Parametric study of the accuracy of an approximate solution for the mild-slope equation

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***Abstract*—In this paper, we call and study a classical equation modeling the refraction and diffraction phenomena of water waves in harbors. This equation is called the mild-slope equation. By using a family of bottom profiles, we construct analytically an approximate solution of this linear equation and perform, with various norm comparisons, a parametric study of this approximate solution. We evaluate $\nabla \cdot \mathrm{u}$ for the resulting irrotational flow using various parameters, such as the mean slope, the averaged water depth...**

***Keywords*- Mild-slope equation; Potential equation; Geometrical optics; Water waves**

I. Introduction

The motion of a fluid in a domain $\tilde{\Omega} = \{(x, y, z), (x, y) \in]0, L[\times]0, l[, -h(x, y) < z < \eta(x, y, t)\}$ with a free surface is characterized by a linearized model deduced from the Euler equation with a free surface and defined in $\Omega = \{(x, y, z), (x, y) \in]0, L[\times]0, l[, -h(x, y) < z < 0\}$. A derivation of this model can be found in Lamb [10] and it is called the potential system of equations:

$$\Delta\phi(x, y, z) = 0 \qquad \text{in } \Omega, \tag{1}$$

$$\frac{\partial \phi}{\partial z}(x, y, z) - \frac{\omega^2}{g}\phi(x, y, z) = 0 \qquad \text{at } z = 0, \tag{2}$$

$$\frac{\partial \phi}{\partial x}(x, y, z)\,\frac{\partial h}{\partial x}(x, y) + \frac{\partial \phi}{\partial y}(x, y, z)\frac{\partial h}{\partial y}(x, y) + \frac{\partial \phi}{\partial z}(x, y, z) = 0 \qquad \text{at } z = -h(x, y), \tag{3}$$

where the unknown ϕ is the velocity potential, h is the given water depth (depending on (x, y)), ω the imposed time frequency and g the gravity acceleration.

In the case where the water depth h is constant, equation (3) reduces to:

$$\frac{\partial \phi}{\partial z}(x, y, z) = 0 \qquad \text{at } z = -h, \tag{4}$$

and a solution of the potential system is:

$$\phi(x, y, z) = Z(h, z)\, \cos(k\, x), \tag{5}$$

where k is the wave number satisfying the following dispersion relation:

$$\omega^2 = g\, k \tanh\Big(k\, h\Big), \tag{6}$$

with

$$Z(h, z) = \frac{\cosh\Big(k\,(z + h)\Big)}{\cosh\Big(k\, h\Big)}. \tag{7}$$

In the general case, one defines $k(x, y)$ as the unique positive solution of:

$$\omega^2 = g\, k(x, y)\, \tanh(k(x, y)\, h(x, y)). \tag{8}$$

One hopes to construct a solution of the potential system similar to (5). Indeed, for a slowly varying bottom:

$$\frac{\nabla h(x, y)}{h(x, y)} << 1, \tag{9}$$

Berkhoff [1, 2] seeks solutions under the form :

$$\phi(x, y, z) = Z\Big(h(x, y), z\Big)\, \varphi(x, y). \tag{10}$$

After algebraic manipulations and approximations that will be detailed hereafter, he obtains an approximate equation in the horizontal domain $]0, L[\times]0, l[$ on φ, called the mild-slope equation [1, 2]:

$$\nabla \cdot \Big(T(x,y) \nabla \varphi(x,y) \Big) + k^2\, T(x,y)\, \varphi(x,y) = 0 \quad \text{in } \Omega_h, \tag{11}$$

where

$$T(x,y) = \frac{\left(h(x,y) + \dfrac{\sinh\Big(2\,k(x,y)\,h(x,y)\Big)}{2\,k(x,y)} \right)}{2\cosh^2\Big(k(x,y)\,h(x,y)\Big)}. \tag{12}$$

Equation (11) has been studied by many authors. Booij [3] computed numerically the solution φ_b of (11), and compared $\varphi_b Z(h(x,y),z)$ with the solution of the potential equation with the same boundary data. In the present paper, we only considered slopes smaller than $\frac{1}{3}$, limiting value obtained by Booij.

