\partial_z \mathbf{V}_1 = \sqrt{\mu}\nabla_X \widetilde{\mathbf{w}} - \boldsymbol{\omega}_h^\perp.$$

Then, integrating with respect to z the previous equation from z to $\varepsilon\zeta(X)$ we get

$$\mathbf{V}_1(X,z) = \int_{s=z}^{\varepsilon\zeta(X)} \boldsymbol{\omega}_h^\perp(X,s)ds + \underline{\mathbf{V}_1}(X) + \mu^{\frac{1}{2}}\mathbf{R}(X,z), \tag{4.69}$$

where $\mathbf{R}$ is a remainder uniformly bounded with respect to μ and

$$\underline{\mathbf{V}_1} = \frac{\underline{\mathbf{V}} - \overline{\mathbf{V}}}{\sqrt{\mu}}.$$

Integrating Equation (4.68) with respect to z from $-1+\beta$ to $\varepsilon\zeta$ we obtain that

$$\int_{z=-1+\beta b(X)}^{\varepsilon\zeta(X)} \mathbf{V}_1(X,z)dz = 0 \ , \forall X \in \mathbb{R}^d.$$

Then, we integrate Equation (4.69) with respect to z from $-1+\beta b$ to $\varepsilon\zeta$ and we get

$$h\underline{\mathbf{V}_1} = -\int_{z'=-1+\beta b}^{\varepsilon\zeta}\int_{s=z'}^{\varepsilon\zeta} \boldsymbol{\omega}_h^{\perp} + \mu^{\frac{1}{2}}\widetilde{\mathbf{R}},$$

where $\widetilde{\mathbf{R}}$ is a remainder uniformly bounded with respect to μ. Plugging the previous expression into Equation (4.69), we get the result. The control of the remainders is straightforward thanks to Lemma 4.2.9 (see also the comments about the notations of [34] in Subsection 4.2.3). □

Remark 4.5.4. *Under the assumptions of the previous Proposition, it is easy to check that*

$$\mathbf{w} = -\mu\nabla_X\cdot\left([1+z-\beta b]\,\overline{\mathbf{V}}\right) + \mu^{\frac{3}{2}}\mathbf{w}_1,$$

with

$$||\mathbf{w}_1\circ\Sigma||_{H^{s,1}} \leq C\left(\frac{1}{\mathrm{h}_{\min}}, \varepsilon\left|\zeta\right|_{H^{t_0+2}}, \beta\left|b\right|_{L^\infty}, \beta\left|\nabla b\right|_{H^{t_0+1}}\right)||\mathbf{V}\circ\Sigma||_{H^{t_0+2,1}}. \tag{4.70}$$

Then, we define the quantity

$$\mathbf{Q} = \mathbf{Q}[\varepsilon\zeta,\beta b](\psi,\boldsymbol{\omega})(t,X) = \frac{1}{h}\int_{z'=-1+\beta b}^{\varepsilon\zeta}\int_{s=z'}^{\varepsilon\zeta}\boldsymbol{\omega}_h^{\perp}. \tag{4.71}$$

The following Proposition shows that $\mathbf{Q}$ satisfies the evolution equation

$$\partial_t\mathbf{Q} + \varepsilon\left(\overline{\mathbf{V}}\cdot\nabla\right)\mathbf{Q} + \varepsilon\left(\mathbf{Q}\cdot\nabla\right)\overline{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}}\mathbf{Q}^{\perp} = 0, \tag{4.72}$$

up to terms of order $\mathcal{O}(\sqrt{\mu})$.

Proposition 4.5.5. *Let $T>0$, $t_0>\frac{d}{2}$, $0\leq s\leq t_0$, $0\leq\mu\leq 1$, $\zeta\in\mathcal{C}^1([0,T];H^{t_0+2}(\mathbb{R}^d))$, $b\in L^\infty\cap\dot{H}^{t_0+2}(\mathbb{R}^d)$. Let $\boldsymbol{\omega},\mathbf{V},\mathbf{w}\in\mathcal{C}^1([0,T];H^{t_0+2}(\mathbb{R}^d))$. Suppose that we are under the assumption of Theorem 4.2.8, that $\boldsymbol{\omega}$ satisfies the third equation of the Castro-Lannes system* (4.20) *(the vorticity equation) and that $\partial_t\zeta+\nabla\cdot\left(h\overline{\mathbf{V}}\right)=0$, on $[0,T]$. Then $\mathbf{Q}$ satisfies*

$$\partial_t\mathbf{Q} + \varepsilon\left(\overline{\mathbf{V}}\cdot\nabla\right)\mathbf{Q} + \varepsilon\left(\mathbf{Q}\cdot\nabla\right)\overline{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}}\mathbf{Q}^{\perp} = \sqrt{\mu}\max\left(\varepsilon,\frac{\varepsilon}{\mathrm{Ro}}\right)\widetilde{\mathbf{R}},$$

and

$$\left|\left|\widetilde{\mathbf{R}}\circ\Sigma\right|\right|_{H^{s,1}} \leq C\left(\frac{1}{\mathrm{h}_{\min}}, \varepsilon\left|\zeta\right|_{H^{t_0+2}}, \beta\left|b\right|_{L^\infty}, \beta\left|\nabla b\right|_{H^{t_0+1}}\right)||\mathbf{V}\circ\Sigma||_{H^{t_0+2,1}}.$$

Proof. This proof is inspired from Subsection 2.3 in [33]. We know that $\boldsymbol{\omega}_h$ satisfies

$$\partial_t\boldsymbol{\omega}_h + \varepsilon\left(\mathbf{V}\cdot\nabla\right)\boldsymbol{\omega}_h + \frac{\varepsilon}{\mu}\mathbf{w}\partial_z\boldsymbol{\omega}_h = \varepsilon\left(\boldsymbol{\omega}_h\cdot\nabla\right)\mathbf{V} + \frac{\varepsilon}{\sqrt{\mu}}\left(\boldsymbol{\omega}_v + \frac{1}{\mathrm{Ro}}\right)\partial_z\mathbf{V}.$$

Using Proposition 4.5.2, Remark 4.5.4 and the fact that $\boldsymbol{\omega}_v = \nabla^{\perp}\cdot\mathbf{V}$, we get

$$\partial_t \boldsymbol{\omega}_h + \varepsilon \left(\overline{\mathbf{V}} \cdot \nabla\right) \boldsymbol{\omega}_h - \varepsilon \nabla_X \cdot \left(\left[1+z-\beta b\right] \overline{\mathbf{V}}\right) \partial_z \boldsymbol{\omega}_h = \varepsilon \left(\boldsymbol{\omega}_h \cdot \nabla\right) \overline{\mathbf{V}} - \varepsilon \left(\nabla^{\perp} \cdot \overline{\mathbf{V}} + \frac{1}{\mathrm{Ro}}\right) \boldsymbol{\omega}_h^{\perp}$$
$$+ \sqrt{\mu} \max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right) \mathbf{R},$$

where $\mathbf{R} \circ \Sigma$ satisfies the same estimate as $\mathbf{w}_1 \circ \Sigma$ in (4.70). If we denote $\mathbf{V}_{sh} = \int_z^{\varepsilon\zeta} \boldsymbol{\omega}_h^{\perp}$, doing the same computations as in Subsection 2.3 [33] and using the fact that $\partial_t \zeta + \nabla \cdot \left(h \overline{\mathbf{V}}\right) = 0$, we get

$$\partial_t \mathbf{V}_{sh} + \varepsilon \left(\overline{\mathbf{V}} \cdot \nabla\right) \mathbf{V}_{sh} + \varepsilon \left(\mathbf{V}_{sh} \cdot \nabla\right) \overline{\mathbf{V}} - \nabla \cdot \left(\left[1+z-\beta b\right] \overline{\mathbf{V}}\right) + \frac{\varepsilon}{\mathrm{Ro}} \mathbf{V}_{sh}^{\perp} = \sqrt{\mu} \max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right) \int_z^{\varepsilon\zeta} \mathbf{R}.$$

Then, integrating this expression with respect to z and using again the fact that $\partial_t \zeta + \nabla \cdot \left(h \overline{\mathbf{V}}\right) = 0$, we get

$$\partial_t \mathbf{Q} + \varepsilon \left(\overline{\mathbf{V}} \cdot \nabla\right) \mathbf{Q} + \varepsilon \left(\mathbf{Q} \cdot \nabla\right) \overline{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}} \mathbf{Q}^{\perp} = \sqrt{\mu} \max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right) \int_{-1+\beta b}^{\varepsilon\zeta} \int_z^{\varepsilon\zeta} \mathbf{R},$$

and the result follows easily. □

4.5.3 Rigorous derivation

The purpose of this part is to prove a rigorous derivation of the water waves equations to the shallow water equations. This part is devoted to the proof of the following theorem. We recall that Σ is defined in (4.40).

Theorem 4.5.6. *Let $N \geq 6$, $0 \leq \mu \leq 1$, $\varepsilon, \beta, \mathrm{Ro}$ satisfying* (4.23). *We assume that we are under the assumptions of Theorem 4.3.6. Then, we can define the following quantity $\boldsymbol{\omega}_0 = \omega_0 \circ \Sigma^{-1}$, $\boldsymbol{\omega} = \omega \circ \Sigma^{-1}$, $\overline{\mathbf{V}}_0 = \overline{\mathbf{V}}[\varepsilon\zeta_0, \beta b](\psi_0, \boldsymbol{\omega}_0)$, $\overline{\mathbf{V}} = \overline{\mathbf{V}}[\varepsilon\zeta, \beta b](\psi, \boldsymbol{\omega})$, $\mathbf{Q}_0 = \mathbf{Q}[\varepsilon\zeta_0, \beta b](\psi_0, \boldsymbol{\omega}_0)$ and $\mathbf{Q} = \mathbf{Q}[\varepsilon\zeta, \beta b](\psi, \boldsymbol{\omega})$and there exists a time $T > 0$ such that*

(i) T has the form

$$T = \min\left(\frac{T_0}{\max(\varepsilon, \beta, \frac{\varepsilon}{Ro})}, \frac{T_0}{|\nabla P|_{L_t^\infty H_X^N}}\right) \text{ and } \frac{1}{T_0} = c^1.$$

(ii) There exists a unique solution $\left(\zeta_{SW}, \overline{\mathbf{V}}_{SW}\right)$ of (4.67) *with initial conditions $\left(\zeta_0, \overline{\mathbf{V}}_0\right)$ on $[0, T]$.*

(iii) There exists a unique solution $\mathbf{Q}_{SW}$ to Equation (4.72) *on $[0, T]$.*

(iv) There exists a unique solution (ζ, ψ, ω) of (4.46) *with initial conditions $(\zeta_0, \psi_0, \omega_0)$ on $[0, T]$.*

(v) The following error estimates hold, for $0 \leq t \leq T$,

$$\left|\left(\zeta, \overline{\mathbf{V}}, \sqrt{\mu}\mathbf{Q}\right) - \left(\zeta_{SW}, \overline{\mathbf{V}}_{SW}, \sqrt{\mu}\mathbf{Q}_{SW}\right)\right|_{L^\infty([0,t]\times\mathbb{R}^d)} \leq \mu\, t c^2,$$

and

$$\left|\underline{\mathbf{V}} - \overline{\mathbf{V}} + \sqrt{\mu}\mathbf{Q}\right|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq \mu\, c^3,$$

with $c^j = C\left(A, \mu_{\max}, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{L^\infty}, |\nabla b|_{H^{N+1}}, |\nabla P|_{W_t^{1,\infty} H_X^N}\right)$.

Remark 4.5.7. *Hence, in shallow waters the rotating Saint-Venant equations are a good model to approximate the water waves equations under a Coriolis forcing. Furthermore, we notice that if we start initially with a irrotational flow, at the order μ, the flow stays irrotational. It means that a Coriolis forcing (not too fast) does not generate a horizontal vorticity in shallow waters and the assumption of a columnar motion, which is the fact that the velocity is horizontal and independent of the vertical variable z, stays valid. It could be interesting to develop an asymptotic model of the water waves equations at the order μ^2 (Green-Naghdi or Boussinesq models) and study the influence a Coriolis forcing in these models. It will be done in Chapter 5.*

Proof. The point *(ii)* follows from Proposition 4.5.1 and the point *(iv)* from Theorem 4.3.6. Since, Equation (4.72) is linear, the point *(iii)* is clear. We only need to show that $\left(\zeta, \overline{\mathbf{V}}\right)$ satisfy the shallow water equations up to a remainder of order μ. Then, a small adaptation of Proposition 6.3 in [80] allows us to prove the point *(v)*. First, we know that

$$\partial_t \psi + \zeta + \frac{\varepsilon}{2} \left| \mathrm{U}^{\mu}_{/\!/} \right|^2 - \frac{\varepsilon}{2\mu} \left(1 + \varepsilon^2 \mu \left|\nabla \zeta\right|^2\right) \underline{\mathbf{w}}^2 + \varepsilon \frac{\nabla}{\Delta} \cdot \left[\left(\underline{\boldsymbol{\omega}} \cdot N^{\mu} + \frac{1}{\mathrm{Ro}} \right) \underline{\mathbf{V}}^{\perp} \right] = -P,$$

and

$$\partial_t \left(\underline{\omega} \cdot N^{\mu} \right) + \varepsilon \nabla \cdot \left(\left[\underline{\omega} \cdot N^{\mu} + \frac{1}{\mathrm{Ro}} \right] \underline{\mathrm{V}} \right) = 0.$$

Since $\mathrm{U}^{\mu}_{/\!/} = \nabla \psi + \frac{\nabla^{\perp}}{\Delta} \left(\underline{\boldsymbol{\omega}} \cdot N^{\mu} \right)$, we get that

$$\partial_t \mathrm{U}^{\mu}_{/\!/} + \nabla \zeta + \frac{\varepsilon}{2} \nabla \left| \mathrm{U}^{\mu}_{/\!/} \right|^2 - \frac{\varepsilon}{2\mu} \nabla \left[\left(1 + \varepsilon^2 \mu \left|\nabla \zeta\right|^2\right) \underline{\mathbf{w}}^2 \right] + \varepsilon \left(\underline{\boldsymbol{\omega}} \cdot N^{\mu} + \frac{1}{\mathrm{Ro}} \right) \underline{\mathbf{V}}^{\perp} = -\nabla P.$$

Then, using Proposition 4.5.3 and plugging the fact that $\mathrm{U}^{\mu}_{/\!/} = \overline{\mathbf{V}} - \sqrt{\mu} \mathbf{Q} + \mu \mathbf{R}$, we get

$$\begin{aligned} \partial_t \overline{\mathbf{V}} + \varepsilon \left(\overline{\mathbf{V}} \cdot \nabla \right) \overline{\mathbf{V}} + \nabla \zeta + \frac{\varepsilon}{\mathrm{Ro}} \overline{\mathbf{V}}^{\perp} + \nabla P - \sqrt{\mu} \Big(\partial_t \mathbf{Q} \\ + \varepsilon \left(\overline{\mathbf{V}} \cdot \nabla \right) \mathbf{Q} + \varepsilon \left(\mathbf{Q} \cdot \nabla \right) \overline{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}} \mathbf{Q}^{\perp} \Big) = -\mu \partial_t \mathbf{R} + \widetilde{\mathbf{R}}, \end{aligned}$$

and using the same idea as Proposition 4.5.3, it is easy to check that

$$\begin{aligned} \left|\left| \widetilde{\mathbf{R}} \circ \Sigma \right|\right|_{H^{2,1}} + \left|\left| \partial_t \mathbf{R} \circ \Sigma \right|\right|_{H^{2,1}} \leq C \left(\frac{1}{\mathrm{h}_{\min}}, \varepsilon \left|\zeta\right|_{H^4}, \varepsilon \left|\partial_t \zeta\right|_{H^4}, \beta \left|b\right|_{L^\infty}, \beta \left|\nabla b\right|_{H^3} \right) \times \\ \left(\left|\left| \mathbf{V} \circ \Sigma \right|\right|_{H^{4,1}} + \left|\left| \partial_t \mathbf{V} \circ \Sigma \right|\right|_{H^{4,1}} \right). \end{aligned}$$

Using Proposition 4.5.5, Theorem 4.3.6, Theorems 4.2.8 and 4.2.11 and Remark 4.2.14, we get the result . □

4.5.4 The Proudman resonance, linear properties of the shallow water equations

In Chapter 2 we studied the Proudman resonance. We recall that it is a linear amplification due to a source term (non constant pressure or moving bottom). The purpose of this part is to study it for the linear shallow water equations

$$\begin{cases} \partial_t \zeta + \nabla \cdot \overline{\mathbf{V}} = 0, \\ \partial_t \overline{\mathbf{V}} + \nabla \zeta + \frac{\varepsilon}{\mathrm{Ro}} \overline{\mathbf{V}}^{\perp} = -\nabla P. \end{cases} \tag{4.73}$$

In the following we take $d = 2$ and we suppose that $\frac{\varepsilon}{\mathrm{Ro}} = 1$ (strong rotation in the sense of [60]). We consider the asymptotic model (in the sense of Definition 4.19 in [80]), for $\delta_0 \leq 1$,

$$\mathcal{A}_{LWW} = \left\{ \left(\varepsilon, \beta, \mu, \frac{\varepsilon}{\mathrm{Ro}} \right), \, 0 < \mu, \, \varepsilon, \, \beta \leq \delta_0, \, \frac{\varepsilon}{\mathrm{Ro}} = 1 \right\}. \tag{4.74}$$

Then we have the following result, which is a small adaptation of Theorem 4.5.6.

Proposition 4.5.8. *Let $d = 2$, $N \geq 6$, $0 \leq \delta_0 \leq 1$, $\varepsilon, \beta, \mu, \mathrm{Ro} \in \mathcal{A}_{LWW}$. We assume that we are under the assumptions of Theorem 4.3.6. Then, we can define the following quantity $\boldsymbol{\omega}_0 = \omega_0 \circ \Sigma^{-1}$, $\boldsymbol{\omega} = \omega \circ \Sigma^{-1}$, $\overline{\mathbf{V}}_0 = \overline{\mathbf{V}}[\varepsilon\zeta_0, \beta b](\psi_0, \boldsymbol{\omega}_0)$, $\overline{\mathbf{V}} = \overline{\mathbf{V}}[\varepsilon\zeta, \beta b](\psi, \boldsymbol{\omega})$, $\mathbf{Q}_0 = \mathbf{Q}[\varepsilon\zeta_0, \beta b](\psi_0, \boldsymbol{\omega}_0)$* and $\mathbf{Q} = \mathbf{Q}[\varepsilon\zeta, \beta b](\psi, \boldsymbol{\omega})$. *Then, there exists a time $T > 0$ such that*

(i) T has the form

$$\frac{1}{T} = c^1.$$

(ii) There exists a unique solution $\left(\zeta_{LSW}, \overline{\mathbf{V}}_{LSW}\right)$ of (4.73) with initial conditions $\left(\zeta_0, \overline{\mathbf{V}}_0\right)$ on $[0, T]$.

(iii) There exists a unique solution (ζ, ψ, ω) of (4.46) with initial conditions $(\zeta_0, \psi_0, \omega_0)$ on $[0, T]$.

(iv) The following error estimates hold, for $0 \leq t \leq T$,

$$\left| \left(\zeta, \overline{\mathbf{V}}\right) - \left(\zeta_{LSW}, \overline{\mathbf{V}}_{LSW}\right) \right|_{L^\infty([0,t] \times \mathbb{R}^2)} \leq \delta_0 \, t c^2,$$

with $c^j = C \left(A, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{L^\infty}, |\nabla b|_{H^{N+1}}, |\nabla P|_{W^{1,\infty}_t H^N_X} \right)$.

We denote in the following $\mathbf{V} = (u, v)^t$. We wonder now if we can catch an elevation of the sea level with the asymptotic model (4.73). We answer to this question in the following proposition.

Proposition 4.5.9. *Let $P \in L^\infty_t W^{2,\infty}_X(\mathbb{R}^2)$, (ζ, u, v) be a solution of (4.73). Then,*

$$|(\zeta, u, v)(t, \cdot)|_{L^\infty} \leq C \ln t,$$

where $C = C \left(\left|(1 + |X|)^2 P\right|_{L^\infty_t H^2_X}, \left|(1 + |X|)^2 \zeta_0\right|_{H^2}, \left|(1 + |X|)^2 u_0\right|_{H^2}, \left|(1 + |X|)^2 v_0\right|_{H^2} \right)$.

Proof. We denote by A the anti-symmetric matrix operator

$$A = \begin{pmatrix} 0 & -\partial_x & -\partial_y \\ -\partial_x & 0 & 1 \\ -\partial_y & -1 & 0 \end{pmatrix}.$$

Using the Duhamel's formula, we have

$$(\zeta, u, v)(t, \cdot) = e^{tA}(\zeta_0, u_0, v_0) + \int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ \nabla P(s, \cdot) \end{pmatrix} ds.$$

Then, we notice that

$$\widehat{e^{tA}}(\xi) = e^{it\sqrt{1+|\xi|^2}} A_1(\xi) + e^{-it\sqrt{1+|\xi|^2}} A_2(\xi) + A_3(\xi),$$

where $A_1, A_2, \nabla A_1, \nabla A_2, \nabla^2 A_1, \nabla^2 A_2 \in L^\infty(\mathbb{R}^2)$ and, if $\xi = (\xi_1, \xi_2)^t$,

$$A_3(\xi) = \frac{1}{1+|\xi|^2} \begin{pmatrix} 1 & i\xi_2 & -i\xi_1 \\ -i\xi_2 & \xi_2^2 & -\xi_1\xi_2 \\ i\xi_1 & -\xi_1\xi_2 & \xi_1^2 \end{pmatrix}.$$

Using the fact that $A_3(\xi) \begin{pmatrix} 0 \\ i\widehat{P}(s,\xi)\xi \end{pmatrix} = 0$, the result follows from the following lemma (see [146] or Corollary 7.2.4 in [69]).

Lemma 4.5.10. *Let* $u_0 \in W^{2,1}(\mathbb{R}^2)$. *Then*

$$\left| \int_{\mathbb{R}} e^{i(x\cdot\xi) \pm t\sqrt{|\xi|^2+1}} u_0(\xi) d\xi \right|_{L^\infty_x} \leq \frac{C}{1+|t|} |u_0|_{W^{2,1}}.$$

□

Hence, we can not expect a resonance from a physical standpoint since the possible amplification is too slow. The dispersive effects due to the Coriolis forcing prevent the Proudman resonance to occur. Notice that our proof is specific to the case $d = 2$. In Section 5.3.2, we study the case $d = 1$.

Chapter 5

Long wave approximation for water waves under a Coriolis forcing

Sommaire

Ce chapitre est basé sur l'article [103]. Nous avons ajouté la sous-section 5.3.2 où nous étudions la résonance de Proudman en présence de la force de Coriolis lorsque $d = 1$. Nous montrons en particulier qu'aucune résonance n'est possible lorsque la tempête se déplace à vitesse fixe.

5.1 Introduction

This paper is devoted to the study of the long wave approximation for water waves under the influence of the gravity and a Coriolis forcing. We start by deriving a generalization of the Boussinesq equations in 1D (in space) and we rigorously justify them as an asymptotic model of the water waves equations. These new Boussinesq equations are not the classical Boussinesq equations. A new term due to the vorticity and the Coriolis forcing appears that can not be neglected. Then, we study the Boussinesq regime and we derive and fully justify different asymptotic models when the bottom is flat : a linear equation linked to the Klein-Gordon equation admitting the so-called Poincaré waves; the Ostrovsky equation, which is a generalization of the KdV equation in presence of a Coriolis forcing, when the rotation is weak; and finally the KdV equation when the rotation is very weak. Therefore, this work provides the first mathematical justification of the Ostrovsky equation. Finally, we derive a generalization of the Green-Naghdi equations in 1D in space for small topography variations and we show that this model is consistent with the water waves equations.

We study the motion of an incompressible, inviscid fluid with a constant density ρ and no surface tension under the influence of the gravity $\boldsymbol{g} = -g\boldsymbol{e_z}$ and the rotation of the Earth with a rotation vector $\mathbf{f} = \frac{f}{2}\boldsymbol{e_z}$. We suppose that the seabed and the surface are graphs above the still water level. The horizontal variable is $X = (x, y) \in \mathbb{R}^2$ and $z \in \mathbb{R}$ is the vertical variable. The water occupies the domain $\Omega_t := \{(X, z) \in \mathbb{R}^3 \ , \ -H + b(X) < z < \zeta(t, X)\}$. The velocity in the fluid domain is denoted $\mathbf{U} = (\mathbf{V}, \mathrm{w})^t$ where $\mathbf{V}$ is the horizontal component of $\mathbf{U}$ and w its vertical component. The equations governing such a fluid are the free surface Euler-Coriolis equations[1]

$$\begin{cases} \partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_{X,z}) \mathbf{U} + \mathbf{f} \times \mathbf{U} = -\dfrac{1}{\rho} \nabla_{X,z} \mathcal{P} - g\boldsymbol{e_z} \text{ in } \Omega_t, \\ \operatorname{div} \mathbf{U} = 0 \text{ in } \Omega_t, \end{cases} \tag{5.1}$$

with the boundary conditions

$$\begin{cases} \mathcal{P}_{|z=\zeta} = P_0, \\ \partial_t \zeta - \underline{\mathrm{U}} \cdot N = 0, \\ \mathrm{U}_b \cdot N_b = 0, \end{cases} \tag{5.2}$$

where P_0 is constant, $N = \begin{pmatrix} -\nabla\zeta \\ 1 \end{pmatrix}$, $N_b = \begin{pmatrix} -\nabla b \\ 1 \end{pmatrix}$, $\underline{\mathrm{U}} = \begin{pmatrix} \underline{\mathrm{V}} \\ \underline{\mathrm{w}} \end{pmatrix} = \mathbf{U}_{|z=\zeta}$ and $\mathrm{U}_b = \begin{pmatrix} \mathrm{V}_b \\ \mathrm{w}_b \end{pmatrix} = \mathbf{U}_{|z=-H+b}$.

Influenced by the works of Zakharov ([153]) and Craig-Sulem-Sulem ([48]), Castro and Lannes in [34] shown that we can express the free surface Euler equations thanks to the unknowns $\left(\zeta, \mathrm{U}_{/\!/}, \boldsymbol{\omega}\right)$[2] where

[1] We consider that the centrifugal potential is constant and included in the pressure term.

[2] In fact, Castro and Lannes used the unknowns $\left(\zeta, \frac{\nabla}{\Delta} \cdot \mathrm{U}_{/\!/}, \boldsymbol{\omega}\right)$. But the unknowns $\left(\zeta, \mathrm{U}_{/\!/}, \boldsymbol{\omega}\right)$ are better to derive shallow water asymptotic models.

$$\mathrm{U}_{/\!\!/} = \underline{\mathbf{V}} + \underline{\mathrm{w}}\nabla\zeta,$$

and they gave a system of three equations on these unknowns. Then, in [101] we proceeded as Castro and Lannes and, taking into account the Coriolis force, we got the following system, called the Castro-Lannes system or the water waves equations,

$$\begin{cases} \partial_t \zeta - \underline{\mathrm{U}} \cdot N = 0, \\ \partial_t \mathrm{U}_{/\!\!/} + \nabla\zeta + \frac{1}{2}\nabla\left|\mathrm{U}_{/\!\!/}\right|^2 - \frac{1}{2}\nabla\left[\left(1+|\nabla\zeta|^2\right)\underline{\mathrm{w}}^2\right] + \underline{\boldsymbol{\omega}}\cdot N\,\underline{\mathbf{V}}^{\perp} + f\underline{\mathbf{V}}^{\perp} = 0, \\ \partial_t \boldsymbol{\omega} + (\mathbf{U}\cdot\nabla_{X,z})\,\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla_{X,z})\,\mathbf{U} + f\partial_z \mathbf{U}, \end{cases} \tag{5.3}$$

where $\mathbf{U} = \begin{pmatrix} \mathbf{V} \\ \mathrm{w} \end{pmatrix} = \mathbf{U}[\zeta, b](\mathrm{U}_{/\!\!/}, \boldsymbol{\omega})$ is the unique solution in $H^1(\Omega_t)$ of

$$\begin{cases} \mathrm{curl}\ \mathbf{U} = \boldsymbol{\omega} \text{ in } \Omega_t, \\ \mathrm{div}\ \mathbf{U} = 0 \text{ in } \Omega_t, \\ \left(\underline{\mathbf{V}} + \underline{\mathrm{w}}\nabla\zeta\right)_{|z=\zeta} = \mathrm{U}_{/\!\!/}, \\ \mathrm{U}_b \cdot N_b = 0, \end{cases} \tag{5.4}$$

and with the following constraint

$$\nabla^{\perp} \cdot \mathrm{U}_{/\!\!/} = \underline{\boldsymbol{\omega}} \cdot N. \tag{5.5}$$

Our principal motivation is the study of the long waves or Boussinesq regime. Hence, we nondimensionalize the previous equations. We have six physical parameters in our problem : the typical amplitude of the surface a, the typical amplitude of the bathymetry a_{bott}, the typical longitudinal scale L_x, the typical transverse scale L_y, the characteristic water depth H and the typical Coriolis frequency f. Then we can introduce five dimensionless parameters

$$\varepsilon = \frac{a}{H},\ \beta = \frac{a_{\mathrm{bott}}}{H},\ \mu = \frac{H^2}{L_x^2},\ \gamma = \frac{L_x}{L_y} \text{ and } \mathrm{Ro} = \frac{a\sqrt{gH}}{HfL_x}.$$

The parameter ε is called the nonlinearity parameter, β is called the bathymetric parameter, μ is called the shallowness parameter, γ is called the transversality parameter and Ro is the Rossby number. Then, we can adimensionalize the Euler equations (5.1) and the Castro-Lannes equations (5.3) (see Part 5.1.1).

We organize our paper in three parts. Section 5.2 is devoted to derive a generalization of the Boussinesq equations in 1D under a Coriolis forcing and to fully justify it. The Boussinesq equations are obtained under the assumption that μ is small, $\varepsilon, \beta = \mathcal{O}(\mu)$ (Boussinesq regime) and by neglecting all the terms of order $\mathcal{O}(\mu^2)$ in the adimensionalized Euler equations or the water waves equations (see for instance [9] in the irrotational framework). It is a system of two equations on the free surface ζ and the vertical average of the horizontal component of the velocity denoted $\overline{\mathbf{V}} = (\overline{u}, \overline{v})^t$ (defined in (5.22)). Our Boussinesq-Coriolis equations are a system of three equations on the surface ζ, the average vertical velocity $\overline{\mathbf{V}}$ and the quantity $\mathbf{V}^{\sharp} = (u^{\sharp}, v^{\sharp})^t$ (defined in (5.29)) which is introduced to catch interactions between the vorticity and the averaged velocity. These equations are the following system

$$
\begin{cases}
\partial_t \zeta + \partial_x \left([1+\varepsilon\zeta - \beta b]\overline{u} \right) = 0, \\
\left(1 - \frac{\mu}{3}\partial_x^2\right) \partial_t \overline{u} + \partial_x \zeta + \varepsilon \overline{u} \partial_x \overline{u} - \frac{\varepsilon}{\text{Ro}} \overline{v} + \frac{\varepsilon}{\text{Ro}} \mu^{\frac{3}{2}} \frac{1}{24} \partial_x^2 \frac{v^\sharp}{h} = 0, \\
\partial_t \overline{v} + \varepsilon \overline{u} \partial_x \overline{v} + \frac{\varepsilon}{\text{Ro}} \overline{u} = 0, \\
\partial_t \frac{\mathbf{V}^\sharp}{h} + \varepsilon \overline{u} \partial_x \frac{\mathbf{V}^\sharp}{h} + \frac{\varepsilon}{\text{Ro}} \frac{\mathbf{V}^\sharp}{h}^{\perp} = 0,
\end{cases}
$$

where $h = 1+\varepsilon\zeta - \beta b$. Then, in Section 5.3 we derive and fully justify different asymptotic models in the Boussinesq regime when the bottom is flat. We first derive in Subsection 5.3.1 a linear system (system (5.38)) linked to the Klein-Gordon equation admitting the so-called Poincaré waves. Then, in Subsection 5.3.3 we study the Ostrovsky equation

$$
\partial_\xi \left(\partial_\tau f + \frac{3}{2} f \partial_\xi f + \frac{1}{6} \partial_\xi^3 f \right) = \frac{1}{2} f.
$$

This equation, derived by Ostrovsky ([113]), is a generalization of the KdV equation in presence of a Coriolis forcing. We offer a rigorous justification of the Ostrovsky approximation under a weak Coriolis forcing, i.e $\frac{\varepsilon}{\text{Ro}} = \mathcal{O}(\sqrt{\mu})$. Notice that this work provides the first mathematical justification of the Ostrovsky equation. In Subsection 5.3.4 we fully justify the KdV approximation (equation (5.52)) when the rotation is very weak, i.e when $\frac{\varepsilon}{\text{Ro}} = \mathcal{O}(\mu)$. Finally, in Section 5.4 we derive a generalization of the Green-Naghdi equations (5.64) in 1D under a Coriolis forcing with small bottom variations and we show that this system is consistent with the water waves equations. The Green-Naghdi equations are originally obtained in the irrotational framework under the assumption that μ is small and by neglecting all the terms of order $\mathcal{O}(\mu^2)$ in the adimensionalized Euler equations or the water waves equations (see for instance [130] or Part 5.1.1.2 in [80] for a derivation in the irrotational framework). They were generalized in [33] in the rotational setting but without a Coriolis forcing. We add one in the paper.