In this paper, we exhibit an analytical approximate solution of the mild-slope equation for a family of slopes depending on two parameters: the mean slope and the averaged water depth defined thereafter. We evaluate various norms for this solution. We restrict ourselves to the case where the mean slope is smaller than $\frac{1}{3}$.

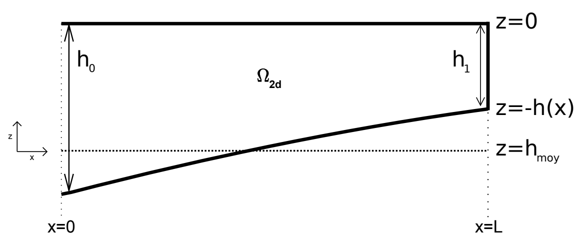

Figure 1. Example of one domain

For simplicity's sake, this study is performed in a 2d domain $\Omega_{2d} = \{(x,z), x \in]0,L[, -h(x) < z < 0\}$. We restrict ourselves to the case where the function $kh(x)$ is linear. Note that in this case $x \to h(x)$ is not linear. This particular form is the key tool for obtaining an analytical approximate solution of the mild-slope equation in particular for an exact expression of the phase θ given in (14). Moreover, for small slopes the water depth is very close to its linear approximation.

This analytical approximate solution deduced from ideas of geometrical optics [13] is:

$$\varphi_a(x) = \frac{1}{\sqrt{k(x)\,T(x)}} \cos\Big(\theta(x) + c\Big), \tag{13}$$

where

$$\theta(x) = \int_0^x k(x')\,dx'. \tag{14}$$

with the constant c is chosen according to the boundary conditions.

Let us define the following operators: $\mathsf{Be}(\varphi)(x) = 0$ is the mild-slope equation. $\mathsf{L}(\varphi)(x,z) = 0$ corresponds to the Laplace equation. $\mathsf{Bo}(\varphi)(x,0) = 0$ is the equation satisfied on $z = -h(x)$.

For the approximate solution φ_a, we study $\mathsf{Be}(\varphi_a)$, $\mathsf{L}(\varphi_a)$ and $\mathsf{Bo}(\varphi_a)$. These are the remainder terms in the corresponding equation

$$\|\mathsf{Be}\big(\varphi\big)\|_{L^2([0,L])} \le C_1 \left\| \left(\frac{\partial h}{\partial x}\right) \right\|_\infty^2, \tag{15}$$

$$\|\mathsf{L}\big(\varphi\big)\|_{L^2([0,L],[-h(x),0])} \le C_2 \left\| \frac{\partial h}{\partial x} \right\|_\infty, \tag{16}$$

$$\|\mathsf{Bo}\big(\varphi\big)\|_{L^2([0,L])} \le C_3 \left\| \frac{\partial h}{\partial x} \right\|_\infty. \tag{17}$$

The main result of our study is that we have the respective order of magnitude of these terms. We observe theoretically or numerically that the remainder terms are small.

II. Model background

The mild-slope equation (11) models the refraction and diffraction phenomena of an harmonic in time wave in harbors and coastal areas and is based on the assumption (9) and we describe its derivation.

A. *Euler equations with free surface*

Let us consider an inviscid and incompressible fluid and an irrotational flow. The velocity $\mathbf{u}$ and the pressure p are given by the Euler equation with free surface:

$$\frac{\partial \mathbf{u}}{\partial t}(x,z,t) + \Big(\mathbf{u}(x,z,t)\cdot\nabla\Big)\mathbf{u}(x,z,t) + \frac{\nabla p(x,z,t)}{\rho} = \boldsymbol{g} \quad \text{in } \Omega_{2d}, \tag{18}$$

$$\nabla \cdot \mathbf{u}(x,z,t) = 0 \quad \text{in } \Omega_{2d}, \tag{19}$$

$$\frac{\partial \eta}{\partial t}(x,t) + \mathbf{u}(x,z,t)\cdot\mathbf{n}_s = 0 \quad \text{at } z = \eta(x,t), \tag{20}$$

$$\mathbf{u}(x,z,t)\cdot\mathbf{n}_b = 0 \quad \text{at } z = -h(x), \tag{21}$$

where ρ is the given density, η the unknown free surface, $\mathbf{n}_s$ and $\mathbf{n}_b$ the outward normals at the free surface and at the bottom.