5.1.1 Nondimensionalization and the Castro-Lannes formulation

We recall the five dimensionless parameter

$$
\varepsilon = \frac{a}{H},\ \beta = \frac{a_{\text{bott}}}{H},\ \mu = \frac{H^2}{L_x^2},\ \gamma = \frac{L_x}{L_y} \text{ and } \text{Ro} = \frac{a\sqrt{gH}}{HfL_x}. \tag{5.6}
$$

We nondimensionalize the variables and the unknowns. We introduce (see [80] or [101] for instance for an explanation of this nondimensionalization)

$$
\begin{cases}
x' = \frac{x}{L_x},\ y' = \frac{y}{L_y},\ z' = \frac{z}{H},\ \zeta' = \frac{\zeta}{a},\ b' = \frac{b}{a_{\text{bott}}},\ t' = \frac{\sqrt{gH}}{L_x} t, \\
\mathbf{V}' = \sqrt{\frac{H}{g}} \frac{\mathbf{V}}{a},\ \text{w}' = H\sqrt{\frac{H}{g}} \frac{\text{w}}{aL_x} \text{ and } \mathcal{P}' = \frac{\mathcal{P}}{\rho g H}.
\end{cases} \tag{5.7}
$$

In this paper, we use the following notations

$$
\nabla^\gamma = \nabla^\gamma_{X'} = \begin{pmatrix} \partial_{x'} \\ \gamma \partial_{y'} \end{pmatrix},\ = \nabla^{\mu,\gamma}_{X',z'} = \begin{pmatrix} \sqrt{\mu}\nabla^\gamma_{X'} \\ \partial_{z'} \end{pmatrix},\ \text{curl}^{\mu,\gamma} = \nabla^{\mu,\gamma}_{X',z'} \times,\ \text{div}^{\mu,\gamma} = \nabla^{\mu,\gamma}_{X',z'} \cdot . \tag{5.8}
$$

We also define

$$\mathbf{U}^{\mu} = \begin{pmatrix} \sqrt{\mu}\mathbf{V}' \\ \mathrm{w}' \end{pmatrix}, \; \boldsymbol{\omega}' = \frac{1}{\mu}\mathrm{curl}^{\mu,\gamma}\mathbf{U}^{\mu}, \; \underline{\mathrm{U}}^{\mu} = \mathbf{U}^{\mu}_{|z'=\varepsilon\zeta'}, \; \mathrm{U}^{\mu}_{b} = \mathbf{U}^{\mu}_{|z'=-1+\beta b'}, \tag{5.9}$$

and

$$N^{\mu,\gamma} = \begin{pmatrix} -\varepsilon\sqrt{\mu}\nabla^{\gamma}\zeta' \\ 1 \end{pmatrix}, \; N^{\mu,\gamma}_{b} = \begin{pmatrix} -\beta\sqrt{\mu}\nabla^{\gamma}b' \\ 1 \end{pmatrix}. \tag{5.10}$$

Notice that our nondimensionalization of the vorticity allows us to consider only weakly sheared flows (see [33], [139], [121]). The nondimensionalized fluid domain is

$$\Omega'_{t'} := \{(X',z') \in \mathbb{R}^3 \; , \; -1 + \beta b'(X') < z' < \varepsilon\zeta'(t',X')\}. \tag{5.11}$$

Finally, if $\mathbf{V} = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$, we define $\mathbf{V}$ by $\mathbf{V}^{\perp} = \begin{pmatrix} -v \\ u \end{pmatrix}$. Then, the Euler-Coriolis equations (5.1) become

$$\begin{cases} \partial_{t'}\mathbf{U}^{\mu} + \dfrac{\varepsilon}{\mu}\left(\mathbf{U}^{\mu}\cdot\nabla^{\mu,\gamma}_{X',z'}\right)\mathbf{U}^{\mu} + \dfrac{\varepsilon\sqrt{\mu}}{\mathrm{Ro}}\begin{pmatrix} \mathbf{V}'^{\perp} \\ 0 \end{pmatrix} = -\dfrac{1}{\varepsilon}\nabla^{\mu,\gamma}_{X',z'}\mathcal{P}' - \dfrac{1}{\varepsilon}\boldsymbol{e_z} \text{ in } \Omega'_t, \\ \mathrm{div}^{\mu,\gamma}_{X',z'}\;\mathbf{U}^{\mu} = 0 \text{ in } \Omega'_t, \end{cases} \tag{5.12}$$

with the boundary conditions

$$\begin{cases} \partial_{t'}\zeta' - \dfrac{1}{\mu}\underline{\mathrm{U}}^{\mu}\cdot N^{\mu,\gamma} = 0, \\ \mathrm{U}^{\mu}_{b}\cdot N^{\gamma,\mu}_{b} = 0. \end{cases} \tag{5.13}$$

We can also nondimensionalize the Castro-Lannes formulation. We introduce the quantity

$$\mathrm{U}^{\mu}_{/\!/} = \underline{\mathbf{V}} + \varepsilon\underline{\mathrm{w}}\nabla^{\gamma}\zeta.$$

Then, the Castro-Lannes formulation becomes (see [34] or [101] when $\gamma = 1$),

$$\begin{cases} \partial_t\zeta - \dfrac{1}{\mu}\underline{\mathrm{U}}^{\mu}\cdot N^{\mu,\gamma} = 0, \\ \partial_t\mathrm{U}^{\mu}_{/\!/} + \nabla^{\gamma}\zeta + \dfrac{\varepsilon}{2}\nabla^{\gamma}\left|\mathrm{U}^{\mu}_{/\!/}\right|^2 - \dfrac{\varepsilon}{2\mu}\nabla^{\gamma}\left[\left(1+\varepsilon^2\mu\,|\nabla^{\gamma}\zeta|^2\right)\underline{\mathrm{w}}^2\right] + \varepsilon\underline{\boldsymbol{\omega}}\cdot N^{\mu,\gamma}\,\underline{\mathbf{V}}^{\perp} + \dfrac{\varepsilon}{\mathrm{Ro}}\underline{\mathbf{V}}^{\perp} = 0, \\ \partial_t\boldsymbol{\omega} + \dfrac{\varepsilon}{\mu}\left(\mathbf{U}^{\mu}\cdot\nabla^{\mu,\gamma}_{X,z}\right)\boldsymbol{\omega} = \dfrac{\varepsilon}{\mu}\left(\boldsymbol{\omega}\cdot\nabla^{\mu,\gamma}_{X,z}\right)\mathbf{U}^{\mu} + \dfrac{\varepsilon}{\mu\mathrm{Ro}}\partial_z\mathbf{U}^{\mu}, \end{cases} \tag{5.14}$$

where $\mathbf{U}^{\mu} = \begin{pmatrix} \sqrt{\mu}\mathbf{V} \\ \mathrm{w} \end{pmatrix} = \mathbf{U}^{\mu}[\varepsilon\zeta,\beta b](\mathrm{U}^{\mu}_{/\!/},\boldsymbol{\omega})$ is the unique solution in $H^1(\Omega_t)$ of

$$\begin{cases} \mathrm{curl}^{\mu,\gamma}\;\mathbf{U}^{\mu} = \mu\boldsymbol{\omega} \text{ in } \Omega_t, \\ \mathrm{div}^{\mu,\gamma}\;\mathbf{U}^{\mu} = 0 \text{ in } \Omega_t, \\ (\underline{\mathbf{V}} + \varepsilon\underline{\mathrm{w}}\nabla^{\gamma}\zeta)_{|z=\varepsilon\zeta} = \mathrm{U}^{\mu}_{/\!/}, \\ \mathrm{U}^{\mu}_{b}\cdot N^{\mu,\gamma}_{b} = 0, \end{cases} \tag{5.15}$$

and with the following constraint

$$\nabla^{\perp}\cdot\mathrm{U}^{\mu}_{/\!/} = \underline{\boldsymbol{\omega}}\cdot N^{\mu,\gamma}. \tag{5.16}$$

Remark 5.1.1. *When, $\boldsymbol{\omega} = 0$ and $Ro = +\infty$, we get the irrotational water waves equations (see Remark 2.4 in [34]). In particular in this situation, when $\gamma = 0$ we can check that the velocity $\mathbf{U}^{\mu}$ becomes two dimensional : $\mathbf{U}^{\mu} = \left(\sqrt{\mu}\mathbf{V}_x, 0, \mathrm{w}\right)^t$. This is not the case when $\boldsymbol{\omega} \neq 0$. Even if $\gamma = 0$, the vorticity transfers energy from $\mathbf{V}_x$ to $\mathbf{V}_y$. The only way to get a two dimensional speed is to assume that $\omega = (0, \omega_y, 0)^t$ (see for instance [82]).*

Remark 5.1.2. *Notice that if $\left(\zeta, \mathrm{U}^{\mu}_{/\!/}, \boldsymbol{\omega}\right)$ is a solution of the Castro-Lannes system* (5.14), *$\nabla^{\perp} \cdot \mathrm{U}^{\mu}_{/\!/}$ satisfies the equation*

$$\partial_t \nabla^{\perp} \cdot \mathrm{U}^{\mu}_{/\!/} + \nabla^{\gamma} \cdot \left(\varepsilon \underline{\boldsymbol{\omega}} \cdot N^{\mu,\gamma} \underline{\mathbf{V}}^{\perp} + \frac{\varepsilon}{\mathrm{Ro}} \underline{\mathbf{V}}\right) = 0.$$

Furthermore, by taking the trace of the third equation of the Castro-Lannes system (5.14)*, we can see that $\underline{\boldsymbol{\omega}} \cdot N^{\mu,\gamma}$ satisfies the equation*

$$\partial_t \left(\underline{\boldsymbol{\omega}} \cdot N^{\mu,\gamma}\right) + \nabla^{\gamma} \cdot \left(\varepsilon \underline{\boldsymbol{\omega}} \cdot N^{\mu,\gamma} \underline{\mathbf{V}}^{\perp} + \frac{\varepsilon}{\mathrm{Ro}} \underline{\mathbf{V}}\right) = 0,$$

Hence, the constraint (5.16) *is propagated by the equations.*

We add a technical assumption. We assume that the water depth is bounded from below by a positive constant

$$\exists\, h_{\min} > 0 \ , \ 1 + \varepsilon\zeta - \beta b \geq h_{\min}. \tag{5.17}$$

We also suppose that the dimensionless parameters satisfy

$$\exists \mu_{\max},\ 0 < \mu \leq \mu_{\max},\ 0 < \varepsilon \leq 1,\ 0 \leq \gamma \leq 1,\ 0 \leq \beta \leq 1 \text{ and } \frac{\varepsilon}{\mathrm{Ro}} \leq 1. \tag{5.18}$$

Remark 5.1.3. *We have $\frac{\varepsilon}{Ro} = \frac{fL}{\sqrt{gH}}$. As said in [101], it is quite reasonable to assume that $\frac{\varepsilon}{Ro} \leq 1$ since for water waves, the typical rotation speed due to the Coriolis forcing is less than the typical water wave celerity (see for instance [117], [61], [84]).*

5.1.2 Notations

- If $\mathbf{A} \in \mathbb{R}^3$, we denote by $\mathbf{A}_h$ its horizontal component.

- If $\mathbf{V} = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$, we define the orthogonal of $\mathbf{V}$ by $\mathbf{V}^{\perp} = \begin{pmatrix} -v \\ u \end{pmatrix}$.

- In this paper, $C(\cdot)$ is a nondecreasing and positive function whose exact value has no importance.

- Consider a vector field $\mathbf{A}$ or a function w defined on Ω. Then, we denote $\underline{\mathbf{A}} = \mathbf{A}_{|z=\varepsilon\zeta}$, $\underline{\mathrm{w}} = \mathrm{w}_{|z=\varepsilon\zeta}$ and $\mathbf{A}_b = \mathbf{A}_{|z=-1+\beta b}$, $\mathrm{w}_b = \mathrm{w}_{|z=-1+\beta b}$.

- If $s \in \mathbb{R}$ and f is a function on $\mathbb{R}^2$, $|f|_{H^s}$ is its H^s-norm, $|f|_2$ is its L^2-norm and $|f|_{L^\infty}$ its $L^\infty(\mathbb{R}^2)$-norm.

- The operator $(\ ,\ )_2$ is the L^2-scalar product in $\mathbb{R}^2$.

- If f is a function defined on $\mathbb{R}^2$, we denote ∇f the gradient of f.

- If w is a function defined on Ω, $\nabla_{X,z}\mathrm{w}$ is the gradient of w and $\nabla_X \mathrm{w}$ its horizontal component.

- If $u = u(X, z)$ is defined in Ω, we define

$$\overline{u}(X) = \frac{1}{1+\varepsilon\zeta-\beta b}\int_{-1+\beta b(X)}^{\varepsilon\zeta(X)} u(X,z)dz \text{ and } u^{*} = u - \overline{u}.$$

5.1.3 Useful results

In this paper, we fully justify different asymptotic models of the water waves equations. Then, we have to define the notion of consistence (see for instance [80]).

Definition 5.1.4. *The Castro-Lannes equations* (5.14) *are consistent of order* $\mathcal{O}\left(\mu^k\right)$ *with a system of equations* S *for* ζ *and* $\overline{\mathbf{V}}$ *if for all sufficiently smooth solutions* $\left(\zeta, \mathrm{U}_{/\!/}^{\mu}, \boldsymbol{\omega}\right)$ *of the Castro-Lannes equations* (5.14) *, the pair* $\left(\zeta, \overline{\mathbf{V}}[\varepsilon\zeta,\beta b]\left(\mathrm{U}_{/\!/}^{\mu}, \boldsymbol{\omega}\right)\right)$ (defined in (5.22)) *solves* S *up to a residual of order* $\mathcal{O}\left(\mu^k\right)$.

We also need an existence result for the Castro-Lannes formulation (5.14). This is the purpose of the next theorem (Theorem 4.3.6 in Section 4.3.4). We recall that the existence of the water waves equations is always under the so-called Rayleigh-Taylor condition assuming the positivity of the Rayleigh-Taylor coefficient $\mathfrak{a}$ (see Sections 2.2.5 and 4.3.3 for the link between $\mathfrak{a}$ and the Rayleigh-Taylor condition) where

$$\mathfrak{a} := \mathfrak{a}[\varepsilon\zeta,\beta b](\mathrm{U}_{/\!/}^{\mu}, \boldsymbol{\omega}) = 1 + \varepsilon\left(\partial_t + \varepsilon\underline{\mathbf{V}}[\varepsilon\zeta,\beta b](\mathrm{U}_{/\!/}^{\mu}, \boldsymbol{\omega})\cdot\nabla\right)\underline{\mathrm{w}}[\varepsilon\zeta,\beta b](\mathrm{U}_{/\!/}^{\mu}, \boldsymbol{\omega}). \tag{5.19}$$

Notice also that in Proposition 4.3.4, we show how we can define initially the Rayleigh-Taylor coefficient $\mathfrak{a}$ (see Equation (4.53)).

Theorem 5.1.5. *Let* $A > 0$, $N \geq 5$, $b \in H^{N+2}(\mathbb{R}^2)$. We assume that $\left(\zeta_0, (\mathrm{U}_{/\!/}^{\mu})_0, \boldsymbol{\omega}_0\right) \in H^N(\mathbb{R}^2)\times H^N(\mathbb{R}^2)\times H^{N-1}(\Omega_0)$ *such that* $\nabla^{\mu,\gamma}\cdot\boldsymbol{\omega}_0 = 0$ *and Condition* (5.16) *is satisfied. We suppose that* $(\varepsilon,\beta,\gamma,\mu,\mathrm{Ro})$ *satisfy* (5.18)*. Finally, we assume that*

$$\exists\, h_{\min}, \mathfrak{a}_{\min} > 0 \ , \ \varepsilon\zeta_0 + 1 - \beta b \geq h_{\min} \text{ and } \mathfrak{a}[\varepsilon\zeta_0,\beta b]((\mathrm{U}_{/\!/}^{\mu})_0, \boldsymbol{\omega}_0) \geq \mathfrak{a}_{\min},$$

and

$$|\zeta_0|_{H^N} + \left|(\mathrm{U}_{/\!/}^{\mu})_0\right|_{H^N} + ||\boldsymbol{\omega}_0||_{H^{N-1}} \leq A.$$

Then, there exists $T > 0$ *and a unique classical solution* $\left(\zeta, \mathrm{U}_{/\!/}^{\mu}, \boldsymbol{\omega}\right)$ *to the Castro-Lannes* (5.14) *with initial data* $\left(\zeta_0, (\mathrm{U}_{/\!/}^{\mu})_0, \boldsymbol{\omega}_0\right)$*. Moreover,*

$$T = \frac{T_0}{\max(\varepsilon,\beta,\frac{\varepsilon}{\mathrm{Ro}})} \ , \ \frac{1}{T_0} = c^1 \text{ and } \max_{[0,T]} |\zeta(t)|_{H^N} + \left|\mathrm{U}_{/\!/}^{\mu}(t)\right|_{H^N} + ||\boldsymbol{\omega}(t)||_{H^{N-1}} = c^2,$$

with $c^j = C\left(A, \mu_{\max}, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{H^{N+2}}\right)$.

Thanks to this theorem, we know that the quantities ζ, $\mathrm{U}_{/\!/}^{\mu}$, ω and then $\overline{\mathbf{V}}$ (defined in (5.22)) remain bounded uniformly with respect to the small parameters during the time evolution of the flow, which will be essential to derive rigorously asymptotic models.

5.2 Boussinesq-Coriolis equations when $\gamma = 0$

This part is devoted to the derivation and the full justification of the Boussinesq-Coriolis equations (5.31) under a Coriolis forcing and with $\gamma = 0$. These equations are an order $\mathcal{O}(\mu^2)$ approximation of the water waves equations under the assumption that $\varepsilon, \beta = \mathcal{O}(\mu)$. The corresponding regime is called *long wave regime* or Boussinesq regime. Contrary to [33], whose approach is based on the averaged Euler equations, our derivation is based on the Castro-Lannes equations (5.14). Then, the asymptotic regime is

$$\mathcal{A}_{\text{Bouss}} = \left\{ (\varepsilon, \beta, \gamma, \mu, \text{Ro}), 0 \leq \mu \leq \mu_0, \frac{\varepsilon}{\text{Ro}} \leq 1, \varepsilon = \mathcal{O}(\mu), \beta = \mathcal{O}(\mu), \gamma = 0 \right\}. \tag{5.20}$$

Remark 5.2.1. *In fact, we can relax the assumption $\gamma = 0$ by only assuming that $\gamma = \mathcal{O}(\mu^2)$ since we neglect all the terms of order $\mathcal{O}(\mu^2)$ in the following.*

We introduce the water depth

$$h(t, X) = 1 + \varepsilon\zeta(t, X) - \beta b(X), \tag{5.21}$$

and the averaged horizontal velocity

$$\overline{\mathbf{V}} = \overline{\mathbf{V}}[\varepsilon\zeta, \beta b](\mathrm{U}^\mu_{/\!/}, \boldsymbol{\omega})(t, X) = \frac{1}{h(t, X)} \int_{z=-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \mathbf{V}[\varepsilon\zeta, \beta b](\mathrm{U}^\mu_{/\!/}, \boldsymbol{\omega})(t, X, z) dz. \tag{5.22}$$

More generally, if u is a function defined in Ω, $\overline{u}$ is its average and $u^* = u - \overline{u}$. In the following we denote $\mathbf{V} = (u, v)^t$. As noticed in [34], we have to introduce the "shear" velocity

$$\mathbf{V}_{\text{sh}} = \mathbf{V}_{\text{sh}}[\varepsilon\zeta, \beta b](\mathrm{U}^\mu_{/\!/}, \boldsymbol{\omega})(t, X) = (u_{\text{sh}}, v_{\text{sh}}) = \int_z^{\varepsilon\zeta} \boldsymbol{\omega}_h^\perp \tag{5.23}$$

and its average

$$\mathbf{Q} = \left(\mathrm{Q}_x, \mathrm{Q}_y\right)^t = \overline{\mathbf{V}}_{\text{sh}} = \frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} \int_{z'}^{\varepsilon\zeta} \boldsymbol{\omega}_h^\perp.$$

When $\gamma = 0$, $\mathrm{U}^\mu_{/\!/} = (\underline{u} + \varepsilon\underline{\mathrm{w}}\partial_x\zeta, \underline{v})^t$. Hence in the following, we denote

$$u_{/\!/} = \underline{u} + \varepsilon\underline{\mathrm{w}}\partial_x\zeta. \tag{5.24}$$

In this section, we do the asymptotic expansion with respect to μ of different quantities. In the following, we denote by R a remainder whose exact value has no importance and which is bounded uniformly with respect to μ.

Remark 5.2.2. *Notice that thanks to Theorem 5.1.5, we know that the quantities ζ, $\mathrm{U}^\mu_{/\!/}$, ω , $\overline{\mathbf{V}}$* and $\mathbf{U}$ *remain bounded uniformly with respect to the small parameters during the time evolution of the flow. Furthermore, $\partial_t\zeta$, $\partial_t\mathrm{U}^\mu_{/\!/}$, $\partial_t\omega$* and $\partial_t\mathbf{U}$ *also remain bounded uniformly with respect to the small parameters during this time.*

5.2.1 Asymptotic expansion for the velocity and useful identities

In this part, we give an expansion of the velocity with respect to μ. First we recall the following fact (See Proposition 4.5.2).

Proposition 5.2.3. *If* $\left(\zeta, \mathrm{U}^{\mu}_{/\!/}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (5.14)*, we have*

$$\underline{\mathrm{U}}^{\mu} \cdot N^{\mu,\gamma} = -\mu \nabla^{\gamma} \cdot \left(h\overline{\mathbf{V}}\right).$$

This proposition, coupled with the first equation of (5.14), gives us an equation that links ζ to $\overline{\mathbf{V}}$. In particular, when $\gamma = 0$, we get an equation that links ζ to $\overline{u}$. We need also an expansion of u and v with respect to μ. The following proposition is for v.

Proposition 5.2.4. *If* $\left(\zeta, \mathrm{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (5.14)*, we have*

$$\begin{aligned} v &= \overline{v} + \sqrt{\mu} v^{*}_{\mathrm{sh}}, \\ \underline{v} &= \overline{v} - \sqrt{\mu} \mathrm{Q}_y, \\ \underline{\boldsymbol{\omega}} \cdot N^{\mu,0} &= \partial_x \underline{v} \end{aligned}$$

and

$$\partial_t \underline{v} + \varepsilon \underline{u} \partial_x \underline{v} + \frac{\varepsilon}{\mathrm{Ro}} \underline{u} = 0.$$

Proof. Since $\mathrm{curl}^{\mu,0}\, \mathbf{U}^{\mu} = \mu \boldsymbol{\omega}$, we get that

$$\sqrt{\mu} \boldsymbol{\omega}_x = -\partial_z v \text{ and } \boldsymbol{\omega}_z = \partial_x v. \tag{5.25}$$

Then, plugging the ansatz $v = \overline{v} + \sqrt{\mu} v_1$ in the first equation and using the fact that the average of v_1 is equal to 0 we get

$$\underline{v} = \overline{v} - \sqrt{\mu} \frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} \int_{z'}^{\varepsilon\zeta} \boldsymbol{\omega}_x.$$

Furthermore, from the equation on the second component of $\mathrm{U}^{\mu,0}_{/\!/}$, we have

$$\partial_t \underline{v} + \varepsilon \boldsymbol{\omega} \cdot N^{\mu,0} \underline{u} + \frac{\varepsilon}{\mathrm{Ro}} \underline{u} = 0.$$

Then, using the second equation of (5.25), we get that $\underline{\boldsymbol{\omega}} \cdot N^{\mu,0} = \partial_x \underline{v}$ and the result follows. □

The expansion of u is more complex and also involves an expansion of w. It is the purpose of the following proposition. But before, we also have to introduce the following operators

$$T\left[\varepsilon\zeta, \beta b\right] f = \int_{z}^{\varepsilon\zeta} \partial_x^2 \int_{-1+\beta b}^{z'} f \text{ and } T^{*}\left[\varepsilon\zeta, \beta b\right] f = \left(T\left[\varepsilon\zeta, \beta b\right] f\right)^{*},$$

When no confusion is possible, we denote $T = T\left[\varepsilon\zeta, \beta b\right]$ and $T^{*} = T^{*}\left[\varepsilon\zeta, \beta b\right]$.

Proposition 5.2.5. *If* $\left(\zeta, \mathrm{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (5.14)*, we have*

$$\begin{aligned} u &= \overline{u} + \sqrt{\mu} u^{*}_{\mathrm{sh}} + \mu T^{*} \overline{u} + \mu^{\frac{3}{2}} T^{*} u^{*}_{\mathrm{sh}} + \mu^2 R, \\ \underline{u} &= \overline{u} - \sqrt{\mu} \mathrm{Q}_x + \mu \underline{T^{*} \overline{u}} - \mu^{\frac{3}{2}} \overline{T u^{*}_{\mathrm{sh}}} + \mu^2 R, \end{aligned}$$

where $T^*\overline{u} = -\frac{1}{2}\left([z+1-\beta b]^2 - \frac{h^2}{3}\right)\partial_x^2\overline{u} + \beta R$*. We also have*

$$\begin{aligned}
\mathrm{w} &= -\mu\partial_x\left(\int_{-1+\beta b}^{z} u\right),\\
\underline{\mathrm{w}} &= -\mu h\partial_x\overline{u} - \mu^{\frac{3}{2}}\partial_x h \mathrm{Q}_x + \max(\mu^2,\beta\mu)R,
\end{aligned}$$

and

$$u_{/\!/} = \overline{u} - \sqrt{\mu}\mathrm{Q}_x - \mu\frac{1}{3h}\partial_x\left(h^3\partial_x\overline{u}\right) - \mu^{\frac{3}{2}}\left(\overline{Tu^*_{\mathrm{sh}}} + \mathrm{Q}_x\left(\partial_x h\right)^2\right) + \max(\mu^2,\beta\mu)R.$$

Proof. This proof is a small adaptation of part 2.2 in [33] and Part 4.2 in [101]. We recall the main steps. Using the fact that the velocity is divergence free and Proposition 5.2.3, we get

$$\mathrm{w} = -\mu\partial_x\left(\int_{-1+\beta b}^{z} u\right).$$

Furthermore, since $\mathrm{curl}^{\mu,0}\ \mathbf{U}^\mu = \mu\boldsymbol{\omega}$, we get that

$$\sqrt{\mu}\boldsymbol{\omega}_y = \partial_z u - \partial_x \mathrm{w}.$$

Then, plugging the ansatz $u = \overline{u} + \sqrt{\mu}u_1$ and using the fact that the average of u_1 is zero, we get

$$u_1 = -\left(\int_z^{\varepsilon\zeta}\boldsymbol{\omega}_y\right)^* - \frac{1}{\sqrt{\mu}}\left(\int_z^{\varepsilon\zeta}\partial_x\mathrm{w}\right)^*$$

and

$$u = \overline{u} + \sqrt{\mu}u^*_{\mathrm{sh}} + \mu T^* u. \tag{5.26}$$

Then, the expansion for u follows by applying $1+\mu T^*$ to the previous equation. Notice that $\underline{T^*u} = -\overline{Tu}$. The computation of $T^*\overline{u}$ follows from the fact that $\overline{u}$ does not depend on z. Finally, the expansion $\underline{\mathrm{w}}$ and $u_{/\!/}$ is the direct consequence for Proposition 5.2.3 and the expansion of u.
□

Thanks to the previous proposition, we can also get an expansion of $\partial_t u$ and $\partial_t \mathrm{w}$.

Proposition 5.2.6. *If* $\left(\zeta, \mathrm{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (5.14)*, we have*

$$\begin{aligned}
&\partial_t\left(u - \overline{u} - \sqrt{\mu}u^*_{\mathrm{sh}} - \mu T^*\overline{u} - \mu^{\frac{3}{2}}T^*u^*_{\mathrm{sh}}\right) = \mu^2 R,\\
&\partial_t\left(\underline{u} - \overline{u} + \sqrt{\mu}\mathrm{Q}_x - \mu\underline{T^*\overline{u}} + \mu^{\frac{3}{2}}\overline{Tu^*_{\mathrm{sh}}}\right) = \mu^2 R,\\
&\partial_t\left(\underline{\mathrm{w}} + \mu h\partial_x\overline{u} + \mu^{\frac{3}{2}}\partial_x h\mathrm{Q}_x\right) = \max(\mu^2,\beta\mu)R.
\end{aligned} \tag{5.27}$$

Proof. From Equality (5.26) we get that

$$u = (1-\mu T^*)\left(\overline{u} + \sqrt{\mu}u^*_{\mathrm{sh}}\right) + \mu^2 T^*T^*u. \tag{5.28}$$

Hence the first and the second equations follows from Remark 5.2.2. For the third equation, we get the result thanks to Proposition (5.2.3) and Remark 5.2.2. □

As [33] noticed, we can not express $\overline{Tu^*_{\text{sh}}}$ in terms of ζ and $\overline{\mathbf{V}}$. Then, we have to introduce

$$\begin{aligned}\mathbf{V}^\sharp = (u^\sharp, v^\sharp)^t &= -\frac{24}{h^3}\int_{-1+\beta b}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta}\int_{-1+\beta b}^{z}(u^*_{\text{sh}}, v^*_{\text{sh}})^t\,,\\ &= \frac{12}{h^3}\int_{-1+\beta b}^{\varepsilon\zeta}(1+z-\beta b)^2\,(u^*_{\text{sh}}, v^*_{\text{sh}})^t\,.\end{aligned} \tag{5.29}$$

Notice that the previous equality follows from a double integration by parts. We have the following Lemma.

Lemma 5.2.7. *We have the following equalities*

$$\begin{aligned}\overline{Tu^*_{sh}} &= -\left(\varepsilon\partial_x\zeta\right)^2 Q_x + \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\partial_x\int_z^{\varepsilon\zeta}\partial_x\int_{-1+\beta b}^{z}u^*_{sh}\\ &= -\left(\partial_x h\right)^2 Q_x - \frac{1}{24h}\partial_x^2\left(h^3 u^\sharp\right) + \beta R.\end{aligned}$$

Proof. We have

$$\partial_x\int_z^{\varepsilon\zeta}\partial_x\int_{-1+\beta b}^{z}u^*_{\text{sh}} = \int_z^{\varepsilon\zeta}\partial_x^2\int_{-1+\beta b}^{z}u^*_{\text{sh}} + \varepsilon\partial_x\zeta\underline{\partial_x\int_{-1+\beta b}^{z}u^*_{\text{sh}}} \tag{5.30}$$

and the first equality follows from the fact that the average of u^*_{sh} is zero and that $\underline{u^*_{\text{sh}}} = -\mathrm{Q}_x$. The second equality follows from the same arguments. □

In the following section, we give equations for Q_x, Q_y $\mathbf{V}^\sharp$ since we can not express these quantities with respect to ζ and $\overline{\mathbf{V}}$. These equations are essential to derive the Boussinesq-Coriolis equations.

5.2.2 Equations for Q_x, Q_y and $\mathbf{V}^\sharp$

In this part we give the equations satisfied by Q_x and Q_y at order $\mathcal{O}\left(\mu^{\frac{3}{2}}\right)$. The computations are similar to Part 5.4.1 in [33]. We start by Q_x.

Proposition 5.2.8. *If* $\left(\zeta, \mathrm{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (5.14)*, then, in the Boussinesq regime* (5.20)*,* Q_x *satisfies the following equation*

$$\partial_t Q_x + \varepsilon\overline{u}\partial_x Q_x + \varepsilon Q_x\partial_x\overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\left(\underline{v} - \overline{v}\right) = \mu^{\frac{3}{2}}R,$$

and u^*_{sh} *satisfies the equation*

$$\partial_t u^*_{\text{sh}} + \varepsilon\overline{u}\partial_x u^*_{\text{sh}} + \varepsilon u^*_{\text{sh}}\partial_x\overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\left(\overline{v} - v\right) = \mu^{\frac{3}{2}}R.$$

Proof. Using the second equation of the vorticity equation of the Castro-Lannes system (5.14), we have

$$\partial_t\boldsymbol{\omega}_y + \varepsilon u\partial_x\boldsymbol{\omega}_y + \frac{\varepsilon}{\mu}\mathrm{w}\partial_z\boldsymbol{\omega}_y = \varepsilon\boldsymbol{\omega}_x\partial_x v + \frac{\varepsilon}{\sqrt{\mu}}\boldsymbol{\omega}_z\partial_z v + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\partial_z v.$$

Since $\boldsymbol{\omega}_x = -\frac{1}{\sqrt{\mu}}\partial_z v$ and $\boldsymbol{\omega}_z = \partial_x v$ we notice that $\varepsilon\boldsymbol{\omega}_x\partial_x v + \frac{\varepsilon}{\sqrt{\mu}}\boldsymbol{\omega}_z\partial_z v = 0$. Using Proposition 5.2.5 we get

$$\partial_t \boldsymbol{\omega}_y + \varepsilon \overline{u} \partial_x \boldsymbol{\omega}_y - \varepsilon \partial_x \left[(1+z-\beta b) \overline{u} \right] \partial_z \boldsymbol{\omega}_y - \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \partial_z v = \mu^{\frac{3}{2}} R,$$

Then, integrating with respect to z, using the fact that $\partial_t \zeta + \partial_x (h\overline{u}) = 0$ and $u_{\mathrm{sh}} = -\int_z^{\varepsilon\zeta} \boldsymbol{\omega}_y$, we get

$$\partial_t u_{\mathrm{sh}} + \varepsilon \overline{u} \partial_x u_{\mathrm{sh}} + \varepsilon u_{\mathrm{sh}} \partial_x \overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} (\underline{v} - v) = \varepsilon \partial_x \left[(1+z-\beta b) \overline{u} \right] \partial_z u_{\mathrm{sh}} + \mu^{\frac{3}{2}} R.$$

Integrating again with respect to z, using the fact that $\partial_t \zeta + \partial_x (h\overline{u}) = 0$ and $Q_x = \overline{u_{\mathrm{sh}}^*}$, we obtain

$$\partial_t Q_x + \varepsilon \overline{u} \partial_x Q_x + \varepsilon Q_x \partial_x \overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} (\underline{v} - \overline{v}) = \mu^{\frac{3}{2}} R.$$

□

We have a similar equation for Q_y.

Proposition 5.2.9. *If* $\left(\zeta, \mathrm{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (5.14)*, then, in the Boussinesq regime* (5.20)*,* Q_x *satisfies the following equation*

$$\partial_t \mathrm{Q}_y + \varepsilon \overline{u} \partial_x \mathrm{Q}_y + \varepsilon \mathrm{Q}_x \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} (\overline{u} - \underline{u}) = \mu^{\frac{3}{2}} R$$

and v_{sh}^* *satisfies the equation*

$$\partial_t v_{\mathrm{sh}}^* + \varepsilon \overline{u} \partial_x v_{\mathrm{sh}}^* + \varepsilon u_{\mathrm{sh}}^* \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} (u - \overline{u}) = \mu^{\frac{3}{2}} R.$$

Proof. Using the first equation of the vorticity equation of the Castro-Lannes system (5.14), we have

$$\partial_t \boldsymbol{\omega}_x + \varepsilon u \partial_x \boldsymbol{\omega}_x + \frac{\varepsilon}{\mu} \mathrm{w} \partial_z \boldsymbol{\omega}_x = \varepsilon \boldsymbol{\omega}_x \partial_x u + \frac{\varepsilon}{\sqrt{\mu}} \boldsymbol{\omega}_z \partial_z u + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \partial_z u.$$

Then, using the fact that $\nabla^{\mu,0} \cdot \boldsymbol{\omega} = 0$ and $\nabla^{\mu,0} \cdot \mathrm{U}^{\mu,\gamma} = 0$, we get

$$\partial_t \boldsymbol{\omega}_x - \frac{\varepsilon}{\sqrt{\mu}} \partial_z (u \boldsymbol{\omega}_z) + \frac{\varepsilon}{\mu} \partial_z (\mathrm{w} \boldsymbol{\omega}_x) = \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \partial_z u.$$

then, we integrate with respect to z and, using the fact that $\partial_t \zeta - \frac{1}{\mu} \underline{\mathrm{U}}^{\mu} \cdot N^{\mu,0} = 0$, $\boldsymbol{\omega}_x = -\frac{1}{\sqrt{\mu}} \partial_z v$ and $\boldsymbol{\omega}_z = \partial_x v$, we obtain

$$\partial_t \left(\int_{-1+\beta b}^{\varepsilon\zeta} \boldsymbol{\omega}_x \right) - \frac{\varepsilon}{\sqrt{\mu}} \underline{u} \partial_x \underline{v} + \frac{\varepsilon}{\sqrt{\mu}} u \partial_x v + \frac{\varepsilon}{\mu^{\frac{3}{2}}} \mathrm{w} \partial_z v + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} (u - \underline{u}) = 0.$$

Then, we integrate again with respect to z and, using Proposition 5.2.4 and the fact that $\partial_t \zeta - \frac{1}{\mu} \underline{\mathrm{U}}^{\mu} \cdot N^{\mu,0} = 0$, $\mathrm{U}_b^{\mu} \cdot N_b^{\mu,0} = 0$, and $\nabla^{\mu,0} \cdot \mathrm{U}^{\mu} = 0$, we get

$$\partial_t \mathrm{Q}_y - \frac{\varepsilon}{\sqrt{\mu}} \underline{u} \partial_x \underline{v} + \frac{\varepsilon}{\sqrt{\mu}} \frac{1}{h} \partial_x \left(\int_{-1+\beta b}^{\varepsilon\zeta} uv \right) + \frac{1}{\sqrt{\mu} h} \partial_t h \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} (\overline{u} - \underline{u}) = 0.$$

Then, thanks to Propositions 5.2.3, 5.2.4 and 5.2.5 we finally obtain that

$$\partial_t \mathrm{Q}_y + \varepsilon \overline{u} \partial_x \mathrm{Q}_y + \varepsilon \mathrm{Q}_x \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} (\overline{u} - \underline{u}) = \mu^{\frac{3}{2}} R.$$

□

Notice that we give in Subsection 5.4.1 a generalization of the two previous propositions to the fully nonlinear Green-Naghdi regime. Furthermore, in the following proposition we give an equation for $\mathbf{V}^\sharp$ up to terms of order $\mathcal{O}\left(\sqrt{\mu}\right)$.

Proposition 5.2.10. *If* $\left(\zeta, \mathrm{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (5.14)*, then* $\mathbf{V}^\sharp$ *satisfies the following equation*

$$\partial_t \mathbf{V}^\sharp + \varepsilon \mathbf{V}^\sharp \partial_x \overline{u} + \varepsilon \overline{u} \partial_x \mathbf{V}^\sharp + \frac{\varepsilon}{\mathrm{Ro}} \mathbf{V}^{\sharp\perp} = \max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right) \sqrt{\mu} R.$$

Proof. The proof is similar to the computation in Part 4.4 in [33]. After multiplying by $(1 + z - \beta b)^2$ and integrating with respect to z the second equations of Propositions 5.2.8 and 5.2.9, we neglect all the term of order $\mathcal{O}(\sqrt{\mu})$. Then, using the fact that $\partial_t \zeta + \partial_x (h\overline{u}) = 0$ and $\mathbf{V} - \overline{\mathbf{V}} = \sqrt{\mu} \mathbf{V}^*_{\mathrm{sh}} + \mu R$, we get the result. □

5.2.3 The Boussinesq-Coriolis equations

We can now establish the Boussinesq-Coriolis equations when $d = 1$. The Boussinesq-Coriolis equations are the following system

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{u}) = 0, \\ \left(1 - \frac{\mu}{3}\partial_x^2\right) \partial_t \overline{u} + \partial_x \zeta + \varepsilon \overline{u} \partial_x \overline{u} - \frac{\varepsilon}{\mathrm{Ro}} \overline{v} + \frac{\varepsilon}{\mathrm{Ro}} \mu^{\frac{3}{2}} \frac{1}{24} \partial_x^2 \frac{v^\sharp}{h} = 0, \\ \partial_t \overline{v} + \varepsilon \overline{u} \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}} \overline{u} = 0, \\ \partial_t \mathbf{V}^\sharp + \varepsilon \mathbf{V}^\sharp \partial_x \overline{u} + \varepsilon \overline{u} \partial_x \mathbf{V}^\sharp + \frac{\varepsilon}{\mathrm{Ro}} \mathbf{V}^{\sharp\perp} = 0, \end{cases} \tag{5.31}$$

where $\mathbf{V}^\sharp$ is defined in (5.29). We can show that the Boussinesq-Coriolis equations are an order $\mathcal{O}(\mu^2)$ approximation of the water waves equations.

Remark 5.2.11. *Inspired by [82], we can renormalize* $\mathbf{V}^\sharp$ *by* h *and, using the first equation of* (5.31)*, we get the following equation*

$$\partial_t \left(\frac{\mathbf{V}^\sharp}{h}\right) + \varepsilon \overline{u} \partial_x \left(\frac{\mathbf{V}^\sharp}{h}\right) + \frac{\varepsilon}{\mathrm{Ro}} \left(\frac{\mathbf{V}^\sharp}{h}\right)^\perp = 0.$$

This remark will be useful for the local existence (Proposition 5.2.15).

Proposition 5.2.12. *In the Boussinesq regime* $\mathcal{A}_{Bouss}$ (5.20)*, the Castro-Lannes equations* (5.14) *are consistent at order* $\mathcal{O}(\mu^2)$ *with the Boussinesq-Coriolis equations* (5.31) *in the sense of Definition 5.1.4.*

Proof. The first equation of the Boussinesq-Coriolis equations is always satisfied for a solution of the Castro-Lannes formulation by Proposition 5.2.3. For the second equation, we use Proposition 5.2.5, Proposition 5.2.8 together with Proposition 5.2.6, Lemma 5.2.7 and Proposition 5.2.10 (we recall that $\varepsilon = \mathcal{O}(\mu)$). Notice the fact that all the terms with Q_x disappear. We also use the fact that

$$h^3 v^\sharp = \frac{v^\sharp}{h} + \mu R.$$

Then, the third equation follows from Proposition 5.2.5, 5.2.5 and 5.2.9 (all the terms with Q_y disappear also). □

We notice that contrary to the classical Boussinesq equations, we have a new term due to the vorticity that we can not neglect in presence of a Coriolis forcing. In our knowledge, this term was not highlighted before in the literature.

Remark 5.2.13. *In the Boussinesq-Coriolis system we could simplify the term* $\partial_x^2 \frac{v^\sharp}{h}$ *by* $\partial_x^2 v^\sharp$ *since these terms are equal up to a remainder of order* $\mathcal{O}(\mu)$*. However, the term* $\partial_x^2 \frac{v^\sharp}{h}$ *will be essential for the local existence* (*see Remark 5.2.16*)*.*

Remark 5.2.14. *If we assume that* $\frac{\varepsilon}{Ro} = \mathcal{O}\left(\sqrt{\mu}\right)$*, we can neglect the term with* $v^\sharp$ *in the second equation of* (5.31) *and we obtain*

$$\begin{cases} \partial_t \zeta + \partial_x \left(h \overline{u}\right) = 0, \\ \left(1 - \frac{\mu}{3}\partial_x^2\right) \partial_t \overline{u} + \partial_x \zeta + \varepsilon \overline{u} \partial_x \overline{u} - \frac{\varepsilon}{\mathrm{Ro}} \overline{v} = 0, \\ \partial_t \overline{v} + \varepsilon \overline{u} \partial_x \overline{v} + \frac{\varepsilon}{\mathrm{Ro}} \overline{u} = 0. \end{cases} \tag{5.32}$$

This system is the classical Boussinesq equations with a standard Coriolis forcing. It is consistent of order $\mathcal{O}(\mu^2)$ *with the Boussinesq-Coriolis equations* (5.31)*. We use this system in Subsections 5.3.3 and 5.3.4.*

5.2.4 Full justification of the Boussinesq-Coriolis equations

In this part, we fully justify the Boussinesq-Coriolis equations (5.31). In the following we denote by u the quantity $\overline{u}$ and by v the quantity $\overline{v}$. We show that the Boussinesq-Coriolis equations are wellposed. We define the energy space

$$X^s(\mathbb{R}) = H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times H^{s+1}(\mathbb{R}), \tag{5.33}$$

endowed with the norm

$$\left|(\zeta, u, v, \mathbf{W})\right|^2_{X^s_\mu} = |\zeta|^2_{H^s} + |u|^2_{H^s} + \mu \left|\partial_x u\right|^2_{H^s} + |v|^2_{H^s} + |\mathbf{W}|^2_{H^s} + \mu \left|\partial_x \mathbf{W}\right|^2_{H^s}. \tag{5.34}$$

Proposition 5.2.15. *Let* $A > 0$*,* $s > \frac{1}{2} + 1$*,* $\left(\zeta_0, u_0, v_0, \mathbf{V}_0^\sharp\right) \in X^s(\mathbb{R})$ *and* $b \in H^{s+1}(\mathbb{R})$*. We suppose that* $(\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}) \in \mathcal{A}_{Bouss}$ *. We assume that*

$$\exists\, h_{\min} > 0 \ , \ \varepsilon\zeta_0 + 1 - \beta b \geq h_{\min}$$

and

$$\left|\left(\zeta_0, u_0, v_0, \frac{\mathbf{V}_0^\sharp}{1 + \varepsilon\zeta_0 - \beta b}\right)\right|_{X^s_\mu} + |b|_{H^{s+1}} \leq A.$$

Then, there exists an existence time $T > 0$ and a unique solution $\left(\zeta, u, v, \mathbf{V}^\sharp\right)$ on $[0,T]$ to the Boussinesq-Coriolis equations (5.31) *with initial data $\left(\zeta_0, u_0, v_0, \mathbf{V}_0^\sharp\right)$ such that we have $\left(\zeta, u, v, \frac{\mathbf{V}^\sharp}{h}\right) \in \mathcal{C}\left([0,T]; X^s(\mathbb{R})\right)$ with $h = 1 + \varepsilon\zeta - \beta b$. Moreover,*

$$T = \frac{T_0}{\max(\mu, \frac{\varepsilon}{\mathrm{Ro}}\sqrt{\mu})} \ , \ \frac{1}{T_0} = c^1 \text{ and } \max_{[0,T]} \left| \left(\zeta, u, v, \frac{\mathbf{V}^\sharp}{h}\right)(t, \cdot) \right|_{X^s_\mu} = c^2,$$

with $c^j = C\left(A, \mu_{\max}, \frac{1}{h_{\min}}\right)$.

Proof. We only give the energy estimates. For the existence see for instance the proof of Theorem 1 in [72]. We assume that $\left(\zeta, u, v, \mathbf{V}^\sharp\right)$ solves (5.31) on $\left[0, \frac{T_0}{\max(\mu, \frac{\varepsilon}{\mathrm{Ro}}\sqrt{\mu})}\right]$ and that

$$1 + \varepsilon\zeta - \beta b \geq \frac{h_{\min}}{2} \text{ on } \left[0, \frac{T_0}{\max(\mu, \frac{\varepsilon}{\mathrm{Ro}}\sqrt{\mu})}\right].$$

We denote $U = (\zeta, u, v)^t$ and we focus first on the first three equations. This part is a small adaptation of the proof of Theorem 1 in [72]. The the first three equations of the Boussinesq-Coriolis equations can be symmetrized, as an hyperbolic system, by multiplying the second and the third equations by $h = 1 + \varepsilon\zeta - \beta b$. Then, we obtain the following system

$$\mathcal{A}_0(U)\partial_t U + \mathcal{A}_1(U)\partial_x U + B_1 U + \frac{\varepsilon}{\mathrm{Ro}} B_2(U) U = \frac{\varepsilon}{\mathrm{Ro}} \mu^{\frac{3}{2}} F(h, v^\sharp),$$

where

$$\mathcal{A}_0(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & h - \mu\frac{h}{3}\partial_x^2 & 0 \\ 0 & 0 & h \end{pmatrix}, \ \mathcal{A}_1(U) = \begin{pmatrix} \varepsilon u & h & h \\ h & \varepsilon h u & 0 \\ h & 0 & \varepsilon h u \end{pmatrix}$$

and

$$B_1 = \begin{pmatrix} 0 & -\beta\partial_x b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \ B_2(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -h \\ 0 & h & 0 \end{pmatrix} \text{ and } F(h, v^\sharp) = \begin{pmatrix} 0 \\ -\frac{h}{24}\partial_x^2 \frac{v^\sharp}{h} \\ 0 \end{pmatrix}.$$

Then we remark that $\mathcal{A}_1$ is symmetric and there exists $c_1, c_2 = C\left(\frac{1}{h_{\min}}, |h|_{L^\infty}\right)$ such that

$$c_1 \left|\partial_x f\right|_2^2 \leq \left(-\frac{1}{3}\partial_x\left(h\partial_x f\right), f\right)_2 \leq c_2 \left|\partial_x f\right|_2^2.$$

Hence we introduce the symmetric matrix operator

$$S(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & h - \frac{\mu}{3}\partial_x\left(h\partial_x \cdot\right) & 0 \\ 0 & 0 & h \end{pmatrix}$$

and the energy associated

$$\mathcal{E}^s(U) = \left(S(U)\Lambda^s U, \Lambda^s U\right)_2.$$

Then, we see that

$$\left(\Lambda^s B_2(U)U, \Lambda^s U\right)_2 = 0$$

and by standard product estimates we get

$$\mu^{\frac{3}{2}} \left| \left(h\Lambda^s \partial_x^2 \frac{v^\sharp}{h}, \Lambda^s u \right)_2 \right| \leq \sqrt{\mu} C(\mathcal{E}^s(U), |b|_{H^{s+1}}) \sqrt{\mu} \left| \frac{v^\sharp}{h} \right|_{H^{s+1}}. \tag{5.35}$$

Furthermore, notice that

$$\begin{aligned}
\mu \left|\partial_t \partial_x u\right|_{H^s} &= \mu \left| \left(1 - \frac{\mu}{3}\partial_x^2\right)^{-1} \partial_x \left(\partial_x \zeta + \varepsilon u \partial_x u - \frac{\varepsilon}{\mathrm{Ro}} v + \frac{\varepsilon}{\mathrm{Ro}} \frac{\mu^{\frac{3}{2}}}{24} \partial_x^2 \frac{v^\sharp}{h} \right) \right|_{H^s}, \\
&\leq C\left(\mu_{\max}, \mathcal{E}^s(U), \sqrt{\mu} \left| \partial_x \frac{v^\sharp}{h} \right|_{H^s} \right).
\end{aligned}$$

and therefore

$$\left(\frac{\mu}{3} \partial_x h \Lambda^s \partial_x \partial_t u, \Lambda^s u \right)_2 \leq \mu C \left(\mathcal{E}^s(U), |b|_{H^{s+1}}, \sqrt{\mu} \left| \partial_x \frac{v^\sharp}{h} \right|_{H^s} \right).$$

Gathering all the previous estimate and proceeding as in [72] we obtain

$$\frac{d}{dt} \mathcal{E}^s(U) \leq \max\left(\mu, \frac{\varepsilon}{\mathrm{Ro}} \sqrt{\mu} \right)_2 C \left(\mathcal{E}^s(U), |b|_{H^{s+1}}, \left| \frac{v^\sharp}{h} \right|_{H^s}, \sqrt{\mu} \left| \partial_x \frac{v^\sharp}{h} \right|_{H^s} \right).$$

Furthermore, using Remark 5.2.11 and the Kato-Ponce estimate, we get

$$\begin{aligned}
&\frac{d}{dt} \left| \frac{\mathbf{V}^\sharp}{h} \right|_{H^s}^2 \leq \mu C \left|u\right|_{H^s} \left| \frac{\mathbf{V}^\sharp}{h} \right|_{H^s}^2, \\
&\frac{d}{dt} \mu \left| \partial_x \frac{\mathbf{V}^\sharp}{h} \right|_{H^s}^2 \leq \mu C \left(\sqrt{\mu} \left|\partial_x u\right|_{H^s} \left| \frac{\mathbf{V}^\sharp}{h} \right|_{H^s}^2 + |u|_{H^s} \sqrt{\mu} \left| \partial_x \frac{\mathbf{V}^\sharp}{h} \right|_{H^s} \right) \sqrt{\mu} \left| \partial_x \frac{\mathbf{V}^\sharp}{h} \right|_{H^s}.
\end{aligned}$$

Then, the result follows. □

Remark 5.2.16. *Notice that the previous energy estimates do not imply that $\mathbf{V}^\sharp \in H^{s+1}(\mathbb{R})$. Hence, it is essential that in Inequality* (5.35) *we have the term $\partial_x^2 \frac{v^\sharp}{h}$ and not simply $\partial_x^2 v^\sharp$ (see Remark 5.2.13).*

Then, we similarly can prove a local wellposedness result for System (5.32).

Corollary 5.2.17. *Let $A > 0$, $s > \frac{1}{2} + 1$, $(\zeta_0, u_0, v_0) \in H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})$ and $b \in H^{s+1}(\mathbb{R})$. We suppose that $(\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}) \in \mathcal{A}_{Bouss}$. We assume that*

$$\exists\, h_{\min} > 0 \;,\; \varepsilon\zeta_0 + 1 - \beta b \geq h_{\min}$$

and

$$|\zeta_0|_{H^s} + |u_0|_{H^s} + \sqrt{\mu}\,|\partial_x u_0|_{H^s} + |v_0|_{H^s} + |b|_{H^{s+1}} \leq A.$$

Then, there exists an existence time $T > 0$ and a unique solution to the Boussinesq-Coriolis equations (5.31) $(\zeta, u, v) \in \mathcal{C}\left([0,T]; H^s(\mathbb{R}) \times H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\right)$ *with initial data* (ζ_0, u_0, v_0)*. Moreover,*

$$T = \frac{T_0}{\mu}\ ,\ \frac{1}{T_0} = c^1 \text{ and } \max_{[0,T]} |\zeta(t,\cdot)|_{H^s} + |u(t,\cdot)|_{H^s} + \sqrt{\mu}\,|\partial_x u(t,\cdot)|_{H^s} + |v(t,\cdot)|_{H^s} = c^2,$$

with $c^j = C\left(A, \mu_{\max}, \frac{1}{h_{\min}}\right)$.

Furthermore, we have a stability result for the Boussinesq-Coriolis system (5.31).

Proposition 5.2.18. *Let the assumptions of Proposition 5.2.15 satisfied. Suppose that there exists* $\left(\tilde{\zeta}, \tilde{u}, \tilde{v}, \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right) \in \mathcal{C}\left(\left[0, \frac{T_0}{\max\left(\mu, \frac{\varepsilon\sqrt{\mu}}{\mathrm{Ro}}\right)}\right]; X^s(\mathbb{R})\right)$ *satisfying*

$$\begin{cases} \partial_t \tilde{\zeta} + \partial_x\left(\tilde{h}\tilde{u}\right) = R_1, \\ \left(1 - \frac{\mu}{3}\partial_x^2\right)\partial_t \tilde{u} + \partial_x \tilde{\zeta} + \varepsilon \tilde{u}\partial_x \tilde{u} - \frac{\varepsilon}{\mathrm{Ro}}\tilde{v} + \frac{\varepsilon}{\mathrm{Ro}}\mu^{\frac{3}{2}}\frac{1}{24}\partial_x \frac{\tilde{v}^\sharp}{\tilde{h}} = R_2, \\ \partial_t \tilde{v} + \varepsilon \tilde{u}\partial_x \tilde{v} + \frac{\varepsilon}{\mathrm{Ro}}\tilde{u} = R_3, \\ \partial_t \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}} + \varepsilon + \varepsilon \tilde{u}\partial_x \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}} + \frac{\varepsilon}{\mathrm{Ro}}\frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}} = R_4, \end{cases}$$

where $\tilde{h} = 1 + \varepsilon\tilde{\zeta} - \beta b$ *and with* $R = (R_1, R_2, R_3, R_4) \in L^\infty\left(\left[0, \frac{T_0}{\max\left(\mu, \frac{\varepsilon\sqrt{\mu}}{Ro}\right)}\right]; X^s(\mathbb{R})\right)$*. Then, if we denote* $\mathfrak{e} = \left(\zeta, u, v, \mathbf{V}^\sharp\right) - \left(\tilde{\zeta}, \tilde{u}, \tilde{v}, \tilde{\mathbf{V}}^\sharp\right)$ *where* $\left(\zeta, u, v, \mathbf{V}^\sharp\right)$ *is the solution given in Proposition 5.2.15, we have*

$$|\mathfrak{e}(t)|_{X_\mu^{s-1}} \leq C\left(A, \mu_{\max}, \frac{1}{h_{\min}}, \left|\left(\tilde{\zeta}, \tilde{u}, \tilde{v}, \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}, R\right)\right|_{L^\infty\left([0,t]; X_\mu^s \times X_\mu^s\right)}\right)\left(\left|\mathfrak{e}_{|t=0}\right|_{X_\mu^{s-1}} + t\,|R|_{X_\mu^s}\right).$$

Proof. This proof is a small adaptation of the one of Proposition 6.5 in [80] (see also [9]). We denote $\tilde{U} = \left(\tilde{\zeta}, \tilde{u}, \tilde{v}\right)$, $\mathfrak{e}_a = U - \tilde{U}$, $R_a = (R_1, R_2, R_3)$ and we keep the notations of the proof of Proposition 5.2.15. Since the Boussinesq-Coriolis equations are symmetrizable, we have

$$\begin{cases} \mathcal{A}_0(U)\partial_t \mathfrak{e}_a + \mathcal{A}_1(U)\partial_x \mathfrak{e}_a + B_1 \mathfrak{e}_a + \frac{\varepsilon}{\mathrm{Ro}} B_2(U)\mathfrak{e}_a = \frac{\varepsilon}{\mathrm{Ro}}\mu^{\frac{3}{2}} F(h, v^\sharp - \tilde{v}^\sharp) + G, \\ \partial_t\left(\frac{\mathbf{V}^\sharp}{h} - \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right) + \varepsilon u \partial_x\left(\frac{\mathbf{V}^\sharp}{h} - \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right) + \frac{\varepsilon}{\mathrm{Ro}}\left(\frac{\mathbf{V}^\sharp}{h} - \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right)^\perp = H, \end{cases}$$

where

$$\begin{aligned} G =& F(h, \tilde{v}^\sharp) - F(\tilde{h}, \tilde{v}^\sharp) - R_a - (\mathcal{A}_0(U) - \mathcal{A}_0(\tilde{U}))\partial_t \tilde{U} \\ &- (\mathcal{A}_1(U) - \mathcal{A}_1(\tilde{U}))\partial_x \tilde{U} - \frac{\varepsilon}{\mathrm{Ro}}(B_2(U) - B_2(\tilde{U}))U, \\ H =& \varepsilon(\tilde{u} - u)\partial_x \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}} + R_4. \end{aligned}$$

Then, using standard products estimates, we get (notice that $s > \frac{1}{2} + 1$)

$$\left(\Lambda^{s-1}G, \Lambda^{s-1}\mathfrak{e}_a\right)_2 \leq \left(|R|_{X^s_\mu} + \mu C\left(\mathcal{E}^s(\tilde{U}), \mathcal{E}^{s-1}(\partial_t \tilde{U}), \left|\frac{\tilde{v}^\sharp}{\tilde{h}}\right|_{H^s}, \sqrt{\mu}\left|\partial_x \frac{\tilde{v}^\sharp}{\tilde{h}}\right|_{H^s}\right)|\mathfrak{e}|_{X^{s-1}}\right)|\mathfrak{e}|_{X^{s-1}}$$

and

$$\left(\Lambda^{s-1}H, \Lambda^{s-1}\left(\frac{\mathbf{V}^\sharp}{h} - \frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right)\right)_2 \leq \left(|R|_{X^s_\mu} + \mu C\left(\mathcal{E}^s(\tilde{U}), \left|\frac{\tilde{\mathbf{V}}^\sharp}{\tilde{h}}\right|_{H^s}, \sqrt{\mu}\left|\partial_x \frac{\tilde{v}^\sharp}{\tilde{h}}\right|_{H^s}\right)|\mathfrak{e}|_{X^{s-1}}\right)|\mathfrak{e}|_{X^{s-1}}.$$

Then, the result follows from energy estimates and the Gronwall's lemma. □

The two previous results and Theorem 5.1.5 allow us to fully justify the Boussinesq-Coriolis equations. We recall that the operators $\overline{\mathbf{V}}[\varepsilon\zeta_0, \beta b](\mathrm{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega})$ and $\mathbf{V}_{\mathrm{sh}}[\varepsilon\zeta, \beta b](\mathrm{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega})(t, X)$ are defined in (5.22) and (5.23) respectively.

Theorem 5.2.19. *Let* $N \geq 7$ and $(\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}) \in \mathcal{A}_{Bouss}$ *. We assume that we are under the assumptions of Theorem 5.1.5. Then, we can define the following quantity* $(u_0, v_0)^t = \overline{\mathbf{V}}[\varepsilon\zeta_0, \beta b]((\mathrm{U}^{\mu,0}_{/\!/})_0, \boldsymbol{\omega}_0)$, $(u, v)^t = \overline{\mathbf{V}}[\varepsilon\zeta, \beta b](\mathrm{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega})$, $\mathbf{V}^\sharp_0 = \mathbf{V}^\sharp[\varepsilon\zeta_0, \beta b]((\mathrm{U}^{\mu,0}_{/\!/})_0, \boldsymbol{\omega}_0)$, $\mathbf{V}^\sharp = \mathbf{V}^\sharp[\varepsilon\zeta, \beta b](\mathrm{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega}_0)$, *and there exists a time* $T > 0$ *such that*

(i) T *has the form*

$$T = \frac{T_0}{\max(\mu, \frac{\varepsilon}{Ro})}, \quad and \quad \frac{1}{T_0} = c^1.$$

(ii) There exists a unique classical solution $\left(\zeta_B, u_B, v_B, \mathbf{V}^\sharp_B\right)$ *of* (5.31) *on* $[0, T]$ *with the initial data* $\left(\zeta_0, u_0, v_0, \mathbf{V}^\sharp_0\right)$ *.*

(iii) There exists a unique classical solution $\left(\zeta, \mathrm{U}^{\mu,0}_{/\!/}, \boldsymbol{\omega}\right)$ *of System* (5.14) *with initial data* $\left(\zeta_0, (\mathrm{U}^{\mu,0}_{/\!/})_0, \boldsymbol{\omega}_0\right)$ *on* $[0, T]$*.*

(iv) The following error estimate holds, for $0 \leq t \leq T$*,*

$$\left|\left(\zeta, u, v, \mathbf{V}^\sharp\right) - \left(\zeta_B, u_B, v_B, \mathbf{V}^\sharp_B\right)\right|_{L^\infty([0,t]\times\mathbb{R})} \leq \mu^2 t\, c^2,$$

with $c^j = C\left(A, \mu_{\max}, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{H^{N+2}}\right)$.

This theorem shows that the solutions of the water waves system (5.14) remain close to the solutions of the Boussinesq-Coriolis equations (5.31) over times $\mathcal{O}\left(\frac{1}{\max(\mu, \frac{\varepsilon}{\mathrm{Ro}})}\right)$ with an accuracy of order $\mathcal{O}(\mu)$. Hence, if one considers a system and wants to show that the solutions of this system remain close to the solutions of the waves equations over times $\mathcal{O}\left(\frac{1}{\max(\mu, \frac{\varepsilon}{\mathrm{Ro}})}\right)$ with an accuracy of order $\mathcal{O}(\mu)$, it is sufficient to compare the solutions of this system with the solutions of the Boussinesq-Coriolis equations (5.31). It is our approach in the following.

5.3 Different asymptotic models in the Boussinesq regime over a flat bottom

The Boussinesq-Coriolis equations (5.31) are particularly interesting for the evolution of offshore water waves. Without vorticity, we get the so-called Boussinesq equations. When we add a rotation, and in particular Coriolis effects, a standard assumption made by physicists is to also assume that the Rossby radius, or Obukhov radius, $\frac{\sqrt{gH}}{f}$ is greater than the typical length of the waves L (see for instance [117], [61], [84]). Then, different regimes for the Coriolis parameter were considered depending on whether the rotation is weak or not ([113], [60], [64]). In this paper, we consider three different regimes (noticed in [60]), a strong rotation ($\frac{\varepsilon}{\mathrm{Ro}} \leq 1$), weak rotation ($\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}(\sqrt{\mu})$) and very weak rotation ($\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}(\mu)$). We derive and fully justify different asymptotic models when the bottom is flat : a linear equation admitting the so-called Poincaré waves (5.39) ; the Ostrovsky equation (5.43), which is a generalization of the KdV equation (5.52) in presence of a Coriolis forcing, when the rotation is weak; and the KdV equation when the rotation is very weak.

5.3.1 Strong rotation, the Poincaré waves

In this part we are interested in the behaviour of long water waves under a strong Coriolis forcing (in the sense of [60]). We suppose that $\frac{\varepsilon}{\mathrm{Ro}}$ is of order 1. The asymptotic regime is

$$\mathcal{A}_{\mathrm{Poin}} = \left\{ (\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}) \, , 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \beta = \gamma = 0, \frac{\varepsilon}{\mathrm{Ro}} = 1 \right\}. \tag{5.36}$$

Then, the Boussinesq-Coriolis equations (5.31) become

$$\begin{cases} \partial_t \zeta + \partial_x \left((1 + \mu \zeta) \, u \right) = 0, \\ \left(1 - \frac{\mu}{3} \partial_x^2 \right) \partial_t u + \partial_x \zeta + \mu u \partial_x u - v + \frac{\mu^{\frac{3}{2}}}{24} \partial_x^2 \frac{v^\sharp}{h} = 0, \\ \partial_t v + \mu u \partial_x v + u = 0, \\ \partial_t \mathbf{V}^\sharp + \mu \mathbf{V}^\sharp \partial_x u + \mu u \partial_x \mathbf{V}^\sharp + \mathbf{V}^{\sharp \perp} = 0. \end{cases} \tag{5.37}$$

Our purpose is to justify the so-called Poincaré waves or Sverdrup waves ([134]), which are inertia-gravity waves. These are linear waves induced by the Coriolis forcing. If we drop all the terms of order $\mathcal{O}(\mu)$ in the Boussinesq-Coriolis equation we get the linear system

$$\begin{cases} \partial_t \zeta + \partial_x u = 0, \\ \partial_t u + \partial_x \zeta - v = 0, \\ \partial_t v + u = 0. \end{cases} \tag{5.38}$$

If we denote $U = (\zeta, u, v)^t$, by taking the Fourier transform, we get

$$\partial_t \widehat{U} = \mathcal{A} \widehat{U} \text{ with } \mathcal{A} = \begin{pmatrix} 0 & -i\xi & 0 \\ -i\xi & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and then,

$$\widehat{U} = \mathcal{S}(t,\xi)\widehat{U}_0 = \begin{pmatrix} \frac{\xi^2\cos(\sqrt{\xi^2+1}t)+1}{\xi^2+1} & -i\xi\frac{\sin(\sqrt{\xi^2+1}t)}{\sqrt{\xi^2+1}} & i\xi\frac{\cos(\sqrt{\xi^2+1}t)-1}{\xi^2+1} \\ -i\xi\frac{\sin(\sqrt{\xi^2+1}t)}{\sqrt{\xi^2+1}} & \cos(\sqrt{\xi^2+1}t) & \frac{\sin(\sqrt{\xi^2+1}t)}{\sqrt{\xi^2+1}} \\ -i\xi\frac{\cos(\sqrt{\xi^2+1}t)-1}{\xi^2+1} & -\frac{\sin(\sqrt{\xi^2+1}t)}{\sqrt{\xi^2+1}} & \frac{\xi^2+\cos(\sqrt{\xi^2+1}t)}{\xi^2+1} \end{pmatrix} \widehat{U}_0. \tag{5.39}$$

Commonly, Poincaré waves are waves of the form

$$U(t,x) = e^{i(xk \pm t\sqrt{k^2+1})}U_0.$$

They are solutions of the Klein-Gordon equation. In this setting, Poincaré waves correspond to solutions of System (5.38) of the form

$$\widehat{U}(t,\xi) = e^{i\pm t\sqrt{\xi^2+1}}\widehat{U}_0(\xi).$$

Therefore, a solution of System (5.38) is a sum of two Poincaré waves if and only if

$$\begin{pmatrix} \frac{1}{\xi^2+1} & 0 & -\frac{i\xi}{\xi^2+1} \\ 0 & 0 & 0 \\ \frac{i\xi}{\xi^2+1} & 0 & \frac{\xi^2}{\xi^2+1} \end{pmatrix} \widehat{U}_0 = 0,$$

which is equivalent to

$$\zeta_0 = \partial_x v_0. \tag{5.40}$$

In the following, we denote by $\mathcal{S}(t)$ the semi-group of the linear Boussinesq-Coriolis equation. The end of this part is devoted to the full justification of Poincaré waves. The following lemma shows that Condition (5.40) is propagated by the flow of System (5.38).

Lemma 5.3.1. *Let (ζ,u,v) be a solution of* (5.38) *such that $(\zeta,u,v)_{|t=0} = 0$ satisfies Condition* (5.40)*. Then, for all $t \in \mathbb{R}$,*

$$\zeta(t,\cdot) = \partial_x v(t,\cdot).$$

We also have the following dispersion result (see for instance [146] and [95] or Corollary 7.2.4 in [69]).

Lemma 5.3.2. *Let $u_0 \in W^{2,1}(\mathbb{R})$. Then*

$$\left| \int_{\mathbb{R}} e^{ix\xi \pm t\sqrt{\xi^2+1}} u_0(\xi) d\xi \right|_{L^\infty_x} \leq \frac{C}{\sqrt{1+|t|}} \left| u_0 \right|_{W^{2,1}}.$$

We can give the main result of this part.

Theorem 5.3.3. *Let $\mu_0 > 0$, $\zeta_0, u_0, v_0, \mathbf{V}_0^\sharp \in H^6(\mathbb{R})$, $x\zeta_0, xu_0, xv_0 \in H^4(\mathbb{R})$, such that ζ_0, v_0 satisfy Condition* (5.40)*, $1+\varepsilon\zeta \geq h_{\min} > 0$ and $0 < \mu < \mu_0$. Then, there exists a time $T > 0$, such that there exists*

(i) a unique classical solution $\left(\zeta_B, u_B, v_B, \mathbf{V}_B^\sharp\right)$ of (5.37) *with initial data $\left(\zeta_0, u_0, v_0, \mathbf{V}_0^\sharp\right)$ on $\left[0, \frac{T}{\sqrt{\mu}}\right]$.*

(ii) a unique solution (ζ, u, v) *of* (5.38) *with initial data* (ζ_0, u_0, v_0) *on* $\left[0, \frac{T}{\sqrt{\mu}}\right]$.

Moreover, we have the following error estimate for all $0 \leq t \leq \frac{T}{\sqrt{\mu}}$,

$$|(\zeta_B, u_B, v_B) - (\zeta, u, v)|_{L^\infty([0,t]\times\mathbb{R})} \leq C\left(\frac{\mu t}{1+\sqrt{t}} + \mu^2 t^2 + \mu^{\frac{3}{2}} t\right).$$

where $C = C\left(T, \frac{1}{h_{\min}}, \mu_0, |\zeta_0|_{H^6}, |u_0|_{H^6}, |v_0|_{H^6}, \left|\mathbf{V}_0^\sharp\right|_{H^6}, |x\zeta_0|_{H^4}, |xu_0|_{H^4}, |xv_0|_{H^4}\right)$.

Proof. The first point follows from Proposition 5.2.15. For the error estimate, if we denote by $U = (\zeta_B, u_B, v_B)^t$, U satisfies the linear Boussinesq-Coriolis equation up to a remainder of order μ and a remainder of order $\mu^{\frac{3}{2}}$. Then, using the Duhamel's formula we get

$$U(t) = \mathcal{S}(t)U_0 + \mu \int_0^t \mathcal{S}(t-\tau)\begin{pmatrix} -\partial_x(\zeta_B u_B)(\tau) \\ -u_B(\tau)\partial_x u_B(\tau) + \frac{1}{3}\partial_x^2\partial_\tau u_B(\tau) \\ -u_B\partial_x v_B \end{pmatrix} + \mu^{\frac{3}{2}} \int_0^t \mathcal{S}(t-\tau)R$$

where R is a remainder bounded uniformly with respect to μ. Then, using again the Duhamel's formula on the first integral we get

$$\begin{aligned} U(t) = \mathcal{S}(t)U_0 - \mu \int_0^t \mathcal{S}(t-\tau)&\begin{pmatrix} \partial_x((\mathcal{S}_1(\tau)U_0)(\mathcal{S}_2(\tau)U_0)) \\ (\mathcal{S}_2(\tau)U_0)\partial_x(\mathcal{S}_2(\tau)U_0) \\ (\mathcal{S}_2(\tau)U_0)\partial_x(\mathcal{S}_3(\tau)U_0) \end{pmatrix} \\ + \mu \int_0^t \mathcal{S}(t-\tau)&\begin{pmatrix} 0 \\ \frac{1}{3}\partial_x^2\partial_\tau \mathcal{S}_2(\tau)U_0 \\ 0 \end{pmatrix} + \mu^2 \int_0^t \int_0^\tau \tilde{R} + \mu^{\frac{3}{2}} \int_0^t \mathcal{S}(t-\tau)\tilde{R} \\ = \mathcal{S}(t)U_0 - \mu I_1(t) + \mu I_2(t) &+ \mu^2 I_3(t) + \mu^{\frac{3}{2}} I_4(t), \end{aligned}$$

where $\mathcal{S}_i(t)$ is the ith row of $\mathcal{S}(t)$. We start by estimating I_1. We have

$$I_1(t) = \int_0^t \mathcal{S}(t-\tau)\begin{pmatrix} \partial_x(\zeta(\tau)u(\tau)) \\ u(\tau)\partial_x u(\tau) \\ u(\tau)\partial_x v(\tau) \end{pmatrix}.$$

Then, we notice that $\partial_x(\zeta(\tau)u(\tau)) = \partial_x(u(\tau)\partial_x v(\tau))$ since $\zeta(\tau) = \partial_x v(\tau)$ by Lemma 5.3.1. Therefore, using Lemma 5.3.2 and products estimates, we get

$$\begin{aligned} |I_1(t)|_{L^\infty} &\leq \int_0^t \frac{1}{\sqrt{1+t-\tau}} \left| \begin{pmatrix} \partial_x((\mathcal{S}_1(\tau)U_0)(\mathcal{S}_2(\tau)U_0)) \\ (\mathcal{S}_2(\tau)U_0)\partial_x(\mathcal{S}_2(\tau)U_0) \\ (\mathcal{S}_2(\tau)U_0)\partial_x(\mathcal{S}_3(\tau)U_0) \end{pmatrix} \right|_{W^{2,1}} \\ &\leq C\left(|\zeta_0|_{H^3}, |u_0|_{H^3}, |v_0|_{H^3}, \left|\mathrm{V}_0^\sharp\right|_{H^3}\right) \frac{t}{\sqrt{1+t}}. \end{aligned}$$

For I_2, using Lemma 5.3.2 we get

$$|I_2| \leq C(|\zeta_0|_{H^4}, |u_0|_{H^4}, |v_0|_{H^4}, |x\zeta_0|_{H^4}, |xu_0|_{H^4}, |xv_0|_{H^4}) \frac{t}{\sqrt{1+t}}.$$

Finally, using Proposition 5.2.15, we have

$$|I_3(t)|_{H^1} \leq C\left(|\zeta_0|_{H^6}, |u_0|_{H^6}, |v_0|_{H^6}, \left|\mathrm{V}_0^\sharp\right|_{H^6}\right) t^2$$
$$|I_4(t)|_{H^1} \leq C\left(|\zeta_0|_{H^4}, |u_0|_{H^4}, |v_0|_{H^4}, \left|\mathrm{V}_0^\sharp\right|_{H^4}\right) t.$$

Gathering these four estimates, we get the result.

□

Hence, using Theorem 5.2.19, we justify that poincaré waves remain close to the solutions of the water waves equations (5.14) over times $\mathcal{O}_\mu(1)$ with an accuracy of order $\mathcal{O}(\mu)$. Furthermore, if one can show that a solution of the water waves equations (5.14), with initial data satisfying Condition (5.40), exists over a time $\mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$, we show that this solution remains close, with an accuracy of order $\mathcal{O}\left(\mu^{\frac{3}{4}}\right)$, to the solution of the linear Boussinesq-Coriolis equations with the same initial data. The reader interested in more linear properties of the water waves equations in shallow water can refer to Chapter 4 in [93].