Due to irrotationality, we have (Hodge theorem [8]):

$$\mathbf{u}(x,z,t) = \nabla\Phi(x,z,t) \quad \text{in } \Omega_{2d}, \tag{22}$$

where Φ is the velocity potential (defined up to a constant).

We deduce the Euler potential equation:

$$\Delta\Phi(x,z,t) = 0 \quad \text{in } \Omega_{2d}, \tag{23}$$

$$\frac{\partial \Phi}{\partial t}(x,z,t) + \frac{1}{2}\Big(\nabla\Phi(x,z,t)\Big)^2 + \frac{p(x,z,t)}{\rho} + g\,z = 0 \quad \text{in } \Omega_{2d}, \tag{24}$$

$$\frac{\partial \eta}{\partial t}(x,t) + \nabla\Phi(x,z,t)\cdot\mathbf{n}_s = 0 \quad \text{at } z = \eta(x,t), \tag{25}$$

$$\nabla\Phi(x,z,t)\cdot\mathbf{n}_b = 0 \quad \text{at } z = -h(x). \tag{26}$$

An additional equation is needed on the unknown free surface η. For this, we consider equation (24) written at the free surface, by assuming $p(x, z = \eta(x,t), t) = 0$:

$$\frac{\partial \Phi}{\partial t}(x,z,t) + \frac{1}{2}\Big(\nabla\Phi(x,z,t)\Big)^2 + g\,\eta(x,t) = 0 \quad \text{at } z = \eta(x,t). \tag{27}$$

B. Potential equation

Let us put $A = \|\varphi\|_\infty$ and assume:

$$\frac{A}{h_{moy}} << 1, \tag{28}$$

$$\frac{A}{h_{moy}}\frac{h_{moy}^2}{L_0^2} << 1, \tag{29}$$

where A is the amplitude, h_{moy} the averaged water depth and L_0 the characteristic length of the wave.

We linearize the equations (25) and (27) under (28-29) and re-write them at $z = 0$:

$$\frac{\partial \eta(x,t)}{\partial t} - \frac{\partial \Phi(x,0,t)}{\partial z} = 0, \tag{30}$$

$$\frac{\partial \Phi(x,0,t)}{\partial t} + g\,\eta(x,t) = 0. \tag{31}$$

We deduce:

$$\frac{\partial^2 \Phi(x,z,t)}{\partial t^2} + g\frac{\partial \Phi(x,z,t)}{\partial z} = 0 \quad \text{at } z = 0. \tag{32}$$

Assume:

$$\Phi(x,z,t) = \phi(x,z)\,\cos(\omega\, t), \tag{33}$$

where ϕ is still called the velocity potential. By replacing (33) in equations (23), (26) and (32), we obtain the potential equation (1-3).

C. Mild-slope equation

Assume in this section that the slope is characterized by:

$$\sigma_* = \max_{D\in[0,L],x\in[0,L-D]}\left(\frac{h(x+D)-h(x)}{D}\right). \tag{34}$$

The value of σ_* is assumed to be small. Note that this is a mathematical characterization of the hypothesis of Berkhoff [1, 2].

Recall that in the case of a flat bottom, (5) is an exact solution of the potential system of equations (1-3). By analogy, in the case of a mild-slope bottom, a solution of the system (1-3) is as following:

$$\phi(x,z) = Z(h(x),z)\,\varphi(x,z). \tag{35}$$

Berkhoff then assumes, using the equation at the free surface (2) that:

$$\Delta\Big(Z(h(x),z)\,\varphi(x,z) - Z(h(x),z)\,\varphi_0(x)\Big) = O(\sigma_*^2), \tag{36}$$

where

$$\varphi(x,0) = \varphi_0(x). \tag{37}$$

Notice that, after integration by parts:

$$\int_{-h(x)}^{0} Z(h(x),z')\,\Delta\Big(Z(h(x),z')\,\varphi_0(x)\Big)dz' = \mathsf{Be}(\varphi_0)(x) + O(h'', h'^2). \tag{38}$$

From:

$$\Delta\Big(Z(h(x),z)\varphi(x,z)\Big) = 0, \tag{39}$$

and the assumption of Berkhoff (36), one obtains:

$$O(\sigma_*^2) + \mathsf{Be}(\varphi_0)(x) + O(h'', h'^2) = 0. \tag{40}$$

Notice that:

$$\|h'\|_\infty \geq c_1\,\sigma_*, \tag{41}$$

$$\|h''\|_\infty \geq c_2\,\sigma_*^2. \tag{42}$$

We observe that the action of the mild-slope operator on the approximate analytical solution is of order of magnitude $O(h'', h^2)$. We also observe that the action of the Laplace operator on $Z\,\varphi_0$ yields an order of magnitude $O(\sigma_*)$, hence it seems that the assumption of Berkhoff (36) is not valid.

It is the consequence of our numerical experiments.

However, if we neglect $O(\sigma_*)$ and $O(h'', h'^2)$ in (38), we get the mild-slope equation:

$$\nabla\Big(T(x)\,\nabla\varphi_0(x)\Big) + k^2\,T(x)\,\varphi_0(x) = 0. \tag{43}$$

Finding an explicit solution of (43) is not easy because k and T are known implicitly through (8). In the literature, many authors studied the dispersion relation and found explicit approximate solutions. These works include the approximations of Hunt [9], Eckart [6], Nielsen [12],Venezian [14]. A review of these different approximations has been made by Fenton and McKee [7].

In our case, we need to know k to compute the approximate analytical solution (see next section). The expression of k is also needed for computing the remainder term $\mathsf{Be}(\varphi)$.

Let us assume:

$$(kh)(x) = k(0)h(0) + \frac{k(L)h(L) - k(0)h(0)}{L}x. \tag{44}$$

The water height is thus given by:

$$h(x) = \frac{g}{\omega^2}\,(kh(x)\tanh\Big((kh)(x)\Big). \tag{45}$$

Hence:

$$k(x) = \frac{\omega^2}{g}\frac{1}{\tanh\Big((kh)(x)\Big)}. \tag{46}$$

Therefore, we have expressions for all the needed quantities thanks to:

$$k(x) = \frac{\omega^2}{g}\frac{(kh)'(x)}{\tan\Big(kh(x)\Big)\,(kh)'(x)}, \tag{47}$$

which implies (see (66) below)

$$\theta(x) = \frac{\omega^2}{g\,(kh)'(x)} \log\big(\frac{\sinh(kh(x))}{\sinh(kh(x))}\big). \tag{48}$$

III. Approximate analytical solution

In this section, we describe a strategy to find an approximate analytical solution at the order 0 for the mild-slope equation. This comes from the geometrical optics method where a description of the method can be find in the book of Rauch [13].

One recalls that, for the partial differential equation

$$T(x)\,\varphi_\epsilon''(x) + T'(x)\,\varphi_\epsilon'(x) + \frac{k^2(x)}{\epsilon^2}T(x)\varphi_\epsilon(x) = 0, \tag{49}$$

depending on ϵ assumed to be small, a classical method to derive asymptotic solutions at the order 0 is to assume that

$$\varphi_\epsilon(x) = a_0(x)\cos\left(\frac{\theta(x)}{\epsilon} + c\right), \tag{50}$$

where, plugging (50) into (49), one gets:

$$\begin{aligned}&\frac{1}{\epsilon^2}\Big(k^2(x) - \theta'(x)^2\Big)T(x)a_0(x)\cos\left(\frac{\theta(x)}{\epsilon}\right)\\ &+\frac{1}{\epsilon}\Big(-a_0'(x)\,\theta'(x)\,T - (a_0(x)\theta''(x) + a_0'(x)\theta'(x))T(x)\\ &-a_0(x)T'(x)\theta'(x)\Big)\sin\left(\frac{\theta(x)}{\epsilon}\right)\\ &+\Big(\Delta a_0(x)\,T(x) + a_0'(x)T'(x)\Big)\cos\left(\frac{\theta(x)}{\epsilon}\right) = 0.\end{aligned} \tag{51}$$

One then solves the cascade of equations:

- coefficients in $\dfrac{1}{\epsilon^2}$:

$$\theta'(x)^2 = k^2, \tag{52}$$

- coefficients in $\dfrac{1}{\epsilon}$:

$$\Big(a_0^2(x)\,T(x)\,k(x)\Big)' = 0. \tag{53}$$

Then, we obtain $a_0(x) = \dfrac{1}{\sqrt{k(x)T(x)}}$.