5.3.2 The Proudman resonance in presence of a Coriolis forcing

In Section 2.3, we studied the Proudman resonance when the Coriolis forcing is negligible. We recall that it is a linear amplification due to a source term (non constant pressure or moving bottom). In subsection 4.5.4 we showed that the Coriolis forcing prevents the Proudman resonance when $d = 2$. In this subsection, we study the case $d = 1$. By proceeding as in Proposition 4.5.8, we can show that the system

$$\begin{cases} \partial_t \zeta + \partial_x u = 0, \\ \partial_t u + \partial_x \zeta - v = -\partial_x P, \\ \partial_t v + u = 0, \end{cases} \tag{5.41}$$

is an asymptotic model of the water waves equations in the regime

$$\mathcal{A}_{LWW} = \left\{ \left(\varepsilon, \beta, \gamma, \mu, \frac{\varepsilon}{\mathrm{Ro}}\right), 0 < \mu,\ \varepsilon,\ \beta \leq \delta_0,\ \frac{\varepsilon}{\mathrm{Ro}} = 1, \gamma \leq \delta_0^2 \right\}, \tag{5.42}$$

with an accuracy $\mathcal{O}(\delta_0 t)$ on times T independent of δ_0. In Section 2.3, we show that a pressure of the form $P(t,x) = P_0(x-t)$ creates a resonance when the Coriolis forcing is negligible. In the next Proposition, we shows that no physical resonance is possible for pressure of the form $P(t,x) = P_0(x - \alpha t)$.

Proposition 5.3.4. *Let $P_0 \in H^3(\mathbb{R}) \cap W^{3,1}(\mathbb{R})$, $\alpha \geq 0$, $P(t,x) = P_0(x-\alpha t)$ and $U = (\zeta, u, v)$ be the solution of* (5.41) *with initial data equal to* 0*. Then, if $\alpha \leq 1$,*

$$|(\zeta, u, v)(t, \cdot)|_{L^\infty} \leq C\, |P_0|_{H^3}.$$

Furthermore, there exists a constant $C > 0$, such that for all $\alpha > 1$,

$$|(\zeta, u, v)(t, \cdot)|_{L^\infty} \leq \frac{C}{\alpha + 1}\left(|P_0|_{H^2} + |P_0|_{W^{3,1}}\right)(1 + \ln(t)).$$

Proof. We keep the notations of Subsection 5.3.1. We begin with the case $\alpha < 1$. Using the Duhamel's formula, we get

$$
\begin{aligned}
U(t,x) =& \int_{\mathbb{R}} \int_0^t S(t-s,\xi) \begin{pmatrix} 0 \\ -i\xi \widehat{P}(s,\xi) \\ 0 \end{pmatrix} e^{ix\xi} ds d\xi \\
=& \int_{\mathbb{R}} \int_{-t}^0 e^{i(x-\alpha t)\xi} e^{i\tau\left(-\alpha\xi + \sqrt{1+\xi^2}\right)} i\xi \widehat{P}_0(\xi) m_1(\xi) + e^{i(x-\alpha t)\xi} e^{-i\tau\left(\alpha\xi+\sqrt{1+\xi^2}\right)} i\xi \widehat{P}_0(\xi) m_2(\xi) d\tau d\xi \\
=& \int_{\mathbb{R}} e^{i(x-\alpha t)\xi} \xi \widehat{P}_0(\xi) \frac{1-e^{it\left(\alpha\xi - \sqrt{1+\xi^2}\right)}}{\sqrt{1+\xi^2}-\alpha\xi} + e^{i(x-t)\xi}\xi\widehat{P}_0(\xi) \frac{e^{it\left(\alpha\xi+\sqrt{1+\xi^2}\right)}-1}{\sqrt{1+\xi^2}+\alpha\xi} m_2(\xi) d\xi,
\end{aligned}
$$

where m_1 and m_2 are bounded vectors in $L^\infty(\mathbb{R})$. Then, the first inequality follows easily from the fact that

$$
\left| \frac{1}{\sqrt{1+\xi^2}-\alpha\xi} \right| \leq C\left(1+|\xi|\right).
$$

For the second inequality, we also have

$$
U(t,x) = \int e^{i(x-\alpha t)\xi} \xi \widehat{P}_0(\xi) \frac{1-e^{it\left(\alpha\xi-\sqrt{1+\xi^2}\right)}}{\sqrt{1+\xi^2}-\alpha\xi} + e^{i(x-t)\xi}\xi\widehat{P}_0(\xi) \frac{e^{it\left(\alpha\xi+\sqrt{1+\xi^2}\right)}-1}{\sqrt{1+\xi^2}+\alpha\xi} m_2(\xi) d\xi.
$$

We focus on the first part of the previous equation since the other part is similar. We begin with the case $1 < \alpha \leq 2$. We denote $g(\xi) = \sqrt{1+\xi^2} - \alpha\xi$ and $\xi_\alpha = \frac{1}{\sqrt{\alpha^2-1}}$ its unique root. The function g is decreasing and we denote by ϕ its inverse. We can compute ϕ explicitly, for $y \in \mathbb{R}$,

$$
\phi(y) = \frac{-\alpha y + \sqrt{\alpha^2 - 1 + y^2}}{\alpha^2-1}.
$$

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a positive, compactly supported in $[-1,1]$, smooth, even function equal to one in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Then, we decompose our integral in two parts

$$
\begin{aligned}
\int_{\mathbb{R}} e^{i(x-\alpha t)\xi} \xi \widehat{P}_0(\xi) \frac{1-e^{it\left(\alpha\xi-\sqrt{1+\xi^2}\right)}}{\sqrt{1+\xi^2}-\alpha\xi} d\xi &= I + J, \\
I = \int_{\mathbb{R}} e^{i(x-\alpha t)\xi} \chi(\xi-\xi_\alpha) \xi \widehat{P}_0(\xi) \frac{1-e^{it\left(\alpha\xi-\sqrt{1+\xi^2}\right)}}{\sqrt{1+\xi^2}-\alpha\xi} d\xi&.
\end{aligned}
$$

Using the change of variable $\xi = \phi(z)$, we get

$$
\begin{aligned}
|I| &= \left| \int_{\xi_\alpha-1}^{\xi_\alpha+1} e^{i(x-\alpha t)\xi} \chi(\xi-\xi_\alpha) \xi \widehat{P}_0(\xi) \frac{1-e^{it\left(\alpha\xi-\sqrt{1+\xi^2}\right)}}{\sqrt{1+\xi^2}-\alpha\xi} d\xi \right| \\
&\leq C \int_{g(\xi_\alpha+1)}^{g(\xi_\alpha-1)} \left| \chi(\phi(z)-\xi_\alpha)\phi'(z)\phi(z)\widehat{P}_0(\phi(z)) \right| \left| \frac{sin(\frac{t}{2}z)}{z} \right| dz
\end{aligned}
$$

Then, we remark that there exists a constant $C(\alpha) > 0$, such that for all $y \in \mathbb{R}$, $|\phi'(y)| \leq C(\alpha)\left(1+|\phi(y)|^2\right)$ and $|g(\xi_\alpha+1)| + |g(\xi_\alpha-1)| \leq C(\alpha)$. We get

$$\begin{aligned} |I| &\leq C(\alpha) \left| \xi \left(1+|\xi|^2\right) \widehat{P}_0 \right|_{L^\infty} \int_{-tC(\alpha)}^{tC(\alpha)} \frac{1}{1+|z|} dz \\ &\leq C(\alpha) \left|P_0\right|_{W^{3,1}} \left(1+\ln(t)\right). \end{aligned}$$

Furthermore, we notice that there exists a constant $C > 0$, $\left| \frac{(1-\chi(\xi-\xi_\alpha))}{g(\xi)} \right| \leq C$. Then

$$\begin{aligned} |J| &= \left| \int_{\mathbb{R}} e^{i(x-\alpha t)\xi} (1-\chi(\xi-\xi_\alpha)) \xi \widehat{P}_0(\xi) \frac{1-e^{it\left(\alpha\xi-\sqrt{1+\xi^2}\right)}}{\sqrt{1+\xi^2}-\alpha\xi} d\xi \right| \\ &\leq C \int_{\mathbb{R}} \left| \xi \widehat{P}_0(\xi) \right| d\xi, \\ &\leq C \left|P_0\right|_{H^2}, \end{aligned}$$

and the result follows. Finally, when $\alpha \geq 2$, we remark that, $\forall y \in \mathbb{R}$, $|\phi'(y)| \leq \frac{1}{\alpha-1}$ and $|\xi_\alpha \pm 1| \leq 2$. We get

$$\begin{aligned} |I| &= \left| \int_{\mathbb{R}} e^{i(x-\alpha t)\xi} \chi(\xi-\xi_\alpha) \xi \widehat{P}_0(\xi) \frac{1-e^{it\left(\alpha\xi-\sqrt{1+\xi^2}\right)}}{\sqrt{1+\xi^2}-\alpha\xi} d\xi \right| \\ &\leq C \int_{-2}^{2} \left| \chi(\phi(z)-\xi_\alpha) \phi'(z) \phi(z) \widehat{P}_0(\phi(z)) \right| \left| \frac{sin(\frac{t}{2}z)}{z} \right| dz \\ &\leq C \frac{1}{\alpha-1} \left| \xi \widehat{P}_0 \right|_{L^\infty} \int_{-2t}^{2t} \frac{1}{1+|z|} dz \\ &\leq \frac{C}{\alpha-1} \left|P_0'\right|_{L^1} \left(1+\ln(t)\right). \end{aligned}$$

The control of J is similar to the previous case and the result follows easily. □

Hence, we see that we can not expect a resonance from a physical standpoint when we consider traveling pressure since the possible amplification is too slow. However, a resonance is possible for particular profiles. Inspired by Subsection 2.3.3, we consider a pressure of the form

$$P(t, \cdot) = e^{-it\sqrt{1+D^2}} P_0.$$

The next proposition shows that with such a profile we get a resonance with a factor of amplification of $\sqrt{t}$.

Proposition 5.3.5. *Let $P_0 \in H^1(\mathbb{R}) \cap W^{3,1}(\mathbb{R})$ such that $xP_0 \in H^1(\mathbb{R})$, $P(t, \cdot) = e^{-it\sqrt{1+D^2}} P_0$ and $U = (\zeta, u, v)$ be the solution of (5.41) with initial data equal to 0. Then,*

$$\left|U(t, \cdot)\right|_{L^\infty} \leq \frac{t}{\sqrt{1+t}} \left(|P|_{H^1} + |xP_0|_{H^1} \right).$$

Furthermore,

$$\varliminf_{t \to \infty} \frac{1}{\sqrt{t}} \left|U(t, \cdot)\right|_{L^\infty} \geq C(P_0) > 0.$$

Proof. We keep the notations of Subsection 5.3.1. Using the Duhamel's formula, we get

$$\begin{aligned}
U(t,x) =& \int_{\mathbb{R}} \int_0^t S(t-s,\xi) \begin{pmatrix} 0 \\ -i\xi \widehat{P}(s,\xi) \\ 0 \end{pmatrix} e^{ix\xi} ds d\xi \\
=& \int_{\mathbb{R}} \int_0^t e^{ix\xi} i\xi \widehat{P}_0(\xi) \left(e^{-is\sqrt{1+\xi^2}} e^{i(t-s)\sqrt{1+\xi^2}} m_1(\xi) + e^{i(s-t)\sqrt{1+\xi^2}} e^{-is\sqrt{1+\xi^2}} m_2(\xi) \right) ds d\xi \\
=& I_1(t,x) + I_2(t,x).
\end{aligned}$$

The first inequality follows from Lemma 5.3.2. For the second inequality, we first notice that

$$\begin{aligned}
|I_1(t,x)| &= \left| \int_{\mathbb{R}} \int_0^t e^{ix\xi} i\xi \widehat{P}_0(\xi) e^{i(t-2s)\sqrt{1+\xi^2}} m_1(\xi) ds d\xi \right| \\
&\leq C \int_{\mathbb{R}} \left| \xi \widehat{P}_0(\xi) \right| d\xi.
\end{aligned}$$

Furthermore, we have

$$I_2(t,x) = t \int_{\mathbb{R}} \int_0^t e^{i\left(x\xi - t\sqrt{1+\xi^2}\right)} i\xi \widehat{P}_0(\xi) m_2(\xi) d\xi.$$

Let $\xi_0 > 0$ such that $\xi_0 \widehat{P}_0(\xi_0) m_2(\xi_0) \neq 0$ and $x = \frac{t\xi_0}{\sqrt{1+\xi_0^2}}$. Then, using a stationary phase argument, we easily get

$$\left| I_2\left(t, \frac{t\xi_0}{\sqrt{1+\xi_0^2}}\right) \right| = \left| \xi_0 \widehat{P}_0(\xi_0) m_2(\xi_0) \right| \sqrt{2\pi t (1+\xi_0)^{\frac{3}{2}}} + o\left(\sqrt{t}\right),$$

and the result follows. □

5.3.3 Weak rotation, the Ostrovsky equation

Without Coriolis forcing and vorticity, it is well-known, that the KdV equation is a good approximation of the water waves equation under the assumption that ε and μ have the same order ([43], [75], [129], [19], Part 7.1 in [80]). When the Coriolis forcing is taken into account, Ostrovsky ([113]) derived an equation for long waves, which is an adaptation of the KdV equation

$$\partial_\xi \left(\partial_\tau f + \frac{3}{2} f \partial_\xi f + \frac{1}{6} \partial_\xi^3 f \right) = \frac{1}{2} f. \tag{5.43}$$

This equation is called the Ostrovsky equation or rKdV-equation in the physical literature. Initially developed for internal water waves, several authors also studied it for surface water waves ([114], [60], [85], [64]). The purpose of this part is to fully justify it. Inspired by [60] we consider the asymptotic regime

$$\mathcal{A}_{\mathrm{Ost}} = \left\{ (\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}), 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \beta = \gamma = 0, \frac{\varepsilon}{\mathrm{Ro}} = \sqrt{\mu} \right\}. \tag{5.44}$$

Then, the Boussinesq-Coriolis equations become (see Remark 5.2.14)

$$
\begin{cases}
\partial_t \zeta + \partial_x \left([1+\mu\zeta]u\right) = 0, \\
\left(1 - \frac{\mu}{3}\partial_x^2\right)\partial_t u + \partial_x \zeta + \mu u \partial_x u - \sqrt{\mu} v = 0, \\
\partial_t v + \mu u \partial_x v + \sqrt{\mu} u = 0.
\end{cases}
\tag{5.45}
$$

In order to motivate our approach, let us recall that we are interested in the one-dimensional propagation of water waves in the long wave regime. If we drop all the terms of order $\mathcal{O}(\sqrt{\mu})$ in the Boussinesq-Coriolis, we see that

$$
\begin{cases}
\partial_t \zeta + \partial_x u = 0, \\
\partial_t u + \partial_x \zeta = 0, \\
\partial_t v = 0.
\end{cases}
$$

Hence, if we assume that v is initially zero, we get a wave equation and the propagation of traveling water waves with speed ± 1. Then it is natural to study how these traveling water waves are perturbed when we add weakly nonlinear effects, i.e when we consider the System (5.45). In this paper, we consider only water waves with speed 1. We consider a WKB expansion for (ζ, u, v). We seek an approximate solution $(\zeta_{app}, u_{app}, v_{app})$ of (5.45) under the form

$$
\begin{aligned}
\zeta_{app}(t,x) &= f(x-t, \mu t) + \mu \zeta_{(1)}(t,x,\mu t), \\
u_{app}(t,x) &= f(x-t, \mu t) + \mu u_{(1)}(t,x,\mu t), \\
v_{app}(t,x) &= \sqrt{\mu} v_{(1/2)}(t,x,\mu t).
\end{aligned}
\tag{5.46}
$$

where $f = f(\xi, \tau)$ is our modulated traveling water waves, and the others terms are correctors. Then, we plug the ansatz in System (5.45) and we get

$$
\begin{aligned}
&\partial_t \zeta_{app} + \partial_x \left([1+\mu\zeta_{app}]u_{app}\right) = \mu R^1_{(1)} + \mu^2 R_1, \\
&\left(1 - \frac{\mu}{3}\partial_x^2\right)\partial_t u_{app} + \partial_x \zeta_{app} + \mu u_{app} \partial_x u_{app} - \sqrt{\mu} v_{app} = \mu R^2_{(1)} + \mu^2 R_2, \\
&\partial_t v_{app} + \mu u_{app} \partial_x v_{app} + \sqrt{\mu} u_{app} = \sqrt{\mu} R^3_{(1/2)} + \mu^{\frac{3}{2}} R_3,
\end{aligned}
\tag{5.47}
$$

where

$$
\begin{aligned}
R^1_{(1)} &= \partial_t \zeta_{(1)} + \partial_x u_{(1)} + \partial_\tau f + 2 f \partial_\xi f, \\
R^2_{(1)} &= \partial_t u_{(1)} + \partial_x \zeta_{(1)} + \partial_\tau f + \frac{1}{3}\partial_\xi^3 f + f \partial_\xi f - v_{(1/2)}, \\
R^3_{(1/2)} &= \partial_t v_{(1/2)} + f,
\end{aligned}
$$

and

$$
\begin{aligned}
R_1 &= \partial_\tau \zeta_{(1)} + \partial_x \left(f u_{(1)} + f \zeta_{(1)} + \mu \zeta_{(1)} u_{(1)}\right), \\
R_2 &= \partial_\tau u_{(1)} - \frac{1}{3}\partial_\xi^3 \partial_\tau f - \frac{1}{3}\partial_x^3 \partial_t u_{(1)} - \mu \frac{1}{3}\partial_x^3 \partial_\tau u_{(1)} + \partial_x \left(f u_{(1)}\right) + \mu u_{(1)} \partial_x u_{(1)}, \\
R_3 &= \partial_\tau v_{(1/2)} + \left(f + \sqrt{\mu} u_{(1)}\right) \partial_x v_{(1/2)} + u_{(1)}.
\end{aligned}
\tag{5.48}
$$

Then, the idea is to choose the correctors with $R^1_{(1)}(t,x,\tau) = R^2_{(1)}(t,x,\tau) = 0$ and $R^3_{(1/2)}(t,x,\tau) = 0$ for all $x \in \mathbb{R}$, $t \in \left[0, \frac{T}{\mu}\right]$ and $\tau \in [0,T]$.

Remark 5.3.6. *In fact, we should add* $\sqrt{\mu}\zeta_{(1/2)}(t,x,\mu t)$*,* $\sqrt{\mu}u_{(1/2)}(t,x,\mu t)$*,* $v_{(0)}(t,x,\mu t)$*, and* $\mu v_{(1)}(t,x,\mu t)$ *to the ansatz* (5.46) *for* ζ_{app}*,* u_{app}*,* v_{app} *and* v_{app} *respectively. However, if we plug them in System* (5.45) *and we want to cancel all the terms of order* $\sqrt{\mu}$ *and* μ*, we get*

$$\begin{aligned}
&\partial_t \zeta_{(1/2)} + \partial_x u_{(1/2)} = 0,\\
&\partial_t u_{(1/2)} + \partial_x \zeta_{(1/2)} + v_{(0)} = 0,\\
&\partial_t v_{(0)} = 0,\\
&\partial_t v_{(1)} + \partial_\tau v_{(0)} + f\partial_x v_{(0)} + u_{(1/2)} = 0,
\end{aligned}$$

which leads to $\zeta_{(1/2)} = u_{(1/2)} = v_{(0)} = v_{(1)} = 0$ *if these quantities are initially zero.*

Then, if we assume that $v_{(1/2)}$ and f vanish at $x=\infty$, the condition $R^3_{(1/2)} = 0$ is equivalent to the equation

$$\partial_t \partial_x v_{(1/2)}(t,x,\tau) + \partial_\xi f(x-t,\tau) = 0.$$

Since, $\partial_t (f(x-t,\tau)) = -\partial_\xi f(x-t,\tau)$, we can take

$$\partial_x v_{(1/2)}(t,x,\tau) = \partial_x v^0_{(1/2)}(x) - f^0(x) + f(x-t,\tau), \tag{5.49}$$

where $v^0_{(1/2)}$ and f^0 are the initial data of $v_{(1/2)}$ and f respectively. Then, we have to introduce the following spaces.

Definition 5.3.7. *For* $s \in \mathbb{R}$*, we define the Hilbert spaces* $\partial_x H^s(\mathbb{R})$ *as*

$$\partial_x H^s(\mathbb{R}) = \left\{ f \in H^{s-1}(\mathbb{R}),\ f = \partial_x \tilde{f} \text{ with } \tilde{f} \in H^s(\mathbb{R}) \right\},$$

and $\tilde{f}$ *is denoted* $\partial_x^{-1} f$ *in the following. In the same way, we define* $\partial_x^2 H^s(\mathbb{R})$*.*

Then, if we assume that $f(\cdot,\tau) \in \partial_x H^s(\mathbb{R})$ for all $\tau \in [0,T]$, we have

$$v_{(1/2)}(t,x,\tau) = v^0_{(1/2)}(x) - \partial_x^{-1} f^0(x) + \partial_x^{-1} f(x-t,\tau),$$

Furthermore, from $R^1_{(1)} = R^2_{(1)} = 0$, if we denote $w_\pm = \zeta_{(1)} \pm u_{(1)}$ we get

$$\begin{aligned}
&(\partial_t + \partial_x)\, w_+ + \left(2\partial_\tau f + 3f\partial_\xi f + \frac{1}{3}\partial_\xi^3 f - \partial_\xi^{-1} f\right)(x-t,\tau) - \left(v^0_{(1/2)} - \partial_\xi^{-1} f^0\right)(x) = 0,\\
&(\partial_t - \partial_x)\, w_- + \left(\frac{1}{2}\partial_\xi f^2 - \frac{1}{3}\partial_\xi^3 f + \partial_\xi^{-1} f\right)(x-t,\tau) + \left(v^0_{(1/2)} - \partial_\xi^{-1} f^0\right)(x) = 0.
\end{aligned} \tag{5.50}$$

The following lemma (Lemma 7.6 in [80]) is the key point to control u and v.

Lemma 5.3.8. *Let* $c_1 \neq c_2$*. Let* $h_1, h_2, h_3 \in L^2(\mathbb{R})$ *with* $h_2 = H_2'$ *and* $H_2 \in L^2(\mathbb{R})$*. We consider the unique solution* g *of*

$$\begin{cases} (\partial_t + c_1 \partial_x) g = h_1(x-c_1 t) + h_2(x-c_2 t) + h_3(x-c_2 t),\\ g_{|t=0} = 0. \end{cases}$$

Then, $\lim_{t\to\infty} \left|\frac{1}{t} g(t,\cdot)\right|_2 = 0$ *if and only if* $h_1 \equiv 0$ *and in that case*

$$|g(t,\cdot)|_2 \leq \frac{C}{|c_1 - c_2|}\left(|H_2|_2 \frac{t}{1+t} + |h_3|_{H^2} \frac{t}{1+\sqrt{t}} \right).$$

Then, in order to avoid a linear growth for the solution of (5.50), we also have to impose that

$$\partial_\tau f + \frac{3}{2} f \partial_\xi f + \frac{1}{6} \partial_\xi^3 f = \frac{1}{2} \partial_\xi^{-1} f, \tag{5.51}$$

which is the Ostrovsky equation. Before giving a full justification of the Ostrovsky equation, we need a local wellposedness result of this equation. The following proposition is a generalization of Theorem 2.1 in [89] and Theorem 2.6 in [143] (see also [87] for weak solutions).

Proposition 5.3.9. *Let $s \geq 3$ and $f_0 \in \partial_x H^s(\mathbb{R})$. Then, there exists a time $T > 0$ and a unique solution $f \in \mathcal{C}\left([0,T]; \partial_x H^s(\mathbb{R})\right))$ to the Ostrovsky equation (5.51) and one has*

$$\left|\partial_\xi^{-1} f(t,\cdot)\right|_{H^s} \leq C\left(T, \left|\partial_\xi^{-1} f_0\right|_{H^s}\right).$$

Moreover, if $f_0 \in \partial_x^2 H^{s+1}(\mathbb{R})$, $f \in \mathcal{C}\left([0,T]; \partial_x^2 H^{s+1}(\mathbb{R})\right))$ and one has

$$\left|\partial_\xi^{-2} f(t,\cdot)\right|_{H^{s+1}} \leq C\left(T, \left|\partial_\xi^{-2} f_0\right|_{H^{s+1}}\right).$$

Proof. We only prove the second point of the Proposition. We denote by $S(t)$ the semi-group of the linearized Ostrovsky equation

$$\partial_\tau f + \frac{1}{6}\partial_\xi^3 f - \frac{1}{2}\partial_\xi^{-1} f = 0,$$

and it is easy to check that this semi-group acts unitary on $H^s(\mathbb{R})$. We denote $\tilde{f} = \partial_\tau f$. Then, $\tilde{f}$ satisfies the equation

$$\partial_\tau \tilde{f} + \frac{3}{2}\partial_\xi\left(\tilde{f} f\right) + \frac{1}{6}\partial_\xi^3 \tilde{f} - \frac{1}{2}\partial_\xi^{-1}\tilde{f} = 0.$$

Using the Duhamel's formula we obtain

$$\partial_\xi^{-1}\tilde{f}(t,\cdot) = S(t)\partial_\xi^{-1}\tilde{f}_0 + \frac{3}{2}\int_0^t S(t-s)\left(f\tilde{f}\right)(s,\cdot)ds.$$

Since $\partial_\xi^{-1}\tilde{f}_0 = -\frac{3}{4}f_0^2 - \partial_\xi^2 f_0 + \partial_\xi^{-2} f_0 \in L^2(\mathbb{R})$, we get the result since we have

$$\frac{1}{2}\partial_\xi^{-2} f = \partial_\xi^{-1}\partial_\tau f + \frac{3}{4}f^2 + \frac{1}{6}\partial_\xi^2 f.$$

□

Notice that contrary to the KdV equation, we do not have a global existence. We can now give the main result of this part.

Theorem 5.3.10. *Let $f^0 \in \partial_x^2 H^{10}(\mathbb{R})$, such that $1 + \varepsilon f \geq h_{\min} > 0$, $v^0 \in \partial_x H^6(\mathbb{R})$ and $\mu_0 > 0$. Then, there exists a time $T > 0$, such that for all $0 < \mu < \mu_0$, we have*
(i) a unique classical solution (ζ_B, u_B, v_B) of (5.45) with initial data $\left(f^0, f^0, \sqrt{\mu}v^0\right)$ on $\left[0, \frac{T}{\mu}\right]$.

(ii) a unique classical solution f of (5.51) with initial data f^0 on $[0,T]$.

(iii) If we define $\left(\zeta_{Ost}, u_{Ostr}\right)(t,x) = \left(f(x-t,\mu t), f(x-t,\mu t)\right)$ we have the following error estimate for all $0 \leq t \leq \frac{T}{\mu}$,

$$\left|(\zeta_B, u_B) - (\zeta_{Ost}, u_{Ost})\right|_{L^\infty([0,t]\times\mathbb{R})} \leq C\left((1+\sqrt{\mu}t)\frac{\mu t}{1+t} + \mu^{\frac{3}{2}}t\right)$$

where $C = C\left(T, \frac{1}{h_{\min}}, \mu_0, \left|\partial_x^{-2} f^0\right|_{H^{10}}, \left|\partial_x^{-1} v^0\right|_{H^6}\right)$.

Proof. In all the proof, C will be a constant as in the theorem. The first and second point follow from Corollary 5.2.17 and 5.3.9. In order to get the error estimate, we have to control the remainders R_1, R_2, R_3, defined in (5.48). First, using Lemma 5.3.8, the fact that we can express the quantities $\frac{1}{2}\partial_\xi f^2 - \frac{1}{3}\partial_\xi^3 f$, $\partial_\xi^{-1} f$ and v_0 as derivatives with respect to x and the fact that f satisfies the Ostrovsky equation (5.51), we have

$$\left|\zeta_{(1)}\right|_2 + \left|u_{(1)}\right|_2 \leq C\frac{t}{1+t}.$$

But we can also control all the derivatives with respect to τ or x of u and v be differentiating (5.50). Hence, we get a control for the remainders R_1, R_2. For R_3, we use the fact that $v = \partial_x^{-1} f$. We finally, obtain

$$|R_1|_{H^2} + |R_2|_{H^2} + |R_3|_{H^2} \leq C\left(\frac{t}{1+t} + \mu t + 1\right),$$

Then, thanks to Proposition 5.2.18 and remark 5.2.14, we get

$$|(\zeta_B, u_B, v_B) - (\zeta_{app}, u_{app}, v_{app})|_{L^\infty([0,t]\times\mathbb{R})} \leq C\mu^{\frac{3}{2}} t\left(\frac{t}{1+t} + \mu t + 1\right).$$

Moreover, we have

$$|(\zeta_{app}, u_{app}) - (\zeta_{Ost}, v_{Ost})|_{L^\infty([0,t]\times\mathbb{R})} \leq \mu\frac{t}{1+t}.$$

Then, the result follows easily.

□

This theorem, combined with Theorem 5.2.19, shows that the solutions of the water waves equations (5.14) is well approximated over times $\mathcal{O}\left(\frac{1}{\sqrt{\mu}}\right)$ with an accuracy of order $\mathcal{O}(\mu)$ by the Ostrovsky approximation if we have a small Coriolis forcing. Notice that contrary to the KdV equation, the Ostrovsky equation does not admit solitons ([154], [58]). Notice also that this approach is similar to the one of the KP equations (see for instance [83], [9] or Part 7.2.1 in [80]). The fact that $f^0 \in \partial_x H^8$ is essential and physical since a solution of the Ostrovsky equation has to be mean free. However, we suppose here that $f^0 \in \partial_x^2 H^9(\mathbb{R})$ and $v^0 \in \partial_x H^5(\mathbb{R})$ which is more restrictive. In fact, using the strategy developed in [15] for the KP approximation we can hope to release this assumption.

5.3.4 Very weak rotation, the KdV equation

As we said before, without Coriolis forcing, it is well-known, that the KdV equation is a good approximation of the water waves equations. In this part we show that if $\frac{\varepsilon}{\mathrm{Ro}}$ is small enough, we get the KdV equation as an asymptotic model. We recall that the KdV equation

$$\partial_\tau f + \frac{3}{2} f\partial_\xi f + \frac{1}{6}\partial_\xi^3 f = 0. \tag{5.52}$$

Inspired by [60], we show that $\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}(\mu)$ is sufficient. we consider the asymptotic regime

$$\mathcal{A}_{KdV} = \left\{(\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}), 0 \leq \mu \leq \mu_0, \varepsilon = \mu, \beta = \gamma = 0, \frac{\varepsilon}{\mathrm{Ro}} = \mu\right\}. \tag{5.53}$$

Then, the Boussinesq-Coriolis equations become (see Remark 5.2.14)

$$\begin{cases} \partial_t \zeta + \partial_x ([1+\mu\zeta]u) = 0, \\ \left(1 - \frac{\mu}{3}\partial_x^2\right)\partial_t u + \partial_x \zeta + \mu u \partial_x u - \mu v = 0, \\ \partial_t v + \mu u \partial_x v + \mu u = 0. \end{cases} \tag{5.54}$$

Proceeding as in the previous part, we seek an approximate solution $(\zeta_{app}, u_{app}, v_{app})$ of (5.54) under the form

$$\begin{aligned} \zeta_{app}(t,x) &= f(x-t,\mu t) + \mu \zeta_{(1)}(t,x,\mu t), \\ u_{app}(t,x) &= f(x-t,\mu t) + \mu u_{(1)}(t,x,\mu t), \\ v_{app}(t,x) &= \mu v_{(1)}(t,x,\mu t). \end{aligned} \tag{5.55}$$

Then, we plug the ansatz in Sytem (5.54) and we get

$$\begin{aligned} &\partial_t \zeta_{app} + \partial_x ([1+\mu\zeta_{app}]u_{app}) = \mu R^1_{(1)} + \mu^2 R_1, \\ &\left(1 - \frac{\mu}{3}\partial_x^2\right)\partial_t u_{app} + \partial_x \zeta_{app} + \mu u_{app} \partial_x u_{app} - \mu v_{app} = \mu R^2_{(1)} + \mu^2 R_2, \\ &\partial_t v_{app} + \mu u_{app} \partial_x v_{app} + \mu u_{app} = \mu R^3_{(1)} + \mu^2 R_3, \end{aligned} \tag{5.56}$$

where

$$\begin{aligned} R^1_{(1)} &= \partial_t \zeta_{(1)} + \partial_x u_{(1)} + \partial_\tau f + 2f\partial_\xi f, \\ R^2_{(1)} &= \partial_t u_{(1)} + \partial_x \zeta_{(1)} + \partial_\tau f + \frac{1}{3}\partial_\xi^3 f + f\partial_\xi f, \\ R^3_{(1)} &= \partial_t v_{(1)} + f, \end{aligned}$$

and

$$\begin{aligned} R_1 &= \partial_\tau \zeta_{(1)} + \partial_x \left(f u_{(1)} + f\zeta_{(1)} + \mu \zeta_{(1)} u_{(1)}\right), \\ R_2 &= \partial_\tau u_{(1)} - \frac{1}{3}\partial_\xi^3 \partial_\tau f - \frac{1}{3}\partial_x^3 \partial_t u_{(1)} - \mu \frac{1}{3}\partial_x^3 \partial_\tau u_{(1)} + \partial_x \left(f u_{(1)}\right) + \mu u_{(1)} \partial_x u_{(1)} - v_{(1)}, \\ R_3 &= \partial_\tau v_{(1)} + \mu \left(f + \mu u_{(1)}\right) \partial_x v_{(1)} + u_{(1)}. \end{aligned}$$

Remark 5.3.11. *We should add $v_{(0)}(t,x,\mu t)$ to the ansatz* (5.55) *for v_{app}. However, if we plug it in System* (5.54) *we get $\partial_t v_{(0)} = 0$ which leads to $v_{(0)} = 0$ if the quantity is initially zero. Hence, we make this assumption in the following.*

As before, we assume that $R^1_{(1)}(t,x,\tau) = R^2_{(1)}(t,x,\tau) = R^3_{(1)}(t,x,\tau) = 0$ for all $x \in \mathbb{R}$, $t \in \left[0, \frac{T}{\mu}\right]$ and $\tau \in [0,T]$ which leads to $v_{(1)} = v^0_{(1)} - \partial_x^{-1} f^0 + \partial_x^{-1} f$ and, if we denote $w_\pm = \zeta_{(1)} \pm u_{(1)}$ we get

$$\begin{aligned} &(\partial_t + \partial_x)\, w_+ + \left(2\partial_\tau f + 3f\partial_\xi f + \frac{1}{3}\partial_\xi^3 f\right)(x-t,\tau) = 0, \\ &(\partial_t - \partial_x)\, w_- + \left(f\partial_\xi f - \frac{1}{3}\partial_\xi^3 f\right)(x-t,\tau) = 0 \end{aligned}$$

and to avoid a linear growth of u or v we need that f satisfies (5.52). We also have a existence result for the KdV equation (see for instance [76]).

Proposition 5.3.12. *Let $s \geq 1$, $f_0 \in H^s(\mathbb{R})$ and $T > 0$. Then, there exists a unique solution to the KdV equation* (5.52) *$f \in \mathcal{C}\left([0,T]; H^s(\mathbb{R})\right)$ and one have*

$$|f|_{H^s} \leq C\left(T, \left|f^0\right|_{H^s}\right).$$

Moreover, if $s \geq 2$ and $f_0 \in \partial_x H^{s+1}(\mathbb{R})$, $f \in \mathcal{C}\left([0,T]; \partial_x H^{s+1}(\mathbb{R})\right)$ and we have

$$\left|\partial_x^{-1} f\right|_{H^{s+1}} \leq C\left(T, \left|\partial_x^{-1} f^0\right|_{H^{s+1}}\right).$$

Then, proceeding as in the previous part, we obtain the following theorem.

Theorem 5.3.13. *Let $f^0 \in \partial_x H^9(\mathbb{R})$, such that such that $1 + \varepsilon f^0 \geq h_{\min} > 0$, $v^0 \in H^5(\mathbb{R})$ and $\mu_0 > 0$. Then, there exists a time $T > 0$, such that for all $0 < \mu < \mu_0$, we have*
(i) a unique classical solution (ζ_B, u_B, v_B) of (5.54) *with initial data $\left(f^0, f^0, \mu v^0\right)$ on $\left[0, \frac{T}{\mu}\right]$.*

(ii) a unique classical solution f of (5.52) *with initial data f^0 on $[0,T]$.*

(iii) If we define $(\zeta_{KdV}, u_{KdV})(t,x) = (f(x-t, \mu t), f(x-t, \mu t))$ we have the following error estimate for all $0 \leq t \leq \frac{T}{\mu}$,

$$|(\zeta_B, u_B) - (\zeta_{KdV}, u_{KdV})|_{L^\infty([0,t]\times\mathbb{R})} \leq C\left(\frac{\mu t}{1+t} + \mu^2 t\right)$$

where $C = C\left(T, \frac{1}{h_{\min}}, \mu_0, \left|\partial_x^{-1} f^0\right|_{H^9}, \left|v^0\right|_{H^5}\right)$.

This theorem, combined with Theorem 5.2.19, shows that the solutions of the water waves equations (5.14) is well approximated over times $\mathcal{O}\left(\frac{1}{\mu}\right)$ with an accuracy of order $\mathcal{O}(\mu)$ by the KdV approximation if we have a very small Coriolis forcing. Notice that contrary to the irrotational case, the transverse velocity v is not zero (noticed also in [60]). Furthermore, in our situation, the initial data for the KdV equation has to be of zero mean which means that we can not expect the propagation of solitons on a large time (they have a constant sign) if $\frac{\varepsilon}{\mathrm{Ro}}$ and μ have the same order.

5.4 Green-Naghdi equations for $\gamma = 0$ and $\beta = \mathcal{O}(\mu)$

This part is devoted to the derivation and justification of the Green-Naghdi equations (5.64) under a Coriolis forcing, with $\gamma = 0$ and for small amplitude topography variations ($\beta = \mathcal{O}(\mu)$). The Green-Naghdi equations are originally obtained in the irrotational framework under the assumption that μ is small (no assumption on ε) and by neglecting all the terms of order $\mathcal{O}(\mu^2)$ in the water waves equations (see for instance [130] or Part 5.1.1.2 in [80]). It is a system of two equations on the surface ζ and the averaged horizontal velocity $\overline{\mathbf{V}}$. These equations were generalized in [33] in presence of vorticity but without a Coriolis forcing. This new system is a cascade of equations that involves a second order tensor and a third order tensor. After deriving these equations, we show that they are an order $\mathcal{O}(\mu^2)$ approximation of the water waves equations. We consider the asymptotic regime for the 1D Green-Naghdi equations

$$\mathcal{A}_{\mathrm{GN}} = \left\{(\varepsilon, \beta, \gamma, \mu, \mathrm{Ro}), 0 \leq \mu \leq \mu_0, 0 \leq \varepsilon, \frac{\varepsilon}{\mathrm{Ro}} \leq 1, \beta = \mathcal{O}(\mu), \gamma = 0\right\}. \quad (5.57)$$

The next subsection is devoted to extending Proposition 5.2.8 and 5.2.9.

5.4.1 Improvements for the equations of Q_x and Q_y

We start by extending Proposition 5.2.8.

Proposition 5.4.1. *If* $\left(\zeta, \mathrm{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (5.14)*, then* Q_x *satisfies the following equation*

$$\begin{aligned}\partial_t Q_x + \varepsilon \overline{u}\partial_x Q_x + \varepsilon Q_x \partial_x \overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\left(\underline{v} - \overline{v}\right) = &- \varepsilon\sqrt{\mu}\frac{1}{h}\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} \left(u_{\mathrm{sh}}^*\right)^2 \\ &+ \varepsilon\sqrt{\mu}\mathrm{Q}_x\partial_x\mathrm{Q}_x + \varepsilon\mu\frac{1}{3}\partial_x\left(h^2\mathrm{Q}_x\partial_x^2\overline{u}\right) \\ &+ \varepsilon\mu\frac{1}{6}h^2 u^\sharp\partial_x^3\overline{u} + \varepsilon\mu\frac{1}{8h}\partial_x\left(h^3 u^\sharp\right)\partial_x^2\overline{u} \\ &+ \varepsilon\max\left(\beta\sqrt{\mu}, \mu^{\frac{3}{2}}\right)R,\end{aligned}$$

and u_{sh}^* *satisfies the equation*

$$\begin{aligned}\partial_t u_{\mathrm{sh}}^* + \varepsilon\overline{u}\partial_x u_{\mathrm{sh}}^* + \varepsilon u_{\mathrm{sh}}^*\partial_x\overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\left(\overline{v} - v\right) = &\varepsilon\sqrt{\mu}\frac{1}{h}\partial_x\int_{-1+\beta b}^{\varepsilon\zeta}\left(u_{\mathrm{sh}}^*\right)^2 - \varepsilon\sqrt{\mu}u_{\mathrm{sh}}^*\partial_x u_{\mathrm{sh}}^* \\ &+ \varepsilon\partial_x\left(\int_{-1+\beta b}^{z}\left[\overline{u} + \sqrt{\mu}u_{\mathrm{sh}}^*\right]\right)\partial_z u_{\mathrm{sh}}^* \\ &+ \varepsilon\mu R.\end{aligned}$$

Proof. Using the second equation of the vorticity equation of the Castro-Lannes system (5.14), we have

$$\partial_t\boldsymbol{\omega}_y + \varepsilon u\partial_x\boldsymbol{\omega}_y + \frac{\varepsilon}{\mu}\mathrm{w}\partial_z\boldsymbol{\omega}_y = \varepsilon\boldsymbol{\omega}_x\partial_x v + \frac{\varepsilon}{\sqrt{\mu}}\boldsymbol{\omega}_z\partial_z v + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\partial_z v.$$

Since $\boldsymbol{\omega}_x = -\frac{1}{\sqrt{\mu}}\partial_z v$ and $\boldsymbol{\omega}_z = \partial_x v$ we notice that $\varepsilon\boldsymbol{\omega}_x\partial_x v + \frac{\varepsilon}{\sqrt{\mu}}\boldsymbol{\omega}_z\partial_z v = 0$. Using Proposition 5.2.5 we get

$$\partial_t\boldsymbol{\omega}_y + \varepsilon\overline{u}\partial_x\boldsymbol{\omega}_y - \varepsilon\partial_x\left[(1+z-\beta b)\overline{u}\right]\partial_z\boldsymbol{\omega}_y - \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\partial_z v + \varepsilon\sqrt{\mu}A_1 + \varepsilon\mu A_2 = \varepsilon\max\left(\mu^{\frac{3}{2}}, \beta\sqrt{\mu}\right)R,$$

where

$$\begin{aligned}A_1 &= u_{\mathrm{sh}}^*\partial_x\boldsymbol{\omega}_y - \partial_x\left(\int_{-1+\beta b}^{z} u_{\mathrm{sh}}^*\right)\partial_z\boldsymbol{\omega}_y, \\ A_2 &= -\frac{1}{2}\left([1+z-\beta b]^2 - \frac{h^2}{3}\right)\partial_x^2\overline{u}\partial_x\boldsymbol{\omega}_y + \frac{1}{2}\partial_x\left(\int_{-1+\beta b}^{z}\left([1+z-\beta b]^2 - \frac{h^2}{3}\right)\partial_x^2\overline{u}\right)\partial_z\boldsymbol{\omega}_y.\end{aligned}$$

Then, integrating with respect to z, using the fact that $\partial_t\zeta + \partial_x\left(h\overline{u}\right) = 0$ and $u_{\mathrm{sh}} = -\int_z^{\varepsilon\zeta}\boldsymbol{\omega}_y$, we get

$$\begin{aligned}\partial_t u_{\mathrm{sh}} + \varepsilon \overline{u} \partial_x u_{\mathrm{sh}} + \varepsilon u_{\mathrm{sh}} \partial_x \overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\underline{v} - v\right) =& \varepsilon \partial_x \left[\left(1 + z - \beta b\right) \overline{u}\right] \partial_z u_{\mathrm{sh}} + \varepsilon \sqrt{\mu} \int_z^{\varepsilon\zeta} A_1 \\ & + \varepsilon \mu \int_z^{\varepsilon\zeta} A_2 + \varepsilon \max\left(\mu^{\frac{3}{2}}, \beta\sqrt{\mu}\right) R.\end{aligned}$$

Integrating again with respect to z, using the fact that $\partial_t \zeta + \partial_x \left(h\overline{u}\right) = 0$ and $Q_x = \overline{u_{\mathrm{sh}}^*}$, we obtain

$$\begin{aligned}\partial_t Q_x + \varepsilon \overline{u} \partial_x Q_x + \varepsilon Q_x \partial_x \overline{u} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}} \left(\underline{v} - \overline{v}\right) =& \varepsilon \sqrt{\mu} \frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} \int_z^{\varepsilon\zeta} A_1 \\ & + \varepsilon \mu \frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} \int_z^{\varepsilon\zeta} A_2 + \varepsilon \max\left(\mu^{\frac{3}{2}}, \beta\sqrt{\mu}\right) R.\end{aligned}$$

The end of the proof is devoted to the computation of the others terms. We have

$$\begin{aligned}\int_z^{\varepsilon\zeta} A_1 &= \int_z^{\varepsilon\zeta} u_{\mathrm{sh}}^* \partial_x \boldsymbol{\omega}_y - \partial_x \left(\int_{-1+\beta b}^{z} u_{\mathrm{sh}}^*\right) \partial_z \boldsymbol{\omega}_y \\ &= \int_z^{\varepsilon\zeta} \partial_x \left(u_{\mathrm{sh}}^* \boldsymbol{\omega}_y\right) - \varepsilon \zeta \mathrm{Q}_x \underline{\boldsymbol{\omega}_y} + \partial_x \left(\int_{-1+\beta b}^{z} u_{\mathrm{sh}}^*\right) \boldsymbol{\omega}_y.\end{aligned}$$

Since $\boldsymbol{\omega}_y = \partial_z u_{\mathrm{sh}}^*$, we obtain

$$\int_z^{\varepsilon\zeta} A_1 = \mathrm{Q}_x \partial_x \mathrm{Q}_x - u_{\mathrm{sh}}^* \partial_x u_{\mathrm{sh}}^* + \partial_x \left(\int_{-1+\beta b}^{z} u_{\mathrm{sh}}^*\right) \partial_z u_{\mathrm{sh}}^*.$$

then, integrating again with respect to z, we obtain

$$\frac{1}{h} \int_{-1+\beta b}^{\varepsilon\zeta} \int_z^{\varepsilon\zeta} A_1 = \mathrm{Q}_x \partial_x \mathrm{Q}_x - \frac{1}{h} \partial_x \int_{-1+\beta b}^{\varepsilon\zeta} \left(u_{\mathrm{sh}}^*\right)^2.$$

Furthermore, we have

$$\begin{aligned}\int_z^{\varepsilon\zeta} A_2 =& -\frac{1}{2} \int_z^{\varepsilon\zeta} \left(\left[1 + z' - \beta b\right]^2 - \frac{h^2}{3}\right) \partial_x^2 \overline{u} \partial_x \boldsymbol{\omega}_y \\ & + \frac{1}{2} \int_z^{\varepsilon\zeta} \partial_x \left(\int_{-1+\beta b}^{z} \left(\left[1 + z' - \beta b\right]^2 - \frac{h^2}{3}\right) \partial_x^2 \overline{u}\right) \partial_z \boldsymbol{\omega}_y \\ =& -\frac{1}{2} \int_z^{\varepsilon\zeta} \partial_x \left[\left(\left[1 + z' - \beta b\right]^2 - \frac{h^2}{3}\right) \partial_x^2 \overline{u} \boldsymbol{\omega}_y\right] - \varepsilon \partial_x \zeta \frac{h^2}{3} \partial_x^2 \overline{u} \underline{\boldsymbol{\omega}_y} \\ & - \frac{1}{2} \partial_x \left(\int_{-1+\beta b}^{z} \left(\left[1 + z' - \beta b\right]^2 - \frac{h^2}{3}\right) \partial_x^2 \overline{u}\right) \boldsymbol{\omega}_y.\end{aligned}$$

Since $\boldsymbol{\omega}_y = \partial_z u_{\mathrm{sh}}^*$, we obtain

$$\int_z^{\varepsilon\zeta} A_2 = \int_z^{\varepsilon\zeta} \partial_x\left([1+z'-\beta b]\,\partial_x^2\overline{u}u_{\mathrm{sh}}^*\right) + \frac{1}{2}\partial_x\left(\left([1+z'-\beta b]^2 - \frac{h^2}{3}\right)\partial_x^2\overline{u}u_{\mathrm{sh}}^*\right)$$
$$- \frac{1}{2}\partial_x\left(\int_{-1+\beta b}^{z}\left([1+z'-\beta b]^2 - \frac{h^2}{3}\right)\partial_x^2\overline{u}\right)\partial_z u_{\mathrm{sh}}^*$$
$$+ \frac{1}{3}\partial_x\left(h^2\partial_x^2\overline{u}\mathrm{Q}_x\right) - \varepsilon\partial_x\zeta h\partial_x^2\overline{u}\mathrm{Q}_x.$$

Then we integrate again with respect to z and we divide h. We obtain

$$\frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta} A_2 = \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta} \partial_x\left([1+z'-\beta b]\,\partial_x^2\overline{u}u_{\mathrm{sh}}^*\right)$$
$$+ \frac{1}{2h}\int_{-1+\beta b}^{\varepsilon\zeta}\partial_x\left(\left([1+z'-\beta b]^2 - \frac{h^2}{3}\right)\partial_x^2\overline{u}u_{\mathrm{sh}}^*\right)$$
$$+ \frac{1}{2h}\int_{-1+\beta b}^{\varepsilon\zeta}\partial_x\left(\left([1+z'-\beta b]^2 - \frac{h^2}{3}\right)\partial_x^2\overline{u}\right)u_{\mathrm{sh}}^*$$
$$+ \frac{1}{3}\partial_x\left(h^2\partial_x^2\overline{u}\mathrm{Q}_x\right) - \frac{4}{3}h\partial_x h\partial_x^2\overline{u}\mathrm{Q}_x + \beta R.$$

Then, using the fact that

$$\int_{-1+\beta b}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta}\int_{-1+\beta b}^{z'}\partial_x u_{\mathrm{sh}}^* = \partial_x\int_{-1+\beta b}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta}\int_{-1+\beta b}^{z'} u_{\mathrm{sh}}^* + \beta R,$$

we finally get

$$\frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta} A_2 = \frac{1}{3}\partial_x\left(h^2\mathrm{Q}_x\partial_x^2\overline{u}\right) + \frac{1}{6}h^2u^\sharp\partial_x^3\overline{u} + \frac{1}{8h}\partial_x\left(h^3u^\sharp\right)\partial_x^2\overline{u} + \beta R,$$

and the first equation follows. The second equation follows similarly using the fact that $u_{\mathrm{sh}}^* = u_{\mathrm{sh}} - \mathrm{Q}_x$. □

We can also extend Proposition 5.2.9.

Proposition 5.4.2. *If $\left(\zeta, \mathrm{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ satisfy the Castro-Lannes system* (5.14)*, then Q_x satisfies the following equation*

$$\partial_t\mathrm{Q}_y + \varepsilon\overline{u}\partial_x\mathrm{Q}_y + \varepsilon\mathrm{Q}_x\partial_x\overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\left(\overline{u} - \underline{u}\right) = \varepsilon\sqrt{\mu}\mathrm{Q}_x\partial_x\mathrm{Q}_y - \varepsilon\sqrt{\mu}\frac{1}{3}h^2\partial_x^2\overline{u}\partial_x\overline{v}$$
$$- \varepsilon\sqrt{\mu}\frac{1}{h}\partial_x\left(\int_{-1+\beta b}^{\varepsilon\zeta} u_{\mathrm{sh}}^*v_{\mathrm{sh}}^*\right)$$
$$- \varepsilon\mu\left(\partial_x h\right)^2\mathrm{Q}_x\partial_x\overline{v} + \varepsilon\mu\frac{h^2}{3}\partial_x^2\overline{u}\partial_x\mathrm{Q}_y$$
$$- \varepsilon\mu\frac{1}{24h}\partial_x^2\left(h^3u^\sharp\right)\partial_x\overline{v} + \varepsilon\mu\frac{1}{24h}\partial_x\left(h^3v^\sharp\partial_x^2\overline{u}\right)$$
$$+ \varepsilon\max\left(\mu^{\frac{3}{2}}, \beta\sqrt{\mu}\right)R,$$

and v_{sh}^* satisfies the equation

$$\partial_t v_{\mathrm{sh}}^* + \varepsilon \overline{u}\partial_x v_{\mathrm{sh}}^* + \varepsilon u_{\mathrm{sh}}^*\partial_x\overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\left(u-\overline{u}\right) = \varepsilon\sqrt{\mu}\frac{1}{h}\partial_x\left(\int_{-1+\beta b}^{\varepsilon\zeta} u_{\mathrm{sh}}^* v_{\mathrm{sh}}^*\right) - \varepsilon\sqrt{\mu}u_{\mathrm{sh}}^*\partial_x v_{\mathrm{sh}}^*$$
$$+ \varepsilon\partial_x\left(\int_{-1+\beta b}^{z}\left[\overline{u}+\sqrt{\mu}u_{\mathrm{sh}}^*\right]\right)\partial_z v_{\mathrm{sh}}^*$$
$$+ \varepsilon\sqrt{\mu}\frac{1}{2}\left(\left[1+z-\beta b\right]^2 - \frac{h^2}{3}\right)\partial_x^2\overline{u}\partial_x\overline{v}$$
$$+ \varepsilon\left(\mu, \beta\sqrt{\mu}\right)R.$$

Proof. Using the first equation of the vorticity equation of the Castro-Lannes system (5.14), we have

$$\partial_t \boldsymbol{\omega}_x + \varepsilon u\partial_x\boldsymbol{\omega}_x + \frac{\varepsilon}{\mu}\mathrm{w}\partial_z\boldsymbol{\omega}_x = \varepsilon\boldsymbol{\omega}_x\partial_x u + \frac{\varepsilon}{\sqrt{\mu}}\boldsymbol{\omega}_z\partial_z u + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\partial_z u.$$

Then, using the fact that $\nabla^{\mu,0}\cdot\boldsymbol{\omega} = 0$ and $\nabla^{\mu,0}\cdot \mathrm{U}^{\mu,\gamma} = 0$, we get

$$\partial_t\boldsymbol{\omega}_x - \frac{\varepsilon}{\sqrt{\mu}}\partial_z\left(u\boldsymbol{\omega}_z\right) + \frac{\varepsilon}{\mu}\partial_z\left(\mathrm{w}\boldsymbol{\omega}_x\right) = \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\partial_z u.$$

then, we integrate with respect to z and, using the fact that $\partial_t\zeta - \frac{1}{\mu}\underline{\mathrm{U}}^{\mu}\cdot N^{\mu,0} = 0$, $\boldsymbol{\omega}_x = -\frac{1}{\sqrt{\mu}}\partial_z v$ and $\boldsymbol{\omega}_z = \partial_x v$, we obtain

$$\partial_t\left(\int_{-1+\beta b}^{\varepsilon\zeta}\boldsymbol{\omega}_x\right) - \frac{\varepsilon}{\sqrt{\mu}}\underline{u}\partial_x\underline{v} + \frac{\varepsilon}{\sqrt{\mu}}u\partial_x v + \frac{\varepsilon}{\mu^{\frac{3}{2}}}\mathrm{w}\partial_z v + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\left(u-\underline{u}\right) = 0.$$

Then, we integrate again with respect to z and, using Proposition 5.2.4 and the fact that $\partial_t\zeta - \frac{1}{\mu}\underline{\mathrm{U}}^{\mu}\cdot N^{\mu,0} = 0$, $\mathrm{U}_b^{\mu}\cdot N_b^{\mu,0} = 0$, and $\nabla^{\mu,0}\cdot\mathrm{U}^{\mu} = 0$, we get

$$\partial_t \mathrm{Q}_y - \frac{\varepsilon}{\sqrt{\mu}}\underline{u}\partial_x\underline{v} + \frac{\varepsilon}{\sqrt{\mu}}\frac{1}{h}\partial_x\left(\int_{-1+\beta b}^{\varepsilon\zeta} uv\right) + \frac{1}{\sqrt{\mu}h}\partial_t h\overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}\left(\overline{u}-\underline{u}\right) = 0.$$

Then, thanks to Propositions 5.2.3, 5.2.4 and 5.2.5 we finally obtain that

$$\partial_t\mathrm{Q}_y + \varepsilon\overline{u}\partial_x\mathrm{Q}_y + \varepsilon\mathrm{Q}_x\partial_x\overline{v} + \frac{\varepsilon}{\mathrm{Ro}\sqrt{\mu}}(\overline{u}-\underline{u}) = \varepsilon\sqrt{\mu}\mathrm{Q}_x\partial_x\mathrm{Q}_y - \varepsilon\sqrt{\mu}\frac{1}{3}h^2\partial_x^2\overline{u}\partial_x\overline{v}$$
$$- \varepsilon\sqrt{\mu}\frac{1}{h}\partial_x\left(\int_{-1+\beta b}^{\varepsilon\zeta} u_{\mathrm{sh}}^* v_{\mathrm{sh}}^*\right)$$
$$+ \varepsilon\mu\overline{Tu_{\mathrm{sh}}^*}\partial_x\overline{v} + \varepsilon\mu\frac{h^2}{3}\partial_x^2\overline{u}\partial_x\mathrm{Q}_y$$
$$+ \varepsilon\mu\frac{1}{2h}\partial_x\left(\int_{-1+\beta b}^{\varepsilon\zeta} v_{\mathrm{sh}}^*\left(\left[1+z-\beta b\right]^2 - \frac{h^2}{3}\right)\partial_x^2\overline{u}\right)$$
$$+ \varepsilon\max\left(\mu^{\frac{3}{2}}, \beta\sqrt{\mu}\right)R.$$

Finally, we can compute that

$$\frac{1}{2}\int_{-1+\beta b}^{\varepsilon\zeta} v_{\mathrm{sh}}^{*}\left([1+z-\beta b]^{2}-\frac{h^{2}}{3}\right)=\frac{1}{24}h^{3}v^{\sharp},$$

and the first equation follows from Lemma 5.2.7. The second equation follows similarly using the fact that $v_{\mathrm{sh}}^{*}=v_{\mathrm{sh}}-\mathrm{Q}_{y}$. □

As noticed in [33], the quantity E defined by

$$E=\begin{pmatrix}E_{xx} & E_{xy}\\ E_{xy} & E_{yy}\end{pmatrix}=\int_{-1+\beta b}^{\varepsilon\zeta}\mathbf{V}_{\mathrm{sh}}^{*}\otimes\mathbf{V}_{\mathrm{sh}}^{*} \tag{5.58}$$

appears in the equations of Q_{x} and Q_{y} and can not be express with respect to ζ, $\overline{\mathbf{V}}$ and $\mathbf{V}^{\sharp}$. The following subsection is devoting to giving an equation for E.

5.4.2 Equations for E

In this part, we derive an equation for E up to terms of order $\mathcal{O}(\mu)$. We have to introduce the quantity F

$$F=(F_{ijk})_{i,j,k}=\int_{-1+\beta b}^{\varepsilon\zeta}\mathbf{V}_{\mathrm{sh}}^{*}\otimes\mathbf{V}_{\mathrm{sh}}^{*}\otimes\mathbf{V}_{\mathrm{sh}}^{*}. \tag{5.59}$$

The following proposition gives an equation for E.

Proposition 5.4.3. *If $\left(\zeta,\mathrm{U}_{/\!/}^{\mu,0},\boldsymbol{\omega}\right)$ satisfy the Castro-Lannes system* (5.14)*, then E satisfies the following equation*

$$\begin{aligned}\partial_{t}E+\varepsilon\overline{u}\partial_{x}E+\varepsilon l\left(E,\partial_{x}\overline{\mathbf{V}}\right)+\varepsilon\sqrt{\mu}\partial_{x}F_{\cdot,\cdot,1}+\frac{\varepsilon}{\mathrm{Ro}}E^{S}&=\left(\varepsilon\sqrt{\mu}\partial_{x}\overline{v}+\frac{\varepsilon\sqrt{\mu}}{\mathrm{Ro}}\right)\mathcal{D}(\mathbf{V}^{\sharp},\overline{u})\\&\quad+\max\left(\varepsilon\mu,\varepsilon\beta\sqrt{\mu},\frac{\varepsilon}{\mathrm{Ro}}\mu\right)R,\end{aligned}$$

where

$$E^{S}=\int_{-1+\beta b}^{\varepsilon\zeta}\mathbf{V}_{\mathrm{sh}}^{\perp}\otimes\mathbf{V}_{\mathrm{sh}}+\mathbf{V}_{\mathrm{sh}}\otimes\mathbf{V}_{\mathrm{sh}}^{\perp}=\begin{pmatrix}-2E_{xy} & E_{xx}-E_{yy}\\ E_{xx}-E_{yy} & 2E_{xy}\end{pmatrix} \tag{5.60}$$

$$l\left(E,\partial_{x}\overline{\mathbf{V}}\right)=\begin{pmatrix}3\partial_{x}\overline{u}E_{xx}+2\partial_{x}\overline{v}E_{xy} & 2\partial_{x}\overline{u}E_{xy}+\partial_{x}\overline{v}E_{yy}\\ 2\partial_{x}\overline{u}E_{xy}+\partial_{x}\overline{v}E_{yy} & \partial_{x}\overline{u}E_{yy}\end{pmatrix} \tag{5.61}$$

and

$$\mathcal{D}(\mathbf{V}^{\sharp},\overline{u})=\partial_{x}^{2}\overline{u}\begin{pmatrix}0 & u^{\sharp}\\ u^{\sharp} & 2v^{\sharp}\end{pmatrix}. \tag{5.62}$$

Proof. The proof is similar to the computation in Part 4.5.2 and Part 5.4.1 in [33]. We compute $\partial_{t}E$ and we use the second equations of Propositions 5.4.1 and 5.4.2 up to terms of order $\mathcal{O}(\mu)$. For the Coriolis contribution, we use the expansion of u and v given in Proposition 5.2.5 and 5.2.4. □

The quantity F appears in the equation of E and can not be expressed with respect to ζ, $\overline{\mathbf{V}}$, $\mathbf{V}^{\sharp}$ and E. The next proposition gives an equation for F up to terms of order $\mathcal{O}(\sqrt{\mu})$.

Proposition 5.4.4. *If* $\left(\zeta, \mathrm{U}_{/\!/}^{\mu,0}, \boldsymbol{\omega}\right)$ *satisfy the Castro-Lannes system* (5.14)*, then* F_{ijk} *satisfies the following equation*

$$\partial_t F_{ijk} + \varepsilon(\overline{u}\partial_x F_{ijk} + \partial_x \overline{u} F_{ijk} + F_{1kj}\partial_x \mathbf{V}_i + F_{i1k}\partial_x \mathbf{V}_j + F_{ij1}\partial_x \mathbf{V}_k) + \frac{\varepsilon}{\mathrm{Ro}} F^S = \max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right)\sqrt{\mu} R,$$

where

$$F^S = \int_{-1+\beta b}^{\varepsilon\zeta} \mathbf{V}_{\mathrm{sh}}^{\perp} \otimes \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}} + \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}}^{\perp} \otimes \mathbf{V}_{\mathrm{sh}} + \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}}^{\perp}. \tag{5.63}$$

Proof. The proof is similar to the computation in Part 4.5.3 and Part 5.4.2 in [33]. We compute $\partial_t F$ and we use the second equations of Propositions 5.4.1 and 5.4.2 up to terms of order $\mathcal{O}(\sqrt{\mu})$. For the Coriolis contribution, we use the expansion of u and v in Proposition 5.2.5 and 5.2.4. □

5.4.3 The Green-Naghdi equations

We can now establish the Green-Naghdi equations when $d = 1$. The Green-Naghdi equations are the following system

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{u}) = 0, \\ (1+\mu\mathcal{T})(\partial_t \overline{u} + \varepsilon\overline{u}\partial_x\overline{u}) + \partial_x\zeta - \frac{\varepsilon}{\mathrm{Ro}}\overline{v} + \varepsilon\mu\mathcal{Q}(\overline{u}) + \varepsilon\mu\partial_x E_{xx} + \varepsilon\mu^{\frac{3}{2}}\mathcal{C}_1\big(u^\sharp, \overline{u}\big) + \frac{\varepsilon}{\mathrm{Ro}}\frac{\mu^{\frac{3}{2}}}{24h}\partial_x^2(h^3 v^\sharp) = 0, \\ \partial_t\overline{v} + \varepsilon\overline{u}\partial_x\overline{v} + \frac{\varepsilon}{\mathrm{Ro}}\overline{u} + \varepsilon\mu\partial_x E_{xy} + \varepsilon\mu^{\frac{3}{2}}\mathcal{C}_2\left(v^\sharp, \partial_x^2\overline{u}\right) = 0, \\ \partial_t \mathbf{V}^\sharp + \varepsilon\mathbf{V}^\sharp\partial_x\overline{u} + \varepsilon\overline{u}\partial_x\mathbf{V}^\sharp + \frac{\varepsilon}{\mathrm{Ro}}\mathbf{V}^{\sharp\perp} = 0, \\ \partial_t E + \varepsilon\overline{u}\partial_x E + \varepsilon\, l\big(E, \partial_x\overline{\mathbf{V}}\big) + \varepsilon\sqrt{\mu}\partial_x F_{\cdot,\cdot,1} + \frac{\varepsilon}{\mathrm{Ro}}E^S = \Big(\varepsilon\sqrt{\mu}\partial_x\overline{v} + \frac{\varepsilon}{\mathrm{Ro}}\sqrt{\mu}\Big)\mathcal{D}(\mathbf{V}^\sharp, \overline{u}), \\ \partial_t F_{ijk} + \varepsilon\overline{u}\partial_x F_{ijk} + \varepsilon\partial_x\overline{u}F_{ijk} + \varepsilon F_{1kj}\partial_x\mathbf{V}_i + \varepsilon F_{i1k}\partial_x\mathbf{V}_j + \varepsilon F_{ij1}\partial_x\mathbf{V}_k + \frac{\varepsilon}{\mathrm{Ro}}F^S = 0. \end{cases} \tag{5.64}$$

where

$$\begin{aligned} &\mathcal{T} = -\frac{1}{3h}\partial_x\left(h^3\partial_x\cdot\right), \\ &\mathcal{Q}(\overline{u}) = \frac{2}{3h}\partial_x\left(h^3\left[\partial_x\overline{u}\right]^2\right), \\ &\mathcal{C}_1\left(u^\sharp, \overline{u}\right) = -\frac{1}{6h}\partial_x\left(2h^3 u^\sharp\partial_x^2\overline{u} + \partial_x(h^3 u^\sharp)\partial_x\overline{u}\right), \\ &\mathcal{C}_2\left(v^\sharp, w\right) = -\frac{1}{24h}\partial_x\left(h^3 v^\sharp w\right), \\ &l\left(E, \partial_x\overline{\mathbf{V}}\right) = \begin{pmatrix} 3\partial_x\overline{u}E_{xx} + 2\partial_x\overline{v}E_{xy} & 2\partial_x\overline{u}E_{xy} + \partial_x\overline{v}E_{yy} \\ 2\partial_x\overline{u}E_{xy} + \partial_x\overline{v}E_{yy} & \partial_x\overline{u}E_{yy} \end{pmatrix}, \\ &\mathcal{D}(\mathbf{V}^\sharp, \overline{u}) = \partial_x^2\overline{u}\begin{pmatrix} 0 & u^\sharp \\ u^\sharp & 2v^\sharp \end{pmatrix} \end{aligned} \tag{5.65}$$

and

$$
\begin{aligned}
E^S &= \int_{-1+\beta b}^{\varepsilon\zeta} \mathbf{V}_{\mathrm{sh}}^{\perp} \otimes \mathbf{V}_{\mathrm{sh}} + \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}}^{\perp} = \begin{pmatrix} -2E_{xy} & E_{xx} - E_{yy} \\ E_{xx} - E_{yy} & 2E_{xy} \end{pmatrix}, \\
F^S &= \int_{-1+\beta b}^{\varepsilon\zeta} \mathbf{V}_{\mathrm{sh}}^{\perp} \otimes \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}} + \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}}^{\perp} \otimes \mathbf{V}_{\mathrm{sh}} + \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}} \otimes \mathbf{V}_{\mathrm{sh}}^{\perp},
\end{aligned}
\tag{5.66}
$$

and $\mathbf{V}^{\sharp}$ is defined in (5.29), E in (5.58) and F in (5.59). Notice that the first, the second and the third equations of System (5.64) are the classical Green-Naghdi equations with new terms due to the vorticity (terms with $\mathbf{V}^{\sharp}$ and E). The last equations are important to get a close system. We can now state that the Green-Naghdi equations are an order $\mathcal{O}(\mu^2)$ approximation of the water waves equations.

Proposition 5.4.5. *In the Green-Naghdi regime with small topography variations $\mathcal{A}_{GN}$, the Castro-Lannes equations* (5.14) *are consistent at order $\mathcal{O}(\mu^2)$ with the Green-Naghdi equations* (5.64) *in the sense of Definition 5.1.4.*

Proof. The proof is similar to the one in Proposition 5.2.12. The first equation of the Green-Naghdi equations is always satisfied for a solution of the Castro-Lannes formulation by Proposition 5.2.3. For the second equation, we use Proposition 5.2.5, Proposition 5.4.1 together with Proposition 5.2.6, Lemma 5.2.7 and Proposition 5.2.10. Notice the fact that all the terms with Q_x disappear. The third equation follows from Proposition 5.2.4, 5.2.5 and 5.4.2 (all the terms with Q_y disappear also). The last equations follows from Propositions 5.2.10, 5.4.3 and 5.4.4. □

Remark 5.4.6. *Notice that even without a Coriolis forcing, we can not decrease the number of equations in the previous Green-Naghdi equations. However, if one suppose also that the vorticity is initially of the form $(0, \omega_y, 0)^t$, which corresponds to the propagation of 2D water waves, we can significantly simplify the Green-Naghdi equations (See Section 4 in [33] and [82]).*

5.4.4 A simplified model in the case of a weak rotation and medium amplitude waves

As noticed in [33], if we assume that $\varepsilon = \mathcal{O}(\sqrt{\mu})$ we can simplify the Green-Naghdi equations. This regime corresponds to medium amplitude waves (in the terminology of [80]). We also assume that $\frac{\varepsilon}{\mathrm{Ro}} = \mathcal{O}(\sqrt{\mu})$. Then, we can simplify the Green-Naghdi system (5.64) by dropping all the terms of $\mathcal{O}(\mu^2)$ and we get

$$
\begin{cases}
\partial_t \zeta + \partial_x (h\overline{u}) = 0, \\
(1 + \mu\mathcal{T}) (\partial_t \overline{u} + \varepsilon \overline{u} \partial_x \overline{u}) + \partial_x \zeta - \dfrac{\varepsilon}{\mathrm{Ro}} \overline{v} + \varepsilon\mu \mathcal{Q}(\overline{u}) + \varepsilon\mu \partial_x E_{xx} = 0, \\
\partial_t \overline{v} + \varepsilon \overline{u} \partial_x \overline{v} + \dfrac{\varepsilon}{\mathrm{Ro}} \overline{u} + \varepsilon\mu \partial_x E_{xy} = 0, \\
\partial_t E + \varepsilon \overline{u} \partial_x E + \varepsilon\, l\big(E, \partial_x \overline{\mathbf{V}}\big) + \dfrac{\varepsilon}{\mathrm{Ro}} E^S = 0.
\end{cases}
\tag{5.67}
$$

Notice that in this regime, we catch effects of the vorticity on $\overline{\mathbf{V}}$ thanks to the second order tensor E. Without vorticity, this regime is particularly interesting since it is related to the Camassa-Holm equation and the Degasperis-Procesi equation (see for instance [41]). It could be interesting to understand how we can adapt these two scalar equations in presence of a Coriolis forcing.

Appendix A

Laplace problems for water waves

Sommaire

This appendix is devoted to the study of Laplace problems that appear in the water waves problems. After obtaining the existence and uniqueness of solutions, we give precise regularity estimates. The second section is devoted to the study of non local operators that appear in the water waves problems. We give regularity estimate and shape derivatives of these operators. Notice that it is very important to obtain precise regularity estimates if one wants to get a local wellposedness for the water waves equations.

We recall that the water occupies the domain $\Omega_t := \{(X,z) \in \mathbb{R}^{d+1} \ , \ -1+\beta b(t,X) < z < \varepsilon\zeta(t,X)\}$. We assume also that the water depth is bounded from below by a positive constant

$$\exists\, h_{\min} > 0 \ , \ \varepsilon\zeta + 1 - \beta b \geq h_{\min}. \tag{A.1}$$

Finally, we suppose that there $\mu_{\max} > 0$, such that

$$0 < \varepsilon \leq 1 \ , \ 0 \leq \beta \leq 1 \text{ and } \mu \leq \mu_{\max}. \tag{A.2}$$

In this chapter, C is a constant and for a function f in a normed space $(X, |\cdot|)$ or a parameter γ, $C(|f|, \gamma)$ is a constant depending on $|f|$ and γ whose exact value has non importance. The norm $|\cdot|_2$ is the L^2-norm and $|\cdot|_\infty$ is the L^∞-norm in $\mathbb{R}^d$. Let $f \in \mathcal{C}^0(\mathbb{R}^d)$ and $m \in \mathbb{N}$ such that $\frac{f}{1+|x|^m} \in L^\infty(\mathbb{R}^d)$. We define the Fourier multiplier $f(D) : H^m(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ as

$$\forall u \in H^m(\mathbb{R}^d), \ \widehat{f(D)u}(\xi) = f(\xi)\widehat{u}(\xi).$$

In $\mathbb{R}^d$ we denote the gradient operator by ∇ and in Ω or $S = \mathbb{R}^d \times (-1,0)$ the gradient operator is denoted $\nabla_{X,z}$. Finally, we denote by $\Lambda := \sqrt{1+|D|^2}$ with $D = -i\nabla$.

In this chapter, $(,)$ is the standard $L^2(\mathbb{R}^d)$ scalar product.