If (a_0, θ) satisfies (53-52), then (50) is an approximate analytical solution of (49) with a remainder term equal to:

$$\nabla\cdot\Big(T(x,y)a_0'(x,y)\Big)\cos\left(\frac{\theta(x,y)}{\epsilon}\right). \tag{54}$$

In the case $\epsilon = 1$, equation (49) reduces to equation (43). Therefore relation (50), with $\epsilon = 1$ defines an approximate solution of equation (43) with a remainder term:

$$\nabla\cdot\Big(T(x)a_0'(x)\Big)\cos\left(\theta(x)\right). \tag{55}$$

One can also consider functions of the form:

$$\varphi(x,y) = a_0(x)\,\cos\Big(\theta(x)\Big) + b_0(x)\,\sin\Big(\theta(x)\Big), \tag{56}$$

where the remainder term is:

$$\begin{aligned}&\nabla\cdot\Big(T(x)a_0'(x)\Big)\cos\left(\theta(x)\right)\\ &+\nabla\cdot\Big(T(x,y)b_0'(x)\Big)\sin\left(\theta(x)\right).\end{aligned} \tag{57}$$

or

$$\varphi(x) = m(x)\cos(\theta(x) + c), \tag{58}$$

where c is a constant. In this case, the remainder term is:

$$\Big(T(x)a_0'(x)\Big)'\,\cos\left(\theta(x) + c\right). \tag{59}$$

IV. Parametric study of the accuracy of the approximate analytical solution

In this section, we study the accuracy of the approximate analytical solution (58). Let us recall that $\mathsf{Be}(\varphi)$ is the remainder term of the mild-slope equation. The expression $\mathsf{L}(\varphi)$ and $\mathsf{Bo}(\varphi)$ represent the remainder terms for the potential equation and the equation of the bottom $z = -h(x)$.

For this, we developed a numerical code taking into account the physical parameter ω (time frequency) and the two extreme values of the water depth $h(0)$ and $h(L)$. Note that, when:

$$\frac{h(L) - h(0)}{L} << 1, \tag{60}$$

$\frac{h(L)+h(0)}{2}$ is a good approximation of the average of the water depth, and $\frac{h(L)-h(0)}{L}$ is a good approximation of the slope.

One denotes, if needed,

$$h_{moy} = \frac{h(L) + h(0)}{2}, \tag{61}$$

and

$$\sigma = \frac{h(L) - h(0)}{L}. \tag{62}$$

The tool returns graphics of the approximate analytical solution and computes the quantities $\mathsf{Be}(\varphi)$, $\mathsf{L}(\varphi)$ and $\mathsf{Bo}(\varphi)$ in order to analyze them.

A. Quantitative study

In this case, we study a domain of length $L = 30$ m and $h_{moy} = 3$ m. We consider an inhomogeneous Dirichlet condition at $x = 0$ and a homogeneous Neumann condition at $x = L$:

$$\varphi(0) = 1, \tag{63}$$

$$\varphi'(L) = 0. \tag{64}$$

1) Approximate geometrical optics solution: In this case, the approximate analytical solution (56) reads:

$$\varphi_{1,b}(x) = \frac{\alpha}{\sqrt{kT(x)}}\cos(\theta(x)) + \frac{\beta}{\sqrt{kT(x)}}\sin(\theta(x)), \tag{65}$$

with

$$\begin{aligned}\theta(x) &= \int_L^x k(x')dx',\\ &= \frac{g}{\omega^2}\frac{L}{kh(L) - kh(0)}\ln\left(\frac{\sinh(kh(L))}{\sinh\left(kh(x)\right)}\right),\end{aligned} \tag{66}$$

and

$$\alpha = \frac{\sqrt{kT(0)}}{\left(\cos(\theta(0)) + \frac{1}{2}\frac{(kT(L))'}{kT(L)\,k(L)}\sin(\theta(0))\right)}, \tag{67}$$

$$\beta = \frac{1}{2}\frac{(kT(L))'}{(kT(L))^{\frac{3}{2}}}\frac{\sqrt{kT(L)}}{k(L)}\alpha. \tag{68}$$