A.1 The Laplace problems

A.1.1 Formulation of the problems

In this part, we extend the results of Chapter 2 in [80] and Section 4 of [71] (see also [78] and [70]). We suppose that the parameters ε, μ and β satisfy Condition (A.2). In this chapter, we have to study two Laplace problems. The first one is the problem

$$\begin{cases} \Delta^\mu \Phi^S = 0 \text{ in } \Omega_t \ , \\ {\Phi^S}_{|z=\varepsilon\zeta} = \psi \ , \ \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_n {\Phi^S}_{|z=-1+\beta b} = 0. \end{cases} \tag{A.3}$$

The second one is the problem

$$\begin{cases} \Delta^\mu \Phi^B = 0 \text{ in } \Omega_t \ , \\ {\Phi^B}_{|z=\varepsilon\zeta} = 0 \ , \ \sqrt{1+\beta^2|\nabla b|^2}\partial_n {\Phi^B}_{|z=-1+\beta b} = B, \end{cases} \tag{A.4}$$

and here $B := \partial_t b$. Notice that n is here the upward normal. We work with Beppo Levi spaces. We define the Beppo Levi spaces as, for $s \geq 0$,

$$\dot{H}^s(\mathbb{R}^d) := \left\{\psi \in L^2_{\text{loc}}(\mathbb{R}^d), \ \nabla\psi \in H^{s-1}(\mathbb{R}^d)\right\}.$$

We refer to [53] and Proposition 2.3 in [80] for general results about these spaces but we recall the following properties of these spaces.

Proposition A.1.1. *Let $s \geq 1$. Then, $H^s(\mathbb{R}^d)$ is dense in $\dot{H}^s(\mathbb{R}^d)$. Furthermore, $\dot{H}^1(\mathbb{R}^d \times (-1,0))/_{\mathbb{R}}$ is a Hilbert space for the norm $|\nabla_{X,z} \cdot |_{L^2}$.*

In order to give precise regularity estimates, we fix the domain. We transform these problems into variable coefficients elliptic problems on $S := \mathbb{R}^d \times (-1,0)$ (the flat strip). We introduce a regularizing diffeomorphism. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a positive, compactly supported, smooth, even function equal to one near 0. For $\delta > 0$ we define

$$\Sigma := \begin{array}{ccc} S & \longrightarrow & \Omega \\ (X,z) & \mapsto & (X, z + [\theta(\delta z|D|)\varepsilon\zeta(X) - \theta(\delta(z+1)|D|)\beta b(X)]\, z + \varepsilon\theta(\delta z|D|)\zeta(X)) \,. \end{array}$$

We omit the dependence on t here. We denote by M_0 a constant of the form

$$M = C\left(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{t_0+1}(\mathbb{R}^d)}, |b|_{H^{t_0+1}(\mathbb{R}^d)}\right).$$

In order to study the Laplace problems in S, we have to treat the regularity in the direction X and in the direction z one at a time. We introduce the following spaces.

Definition A.1.2. *Let $s \in \mathbb{R}$. We define the Banach spaces $\left(H^{s,k}(S), |\cdot|_{s,k}\right)$, $k = 0,1$, by*

$$H^{s,1}(S) := L^2_z H^s_X(S) \cap H^1_z H^{s-1}_X(S), \text{ and } |u|^2_{s,1} = |\Lambda^s u|^2_2 + |\Lambda^{s-1}\partial_z u|^2_2,$$

and

$$H^{s,0}(S) := L^2_z H^s_X(S), \text{ and } |u|^2_{s,0} = |\Lambda^s u|^2_2.$$

Remark A.1.3. *We have the following inclusion (see Proposition 2.10 in [80]) for $s \in \mathbb{R}$*

$$H^{s+\frac{1}{2},1}(S) \subset L^\infty_z H^s_X(S).$$

Σ satisfies the following properties (see Propositions 2.16 and 2.18 in [80] for the proof).

Proposition A.1.4. *Let $t_0 > \frac{d}{2}$, ζ, $b \in H^{t_0+1}(\mathbb{R}^d)$ such that Condition* (A.1) *is satisfied and $s \leq t_0 + \frac{1}{2}$. Then, we can extend continuously Σ as a mapping $\overline{S} \to \overline{\Omega}$. Furthermore, for $\delta > 0$ small enough, Σ is a regularizing diffeomorphism and we have*

$$|J_\Sigma|_{L^\infty(S)} \leq M \ , \ |\ det\ (J_\Sigma)|_{L^\infty(S)} \geq \frac{h_{\min}}{2} \text{ and } \partial_z \Sigma \geq \frac{1}{2}.$$

Finally, if we denote by

$$\sigma(X,z) := [\theta(\delta z|D|)\varepsilon\zeta(X) - \theta(\delta(z+1)|D|)\beta b(X)]\, z + \varepsilon\theta(\delta z|D|)\zeta(X),$$

we have

$$\begin{aligned}
\left|\nabla^\mu_{X,z}\sigma\right|_{H^{s,1}(S)} &\leq C\left(\mu_{\max}, \frac{1}{\delta}\right)\left(|\varepsilon\zeta|_{H^{s+\frac{1}{2}}} + |\beta b|_{H^{s+\frac{1}{2}}}\right), \\
\left|\Lambda^{s-\frac{1}{2}}\nabla^\mu_{X,z}\sigma\right|_{L^\infty_z L^2_X(S)} &\leq C\left(\mu_{\max}, \frac{1}{\delta}\right)\left(|\varepsilon\zeta|_{H^{s+\frac{1}{2}}} + |\beta b|_{H^{s+\frac{1}{2}}}\right), \\
\left|\Lambda^{s-\frac{3}{2}}\partial_z\nabla^\mu_{X,z}\sigma\right|_{L^\infty_z L^2_X(S)} &\leq C\left(\mu_{\max}, \frac{1}{\delta}\right)\left(|\varepsilon\zeta|_{H^{s+\frac{1}{2}}} + |\beta b|_{H^{s+\frac{1}{2}}}\right).
\end{aligned}$$

Remark A.1.5. *Notice that if* $h \in H^s(\mathbb{R}^d)$*, then*

$$\theta(\delta z|D|)h \in L_z^\infty H_X^s(S) \cap L_z^2 H_X^{s+\frac{1}{2}}(S).$$

See Lemma 2.20 in [80].

Remark A.1.6. *The smallness of* δ *depends on the inverse of* $(|\varepsilon\zeta|_{H^{t_0+1}} + |\beta b|_{H^{t_0+1}})$.

In the following, we fix $\delta > 0$ such that Proposition A.1.4 is valid. Then, we can transform our equations. We denote by $\phi^S := \Phi^S \circ \Sigma$ and $\phi^B := \Phi^B \circ \Sigma$. We obtain that ϕ^S and ϕ^B satisfy

$$\begin{cases} \nabla^\mu_{X,z} \cdot P(\Sigma)\nabla^\mu_{X,z}\phi^S = 0 \text{ in } S, \\ \phi^S_{|z=0} = \psi \ , \ \partial_n \phi^S_{|z=-1} = 0, \end{cases} \tag{A.5}$$

and

$$\begin{cases} \nabla^\mu_{X,z} \cdot P(\Sigma)\nabla^\mu_{X,z}\phi^B = 0 \text{ in } S, \\ \phi^B_{|z=0} = 0 \ , \ \partial_n \phi^B_{|z=-1} = B, \end{cases} \tag{A.6}$$

with $P(\Sigma) = I_{d+1\times d+1} + Q(\Sigma) \in S^{++}_{d+1}(\mathbb{R})$ and

$$Q(\Sigma) := \begin{pmatrix} \partial_z \sigma I_{d\times d} & -\sqrt{\mu}\nabla_X \sigma \\ -\sqrt{\mu}\nabla_X \sigma^t & \frac{-\partial_z\sigma + \mu|\nabla_X\sigma|^2}{1+\partial_z\sigma}. \end{pmatrix}. \tag{A.7}$$

Notice that $P(\Sigma)$ is well defined thanks to Proposition A.1.4 and that $\partial_n := \mathbf{e}_z \cdot (P(\Sigma)\nabla^\mu \cdot\ )$. We have to know the regularity of $P(\Sigma)$. It is the subject of the next proposition.

Proposition A.1.7. *Let* $t_0 > \frac{d}{2}$*,* $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ *such that Condition* (A.1) *is satisfied. Then,*

$$|Q(\Sigma)|_{H^{t_0+\frac{1}{2},1}} \ , \left|\Lambda^{t_0} Q(\Sigma)\right|_{L^\infty_z L^2_X(S)} , \left|\Lambda^{t_0-1}\partial_z Q(\Sigma)\right|_{L^\infty_z L^2_X(S)} \leq M.$$

Furthermore, $P(\Sigma)$ *is coercive. There exist a constant* $k(\Sigma) > 0$ *such that* $\frac{1}{k(\Sigma)} \leq M$ *and*

$$\forall \Theta \in \mathbb{R}^{d+1} \ , \forall (X,z) \in \mathcal{S} \ , \ P(\Sigma)(X,z)\Theta \cdot \Theta \geq k(\Sigma)|\Theta|^2.$$

Proof. By the product estimate B.2.3, $|\nabla_X \sigma|^2 \in H^{t_0+\frac{1}{2},1}(S)$. Then, using Products estimates B.2.2 and B.2.5 we obtain

$$\begin{aligned}
\left|\Lambda^{t_0+\frac{1}{2}} Q(\Sigma)\right|_{L^2(S)} &\leq \max(\sqrt{\mu_{\max}}, 1)\ C\left(\left|\Lambda^{t_0+\frac{1}{2}}\nabla^\mu_{X,z}\sigma\right|_{L^2(S)}\right), \\
\left|\Lambda^{t_0-\frac{1}{2}}\partial_z Q(\Sigma)\right|_{L^2(S)} &\leq \max(\sqrt{\mu_{\max}}, 1)\ C\left(\left|\Lambda^{t_0-\frac{1}{2}}\partial_z\nabla^\mu_{X,z}\sigma\right|_{L^2(S)}\right), \\
\left|\Lambda^{t_0} Q(\Sigma)\right|_{L^\infty_z L^2_X(S)} &\leq \max(\sqrt{\mu_{\max}}, 1)\ C\left(\left|\Lambda^{t_0}\nabla^\mu_{X,z}\sigma\right|_{L^\infty_z L^2_X(S)}\right), \\
\left|\Lambda^{t_0-1}\partial_z Q(\Sigma)\right|_{L^\infty_z L^2_X(S)} &\leq \max(\sqrt{\mu_{\max}}, 1)\ C\left(\left|\Lambda^{t_0-1}\partial_z\nabla^\mu_{X,z}\sigma\right|_{L^\infty_z L^2_X(S)}\right).
\end{aligned}$$

We get the first estimates thanks to Proposition A.1.4. A straightforward computing gives the second point (see Lemma 2.26 in [80]). □

In order to define variational formulations of the Laplace problems, it is useful to introduce an extension result.

Definition A.1.8. *Let χ be a smooth compactly supported real function that is equal to* 1 *near* 0 *and $\psi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$. We denote by*

$$\psi^{\dagger}(.,z) := \psi + \int_0^1 \chi'(s\sqrt{\mu}z|D|)ds\sqrt{\mu}z|D|\psi, \tag{A.8}$$

where $-1 < z < 0$. We introduce also $\mathfrak{P} := \frac{|D|}{\sqrt{1+\sqrt{\mu}|D|}}$. We will see that $\mathfrak{P}$ acts as the square root of the Dirichlet-Neumann operator in $\dot{H}^s(\mathbb{R}^d)$. Lemma 2.34 in [80] gives the following regularity result for $\psi^{\dagger}$.

Proposition A.1.9. *Let $s \in \mathbb{R}$ and $\psi \in \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$. Then, $\nabla^{\mu}_{X,z}\psi^{\dagger} \in H^{s,1}(S)$ and*

$$|\Lambda^s \nabla^{\mu}_{X,z}\psi^{\dagger}|_{L^2(S)} \leq C\sqrt{\mu}|\mathfrak{P}\psi|_{H^s(\mathbb{R}^d)} \text{ and } |\Lambda^{s-1}\partial_z \nabla^{\mu}_{X,z}\psi^{\dagger}|_{L^2(S)} \leq C\mu|\mathfrak{P}\psi|_{H^s(\mathbb{R}^d)}.$$

We can now define the variational formulations of Problem (A.5) and (A.6). We introduce

$$H^1_{0,surf}(S) := \overline{\mathcal{D}(S \cup \{z=-1\})}^{|\,|_{H^1(S)}} = \overline{\mathcal{D}(S \cup \{z=-1\})}^{|\,|_{\dot{H}^1(S)}}.$$

See Proposition 2.3 (3) in [80] for a proof of the second equality.

Definition A.1.10. *Let $\psi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$. We say that ϕ is a variational solution of* (A.5) *if $\phi = \widetilde{\phi} + \psi^{\dagger}$ with $\widetilde{\phi} \in H^1_{0,surf}(S)$ and for all $\varphi \in H^1_{0,surf}(S)$,*

$$\int_S \nabla^{\mu}\widetilde{\phi} \cdot P(\Sigma)\nabla^{\mu}\varphi = -\int_S \nabla^{\mu}\psi^{\dagger} \cdot P(\Sigma)\nabla^{\mu}\varphi.$$

Definition A.1.11. *Let $B \in H^{-\frac{1}{2}}(\mathbb{R}^d)$. We say that $\phi \in H^1_{0,surf}(S)$ is a variational solution of* (A.6) *if for all $\varphi \in H^1_{0,surf}(S)$,*

$$\int_S \nabla^{\mu}\phi \cdot P(\Sigma)\nabla^{\mu}\varphi = -\langle B, \varphi_{|z=-1}\rangle_{H^{-\frac{1}{2}}-H^{\frac{1}{2}}}.$$

We have the following trace result.

Lemma A.1.12. *For all $\varphi \in H^1_{0,surf}(S)$ we have*

$$\left|\sqrt{1+\sqrt{\mu}|D|}\,\varphi_{|z=-1}\right|_{L^2(\mathbb{R}^d)} \leq 2\left|\nabla^{\mu}_{X,z}\varphi\right|_{L^2(S)}.$$

Proof. Let $u \in \mathcal{D}(S \cup \{z=-1\})$. We have

$$\begin{aligned}
\int_{\xi}(1+\sqrt{\mu}|\xi|)|\widehat{u}(\xi,-1)|^2 d\xi &\leq 2\int_{\xi}(1+\sqrt{\mu}|\xi|)\int_{z=-1}^0 |\partial_z\widehat{u}(\xi,z)||\widehat{u}(\xi,z)|dzd\xi \\
&\leq \int_{\xi}\int_{z=-1}^0 |\partial_z\widehat{u}(\xi,z)|^2 dz + \int_{\xi}\int_{z=-1}^0 (1+\sqrt{\mu}|\xi|)^2|\widehat{u}(\xi,z)|^2 dzd\xi \\
&\leq \int_{\xi}\int_{z=-1}^0 |\partial_z\widehat{u}(\xi,z)|^2 dzd\xi + 2\int_{\xi}\int_{z=-1}^0 (1+\mu|\xi|^2)|\widehat{u}(\xi,z)|^2 dzd\xi
\end{aligned}$$

Furthermore, one can check that

$$\int_{\xi}\int_{z=-1}^0 |\widehat{u}(\xi,z)|^2 dzd\xi \leq \frac{1}{2}\int_{\xi}\int_{z=-1}^0 |\partial_z\widehat{u}(\xi,z)|^2 dzd\xi.$$

Finally,

$$\int_\xi (1+\sqrt{\mu}|\xi|)|\widehat{u}(\xi,-1)|^2 d\xi \leq 2\int_\xi \int_{z=-1}^0 |\partial_z \widehat{u}(\xi,z)|^2 dz d\xi + 2\int_\xi \int_{z=-1}^0 \mu|\xi|^2 |\widehat{u}(\xi,z)|^2 dz d\xi \ ,$$

and the result follows by density. ☐

We can now establish existence and uniqueness results.

Proposition A.1.13. *Let $\psi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$, $B \in H^{-\frac{1}{2}}(\mathbb{R}^d)$ and ζ, $b \in H^{t_0+1}(\mathbb{R}^d)$ satifying* (A.1). *Then, problems* (A.5) *and* (A.6) *have a unique variational solution named respectively $\psi^{\mathfrak{h}} \in \dot{H}^1(\mathbb{R}^d)$ and $B^{\mathfrak{d}} \in H^1_{0,surf}(S)$.*

Proof. Because S is bounded in the direction z and that $P(\Sigma)$ is uniformly coercive, the results follow from the Lax-Milgram's theorem, Proposition (A.1.1) and Poincaré's inequality in $H^1_{0,surf}(S)$. ☐

It is tempting to extend the variational formulations A.1.10 and A.1.11 for $\phi \in \dot{H}^1(S)/_{\mathbb{R}}$. It is worked for $\psi^{\mathfrak{h}}$ but it requires that $B \in \left(\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)/_{\mathbb{R}}\right)'$ for $B^{\mathfrak{d}}$. In fact, we can identify $\left(\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)/_{\mathbb{R}}\right)'$ as the Banach space $H_*^{-\frac{1}{2}}(\mathbb{R}^d)$, where

$$H_*^{-\frac{1}{2}}(\mathbb{R}^d) := \left\{ u \in H^{-\frac{1}{2}}(\mathbb{R}^d),\, \exists \underline{u} \in H^{\frac{1}{2}}(\mathbb{R}^d),\, u = |D|\underline{u} \right\}, \tag{A.9}$$

endowed with norm $|\cdot|_{H_*^{-\frac{1}{2}}} := \left| \frac{\cdot}{|D|} \right|_{H^{\frac{1}{2}}}$. See for instance [29] for a proof of this result. Notice that $\underline{u} \in H^{\frac{1}{2}}(\mathbb{R}^d)$ is unique since

$$\ker(|D|) = \ker(-\Delta) = \{0\} \text{ in } L^2(\mathbb{R}^d).$$

Then, for $u \in H_*^{-\frac{1}{2}}(\mathbb{R}^d)$, $\frac{u}{|D|}$ makes sense and we can define, for $v \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$

$$(u,v)_{H_*^{-\frac{1}{2}}(\mathbb{R}^d) - \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)} = \left(|D|v, \frac{u}{|D|} \right)_{H^{-\frac{1}{2}}(\mathbb{R}^d) - H^{\frac{1}{2}}(\mathbb{R}^d)}.$$

Notice that if $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^d)$, we have

$$(u,\varphi)_{H_*^{-\frac{1}{2}} - \dot{H}^{1/2}} = (u,\varphi)_{H^{-1/2} - H^{1/2}} \, .$$

We have also a density result for $H_*^{-\frac{1}{2}}(\mathbb{R}^d)$. For $s \geq 0$, we denote by $H_*^{s-\frac{1}{2}}(\mathbb{R}^d)$ the space $H_*^{-\frac{1}{2}}(\mathbb{R}^d) \cap H^{s-\frac{1}{2}}(\mathbb{R}^d)$. Notice that

$$H_*^{s-\frac{1}{2}}(\mathbb{R}^d) := \left\{ u \in H^{s-\frac{1}{2}}(\mathbb{R}^d),\, \exists \underline{u} \in H^{s+\frac{1}{2}}(\mathbb{R}^d),\, u = |D|\underline{u} \right\}. \tag{A.10}$$

Lemma A.1.14. *Let $s > 0$. Then, $\mathcal{S}_0(\mathbb{R}^d) := \left\{ u \in \mathcal{S}(\mathbb{R}^d),\, \widehat{u}(0) = 0 \right\}$ is dense in $H_*^{-\frac{1}{2}}(\mathbb{R}^d)$ and $H^{s-\frac{1}{2}}(\mathbb{R}^d)$ is dense in $H_*^{-\frac{1}{2}}(\mathbb{R}^d)$.*

Proof. If $u \in H_*^{-\frac{1}{2}}(\mathbb{R}^d)$, $u = |D|\underline{u}$ with $\underline{u} \in H^{\frac{1}{2}}(\mathbb{R}^d)$ and we take

$$u_n := \left(\sqrt{\frac{1}{n^2} + |D|^2} - \frac{1}{n} \right) \underline{v}_n \ ,$$

with $\underline{v}_n \in \mathcal{S}(\mathbb{R}^d)$ converging to $\underline{u}$ in $H^{\frac{1}{2}}(\mathbb{R}^d)$. The result follows from the dominated convergence theorem. Furthermore, if $u \in H_*^{-\frac{1}{2}}(\mathbb{R}^d)$, we take

$$\underline{u}_n := \frac{\underline{u}}{\left(1 + \frac{|D|^2}{n^2}\right)^{\frac{s}{2}}} \ ,$$

and $u_n = |D|\underline{u}_n \in H_*^{s-\frac{1}{2}}(\mathbb{R}^d)$ converges to u in $H_*^{-\frac{1}{2}}(\mathbb{R}^d)$ thanks to the dominated convergence theorem. □

Remark A.1.15. *The density of $\mathcal{S}_0(\mathbb{R}^d)$ implies that $\int_{\mathbb{R}^d} u$ makes sense and is equal to 0 for all $u \in H_*^{-\frac{1}{2}}(\mathbb{R}^d)$.*

In the case where $B \in H_*^{-\frac{1}{2}}(\mathbb{R}^d)$, $P(\Sigma)\nabla^\mu B^{\mathfrak{d}}$ has the same property.

Proposition A.1.16. *Let $B \in H_*^{-\frac{1}{2}}(\mathbb{R}^d)$. Then $\dfrac{P(\Sigma)\nabla^\mu B^{\mathfrak{d}}}{|D|} \in H^{1,0}(S)$.*

Proof. Let $\varphi \in H^1_{0,surf}(S)$. Then, $\widetilde{\varphi} := \frac{\sqrt{1+|D|^2}}{\sqrt{\frac{1}{n}+|D|^2}}\varphi \in H^1_{0,surf}(S)$, and applying the variational formulation (A.1.11) to $\widetilde{\varphi}$ we have, thanks to the trace result A.1.12,

$$\begin{aligned}
\left| \int_S \frac{\sqrt{1+|D|^2}}{\sqrt{\frac{1}{n}+|D|^2}} P(\Sigma)\nabla^\mu B^{\mathfrak{d}} \cdot \nabla^\mu \varphi \right| &= \left| \int_{\mathbb{R}^d} \frac{\sqrt{1+|D|^2}}{\sqrt{\frac{1}{n}+|D|^2}} B \cdot \varphi_{|z=-1} \right| , \\
&\leq \left| \frac{\sqrt{1+|D|^2}}{\sqrt{1+\sqrt{\mu}|D|}} \frac{B}{\sqrt{\frac{1}{n}+|D|^2}} \right|_{L^2} \left| \sqrt{1+\sqrt{\mu}|D|}\varphi_{|z=-1} \right|_{L^2} , \\
&\leq C \left| \frac{1}{\sqrt{1+\sqrt{\mu}|D|}} \frac{B}{|D|} \right|_{H^1} |\nabla^\mu \varphi|_{L^2(S)} , \\
&\leq C \left| \frac{1}{\sqrt{1+\sqrt{\mu}|D|}} B \right|_{H^0_*} |\nabla^\mu \varphi|_{L^2(S)} .
\end{aligned}$$

Then, by duality, we obtain that

$$\left| \frac{P(\Sigma)\nabla^\mu B^{\mathfrak{d}}}{\sqrt{\frac{1}{n}+|D|^2}} \right|_{H^{1,0}(S)} \leq C \left| \frac{1}{\sqrt{1+\sqrt{\mu}|D|}} B \right|_{H^0_*(\mathbb{R}^d)} ,$$

and the result follows. □

A.1.2 Regularity estimates of the solutions

We need some regularity results for the solutions of the Laplace problems. The first result is Corollary 2.40 in [80]. We recall that $M = C\left(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{t_0+1}(\mathbb{R}^d)}, |b|_{H^{t_0+1}(\mathbb{R}^d)}\right)$.

Theorem A.1.17. *Let $t_0 > \frac{d}{2}$ and $0 \leq s \leq t_0 + \frac{1}{2}$. Let $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ be such that Condition* (A.1) *is satisfied. Then, for all $\psi \in \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$, we have*

$$\left|\Lambda^s \nabla^{\mu}_{X,z} \psi^{\mathfrak{h}}\right|_{L^2(S)} \leq \sqrt{\mu} M \left|\mathfrak{P}\psi\right|_{H^s(\mathbb{R}^d)} .$$

Futhermore, if $s \geq \max(0, 1-t_0)$, we have

$$\left|\Lambda^{s-1} \partial_z \nabla^{\mu}_{X,z} \psi^{\mathfrak{h}}\right|_{L^2(S)} \leq \sqrt{\mu} M \left|\mathfrak{P}\psi\right|_{H^s(\mathbb{R}^d)} .$$

Remark A.1.18. *If we summarize the previous theorem in a Sobolev framework, we see that for a domain with a regularity $H^{t_0+1}(\mathbb{R}^d)$, we can expect at most a $H^{t_0+\frac{3}{2}}$ regularity for the Laplace solution. This gain of one half derivative will be crucial for the local wellposedness of the water waves equations.*

We can prove the same estimates for $B^{\mathfrak{d}}$. Notice that the next result is an extension of Proposition 4.15 in [71].

Theorem A.1.19. *Let $t_0 > \frac{d}{2}$ and $0 \leq s \leq t_0 + \frac{1}{2}$. Let $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ be such that Condition* (A.1) *is satisfied. Then, for all $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$, we have*

$$\left|\Lambda^s \nabla^{\mu}_{X,z} B^{\mathfrak{d}}\right|_{L^2(S)} \leq M \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}} B\right|_{H^s(\mathbb{R}^d)} .$$

Futhermore, if $s \geq \max(0, 1-t_0)$, we have

$$\left|\Lambda^{s-1} \partial_z \nabla^{\mu}_{X,z} B^{\mathfrak{d}}\right|_{L^2(S)} \leq M \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}} B\right|_{H^s(\mathbb{R}^d)} .$$

Finally, if $B \in H^{s-\frac{1}{2}}_*(\mathbb{R}^d)$, $\dfrac{P(\Sigma)\nabla^{\mu} B^{\mathfrak{d}}}{|D|} \in H^{s+1,0}(S)$ (and in $H^{s+1,1}(S)$ if $s \geq 1 - t_0$).

Remark A.1.20. *Notice that, if $\mu > 0$, $\nabla^{\mu}_{X,z} B^{\mathfrak{d}}$ is a half more regular than B, whereas when μ goes to 0, $\partial_z B^{\mathfrak{d}}$ has the same regularity than B. This means that in the shallow water limit (μ goes to zero), we lose the regularity we gained. However, we will see in the next part that we can regain this loss of regularity for the Neumann-Neumann operator and that it will not a problem after all.*

Proof. Let $\delta > 0$ and χ be a smooth compactly supported real function that is equal to 1 near 0. We introduce the smoothing operator $\Lambda^s_{\delta} := \chi(\delta\Lambda)\Lambda^s$. We know that $B^{\mathfrak{d}} \in H^1_{0,surf}(S)$. Therefore, using $\Lambda^{2s}_{\delta} B^{\mathfrak{d}}$ a test function, we have

$$\int_S \nabla^{\mu} B^{\mathfrak{d}} \cdot P(\Sigma) \nabla^{\mu} \Lambda^{2s}_{\delta} B^{\mathfrak{d}} = -\int_{\mathbb{R}^d} B (\Lambda^{2s}_{\delta} B^{\mathfrak{d}})_{|z=-1}.$$

Since $P(\Sigma)$ is symmetric, Λ^s_{δ} commutes with ∇^{μ} and is independent of z we obtain that

$$\int_S P(\Sigma)\Lambda_\delta^s \nabla^\mu B^{\mathfrak{d}} \cdot \nabla^\mu \Lambda_\delta^s B^{\mathfrak{d}} = -\int_S [\Lambda_\delta^s, Q(\Sigma)]\nabla^\mu B^{\mathfrak{d}} \cdot \nabla^\mu \Lambda_\delta^s B^{\mathfrak{d}}$$
$$- \int_{\mathbb{R}^d} \frac{\Lambda_\delta^s}{\sqrt{1+\sqrt{\mu}|D|}} B \left(\sqrt{1+\sqrt{\mu}|D|}\Lambda_\delta^s B^{\mathfrak{d}} \right)_{|z=-1}.$$

Then by coercivity of $P(\Sigma)$ and trace inequality A.1.12

$$k(\Sigma) \left|\Lambda_\delta^s \nabla^\mu B^{\mathfrak{d}}\right|_2^2 \leq \left|[\Lambda_\delta^s, Q(\Sigma)]\nabla^\mu B^{\mathfrak{d}}\right|_2 \left|\Lambda_\delta^s \nabla^\mu B^{\mathfrak{d}}\right|_2$$
$$+ 2\left|\Lambda_\delta^s \nabla^\mu B^{\mathfrak{d}}\right|_2 \left|\frac{\Lambda_\delta^s}{\sqrt{1+\sqrt{\mu}|D|}} B\right|_{L^2(\mathbb{R}^d)},$$

and

$$k(\Sigma) \left|\Lambda_\delta^s \nabla^\mu B^{\mathfrak{d}}\right|_2 \leq \left|[\Lambda_\delta^s, Q(\Sigma)]\nabla^\mu B^{\mathfrak{d}}\right|_2 + 2\left|\frac{\Lambda_\delta^s}{\sqrt{1+\sqrt{\mu}|D|}} B\right|_{L^2(\mathbb{R}^d)}.$$

We have to distinguish two cases.

a) $0 \leq s \leq t_0$ **:**

The commutator estimate B.3.2 (with $T_0 = t_0$) and Proposition A.1.7 give

$$k(\Sigma) \left|\Lambda_\delta^s \nabla^\mu B^{\mathfrak{d}}\right|_2 \leq C \left|Q(\Sigma)\right|_{L^\infty H^{t_0}} \left|\Lambda_\delta^{s-\varepsilon} \nabla^\mu B^{\mathfrak{d}}\right|_2 + 2\left|\frac{\Lambda_\delta^s}{\sqrt{1+\sqrt{\mu}|D|}} B\right|_{L^2(\mathbb{R}^d)}$$
$$\leq M \left|\Lambda_\delta^{s-\varepsilon} \nabla^\mu B^{\mathfrak{d}}\right|_2 + 2\left|\frac{\Lambda_\delta^s}{\sqrt{1+\sqrt{\mu}|D|}} B\right|_{L^2(\mathbb{R}^d)}$$

for some $\varepsilon > 0$ small enough ($\varepsilon < t_0 - \frac{d}{2}$). Using a finite induction on s and taking the limit when δ goes to 0, the first inequality follows. For the second estimate, we only need to give a control of $\partial_z^2 B^{\mathfrak{d}}$. We use Equation (A.6) satisfied by $B^{\mathfrak{d}}$. We express $P(\Sigma)$ as

$$P(\Sigma) := \begin{pmatrix} (1+a(X,z))I_{d\times d} & \mathbf{q}(X,z) \\ \mathbf{q}^t(X,z) & 1+q_{d+1}(X,z) \end{pmatrix}.$$

A simple computing gives

$$(1+q_{d+1})\partial_z^2 B^{\mathfrak{d}} = -\sqrt{\mu}\nabla_X \cdot \left((1+a)\sqrt{\mu}\nabla_X B^{\mathfrak{d}}\right) - \sqrt{\mu}\nabla_X \cdot \left(\partial_z B^{\mathfrak{d}} \mathbf{q}\right)$$
$$- \sqrt{\mu}\partial_z \mathbf{q} \cdot \nabla_X B^{\mathfrak{d}} - \sqrt{\mu}\partial_z \nabla_X B^{\mathfrak{d}} \cdot \mathbf{q} - \partial_z q_{d+1}\partial_z B^{\mathfrak{d}}.$$

We have a, $\mathbf{q}$, $q_{d+1} \in L_z^\infty H_X^{t_0}(S)$, $\partial_z \mathbf{q}$, $\partial_z q_{d+1} \in L_z^\infty H_X^{t_0-1}(S)$ and $1 + q_{d+1} \geq k(\Sigma)$. Then, since $s \geq 1 - t_0$ and $\nabla_X B^{\mathfrak{d}} \in H^{s,1}(S)$, by the product estimates B.2.2 and B.2.4 (with $T_0 = t_0$), we obtain the result.

b) $t_0 \leq s \leq t_0 + \frac{1}{2}$ **:**

The commutator estimate B.3.3 (with $T_0 = t_0 + \frac{1}{2}$ and $t_1 > \frac{1}{2}$) and Proposition A.1.7 give

$$k(\Sigma) \left|\Lambda_\delta^s \nabla^\mu B^\mathfrak{d}\right|_{L^2} \leq M \left[\left|\Lambda_\delta^{s+\frac{1}{2}-t_1} \nabla^\mu B^\mathfrak{d}\right|_{L^2} + \left|\Lambda_\delta^{s-1+\frac{1}{2}-t_1} \partial_z \nabla^\mu B^\mathfrak{d}\right|_{L^2} + 2\left|\frac{\Lambda_\delta^s}{\sqrt{1+\sqrt{\mu}|D|}} B\right|_{L^2}\right].$$

We denote by $\varepsilon := \frac{1}{2} - t_1$. We obtain the first inequality for $t_0 \leq s \leq t_0 + \varepsilon$ thanks to the previous case. Furthermore, we saw that

$$\begin{aligned}(1+q_{d+1})\partial_z^2 B^\mathfrak{d} = &- \sqrt{\mu}\nabla_X^\mu \cdot \left((1+a)\sqrt{\mu}\nabla_X B^\mathfrak{d}\right) - \sqrt{\mu}\nabla_X \cdot \left(\partial_z B^\mathfrak{d} \mathbf{q}\right) \\ &- \sqrt{\mu}\partial_z \mathbf{q} \cdot \nabla_X B^\mathfrak{d} - \sqrt{\mu}\partial_z \nabla_X B^\mathfrak{d} \cdot \mathbf{q} - \partial_z q_{d+1} \partial_z B^\mathfrak{d}.\end{aligned}$$

We have a, $\mathbf{q}$, $q_{d+1} \in L_z^2 H_X^{t_0+\frac{1}{2}}(S)$, $\partial_z \mathbf{q}$, $\partial_z q_{d+1} \in L_z^2 H_X^{t_0-\frac{1}{2}}(S)$ and $1+q_{d+1} \geq k(\Sigma)$. Then, since $s \geq 1 - t_0$ and $\nabla_X B^\mathfrak{d} \in L_z^\infty H_X^{s-\frac{1}{2}}(S)$, by the product estimates B.2.2 and B.2.5 (with $T_0 = t_0$), and we obtain the second inequality for $t_0 \leq s \leq t_0 + \varepsilon$. Using a finite induction, we obtain the first and the second inequality.

Finally, if $B \in H_*^{s-\frac{1}{2}}(\mathbb{R}^d)$, we proceed as in the proof of Proposition A.1.16. □

Remark A.1.21. *Theorems A.1.17 and A.1.19 show that if $s \geq 1$, which corresponds to $\psi \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$ and $B \in H^{\frac{1}{2}}(\mathbb{R}^d)$, we have $B^\mathfrak{d} \in H^2(S)$ and $\psi^\mathfrak{h} \in \dot{H}^2(S)$. Hence, we can check that Φ^S solution* (A.3) *and Φ^B solution* (A.4) *are in $H^2(\Omega)$.*

A.1.3 Shape derivatives of the $B^\mathfrak{d}$

In order to get shape derivatives of the Dirichlet-Neumann and the Neumann operator, we need some for the solutions of the Laplace problems. For the Laplace problem (A.5), we refer to Appendix A in [80] and we only study the shape derivative of $B^\mathfrak{d}$. Let $t_0 > \frac{d}{2}$, $s \geq 0$ and $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$. We denote by Γ the set of functions (ζ, b) in $H^{t_0+1}(\mathbb{R}^d)$ satisfying (A.1). We introduce the map :

$$\mathfrak{D}(B) := \begin{cases} \Gamma & \to L_z^2 H_X^{s+1}(S) \\ [\zeta, b] & \mapsto B^\mathfrak{d}. \end{cases} \tag{A.11}$$

We begin with a result for $P(\Sigma)$.