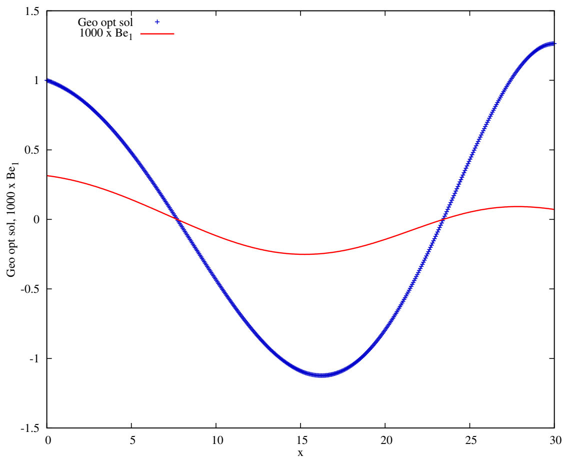

Figure 2. Functions $\varphi_{1,b}(x)$ and $1000\times \mathsf{Be}\big(\varphi_{1,b}(x)\big)$ — $\frac{\omega^2}{g} = 0.1$ m^{-1}

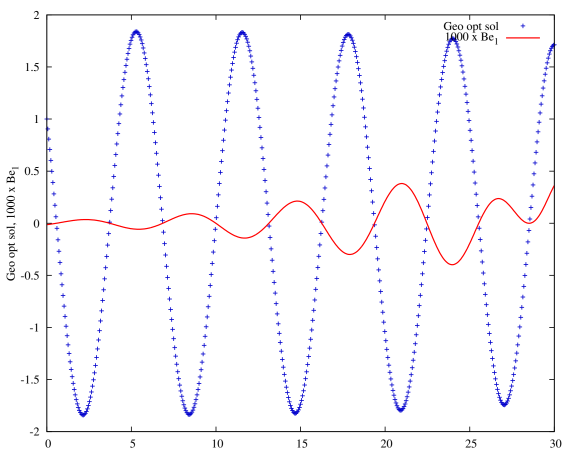

Figure 3. Functions $\varphi_{1,b}(x)$ and $1000 \times \mathsf{Be}\big(\varphi_{1,b}(x)\big)$ — $\frac{\omega^2}{g} = 1$ m^{-1}

Figures (2) and (3) represent $\varphi_{1,b}$ (crosses) and $1000 \times \mathsf{Be}\big(\varphi_{1,b}(x)\big)$ (line) for different time frequencies ω. On both test cases, we observe that the remainder term $\mathsf{Be}\big(\varphi_{1,b}\big)$ is extremely small compared with the function $\varphi_{1,b}$ (see figure (2,3)). We deduce from this observation that the analytical approximate solution is a good approximation of a solution of the mild-slope equation. Note that the functions $-\mathsf{Be}\big(\varphi_{1,b}\big)$ and $\varphi_{1,b}$ have approximatively the same phase. These results are coherent with the expression of the remainder term (57) for which we can prove that the order is order that is $O(h'', h'^2)$ as described below.

Slope	$\|\mathsf{Be}(\varphi_{1,b})\|_{L^2}$	$\|\mathsf{L}(\varphi_{1,b})\|_{L^2}$	$\|\mathsf{Bo}(\varphi_{1,b})\|_{L^2}$
10^{-1}	9.22×10^{-4}	1.25×10^{-1}	1.21×10^{-1}
10^{-2}	1.72×10^{-5}	1.65×10^{-2}	1.29×10^{-2}
10^{-3}	1.72×10^{-7}	1.67×10^{-3}	1.30×10^{-3}
10^{-4}	1.72×10^{-9}	1.66×10^{-4}	1.30×10^{-4}

Table I
REMAINDER TERMS IN L^2-NORM COMPARED TO SLOPES WITH $\varphi_{1,b}$ - $\frac{\omega^2}{g} = 1$ M^{-1} - $k_{moy} = 1.0048$ M^{-1}

Slope	$\|\mathsf{Be}(\varphi_{1,b})\|_{L^2}$	$\|\mathsf{L}(\varphi_{1,b})\|_{L^2}$	$\|\mathsf{Bo}(\varphi_{1,b})\|_{L^2}$
10^{-1}	9.20×10^{-4}	4.67×10^{-2}	7.68×10^{-2}
10^{-2}	9.85×10^{-6}	4.19×10^{-3}	7.64×10^{-3}
10^{-3}	9.81×10^{-8}	4.16×10^{-4}	7.58×10^{-4}
10^{-4}	9.80×10^{-10}	4.16×10^{-5}	7.58×10^{-5}