Proposition A.1.22. *Let $t_0 > \frac{d}{2}$, $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ and $0 \leq s \leq t_0 + \frac{1}{2}$. Then, the map*

$$Q(\cdot) := \begin{cases} \Gamma & \to L_z^\infty H_X^{t_0}(S) \cap L_z^2 H_X^{t_0+\frac{1}{2}}(S) \\ [\zeta, b] & \mapsto Q(\Sigma) \end{cases}$$

is smooth. Furthermore, for all $j \in \mathbb{N}^*$, and $(\boldsymbol{h}, \boldsymbol{k}) := (h_1, .., h_j, k_1, .., k_j) \in H^{t_0+1}(\mathbb{R}^d)^{2j}$, we have

$$\left|d^jQ_{(\zeta,b)}.(\boldsymbol{h},\boldsymbol{k})\right|_{H^{t_0+\frac{1}{2},1}} \leq M\prod_{i\geq 1}\left|(\varepsilon h_i,\beta k_i)\right|_{H^{t_0+1}},$$

$$\left|\Lambda^{t_0}d^jQ_{(\zeta,b)}.(\boldsymbol{h},\boldsymbol{k})\right|_{L^\infty_z L^2_X(S)} \leq M\prod_{i\geq 1}\left|(\varepsilon h_i,\beta k_i)\right|_{H^{t_0+1}},$$

$$\left|\Lambda^{s}d^jQ_{(\zeta,b)}.(\boldsymbol{h},\boldsymbol{k})\right|_{L^2(S)} \leq M\ \left|(\varepsilon h_1,\beta k_1)\right|_{H^{s+\frac{1}{2}}}\prod_{i\geq 2}\left|(\varepsilon h_i,\beta k_i)\right|_{H^{t_0+1}},$$

$$\left|\Lambda^{s-\frac{1}{2}}d^jQ_{(\zeta,b)}.(\boldsymbol{h},\boldsymbol{k})\right|_{L^\infty_z L^2_X(S)} \leq M\ \left|(\varepsilon h_1,\beta k_1)\right|_{H^{s+\frac{1}{2}}}\prod_{i\geq 2}\left|(\varepsilon h_i,\beta k_i)\right|_{H^{t_0+1}}.$$

Proof. The fact that Q is smooth is clear. Since,

$$Q(\Sigma)=\begin{pmatrix}\partial_z\sigma I_{d\times d} & -\sqrt{\mu}\nabla_X\sigma\\ -\sqrt{\mu}\nabla_X\sigma^t & \frac{-\partial_z\sigma+\mu|\nabla_X\sigma|^2}{1+\partial_z\sigma}\end{pmatrix},$$

with $\sigma(X,z):=[\theta(\delta z|D|)\varepsilon\zeta(X)-\theta(\delta(z+1)|D|)\beta b(X)]\,z+\varepsilon\theta(\delta z|D|)\zeta(X)$. All the terms excepted $Q_{d+1,d+1}$ are linear and then, if we differentiate more than twice ($j\geq 2$), this terms disappear. We only have to consider

$$Q_{d+1,d+1}:=\frac{-\partial_z\sigma+\mu\left|\nabla_X\sigma\right|^2}{1+\partial_z\sigma}.$$

We notice that if we differentiate j times, we obtain a sum of products of terms of the form $|D|\theta(\delta z|D|)h_i$ and $|D|\theta(\delta(z+1)|D|)k_i$ which are $L^\infty_z H^{t_0}_X\cap L^2_z H^{t_0+\frac{1}{2}}_X$-functions. Then, using Remark A.1.5, we have, for $i\geq 2$, $|D|\theta(\delta z|D|)h_i$ and $|D|\theta(\delta(z+1)|D|)k_i$ in $L^\infty_z H^{t_0}_z\cap L^2_z H^{t_0+\frac{1}{2}}_z(\mathbb{R}^d)$ and, by the product estimate B.2.3, product of terms of this form stays in $L^\infty_z H^{t_0}_z\cap L^2_z H^{t_0+\frac{1}{2}}_z$. Furthermore, if we take (h_1,k_1) in $H^{s+\frac{1}{2}}(\mathbb{R}^d)$, $|D|\theta(\delta z|D|)h_1$ and $|D|\theta(\delta(z+1)|D|)k_1$ in $L^\infty_z H^{s-\frac{1}{2}}_X\cap L^2_z H^s_X(\mathbb{R}^d)$, by Products estimates B.2.3 and B.2.2 we obtain the third and the fourth inequality. However, if we take (h_1,k_1) in $H^{t_0+1}(\mathbb{R}^d)$, we obtain the first and the second inequality. ☐

We can now prove shape derivatives estimates for $B^{\mathfrak{d}}$.

Theorem A.1.23. *Let $t_0>\frac{d}{2}$, $\zeta,b\in H^{t_0+1}(\mathbb{R}^d)$, $0\leq s\leq t_0+\frac{1}{2}$ and $B\in H^{s-\frac{1}{2}}(\mathbb{R}^d)$. Then, $\mathfrak{D}(B)$ is smooth. Furthermore, for all $j\in\mathbb{N}^*$, and $(\boldsymbol{h},\boldsymbol{k}):=(h_1,..,h_j,k_1,..,k_j)\in H^{t_0+1}(\mathbb{R}^d)^{2j}$, we have*

$$\left|\Lambda^s\nabla^\mu_{X,z}d^j\mathfrak{D}(B)_{(\zeta,b)}.(\boldsymbol{h},\boldsymbol{k})\right|_{L^2(S)}\leq M\prod_i\left|(\varepsilon h_i,\beta k_i)\right|_{H^{t_0+1}}\left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\ B\right|_{H^s(\mathbb{R}^d)},$$

Futhermore, if $s\geq\max(0,1-t_0)$, we have

$$\left|\Lambda^{s-1}\partial_z\nabla^\mu_{X,z}d^j\mathfrak{D}(B)_{(\zeta,b)}.(\boldsymbol{h},\boldsymbol{k})\right|_{L^2(S)}\leq M\prod_i\left|(\varepsilon h_i,\beta k_i)\right|_{H^{t_0+1}}\left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\ B\right|_{H^s(\mathbb{R}^d)}.$$

Proof. We recall that $B^{\mathfrak{d}}$ satisfies the variational formulation of Definition A.1.11. We differentiate formally with respect to $(\mathbf{h}, \mathbf{k})$ this formulation and we obtain, for all $\varphi \in H^1_{0,surf}(S)$,

$$\int_S \sum_{\substack{j_1+j_2=j \\ I_1 \sqcup I_2 = [1,j]}} C_{j_1,j_2,I_1,I_2}\, d^{j_1} P_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1} \nabla^\mu d^{j_2} \mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2} \cdot \nabla^\mu \varphi \, dx = 0,$$

where C_{j_1,j_2,I_1,I_2} are constants. Notice that the coordinates of $(\mathbf{h},\mathbf{k})_{I_1}$ and $(\mathbf{h},\mathbf{k})_{I_2}$ form a permutation of the coordinates of $(\mathbf{h},\mathbf{k})$. We denote by $\mathbf{g}$ and v

$$\mathbf{g} := \sum_{\substack{j_1+j_2=j,\ j_2<j \\ I_1 \sqcup I_2 = [1,j]}} C_{j_1,j_2,I_1,I_2}\, d^{j_1} Q_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1} \nabla^\mu d^{j_2} \mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2},$$

and

$$v := d^j \mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k}).$$

Notice also that $C_{0,j,\emptyset,[1,j]} = 1$. Then we have

$$\int_S P(\Sigma) \nabla^\mu v \cdot \nabla^\mu \varphi \, dx = - \int_S \mathbf{g} \cdot \nabla^\mu \varphi \, dx.$$

Therefore, we can use the same method that Theorem A.1.19. We take $\varphi = \Lambda^{2s}_\delta B^{\mathfrak{d}}$ and a simple computing gives

$$k(\Sigma) \left| \Lambda^s_\delta \nabla^\mu v \right|_{L^2} \leq \left| [\Lambda^s_\delta, Q(\Sigma)] v \right|_2 + \left| \Lambda^s_\delta \, \mathbf{g} \right|_{L^2(S)},$$

We have to distinguish two cases.

a) $0 \leq s \leq t_0$ **:**

The commutator estimate B.3.2 (with $T_0 = t_0$), Proposition A.1.7 and a finite induction on s gives

$$\left| \Lambda^s \nabla^\mu v \right|_{L^2(S)} \leq M \left| \Lambda^s \, \mathbf{g} \right|_{L^2(S)}.$$

Furthermore, using the product estimate B.2.2 we obtain

$$\left| \Lambda^s \, \mathbf{g} \right|_{L^2(S)} \leq C \sum_{\substack{j_1+j_2=j,\ j_2<j \\ I_1 \sqcup I_2 = [1,j]}} \left| d^{j_1} Q_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1} \right|_{L^\infty_z H^{t_0}_X} \left| \Lambda^s \nabla^\mu d^{j_2} \mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2} \right|_{L^2(S)}.$$

Using the first inequality of Proposition A.1.22, Theorem A.1.19 and an induction on j, we obtain the first inequality. Furthermore, in order to obtain the second inequality, we only need to estimate $\partial_z^2 v$. We know that v satisfies,

$$\nabla^\mu_{X,z} \cdot P(\Sigma) \nabla^\mu_{X,z} v = \nabla^\mu_{X,z} \cdot \mathbf{g}.$$

Using the same notation than Theorem A.1.19, we have

$$(1+q_{d+1})\partial_z^2 v = -\sqrt{\mu}\nabla_X^\mu\cdot(a\sqrt{\mu}\nabla_X v)-\sqrt{\mu}\nabla_X\cdot(\partial_z v\,\mathbf{q})$$
$$-\sqrt{\mu}\partial_z\mathbf{q}\cdot\nabla_X v-\sqrt{\mu}\partial_z\nabla_X v\cdot\mathbf{q}-\partial_z q_{d+1}\partial_z v-\nabla_{X,z}\cdot\mathbf{g},$$

and $\nabla_{X,z}\cdot\mathbf{g}\in L_z^2H_X^{s-1}(\mathbb{R}^d)$ thanks to the first inequality. Then, the second inequality follows (same estimate as Theorem A.1.19).

b) $s=t_0+\frac{1}{2}$ **:**

The commutator estimate B.3.3 (with $T_0=t_0+\frac{1}{2}$ and $t_1>\frac{1}{2}$) and Proposition A.1.7 give

$$k(\Sigma)\left|\Lambda_\delta^{t_0+\frac{1}{2}}\nabla^\mu v\right|_{L^2}\leq M\left[\left|\Lambda_\delta^{t_0+1-t_1}\nabla^\mu v\right|_{L^2}+\left|\Lambda_\delta^{t_0-t_1}\partial_z\nabla^\mu v\right|_{L^2}+\left|\Lambda^{t_0+\frac{1}{2}}\mathbf{g}\right|_{L^2}\right].$$

Furthermore, by the product estimates B.2.3,

$$\left|\Lambda^{t_0+\frac{1}{2}}\,\mathbf{g}\right|_{L^2(S)}\leq C\sum_{\substack{j_1+j_2=j,\ j_2<j\\ I_1\sqcup I_2=[1,j]}}\left|d^{j_1}Q_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1}\right|_{L_z^\infty H_X^{t_0}}\left|\Lambda^{t_0+\frac{1}{2}}\nabla^\mu d^{j_2}\mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2}\right|_{L^2(S)}$$
$$+\left|\Lambda^{t_0+\frac{1}{2}}d^{j_1}Q_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1}\right|_{L^2}\left|\nabla^\mu d^{j_2}\mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2}\right|_{L_z^\infty H_X^{t_0}}$$

and hence

$$\left|\Lambda^{t_0+\frac{1}{2}}\,\mathbf{g}\right|_{L^2(S)}\leq C\sum_{\substack{j_1+j_2=j,\ j_2<j\\ I_1\sqcup I_2=[1,j]}}\left|d^{j_1}Q_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1}\right|_{L_z^\infty H_X^{t_0}}\left|\Lambda^{t_0+\frac{1}{2}}\nabla^\mu d^{j_2}\mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2}\right|_{L^2(S)}$$
$$+\left|\Lambda^{t_0+\frac{1}{2}}d^{j_1}Q_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1}\right|_{L^2}\left|\nabla^\mu d^{j_2}\mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2}\right|_{H_X^{t_0+\frac{1}{2},1}}.$$

Then, the result follows using small adaptations of the proof of Theorem A.1.19.

c) $0<s<t_0+\frac{1}{2}$ **:**

The result follows from the previous case by interpolations.

□

The following theorem gives also shape derivatives estimates for $B^{\mathfrak{d}}$.

Theorem A.1.24. *Let $t_0>\frac{d}{2}$, $\zeta,b\in H^{t_0+1}(\mathbb{R}^d)$, $0\leq s\leq t_0$ and $B\in H^{t_0}(\mathbb{R}^d)$. Then, $\mathfrak{D}(B)$ is smooth. Furthermore, for all $j\in\mathbb{N}^*$, and $(\boldsymbol{h},\boldsymbol{k}):=(h_1,..,h_j,k_1,..,k_j)\in H^{t_0+1}(\mathbb{R}^d)^{2j}$, we have*

$$\left|\Lambda^s\nabla_{X,z}^\mu d^j\mathfrak{D}(B)_{(\zeta,b)}.(\boldsymbol{h},\boldsymbol{k})\right|_{L^2}\leq M|(\varepsilon h_1,\beta k_1)|_{H^{s+\frac{1}{2}}}\prod_{i\geq 2}|(\varepsilon h_i,\beta k_i)|_{H^{t_0+1}}\left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\,B\right|_{H^{t_0+\frac{1}{2}}}.$$

Futhermore, if $s\geq\max(0,1-t_0)$, we have

$$\left|\Lambda^{s-1}\partial_z\nabla_{X,z}^\mu d^j\mathfrak{D}(B)_{(\zeta,b)}.(\boldsymbol{h},\boldsymbol{k})\right|_{L^2}\leq\sqrt{\mu}M|(\varepsilon h_1,\beta k_1)|_{H^{s+\frac{1}{2}}}\prod_{i\geq 2}|(\varepsilon h_i,\beta k_i)|_{H^{t_0+1}}\left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\,B\right|_{H^{t_0+\frac{1}{2}}}.$$

Proof. We use the same notation that the previous theorem. By the product estimate B.2.2, we have

$$\begin{aligned}|\Lambda^s \, \mathbf{g}|_{L^2(S)} \leq C \sum_{\substack{j_1+j_2=j,\ j_2<j \\ I_1 \sqcup I_2=[1,j] \\ (1,1)\in I_1}} & \left|\Lambda^s d^{j_1} Q_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1}\right|_{L^2(S)} \left|\nabla^\mu d^{j_2} \mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2}\right|_{L^\infty_z H^{t_0}(S)} \\ + C \sum_{\substack{j_1+j_2=j,\ j_2<j \\ I_1 \sqcup I_2=[1,j] \\ (1,1)\in I_2}} & \left|d^{j_1} Q_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1}\right|_{L^\infty_z H^{t_0}_X} \left|\Lambda^s \nabla^\mu d^{j_2} \mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2}\right|_{L^2(S)} .\end{aligned}$$

Since $|\cdot|_{L^\infty_z H^{t_0}_X} \leq C|\cdot|_{H^{t_0+\frac{1}{2},1}}$, we can control the first sum with the estimate of the theorem A.1.23 and the result follow by the second inequality of Proposition A.1.22 and a finite induction. ☐

A.2 The Dirichlet-Neumann and the Neumann-Neumann operators

We refer to Chapter 3 of [80] for more details about the Dirichlet-Neumann operator. We extend the result of Section 3 in [71]. We define the Dirichlet-Neumann operator $G_\mu[\varepsilon\zeta, \beta b]$ by

$$G_\mu[\varepsilon\zeta, \beta b](\psi) := \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_{\mathbf{n}} \Phi^S{}_{|z=\varepsilon\zeta} = -\mu\varepsilon\nabla\zeta \cdot \nabla_X \Phi^S{}_{|z=\varepsilon\zeta} + \partial_z \Phi^S{}_{|z=\varepsilon\zeta} \ , \tag{A.12}$$

where Φ^S satisfies (A.3). We define also the Neumann-Neumann operator $G^{NN}_\mu[\varepsilon\zeta, \beta b]$ by

$$G^{NN}_\mu[\varepsilon\zeta, \beta b](B) := \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_{\mathbf{n}} \Phi^B{}_{|z=\varepsilon\zeta} = -\mu\nabla(\varepsilon\zeta) \cdot \nabla_X \Phi^B{}_{|z=\varepsilon\zeta} + \partial_z \Phi^B{}_{|z=\varepsilon\zeta} \ , \tag{A.13}$$

where Φ^B satisfies (A.4).

Remark A.2.1. . *Notice that*

$$\frac{1}{\mu} G_\mu[0,0](\psi) = |D|^2 \frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu}|D|}\psi \ \textit{and}\ G^{NN}_\mu[0,0](B) = \frac{1}{\cosh(\sqrt{\mu}|D|)} B.$$

A.2.1 Main properties

We can express the Dirichlet-Neumann and the Neumann-Neumann operator with the formalism of the previous section. For $\psi \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$ and $B \in H^{\frac{1}{2}}(\mathbb{R}^d)$ we have

$$G_\mu[\varepsilon\zeta, \beta b](\psi) = \left(\mathbf{e}_z \cdot P(\Sigma)\nabla^\mu \psi^{\mathfrak{h}}\right)_{|z=0} , \tag{A.14}$$

and

$$G^{NN}_\mu[\varepsilon\zeta, \beta b](B) = \left(\mathbf{e}_z \cdot P(\Sigma)\nabla^\mu B^{\mathfrak{d}}\right)_{|z=0} . \tag{A.15}$$

The following two results is a summarize of Paragraph **3.1** in [80]. It gives some basic properties of the Dirichlet-Neumann operator. The first result is a symmetry property and a dual formulation of the Dirichlet-Neumann operator.

Proposition A.2.2. *Let $\psi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that* (A.1) *is satisfied. $G_\mu[\varepsilon\zeta, \beta b](\cdot)$ can be extended to $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ with the dual formulation*

$$G_\mu[\varepsilon\zeta, \beta b](\psi) = \begin{cases} \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)/_{\mathbb{R}} & \longrightarrow \qquad \mathbb{R} \\ \qquad \varphi & \longmapsto \displaystyle\int_S \nabla^\mu_{X,z}\psi^\mathfrak{h} \cdot P(\Sigma)\nabla^\mu_{X,z}\varphi^\dagger dx, \end{cases} \tag{A.16}$$

where $\varphi^\dagger$ is defined in (A.8)*. Furthermore, $G_\mu[\varepsilon\zeta, \beta b](\cdot)$ is a symmetric positive operator and, for all ψ_1, ψ_2, ψ in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$,*

$$\begin{aligned} &(G_\mu[\varepsilon\zeta, \beta b](\psi_1), \psi_2)_{(\dot{H}^{1/2})' - \dot{H}^{1/2}} = (G_\mu[\varepsilon\zeta, \beta b](\psi_2), \psi_1)_{(\dot{H}^{1/2})' - \dot{H}^{1/2}}, \\ &|(G_\mu[\varepsilon\zeta, \beta b](\psi_1), \psi_2)| \leq \sqrt{(G_\mu[\varepsilon\zeta, \beta b](\psi_1), \psi_1)} \sqrt{(G_\mu[\varepsilon\zeta, \beta b](\psi_2), \psi_2)}, \\ &(G_\mu[\varepsilon\zeta, \beta b](\psi), \psi) \geq 0, \\ &G_\mu[\varepsilon\zeta, \beta b](\psi) \in H^{-\frac{1}{2}}(\mathbb{R}^d). \end{aligned}$$

Finally, if $G_\mu[\varepsilon\zeta, \beta b](\psi_1)$, $G_\mu[\varepsilon\zeta, \beta b](\psi_2) \in L^2(\mathbb{R}^d)$, then

$$(G_\mu[\varepsilon\zeta, \beta b](\psi_1), \psi_2)_{(\dot{H}^{1/2})' - \dot{H}^{1/2}} = (G_\mu[\varepsilon\zeta, \beta b](\psi_1), \psi_2)_2 .$$

Remark A.2.3. *Notice that, instead of $\varphi^\dagger$, we can take $\varphi^\mathfrak{h}$ in the dual formulation.*

We see that $G_\mu[\varepsilon\zeta, \beta b]$ maps continuously $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ into $H_*^{-\frac{1}{2}}(\mathbb{R}^d)$. Furthermore, we can also compute explicitly an operator whose norm is equivalent to the square root of the Dirichlet-Neumann operator. We recall that $\mathfrak{P} := \frac{|D|}{\sqrt{1+\sqrt{\mu}|D|}}$ and that

$$M = C\left(\frac{1}{h_{\min}}, \mu_{\max}, |\zeta|_{H^{t_0+1}}, |b|_{H^{t_0+1}}\right).$$

Proposition A.2.4. *Let $t_0 > \frac{d}{2}$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that* (A.1) *is satisfied. Then, for all $\psi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$, we have*

$$\left(\frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b](\psi), \psi\right) \leq M\,|\mathfrak{P}\psi|_2^2 \;\; \text{and} \;\; |\mathfrak{P}\psi|_2^2 \leq M\left(\frac{1}{\mu}G_\mu[\varepsilon\zeta, \beta b](\psi), \psi\right).$$

Conversely the Neumann-Neumann operator is not symmetric but it does not have a loss of regularity. We need an extension result in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ in order to give a dual formulation of the Neumann-Neumann operator.

Definition A.2.5. *Let $\varphi \in H^{-\frac{1}{2}}(\mathbb{R}^d)$. We define $\varphi^\#$ as*

$$\varphi^\# = \frac{\sinh([z+1]\sqrt{\mu}|D|)}{\sinh(\sqrt{\mu}|D|)}\varphi.$$

Remark A.2.6. *$\varphi^\#$ satisfies weakly*

$$\begin{cases} \Delta^\mu \varphi^\# = 0 \text{ in } S, \\ \varphi^\#_{|z=-1} = \varphi \;;\; \varphi^\#_{|z=0} = 0. \end{cases}$$

We can prove easily regularity results for $\varphi^\#$ similar to $\varphi^\mathfrak{h}$.

Proposition A.2.7. *Let $s \geq 0$ and $\varphi \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$. Then,*

$$\left|\Lambda^{s-1}\nabla^{\mu}_{X,z}\varphi^{\#}\right|_{L^2(S)} \leq C(\mu_{\max})\left|\sqrt{1+\sqrt{\mu}|D|}\varphi\right|_{H^{s-1}(\mathbb{R}^d)},$$
$$\left|\Lambda^{s-2}\partial_z\nabla^{\mu}_{X,z}\varphi^{\#}\right|_{L^2(S)} \leq C(\mu_{\max})\mu^{\frac{1}{4}}\left|\sqrt{1+\sqrt{\mu}|D|}\varphi\right|_{H^{s-1}(\mathbb{R}^d)}.$$

We can now give a dual formulation of the Neumann-Neumann operator. We introduce the Dirichlet-Dirichlet operator, for $\psi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$,

$$G^{DD}_{\mu}[\varepsilon\zeta, \beta b](\psi) := \left(\psi^{\mathfrak{h}}\right)_{|z=-1}. \tag{A.17}$$

The following result extends Proposition 3.3 in [71].

Proposition A.2.8. *Let $t_0 > \frac{d}{2}$, $B \in H^{-\frac{1}{2}}_{*}(\mathbb{R}^d)$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that* (A.1) *is satisfied. $G^{NN}_{\mu}[\varepsilon\zeta, \beta b](\cdot)$ can be extended to $H^{-\frac{1}{2}}_{*}(\mathbb{R}^d)$ with the dual formulation*

$$G^{NN}_{\mu}[\varepsilon\zeta, \beta b](B) = \begin{cases} \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)/\mathbb{R} & \longrightarrow \quad \mathbb{R} \\ \psi & \longmapsto \left(\dfrac{P(\Sigma)\nabla^{\mu}_{X,z}B^{\mathfrak{d}}}{|D|}, \nabla^{\mu}_{X,z}(|D|\psi)^{\#}\right)_{H^{1,0}-H^{-1,0}} \end{cases}. \tag{A.18}$$

Furthermore, the adjoint of $G^{NN}_{\mu}[\varepsilon\zeta, \beta b]$ is $G^{DD}_{\mu}[\varepsilon\zeta, \beta b]$. For all $B \in H^{-\frac{1}{2}}_{*}(\mathbb{R}^d)$ and $\psi \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$,

$$\left(G^{NN}_{\mu}[\varepsilon\zeta, \beta b](B), \psi\right)_{H^{-\frac{1}{2}}_{*}-\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)} = \left(B, G^{DD}_{\mu}[\varepsilon\zeta, \beta b](\psi)\right)_{H^{-1/2}_{*}-\dot{H}^{1/2}}.$$

Remark A.2.9. *We see that $G^{NN}_{\mu}[\varepsilon\zeta, \beta b](\cdot)$ maps continuously $H^{-\frac{1}{2}}_{*}(\mathbb{R}^d)$ into itself. Furthermore, if $B \in H^{-\frac{1}{2}}(\mathbb{R}^d)$ and $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^d)$, we have*

$$\left(G^{NN}_{\mu}[\varepsilon\zeta, \beta b](B), \varphi\right)_{H^{-\frac{1}{2}}-H^{\frac{1}{2}}} = \int_S P(\Sigma)\nabla^{\mu}_{X,z}B^{\mathfrak{d}} \cdot \nabla^{\mu}_{X,z}\varphi^{\#}dx.$$

Hence, $G^{NN}_{\mu}[\varepsilon\zeta, \beta b](\cdot)$ maps continuously $H^{-\frac{1}{2}}(\mathbb{R}^d)$ into itself.

Remark A.2.10. *Notice that the formulation* (A.18) *makes sense thanks to Proposition A.1.16.*

Proof. We take $B \in H^{\frac{1}{2}}_{*}(\mathbb{R}^d)$ and $\psi \in H^{\frac{3}{2}}(\mathbb{R}^d)$. We compute

$$\begin{aligned}\int_{\mathbb{R}^d} G^{NN}_{\mu}[\varepsilon\zeta, \beta b](B)\,\psi\,dx &= \int_{\mathbb{R}^d}\left(\mathbf{e}_z \cdot P(\Sigma)\nabla^{\mu}B^{\mathfrak{d}}\right)_{|z=0}\left(\psi^{\#}\right)_{|z=0}dx \\ &\quad - \int_{\mathbb{R}^d}\left(\mathbf{e}_z \cdot P(\Sigma)\nabla^{\mu}B^{\mathfrak{d}}\right)_{|z=-1}\left(\psi^{\#}\right)_{|z=-1}dx, \\ &= \int_S P(\Sigma)\nabla^{\mu}_{X,z}B^{\mathfrak{d}} \cdot \nabla^{\mu}_{X,z}\psi^{\#}dx + \int_S \left(\nabla^{\mu}_{X,z} \cdot P(\Sigma)\nabla^{\mu}_{X,z}B^{\mathfrak{d}}\right)\varphi^{\#}dx, \\ &= \int_S P(\Sigma)\nabla^{\mu}_{X,z}B^{\mathfrak{d}} \cdot \nabla^{\mu}_{X,z}\psi^{\#}dx, \\ &= \int_S \frac{P(\Sigma)\nabla^{\mu}_{X,z}B^{\mathfrak{d}}}{|D|} \cdot \nabla^{\mu}_{X,z}(|D|\psi)^{\#}dx,\end{aligned}$$

since $(\cdot)^{\#}$ and D commute. Furthermore,

$$
\begin{aligned}
\int_{\mathbb{R}^d} G_\mu^{NN}[\varepsilon\zeta,\beta b](B)\,\psi \;-B\, G_\mu^{DD}[\varepsilon\zeta,\beta b](\psi) &= \int_{\mathbb{R}^d} \left(\mathbf{e}_z \cdot P(\Sigma)\nabla^\mu B^{\mathfrak{d}}\right)_{|z=0} \left(\psi^{\mathfrak{h}}\right)_{|z=0} dx \\
&\quad - \int_{\mathbb{R}^d} \left(\mathbf{e}_z \cdot P(\Sigma)\nabla^\mu B^{\mathfrak{d}}\right)_{|z=-1} \left(\psi^{\mathfrak{h}}\right)_{|z=-1} dx, \\
&= \int_S \left(\nabla^\mu_{X,z} \cdot P(\Sigma)\nabla^\mu_{X,z} B^{\mathfrak{d}}\right) \psi^{\mathfrak{h}} dx \\
&\quad + \int_S P(\Sigma)\nabla^\mu_{X,z} B^{\mathfrak{d}} \cdot \nabla^\mu_{X,z}\psi^{\mathfrak{h}} dx, \\
&= \int_S P(\Sigma)\nabla^\mu_{X,z} B^{\mathfrak{d}} \cdot \nabla^\mu_{X,z}\psi^{\mathfrak{h}} dx,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_S P(\Sigma)\nabla^\mu_{X,z} B^{\mathfrak{d}} \cdot \nabla^\mu_{X,z}\psi^{\mathfrak{h}} dx &= \int_S B^{\mathfrak{d}} \left(\nabla^\mu_{X,z} \cdot P(\Sigma)\nabla^\mu_{X,z}\psi^{\mathfrak{h}}\right) dx \\
&\quad + \int_{\mathbb{R}^d} \left(\mathbf{e}_z \cdot P(\Sigma)\nabla^\mu \psi^{\mathfrak{h}}\right)_{|z=0} \left(B^{\mathfrak{d}}\right)_{|z=0} dx \\
&\quad - \int_{\mathbb{R}^d} \left(\mathbf{e}_z \cdot P(\Sigma)\nabla^\mu \psi^{\mathfrak{h}}\right)_{|z=-1} \left(B^{\mathfrak{d}}\right)_{|z=-1} dx, \\
&= 0.
\end{aligned}
$$

Finally, thanks to Proposition A.1.16 and Proposition A.2.7, we have

$$
\begin{aligned}
\left| \left(\frac{P(\Sigma)\nabla^\mu_{X,z} B^{\mathfrak{d}}}{|D|}, \nabla^\mu_{X,z}(|D|\psi)^{\#} \right)_{H^{1,0}-H^{-1,0}} \right| &\leq \left| \frac{P(\Sigma)\nabla^\mu_{X,z} B^{\mathfrak{d}}}{|D|} \right|_{H^{1,0}} \left| \nabla^\mu_{X,z}(|D|\psi)^{\#} \right|_{H^{-1,0}}, \\
&\leq C \left| \frac{1}{\sqrt{1+\sqrt{\mu}|D|}} B \right|_{H^0_*} \left| \sqrt{1+\sqrt{\mu}|D|}|D|\psi \right|_{H^{-1}}, \\
&\leq C(\mu_{\max}) \left| \frac{1}{\sqrt{1+\sqrt{\mu}|D|}} B \right|_{H^0_*} |\mathfrak{P}\psi|_{L^2}\,.
\end{aligned}
$$

Since $H^{\frac{3}{2}}(\mathbb{R}^d)$ is dense in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)/_{\mathbb{R}}$ (Proposition A.1.1) and $H^{\frac{1}{2}}_*(\mathbb{R}^d)$ is dense in $H^{-\frac{1}{2}}_*(\mathbb{R}^d)$ (Lemma A.1.14), we get the result by density.

□

In order to study shape derivatives of Dirichlet-Neumann and Neumann-Neumann operators, we have to introduce the Neumann-Dirichlet operator. For $B \in H^{-\frac{1}{2}}(\mathbb{R}^d)$, we define

$$
G_\mu^{ND}[\varepsilon\zeta,\beta b](B) := \left(B^{\mathfrak{d}}\right)_{|z=-1}. \tag{A.19}
$$

The following result is a symmetry property and a dual formulation of the Neumann-Dirichlet operator which extends Proposition 3.3 in [71].

Proposition A.2.11. *Let* $B \in H^{-\frac{1}{2}}(\mathbb{R}^d)$ *and* $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ *such that* (A.1) *is satisfied.* $G_\mu^{ND}[\varepsilon\zeta, \beta b](B)$ *can be view as*

$$G_\mu^{ND}[\varepsilon\zeta, \beta b](B) = \begin{cases} H^{-\frac{1}{2}}(\mathbb{R}^d) & \longrightarrow \quad \mathbb{R} \\ C & \longmapsto -\int_S P(\Sigma)\nabla_{X,z}^\mu B^{\mathfrak{d}} \cdot \nabla_{X,z}^\mu C^{\mathfrak{d}} dx. \end{cases} \tag{A.20}$$

Furthermore, $G_\mu^{ND}[\varepsilon\zeta, \beta b]$ *is a negative operator and, for all* B_1*,* B_2*,* B *in* $H^{-\frac{1}{2}}(\mathbb{R}^d)$*,*

$$\begin{aligned} &\left(G_\mu^{ND}[\varepsilon\zeta, \beta b](B_1), B_2\right)_{(H^{-1/2})'-H^{1/2}} = \left(G_\mu^{ND}[\varepsilon\zeta, \beta b](B_2), B_1\right)_{(H^{-1/2})'-H^{1/2}}, \\ &\left|\left(G_\mu^{ND}[\varepsilon\zeta, \beta b](B_1), B_2\right)\right| \leq \sqrt{\left(G_\mu^{ND}[\varepsilon\zeta, \beta b](B_1), B_1\right)} \sqrt{\left(G_\mu^{ND}[\varepsilon\zeta, \beta b](B_2), B_2\right)}, \\ &\left(G_\mu^{ND}[\varepsilon\zeta, \beta b](B), B\right) \geq 0. \end{aligned}$$

Proof. We take $B \in H^{\frac{1}{2}}(\mathbb{R}^d)$. Since $B_1^{\mathfrak{d}}$ satisfies (A.6), we have

$$\begin{aligned} \left(G_\mu^{ND}[\varepsilon\zeta, \beta b](B_1), B_2\right) &= \int_{\mathbb{R}^d} \left(B_1^{\mathfrak{d}}\right)_{|z=-1} \left(\partial_n B_2^{\mathfrak{d}}\right)_{|z=-1} - \int_{\mathbb{R}^d} \left(B_1^{\mathfrak{d}}\right)_{|z=0} \left(\partial_n B_2^{\mathfrak{d}}\right)_{|z=0}, \\ &= -\int_S P(\Sigma)\nabla^\mu B_1^{\mathfrak{d}} \cdot \nabla^\mu B_2^{\mathfrak{d}} dx. \end{aligned}$$

This expression is symmetric and $P(\Sigma)$ is coercive so it is clear that $G_\mu^{ND}[\varepsilon\zeta, \beta b](\cdot)$ is a symmetric positive operator. Furthermore, we have

$$\begin{aligned} \left(G_\mu^{ND}[\varepsilon\zeta, \beta b](B_1), B_2\right) &= -\int_S P(\Sigma)\nabla^\mu B_1^{\mathfrak{d}} \cdot B_2^{\mathfrak{d}} dx, \\ &= -\int_S P(\Sigma)^{1/2}\nabla^\mu B_1^{\mathfrak{d}} \cdot P(\Sigma)^{1/2}\nabla^\mu B_2^{\mathfrak{d}} dx, \\ &\leq \left|P(\Sigma)^{1/2}\nabla^\mu B_1^{\mathfrak{d}}\right|_{L^2} \left|P(\Sigma)^{1/2}\nabla^\mu B_2^{\mathfrak{d}}\right|_{L^2}. \end{aligned}$$

The fact that

$$|P(\Sigma)^{1/2}\nabla^\mu B_1^{\mathfrak{d}}|_{L^2}^2 = -\left(G_\mu^{ND}[\varepsilon\zeta, \beta b](B_1), B_1\right)$$

implies the second inequality.

□

A.2.2 Regularity Estimates

In this part we give some controls of the Dirichlet-Neumann and the Neumann-Neumann operators. The first Proposition is Theorem 3.15 in [80].

Proposition A.2.12. *Let* $t_0 > \frac{d}{2}$*,* $0 \leq s \leq t_0 + \frac{1}{2}$ *and* $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$*. Then,* $G_\mu[\varepsilon\zeta, \beta b]$ *maps continuously* $\dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$ *into* $H_*^{s-\frac{1}{2}}(\mathbb{R}^d)$

$$|G_\mu[\varepsilon\zeta, \beta b](\psi)|_{H^{s-\frac{1}{2}}} \leq \mu^{\frac{3}{4}} M |\mathfrak{P}\psi|_{H^s}.$$

Furthermore, if $0 \leq s \leq t_0$*,* $G_\mu[\varepsilon\zeta, \beta b]$ *maps continuously* $\dot{H}^{s+1}(\mathbb{R}^d)$ *into* $H_*^{s-\frac{1}{2}}(\mathbb{R}^d)$

$$|G_\mu[\varepsilon\zeta, \beta b](\psi)|_{H^{s-\frac{1}{2}}} \leq \mu M |\mathfrak{P}\psi|_{H^{s+\frac{1}{2}}}.$$

We have a similar estimation for the Neumann-Neumann operator.