Table II
REMAINDER TERMS IN L^2-NORM COMPARED TO SLOPES WITH $\varphi_{1,b}$ - $\frac{\omega^2}{g} = 0.1$ M^{-1} - $k_{moy} = 0.1922$ M^{-1}

Tables (I) and (II) represent the L^2-norm of $\mathsf{Be}(\varphi_{1,b})$, $\mathsf{L}(\varphi_{1,b})$ and $\mathsf{Bo}(\varphi_{1,b})$ for different slopes σ. We observe that mild-slope operator applied to the approximate analytical solution is of size $O(h'', h'^2)$ in L^2-norm. This result is coherent with the expression of the remainder term (57) for which we can prove order that is the same. The operators of the potential system $\mathsf{L}(\varphi_{1,b})$ and $\mathsf{Bo}(\varphi_{1,b})$ are size of $O(h')$.

2) Solution of flat bottom: Let us consider a classical solution of the mild-slope equation with a flat bottom:

$$\varphi''(x) + k_0^2\,\varphi(x) = 0, \tag{69}$$

such as

$$\varphi'(0) = 1, \tag{70}$$
$$\varphi'(L) = 0. \tag{71}$$

One has:

$$\varphi_{1,f}(x) = \cos(k_0\,x) + \frac{\sin(k_0\,L)}{\cos(k_0\,L)}\sin(k_0\,x). \tag{72}$$

One replaces $\varphi_{1,f}$ in the mild-slope operator $\mathsf{Be}(\varphi)$ and one obtains the following results.

The tables (III) and (IV) represent the L^2-norm of remainder terms $\mathsf{Be}(\varphi_{1,f})$, $\mathsf{L}(\varphi_{1,f})$ and $\mathsf{Bo}(\varphi_{1,f})$ for different slopes σ. In this case, all the remainder terms are of size $O(h')$ in L^2-norm. The order of magnitude given by the

Slope	$\|\mathsf{Be}(\varphi_{1,f})\|_{L^2}$	$\|\mathsf{L}(\varphi_{1,f})\|_{L^2}$	$\|\mathsf{Bo}(\varphi_{1,f})\|_{L^2}$
10^{-1}	1.75×10^{-1}	2.70×10^{-1}	1.34×10^{-1}
10^{-2}	2.03×10^{-2}	3.30×10^{-2}	1.30×10^{-2}
10^{-3}	2.04×10^{-3}	3.48×10^{-3}	1.29×10^{-3}
10^{-4}	2.03×10^{-4}	3.67×10^{-4}	1.29×10^{-4}

Table III
REMAINDER TERMS IN L^2-NORM COMPARED TO SLOPES WITH $\varphi_{1,f}$ - $\dfrac{\omega^2}{g} = 1\ \text{M}^{-1}$ - $k_{moy} = 1.0048\ \text{M}^{-1}$

Slope	$\|\mathsf{Be}(\varphi_{1,f})\|_{L^2}$	$\|\mathsf{L}(\varphi_{1,f})\|_{L^2}$	$\|\mathsf{Bo}(\varphi_{1,f})\|_{L^2}$
10^{-1}	1.41×10^{-1}	9.10×10^{-2}	5.58×10^{-2}
10^{-2}	1.80×10^{-2}	1.13×10^{-2}	7.45×10^{-3}
10^{-3}	1.81×10^{-3}	1.14×10^{-3}	7.56×10^{-4}
10^{-4}	1.80×10^{-4}	1.08×10^{-4}	7.57×10^{-5}

Table IV
REMAINDER TERMS IN L^2-NORM COMPARED TO SLOPES WITH $\varphi_{1,f}$ - $\dfrac{\omega^2}{g} = 0.1\ \text{M}^{-1}$ - $k_{moy} = 0.1922\ \text{M}^{-1}$

mild-slope equation is not obtained in the Laplace equation and in the equation at $z = -h(x)$.

In this quantitative study, the approximate analytical solution (65) yields global remainder terms ($\mathsf{L}(\varphi) - \mathsf{Bo}(\varphi)$) of the same accuracy as the flat bottom solution (72). However, one observes in the next section that the functions obtained do not have the same qualitative behavior.

B. Qualitative study

In this section, the study is done for the following physical parameters:

- $L = 70$ m,
- $h_{moy} = 0.6$ m,
- $\sigma = 0.01$,
- $\dfrac{\omega^2}{g} = 0.5\ \text{m}^{-1}$.

At $x = 0$, we consider the following boundary conditions:

$$\varphi_2(0) = \frac{1}{\sqrt{k(0)\,T(0)}}, \tag{73}$$

$$\varphi_2'(0) = 0. \tag{74}$$

We consider an approximate geometrical optic solution:

$$\varphi_{2,b}(x) = \frac{1}{\sqrt{k(x)\,T(x)}} \cos(\theta(x)), \tag{75}$$

where

$$\theta(x) = \int_0^x k(x')\,dx'. \tag{76}$$

On a flat bottom h_0, we define:

$$\varphi_{2,f}(x) = \frac{1}{\sqrt{k(0)\,T(0)}} \cos(k_0\,x). \tag{77}$$

Let us define the free surface functions:

$$\eta_{2,b}(x,t) = \mathrm{Re}\left(i\frac{\omega^2}{g} Z(x,0)\varphi_{2,b}(x)\mathrm{e}^{i\omega t}\right), \tag{78}$$

$$\eta_{2,f}(x,t) = \mathrm{Re}\left(i\frac{\omega^2}{g} Z(x,0)\varphi_{2,f}(x)\mathrm{e}^{i\omega t}\right). \tag{79}$$

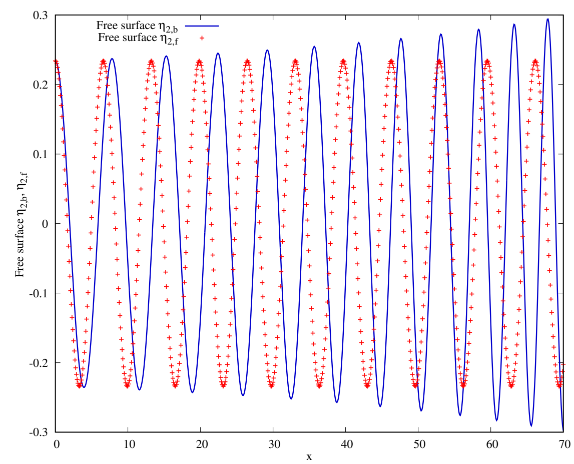

Figure 4. Functions $\eta_{2,b}(x,10)$ and $\eta_{2,f}(x,10))$

Figure (4) represents $\eta_{2,b}(x,10)$ (line) and $\eta_{2,f}(x,10)$ (cross) in function of x. In the case of the approximate analytical solution (75), the phase evolution due to the variation of the bottom is correctly taken into account and the amplitude of the solution increases. These two phenomena are closer to the physical observations.

CONCLUSION

In this paper, we studied the mild-slope equation allowing to model the refraction and diffraction phenomena of water waves in harbors. Thus, we developed an approximate analytical solution deduced from ideas of geometrical optics and perform a parametric study of this approximate solution in various norm comparisons. A quantitative and qualitative study have been performed. Hence, we observed that for a approximative analytical solution φ, $\mathsf{Be}(\varphi)$ is size of $O(h'^2, h'')$. However, the accuracy of the remainder term in the Laplace equation as well as in the relation on $z = -h(x)$ is worse that was assumed by Berkhoff for the global equation. Recall that the expression,

$$\Delta\Big(Z(h(x),z)\,\varphi(x,z) - Z(h(x),z)\,\varphi_0(x)\Big)$$

was assumed to be of order $O(\sigma_*)$. We deduce that a solution of the mild-slope equation is not appropriate to compare with a solution of the potential system of equations, or more generally with the Euler equation. Qualitatively, we observe that the variations of the amplitude and the phase are correctly taken into account. These phenomena are closer to the physical observations and appear due to the variation of the bottom. This implies an approximate solution of the mild-slope equation is an interesting solution to represent physical phenomena in the case of slowly varying bottom.

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