Proposition A.2.13. *Let $t_0 > \frac{d}{2}$, $0 \leq s \leq t_0 + \frac{1}{2}$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that Condition* (A.1) *is satisfied. Then, $G_\mu^{NN}[\varepsilon\zeta, \beta b]$ maps continuously $H_*^{s-\frac{1}{2}}(\mathbb{R}^d)$ into itself*

$$\left| G_\mu^{NN}[\varepsilon\zeta, \beta b](B) \right|_{H_*^{s-\frac{1}{2}}} \leq M \left| B \right|_{H_*^{s-\frac{1}{2}}}.$$

Remark A.2.14. *Notice for all μ (even μ goes to 0), $G_\mu^{NN}[\varepsilon\zeta, \beta b](B)$ has the same regularity than B.*

Proof. The dual formulation of $G_\mu^{NN}[\varepsilon\zeta, \beta b]$ shows that it maps continuously $H_*^{-\frac{1}{2}}(\mathbb{R}^d)$ into itself. We have to prove that the regularity of B is kept by G_μ^{NN}. We argue by duality. Let take $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$ and φ in $H^{s-\frac{1}{2}}(\mathbb{R}^d)$. We have

$$\left(\Lambda^{s-1/2} G_\mu^{NN}[\varepsilon\zeta, \beta b](B), \varphi \right) = \int_S P(\Sigma) \nabla_{X,z}^\mu B^{\mathfrak{d}} \cdot \nabla_{X,z}^\mu \left(\Lambda^{s-1/2} \varphi \right)^\# dx.$$

Since $(\cdot)^\#$, Λ and $\nabla_{X,z}^\mu$ commute, we obtain, using Proposition A.2.7,

$$\begin{aligned} \left| \left(\Lambda^{s-1/2} G_\mu^{NN}[\varepsilon\zeta, \beta b](B), \varphi \right) \right| &= \left| \int_S \frac{\sqrt{1+\sqrt{\mu}|D|}}{(1+|D|^2)^{\frac{1}{4}}} \Lambda^s P(\Sigma) \nabla_{X,z}^\mu B^{\mathfrak{d}} \cdot \nabla_{X,z}^\mu \left(\frac{1}{\sqrt{1+\sqrt{\mu}|D|}} \varphi \right)^\# dx \right|, \\ &\leq C(\mu_{\max}) \left| \varphi \right|_{L^2} \left| \sqrt{1+\sqrt{\mu}|D|} \Lambda^{s-\frac{1}{2}} P(\Sigma) \nabla_{X,z}^\mu B^{\mathfrak{d}} \right|_{L^2}. \end{aligned}$$

Then, we have

$$\begin{aligned} \left| \Lambda^{s-1/2} G_\mu^{NN}[\varepsilon\zeta, \beta b](B) \right|_{L^2(S)} \leq & C(\mu_{\max}) \left| \left[\sqrt{1+\sqrt{\mu}|D|} \Lambda^{s-\frac{1}{2}}, Q(\Sigma) \right] \nabla_{X,z}^\mu B^{\mathfrak{d}} \right|_{L^2} \\ & + C(\mu_{\max}) \left| Q(\Sigma) \sqrt{1+\sqrt{\mu}|D|} \Lambda^{s-\frac{1}{2}} \nabla_{X,z}^\mu B^{\mathfrak{d}} \right|_{L^2}. \end{aligned}$$

If $s \leq t_0$, we use Proposition B.3.2 (with a slight modification) and the product estimate B.2.2. We obtain

$$\left| \Lambda^{s-1/2} G_\mu^{NN}[\varepsilon\zeta, \beta b](B) \right|_{L^2(S)} \leq C \left(\left| Q(\Sigma) \right|_{L^\infty H^{t_0}} \left| \sqrt{1+\sqrt{\mu}|D|} \Lambda^{s-\frac{1}{2}} \nabla_{X,z}^\mu B^{\mathfrak{d}} \right|_2 \right).$$

However, if $t_0 \leq s \leq t_0 + \frac{1}{2}$, we use Proposition B.3.3 (with a slight modification) and the product estimate B.2.2. We obtain,

$$\begin{aligned} \left| \Lambda^{s-1/2} G_\mu^{NN}[\varepsilon\zeta, \beta b](B) \right|_{L^2(S)} \leq \; & C \left| Q(\Sigma) \right|_{L^2 H^{t_0+\frac{1}{2}}} \left| \sqrt{1+\sqrt{\mu}|D|} \Lambda^{s-1} \nabla_{X,z}^\mu B^{\mathfrak{d}} \right|_{L_z^\infty L_X^2} \\ & + C \left| Q(\Sigma) \right|_{L_z^\infty H_X^{t_0}} \left| \sqrt{1+\sqrt{\mu}|D|} \Lambda^{s-\frac{1}{2}} \nabla_{X,z}^\mu B^{\mathfrak{d}} \right|_{L^2}. \end{aligned}$$

In any case, using Theorem A.1.19, we get the result.

□

We can also give some regularity estimates for $G_\mu^{DD}[\varepsilon\zeta,\beta b]$ since it is the adjoint of $G_\mu^{NN}[\varepsilon\zeta,\beta b]$.

Proposition A.2.15. *Let $t_0 > \frac{d}{2}$, $0 \leq s \leq t_0 + \frac{1}{2}$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that Condition* (A.1) *is satisfied. Then, $G_\mu^{DD}[\varepsilon\zeta,\beta b]$ maps continuously $\dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$ into itself*

$$\left|\nabla G_\mu^{DD}[\varepsilon\zeta,\beta b](\psi)\right|_{H^{s-\frac{1}{2}}} \leq M \left|\nabla\psi\right|_{H^{s-\frac{1}{2}}}.$$

Finally, we can give some regularity estimates for $G_\mu^{ND}[\varepsilon\zeta,\beta b]$.

Proposition A.2.16. *Let $t_0 > \frac{d}{2}$, $0 \leq s \leq t_0 + \frac{1}{2}$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that Condition* (A.1) *is satisfied. Then, $G_\mu^{ND}[\varepsilon\zeta,\beta b]$ maps continuously $H^{s-\frac{1}{2}}(\mathbb{R}^d)$ into $H^{s+\frac{1}{2}}(\mathbb{R}^d)$*

$$\left|G_\mu^{ND}[\varepsilon\zeta,\beta b](B)\right|_{H^{s+\frac{1}{2}}} \leq M \left|B\right|_{H^{s-\frac{1}{2}}}.$$

We can extend these estimates to $\underline{w}[\varepsilon\zeta,\beta b]$, the vertical velocity at the surface and to $\underline{V}[\varepsilon\zeta,\beta b]$ the horizontal velocity at the surface. These operators appear naturally when we differentiate the Dirichlet-Neumann and the Neumann-Neumann operator with respect to the surface ζ. We define

$$\underline{w}[\varepsilon\zeta,\beta b] := \begin{cases} \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d) \times H_*^{s-\frac{1}{2}}(\mathbb{R}^d) \to & H^{s-\frac{1}{2}}(\mathbb{R}^d) \\ \quad (\psi, B) \mapsto & \dfrac{G_\mu[\varepsilon\zeta,\beta b](\psi) + \mu G_\mu^{NN}[\varepsilon\zeta,\beta b](B) + \varepsilon\mu\nabla\zeta\cdot\nabla\psi}{1+\varepsilon^2\mu|\nabla\zeta|^2}, \end{cases} \tag{A.21}$$

and

$$\underline{V}[\varepsilon\zeta,\beta b] := \begin{cases} \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d) \times H_*^{s-\frac{1}{2}}(\mathbb{R}^d) \to & H^{s-\frac{1}{2}}(\mathbb{R}^d) \\ \quad (\psi, B) \mapsto & \nabla\psi - \varepsilon\underline{w}[\varepsilon\zeta,\beta b](\psi,B)\nabla\zeta. \end{cases} \tag{A.22}$$

Proposition A.2.17. *Let $t_0 > \frac{d}{2}$, $0 \leq s \leq t_0 + \frac{1}{2}$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that Condition* (A.1) *is satisfied. Then, $\underline{w}[\varepsilon\zeta,\beta b]$ maps continuously $\dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s-\frac{1}{2}}(\mathbb{R}^d)$ into $H^{s-\frac{1}{2}}(\mathbb{R}^d)$*

$$\left|\underline{w}[\varepsilon\zeta,\beta b](\psi,B)\right|_{H^{s-\frac{1}{2}}} \leq M\left(\mu^{\frac{3}{4}}|\mathfrak{P}\psi|_{H^s} + \mu|B|_{H^{s-\frac{1}{2}}}\right).$$

Furthermore, if $1 \leq s \leq t_0$, $\underline{w}[\varepsilon\zeta,\beta b]$ maps continuously $\dot{H}^{s+1}(\mathbb{R}^d) \times H_*^{s-\frac{1}{2}}(\mathbb{R}^d)$ into $H^{s-\frac{1}{2}}(\mathbb{R}^d)$

$$\left|\underline{w}[\varepsilon\zeta,\beta b](\psi,B)\right|_{H^{s-\frac{1}{2}}} \leq M\mu\left(|\mathfrak{P}\psi|_{H^{s+\frac{1}{2}}} + |B|_{H^{s-\frac{1}{2}}}\right).$$

Finally, we have the same estimate holds for $\underline{V}[\varepsilon\zeta,\beta b]$.

In the same way, we can extend also these estimates to $\underline{\widetilde{w}}[\varepsilon\zeta,\beta b]$, the vertical velocity at the bottom and to $\underline{\widetilde{V}}[\varepsilon\zeta,\beta b]$ the horizontal velocity at the bottom. These operators appear naturally when we differentiate the Dirichlet-Neumann and the Neumann-Neumann operator with respect to the bottom b. We define

$$\underline{\widetilde{w}}[\varepsilon\zeta,\beta b] := \begin{cases} \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d) \times H^{s-\frac{1}{2}}(\mathbb{R}^d) \to & H^{s-\frac{1}{2}}(\mathbb{R}^d) \\ \quad (\psi, B) \mapsto & \dfrac{\mu B + \beta\mu\nabla b\cdot\nabla\left(G_\mu^{DD}[\varepsilon\zeta,\beta b](\psi) + \mu G_\mu^{ND}[\varepsilon\zeta,\beta b](B)\right)}{1+\beta^2\mu|\nabla b|^2}, \end{cases} \tag{A.23}$$

and

$$\underline{\widetilde{V}}[\varepsilon\zeta,\beta b] := \begin{cases} \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)\times H^{s-\frac{1}{2}}(\mathbb{R}^d) \to \quad H^{s-\frac{1}{2}}(\mathbb{R}^d) \\ \quad (\psi,B) \quad\quad \mapsto \nabla\left(G_\mu^{DD}[\varepsilon\zeta,\beta b](\psi)+\mu G_\mu^{ND}[\varepsilon\zeta,\beta b](B)\right)-\beta\underline{\widetilde{w}}[\varepsilon\zeta,\beta b](\psi,B)\nabla b. \end{cases} \tag{A.24}$$

Proposition A.2.18. *Let $t_0 > \frac{d}{2}$, $0 \leq s \leq t_0 + \frac{1}{2}$ and $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$ such that Condition* (A.1) *is satisfied. Then, $\underline{\widetilde{w}}[\varepsilon\zeta,\beta b]$ maps continuously $\dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)\times H^{s-\frac{1}{2}}(\mathbb{R}^d)$ into $H^{s-\frac{1}{2}}(\mathbb{R}^d)$*

$$|\underline{\widetilde{w}}[\varepsilon\zeta,\beta b](\psi,B)|_{H^{s-\frac{1}{2}}} \leq M\left(|\nabla\psi|_{H^{s-\frac{1}{2}}}+\mu|B|_{H^{s-\frac{1}{2}}}\right).$$

Finally, we have the same continuity result for $\underline{\widetilde{V}}[\varepsilon\zeta,\beta b]$.

We end this section with a commutator estimate for the Dirichlet-Neumann operator.

Proposition A.2.19. *Let $t_0 > \frac{d}{2}$ and $\zeta, b \in H^{t_0+2}(\mathbb{R}^d)$ such that Condition* (A.1) *is satisfied. Then, for all $V \in H^{t_0+1}(\mathbb{R}^d)^2$ and $u \in H^{\frac{1}{2}}(\mathbb{R}^d)$, we have*

$$\left(V\cdot\nabla u, \frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b](u)\right) \leq C\left(M,\varepsilon\left|\zeta\right|_{H^{t_0+2}},\beta\left|b\right|_{H^{t_0+2}}\right)|V|_{W^{1,\infty}}\left|\mathfrak{P}u\right|_2^2.$$

A.2.3 Shape derivatives

Let $t_0 > \frac{d}{2}$. Given $\psi \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$ and $B \in H_*^{\frac{1}{2}}(\mathbb{R}^d)$. We denote by Γ the set of functions (ζ, b) in $H^{t_0+1}(\mathbb{R}^d)$ satisfying (A.1). We introduce two maps :

$$G_\mu(\psi) := \begin{cases} \quad\Gamma \quad \to \; H_*^{\frac{1}{2}}(\mathbb{R}^d) \\ (\zeta,b) \mapsto G_\mu[\varepsilon\zeta,\beta b](\psi), \end{cases} \tag{A.25}$$

which is the Dirichlet-Neumann operator and

$$G_\mu^{NN}(B) := \begin{cases} \quad\Gamma \quad \to \; H_*^{\frac{1}{2}}(\mathbb{R}^d) \\ (\zeta,b) \mapsto G_\mu^{NN}[\varepsilon\zeta,\beta b](B), \end{cases} \tag{A.26}$$

which is the Neumann-Neumann operator. We can also define $\underline{w}(\psi,B)$ and $\underline{V}(\psi,B)$.

In order to linearize the water waves equations, we need a shape derivative formula for the Dirichlet-Neumann and the Neumann-Neumann operators. The following proposition is a summarize of Theorems 3.5 and 3.6 in [71] and Theorem 3.21 in [80].

Proposition A.2.20. *Let $t_0 > \frac{d}{2}$, $\zeta, b \in H^{t_0+1}(\mathbb{R}^d)$, $\psi \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$ and $B \in H^{\frac{1}{2}}(\mathbb{R}^d)$. Then, $G_\mu(\psi)$ and $G_\mu^{NN}(B)$ are Fréchet differentiable. For $(h,k) \in H^{t_0+1}(\mathbb{R}^d)$, we have*

$$dG_\mu(\psi).(h,0)+\mu dG_\mu^{NN}(B).(h,0) = -\varepsilon G_\mu[\varepsilon\zeta,\beta b](h\,\underline{w}[\varepsilon\zeta,\beta b](\psi,B)) \tag{A.27}$$

$$\qquad -\varepsilon\mu\nabla\cdot(h\,\underline{V}[\varepsilon\zeta,\beta b](\psi,B)), \tag{A.28}$$

and

$$dG_\mu(\psi).(0,k)+\mu dG_\mu^{NN}(B).(0,k) = -\beta\mu G_\mu^{NN}[\varepsilon\zeta,\beta b]\left(\nabla\cdot\left(k\,\underline{\widetilde{V}}[\varepsilon\zeta,\beta b](\psi,B)\right)\right),$$

Furthermore,

$$dG_\mu^{DD}(\psi).(h,0) + \mu dG_\mu^{ND}(B).(h,0) = -\varepsilon G_\mu^{DD}[\varepsilon\zeta,\beta b](h\,\underline{w}[\varepsilon\zeta,\beta b](\psi,B)).$$

Remark A.2.21. *Notice that each term in the r.h.s of the formula* (A.27) *lies on* $H^{-\frac{1}{2}}(\mathbb{R}^d)$ *where as* $dG_\mu(\psi)$ *and* $dG_\mu^{NN}(B)$ *are in* $H^{\frac{1}{2}}(\mathbb{R}^d)$*. In fact, there are a cancellation of the singular terms.*

Thanks to these formulae we can give some controls to the first shape derivatives of the operators. For instance, we give an estimate for $d\underline{\widetilde{w}}$ and $d\underline{\widetilde{V}}$.

Proposition A.2.22. *Let* $t_0 > \frac{d}{2}$ *and* $(\zeta,b) \in H^{t_0+1}(\mathbb{R}^d)$ *such that Condition* (A.1) *is satisfied. Then, for* $0 \leq s \leq t_0 + \frac{1}{2}$*, for* $\psi \in \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$ *and* $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$*, we have*

$$\left| d\underline{\widetilde{V}}(\psi,B).(h,k) \right|_{H^{s-\frac{1}{2}}},\ \left| d\underline{\widetilde{w}}(\psi,B).(h,k) \right|_{H^{s-\frac{1}{2}}} \leq M\,|(h,k)|_{H^{t_0+1}} \left(|\nabla\psi|_{H^{s-\frac{1}{2}}} + |B|_{H^{s-\frac{1}{2}}} \right).$$

Proof. This result follows from Proposition A.2.20 and Proposition A.2.13. □

We end this part by giving some controls of the shape derivatives of G_μ and G_μ^{NN}. We do not use the previous method, we differentiate j times directly the dual formulation of both operators. The following proposition is Proposition 3.28 in [80].

Proposition A.2.23. *Let* $t_0 > \frac{d}{2}$ *and* $(\zeta,b) \in H^{t_0+1}(\mathbb{R}^d)$ *such that Condition* (A.1) *is satisfied. Then*

- *(1) For all* $0 \leq s \leq t_0 + \frac{1}{2}$ *and* $\psi \in \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$

$$|d^j G_\mu.(\boldsymbol{h},\boldsymbol{k})(\psi)|_{H^{s-\frac{1}{2}}} \leq M\mu^{\frac{3}{4}} \prod_{i\geq 1} |(\varepsilon h_i,\beta k_i)|_{H^{t_0+1}} |\mathfrak{P}\psi|_{H^s},$$

- *(2) For all* $0 \leq s \leq t_0$ *and* $\psi \in \dot{H}^{s+1}(\mathbb{R}^d)$

$$|d^j G_\mu.(\boldsymbol{h},\boldsymbol{k})(\psi)|_{H^{s-\frac{1}{2}}} \leq M\mu \prod_{i\geq 1} |(\varepsilon h_i,\beta k_i)|_{H^{t_0+1}} |\mathfrak{P}\psi|_{H^{s+\frac{1}{2}}},$$

- *(3) For all* $0 \leq s \leq t_0$ *and* $\psi \in \dot{H}^{t_0+1}(\mathbb{R}^d)$

$$|d^j G_\mu.(\boldsymbol{h},\boldsymbol{k})(\psi)|_{H^{s-\frac{1}{2}}} \leq M\mu^{\frac{3}{4}} |(\varepsilon h_1,\beta k_1)|_{H^{s+\frac{1}{2}}} \prod_{i\geq 2} |(\varepsilon h_i,\beta k_i)|_{H^{t_0+1}} |\mathfrak{P}\psi|_{H^{t_0+\frac{1}{2}}}.$$

- *(4) For all* $0 \leq s \leq t_0 - \frac{1}{2}$ *and* $\psi \in \dot{H}^{t_0+1}(\mathbb{R}^d)$

$$|d^j G_\mu.(\boldsymbol{h},\boldsymbol{k})(\psi)|_{H^{s-\frac{1}{2}}} \leq M\mu |(\varepsilon h_1,\beta k_1)|_{H^{s+1}} \prod_{i\geq 2} |(\varepsilon h_i,\beta k_i)|_{H^{t_0+1}} |\mathfrak{P}\psi|_{H^{t_0+\frac{1}{2}}}.$$

Proposition A.2.24. *Let* $t_0 > \frac{d}{2}$ *and* $(\zeta,b) \in H^{t_0+1}(\mathbb{R}^d)$ *such that Condition* (A.1) *is satisfied. Then for all* $0 \leq s \leq t_0 + \frac{1}{2}$ *and* $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$*, we have*

$$\left| d^j G_\mu^{NN}.(\boldsymbol{h},\boldsymbol{k})(B) \right|_{H^{s-\frac{1}{2}}} \leq M \prod_{i\geq 1} |(\varepsilon h_i,\beta k_i)|_{H^{t_0+1}} |B|_{H^{s-\frac{1}{2}}}.$$

Furthermore, if $0 \leq s \leq t_0$ *and* $B \in H^{t_0}(\mathbb{R}^d)$,

$$\left|d^j G^{NN}_\mu.(\boldsymbol{h},\boldsymbol{k})(B)\right|_{H^{s-\frac{1}{2}}} \leq M\left|(\varepsilon h_1, \beta k_1)\right|_{H^{s+\frac{1}{2}}} \prod_{i\geq 2} |(\varepsilon h_i, \beta k_i)|_{H^{t_0+1}} |B|_{H^{t_0}}.$$

Proof. Differentiating the dual formulation of $G^{NN}_\mu[\varepsilon\zeta, \beta b]$ for $B \in H^{s-\frac{1}{2}}(S)$, $\Lambda^{s-\frac{1}{2}}\varphi \in L^2(S)$ and arguing by duality, we obtain that

$$\left|d^j G^{NN}_\mu.(\mathbf{h},\mathbf{k})(B)\right|_{H^{s-\frac{1}{2}}} \leq C(\mu_{\max}) \left| \sqrt{1+\sqrt{\mu}|D|}\Lambda^{s-\frac{1}{2}} \sum_{\substack{j_1+j_2=j \\ I_1 \sqcup I_2 = [1,j]}} d^{j_1} P_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1} \nabla^\mu d^{j_2}\mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2} \right|_{L^2}.$$

Notice that the coordinates of $(\mathbf{h},\mathbf{k})_{I_1}$ and $(\mathbf{h},\mathbf{k})_{I_2}$ form a permutation of the coordinates of $(\mathbf{h},\mathbf{k})$. If $0 \leq s \leq t_0$, using Commutator estimate B.3.2 and the product estimate B.2.2 we get

$$\left|d^j G^{NN}_\mu.(\mathbf{h},\mathbf{k})(B)\right|_{H^{s-\frac{1}{2}}} \leq C \sum_{\substack{j_1+j_2=j \\ I_1 \sqcup I_2 = [1,j]}} \left|d^{j_1} P_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1}\right|_{L^\infty_z H^{t_0}_X} \left|\sqrt{1+\sqrt{\mu}|D|}\Lambda^{s-\frac{1}{2}}\nabla^\mu d^{j_2}\mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2}\right|_{L^2}.$$

Then, the first estimate of Proposition A.1.22 and Theorem A.1.23 give the first inequality. For the second inequality, we have to distinguish two cases : if $(1,1) \in I_1$ we use the third estimate of Proposition A.1.22 and Theorem A.1.23 whereas if $(1,1) \in I_2$ we use the first estimate of Proposition A.1.22 and Theorem A.1.24. If now $s = t_0 + \frac{1}{2}$, using Commutator estimate B.3.3 (with $t_1 = \frac{1}{2}$) and the product estimate B.2.2 (with Remark A.1.3) we get

$$\left|d^j G^{NN}_\mu.(\mathbf{h},\mathbf{k})(B)\right|_{H^{s-\frac{1}{2}}} \leq C \sum_{\substack{j_1+j_2=j \\ I_1 \sqcup I_2 = [1,j]}} \left|d^{j_1} P_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_1}\right|_{H^{t_0+\frac{1}{2},1}} \left|\sqrt{1+\sqrt{\mu}|D|}\nabla^\mu d^{j_2}\mathfrak{D}(B)_{(\zeta,b)}.(\mathbf{h},\mathbf{k})_{I_2}\right|_{H^{t_0,1}}.$$

Using, the first and the second estimate of Proposition A.1.22 and Theorem A.1.23, we obtain the first inequality for $s = t_0 + \frac{1}{2}$. The case $t_0 \leq s \leq t_0 + \frac{1}{2}$ follows by interpolation. □

A straightforward corollary is a control of $d\underline{w}$ and $d\underline{V}$.

Corollary A.2.25. *Let* $t_0 > \frac{d}{2}$ *and* $(\zeta, b) \in H^{t_0+1}(\mathbb{R}^d)$ *such that Condition* (A.1) *is satisfied. Then*

- *(1) For all* $0 \leq s \leq t_0 + \frac{1}{2}$, $\psi \in \dot{H}^{s+\frac{1}{2}}(\mathbb{R}^d)$ *and* $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$,

$$|d^j \underline{w}.(\boldsymbol{h},\boldsymbol{k})(\psi)|_{H^{s-\frac{1}{2}}} \leq M \prod_{i\geq 1} |(\varepsilon h_i, \beta k_i)|_{H^{t_0+1}} \left(\mu^{\frac{3}{4}} |\mathfrak{P}\psi|_{H^s} + \mu |B|_{H^{s-\frac{1}{2}}}\right),$$

- *(2) For all* $0 \leq s \leq t_0$, $\psi \in \dot{H}^{s+1}(\mathbb{R}^d)$ *and* $B \in H^{s-\frac{1}{2}}(\mathbb{R}^d)$,

$$|d^j \underline{w}.(\boldsymbol{h},\boldsymbol{k})(\psi)|_{H^{s-\frac{1}{2}}} \leq M \prod_{i\geq 1} |(\varepsilon h_i, \beta k_i)|_{H^{t_0+1}} \left(\mu |\mathfrak{P}\psi|_{H^{s+\frac{1}{2}}} + \mu |B|_{H^{s-\frac{1}{2}}}\right),$$

- *(3) For all* $0 \leq s \leq t_0$, $\psi \in \dot{H}^{t_0+1}(\mathbb{R}^d)$ *and* $B \in H^{t_0}(\mathbb{R}^d)$,

$$|d^j \underline{w}.(\boldsymbol{h},\boldsymbol{k})(\psi)|_{H^{s-\frac{1}{2}}} \leq M |(\varepsilon h_1, \beta k_1)|_{H^{s+\frac{1}{2}}} \prod_{i\geq 2} |(\varepsilon h_i, \beta k_i)|_{H^{t_0+1}} \left(\mu^{\frac{3}{4}} |\mathfrak{P}\psi|_{H^{t_0+\frac{1}{2}}} + \mu |B|_{H^{t_0}}\right).$$

- *(4) For all* $0 \leq s \leq t_0 - \frac{1}{2}$, $\psi \in \dot{H}^{t_0+1}(\mathbb{R}^d)$ *and* $B \in H^{t_0}(\mathbb{R}^d)$,

$$|d^j \underline{w}.(\boldsymbol{h}, \boldsymbol{k})(\psi)|_{H^{s-\frac{1}{2}}} \leq M|(\varepsilon h_1, \beta k_1)|_{H^{s+1}} \prod_{i\geq 2} |(\varepsilon h_i, \beta k_i)|_{H^{t_0+1}} \left(\mu |\mathfrak{P}\psi|_{H^{t_0+\frac{1}{2}}} + \mu |B|_{H^{t_0}} \right).$$

Finally, the same estimates holds for $d^j \underline{V}.(\boldsymbol{h}, \boldsymbol{k})(\psi)$.

Appendix B

Useful estimates

Sommaire

In this part, we give some useful estimates, product and commutator estimates. We refer to Appendix B in [80], [79] and Chapter II in [8] for the proofs.

B.1 Estimate for $\mathfrak{P}$

The first estimates are useful to control $\mathfrak{P}f$. We recall that $\mathfrak{P} = \frac{|D|}{\sqrt{1+\sqrt{\mu}|D|}}$.

Proposition B.1.1. *Let $f \in H^1(\mathbb{R}^d)$ and $g \in H^{\frac{1}{2}}(\mathbb{R}^d)$. Then,*

$$|\mathfrak{P}g|_2 \leq \mu^{-\frac{1}{4}}|g|_{H^{\frac{1}{2}}}, \ |\mathfrak{P}f|_{H^{\frac{1}{2}}} \leq \max(1, \mu^{-\frac{1}{4}})|\nabla f|_2 \text{ and } |\nabla\psi|_{H^s} \leq \max(1, \mu^{\frac{1}{4}})\,|\mathfrak{P}\psi|_{H^{s+\frac{1}{2}}}$$

We aslo need a product estimate for $\mathfrak{P}$.

Lemma B.1.2. *Let $u \in W^{1,\infty}(\mathbb{R}^d)$ and $v \in H^{\frac{1}{2}}(\mathbb{R}^d)$. Then,*

$$|\sqrt{\mu}\mathfrak{P}\,(uv)|_2 \leq C\,(\mu_{\max})\,|u|_{W^{1,\infty}(\mathbb{R}^d)}\left|\sqrt{1+\sqrt{\mu}|D|}v\right|_2.$$

The following proposition is commutator estimate for $\mathfrak{P}$.

Lemma B.1.3. *Let $t_0 > \frac{d}{2}$, $u \in H^{t_0+1}(\mathbb{R}^d)$ and $v \in H^{\frac{1}{2}}(\mathbb{R}^d)$. Then,*

$$|[\mathfrak{P}, u]\,v|_2 \leq C\,|u|_{H^{t_0+1}}\,(|v|_2 + |\mathfrak{P}v|_2)\,.$$

We also give a regularity estimate for functions in $H_*^{-\frac{1}{2}}(\mathbb{R}^d)$. We recall that

$$H_*^{-\frac{1}{2}}(\mathbb{R}^d) = \left\{u \in H^{-\frac{1}{2}}(\mathbb{R}^d)\ ,\ \exists v \in H^{\frac{1}{2}}(\mathbb{R}^d)\ ,\ u = |D|v\right\},$$

and we denote v by $v = \frac{1}{|D|}u$.

Lemma B.1.4. *Let $s \geq 0$ and $u \in H_*^{-\frac{1}{2}}(\mathbb{R}^d) \cap H^{s-\frac{1}{2}}(\mathbb{R}^d)$. Then $u \in H_*^{s-\frac{1}{2}}(\mathbb{R}^d)$ and*

$$\left|\frac{1}{\mathfrak{P}}u\right|_{H^s} \leq \left|\frac{1}{\mathfrak{P}}u\right|_2 + \left|\sqrt{1+\sqrt{\mu}|D|}u\right|_{H^{s-1}}.$$

B.2 Product estimates

We need some product estimates in $\mathbb{R}^d$. The following Proposition is Proposition B.2 in [80].

Proposition B.2.1. *Let $s, s_1, s_2 \in \mathbb{R}$ such that $s \leq s_1$, $s \leq s_2$, $s_1 + s_2 \geq 0$ and $s < s_1 + s_2 - \frac{d}{2}$. Then, there exists a constant $C > 0$ such that for all $f \in H^{s_1}(\mathbb{R}^d)$ and for all $g \in H^{s_2}(\mathbb{R}^d)$, we have $fg \in H^s(\mathbb{R}^d)$ and*

$$|fg|_{H^s} \leq C|f|_{H^{s_1}}|g|_{H^{s_2}}.$$

We also need some product estimates in $S = \mathbb{R}^d \times (-1, 0)$. The following Propositions are the Corollary B.5 in [80]. We recall that Λ^s is the Fourier multiplier

$$\Lambda^s = \left(1 + |D|^2\right)^{\frac{s}{2}}.$$

Proposition B.2.2. *Let $s, s_1, s_2 \in \mathbb{R}$ such that $s \leq s_1$, $s \leq s_2$, $s_1 + s_2 \geq 0$, $s < s_1 + s_2 - \frac{d}{2}$ and $p \in \{2, +\infty\}$. Then, there exists a constant $C > 0$ such that for all $f \in L_z^\infty H_X^{s_1}(S)$ and for all $g \in L_z^p H_X^{s_2}(S)$, we have $fg \in L_z^p H_X^s(S)$ and*

$$|\Lambda^s (fg)|_{L_z^p L_X^2(S)} \leq C |\Lambda^{s_1} f|_{L_z^\infty L_X^2(S)} |\Lambda^{s_2} g|_{L_z^p L_X^2(S)}.$$

Proposition B.2.3. *Let $T_0 > \frac{d}{2}$ and $s \geq -T_0$. Then, there exists a constant $C > 0$ such that for all f,$g \in L_z^\infty H_X^{T_0}(S) \cap H^{s,0}(S)$, we have $fg \in H^{s,0}(S)$ and*

$$|\Lambda^s (fg)|_{L^2(S)} \leq C. \left(|f|_{L_z^\infty H_X^{T_0}(S)} |\Lambda^s g|_{L^2(S)} + \mathbf{1}_{\{s > T_0\}} |g|_{L_z^\infty H_X^{T_0}(S)} |\Lambda^s f|_{L^2(S)} \right).$$

Notice that if $s \leq T_0$, $g \in H^{s,0}(S)$ is enough.

The following propositions gives estimates for $1/(1+g)$ in the flat strip S. We refers to Corollary B.6 in [80].

Proposition B.2.4. *Let $T_0 > \frac{d}{2}$, $-T_0 \leq s \leq T_0$, $k_0 > 0$ and $p \in \{2, +\infty\}$. Then, for all $f \in L_z^p H_X^s(S)$ and $g \in L_z^\infty H_X^{T_0}(S)$ with $1 + g \geq k_0$, we have*

$$\left| \Lambda^s \frac{f}{1+g} \right|_{L_z^p L^2(S)} \leq C \left(\frac{1}{k_0}, |g|_{L_z^\infty H_X^{T_0}} \right) |\Lambda^s f|_{L_z^p L^2(S)}.$$

Proposition B.2.5. *Let $T_0 > \frac{d}{2}$, $s \geq -T_0$ and $k_0 > 0$. Then, for all f,$g \in L_z^\infty H_X^{T_0}(S) \cap H^{s,0}(S)$ with $1 + g \geq k_0$, we have*

$$\left| \frac{f}{1+g} \right|_{H^{s,0}} \leq C \left(\frac{1}{k_0}, |g|_{L_z^\infty H_X^{T_0}} \right) \left(|f|_{H^{s,0}} + \mathbf{1}_{\{s > T_0\}} |f|_{L_z^\infty H_X^{T_0}} |g|_{H^{s,0}} \right).$$

Notice that if $s \leq T_0$, $f \in H^{s,0}(S)$ is enough.

B.3 Commutator estimates

We also need some commutator estimates. We recall the classical Coifman-Meyer estimate.

Lemma B.3.1. *Let $s > \frac{d}{2} + 1$. Then, for $f, g \in L^2(\mathbb{R}^d)$,*

$$|[\Lambda^s, f] g|_2 \leq C |f|_{H^s} |g|_{H^{s-1}}.$$

We need some commutator estimates in S. The following Propositions are Corollary B.17 in [80]. We denote by Λ_δ^s, the Fourier multiplier

$$\Lambda_\delta^s = \chi(\delta D) \Lambda^s,$$

where $\delta > 0$ and χ a smooth positive compactly supported function.

Proposition B.3.2. *Let $T_0 > \frac{d}{2}$, $\delta \geq 0$, $0 < t_1 \leq 1$ with $t_1 < T_0 - \frac{d}{2}$ and $-\frac{d}{2} < s \leq T_0 + t_1$. Then for all $u \in L_z^\infty H_X^{T_0}$ and $v \in H^{s-t_1,0}(S)$ we have*

$$|[\Lambda_\delta^s, u] v|_{L^2(S)} \leq C \left| \Lambda_\delta^{T_0} u \right|_{L_z^\infty L_X^2(S)} \left| \Lambda_\delta^{s-t_1} v \right|_{L^2(S)}.$$

Proposition B.3.3. *Let $T_0 > \frac{d}{2}$, $\delta \geq 0$, $0 < t_1 \leq 1$ with $t_1 < T_0 - \frac{d}{2}$ and $-\frac{d}{2} < s \leq T_0 + t_1$. Then for all $u \in H^{T_0,0}$ and $v \in L_z^\infty H_X^{s-t_1}$ we have*

$$|[\Lambda_\delta^s, u] v|_{L^2(S)} \leq C \left| \Lambda_\delta^{T_0} u \right|_{L^2(S)} \left| \Lambda_\delta^{s-t_1} v \right|_{L_z^\infty L_X^2(S)}.$$

B.4 Estimates for $\mathcal{H}_\mu$

In this part, we give some estimate for the operator $\mathcal{H}_\mu$ and some standard product and commutator estimates. For the estimate for $\mathcal{H}_\mu$, we refer to part III in [125]. For the others estimates we refer to [8] and [79]. We recall that $\mathcal{H}_\mu$ is defined by

$$\mathcal{H}_\mu = -\frac{\tanh(\sqrt{\mu}D)}{D}\partial_x.$$

First, we show that $\mathcal{H}_\mu$ is a zero-order operator.

Proposition B.4.1. *Let $s \geq 0$ and μ such that $0 < \mu_{\min} \leq \mu \leq \mu_{\max}$. Then,*

$$|\mathcal{H}_\mu u|_{H^s} \leq C\left(\mu_{\max}\right)|u|_{H^s}.$$

Furthermore, for all $s \geq r \geq 0$,

$$\left|\left(\mathcal{H}_\mu^2 + 1\right)u\right|_{H^s} \leq C_r\left(\frac{1}{\mu_{\min}}\right)|u|_{H^r}.$$

Then, we give a commutator estimate for $\mathcal{H}_\mu$.

Proposition B.4.2. *Let $s \geq 0$, $t_0 > \frac{1}{2}$, $r \geq 0$, and $\mu \geq \mu_{\min} > 0$. Then,*

$$|[\mathcal{H}_\mu, a]\,u|_2 \leq C\,|a|_{H^{t_0}}\,|f|_2,$$

$$\left||\xi|^s\widehat{[\mathcal{H}_\mu, a]\,u}\right|_2 \leq C\left(\frac{1}{\mu_{\min}}\right)|a|_{H^{r+s}}\left|\frac{(1+|\xi|)^{t_0}}{|\xi|^r}\widehat{u}\right|_2,$$

and

$$\left||\xi|^s\widehat{[\mathcal{H}_\mu, a]\,u}\right|_2 \leq C\left(\frac{1}{\mu_{\min}}\right)|a|_{H^{r+s+t_0}}\left|\frac{1}{|\xi|^r}\widehat{u}\right|_2.$$

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