Some infinite dimensionals rough-paths

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Rough paths

- Nonlinear systems $y_t$ driven by a (non-differentiable) noise $x_t$
  \[ dy = f(y)dx \]
- The output $y$ is a nice function of the iterated integrals of $x$:
  \[ (x, \int dx dx, \ldots, \int dx \otimes^n) \xrightarrow{\Phi} y \]
  We can consider only a finite number of them. No need of formal series.
Algebraically

The increment $\delta y_{ts} = y_t - y_s$ of the solution of the integral equation
$y = \int f(y)dx$ has a natural expansion

$$\delta y_{ts} = \int_s^t f(y_u)dx_u = f(y_s) \int_s^t dx_u + f(y_s)f'(y_s) \int_s^t \int_s^u dx_w dx_u + \text{h.o.t.}$$

If we neglect the h.o.t. it belongs to the linear span of the iterated integrals

$$X_{ts}^1 = x_t - x_s \quad \ldots \quad X_{ts}^n = \int_s^t X_{us}^{n-1} dx_u$$

and we write it as

$$[ (fX)_{ts} = f_s X_{ts} ]$$

$$\delta y = y^1 X^1 + y^2 X^2 + \cdots$$

where $y_t^1 = f(y_t)$, $y_t^2 = f(y_t)f'(y_t)$, and so forth.

The solution of the ODE is identified with the fixed point of

$$y \xrightarrow{\text{nonlinear map}} f(y) \xrightarrow{\text{integration}} I(f(y))$$

for $ys$ whose increments can be (partially) expanded on the $\{X^n\}_{n}$
Phenomenology of rough-paths
Trees

\( \mathcal{L} \) finite set. Trees labeled by \( \mathcal{L}, \mathcal{T}_\mathcal{L} \)

\[
\begin{array}{cccc}
\bullet_2 & \bullet_3 & \square_1 & \triangle_2 \\
& & \& & \\
\end{array}
\]

\( (\tau_1, \cdots, \tau_k) \xrightarrow{B^a_+} \tau = [\tau_1, \cdots, \tau_k]_a \)

\[
[\bullet] = \begin{array}{c} \bullet \\
\end{array}, \quad [\bullet, [\bullet]] = \begin{array}{c} \triangle \\
\end{array}, \quad \text{etc.} \ldots
\]

\[
\Delta(\tau) = 1 \otimes \tau + \sum_{a \in \mathcal{L}} (B^a_+ \otimes \text{id})[\Delta(B^a_- (\tau))] = \tau_1 \cdots \tau_n \text{ if } a = b
\]

\[
B^a_-(B^b_+ (\tau_1 \cdots \tau_n)) = \begin{cases} 
\tau_1 \cdots \tau_n & \text{if } a = b \\
0 & \text{otherwise}
\end{cases}
\]
Differential equations (à la Butcher)

The solution $y$ of the differential equation

$$dy = f(y)dt, \quad y_0 = \eta$$

has the B-series representation

$$y_t = \eta + \sum_{\tau \in \mathcal{T}} \psi^f(\tau)(\eta) \frac{t^{|	au|}}{\sigma(\tau)\tau!}$$

**Elementary differentials** $\psi^f$ defined as

$$\psi^f(\bullet)(\xi) = f(\xi), \quad \psi^f([\tau^1 \cdots \tau^k]) = f_b(\eta)\psi^f(\tau^1)(\xi)^{b_1} \cdots \psi^f(\tau^k)(\xi)^{b_k}$$

where $f_0(\xi) = f(\xi)$ and $f_b(\xi) = \prod_{i=1}^{|b|} \partial_{\xi^b_i} f(\xi)$ derivatives of the vectorfields.
Driven differential equations

Given a collection of paths \( \{x^a \in C^1([0, T], \mathbb{R})\}_{a \in \mathcal{L}} \), \( \eta \in \mathbb{R}^n \)

Analytic vectorfields \( \{f_a : \mathbb{R}^n \rightarrow \mathbb{R}^n\}_{a \in \mathcal{L}} \)

**Theorem**

The differential equation

\[
dy_t = f_a(y_t) dx_t^a, \quad y_0 = \eta
\]

admit locally the series solution

\[
y_t = y_s + \sum_{\tau \in \mathcal{T}_\mathcal{L}} \frac{1}{\sigma(\tau)} \phi^f(\tau)(y_s) X_{\tau s}, \quad y_0 = \eta
\]

where \( \phi^f(a)(\xi) = f_a(\xi), \phi^f([\tau^1 \cdots \tau^k]_a)(\xi) = f_{a;b_1 \cdots b_k}(\xi) \prod_{i=1}^k [\phi^f(\tau^i)(\xi)]^{b_i} \).
Smooth iterated integrals

Let $X : \mathcal{T}_\mathcal{L} \rightarrow \mathcal{C}_2 \subset C([0, T]^2; \mathbb{R})$

$$X^{\bullet}_a = \int_s^t dx^a, \quad X^{[\tau^1 \cdots \tau^k]}_a = \int_s^t \prod_{i=1}^k X^{\tau^i}_{tu} dx^a. \quad (1)$$

Extend $X$ to $\mathcal{A}\mathcal{T}_\mathcal{L}$ considering $\mathcal{C}_2$ as an algebra with (commutative) product $(a \circ b)_a = a_{ts} b_{ts}$ for $a, b \in \mathcal{C}_2$. We let $X^1 = 1$.

$$X^{\tau_1 \cdots \tau_n}_t = X^{\tau_1}_t X^{\tau_2}_t \cdots X^{\tau_n}_t, \quad X^{[\tau_1 \cdots \tau_n]} = \int X^{\tau_1 \cdots \tau_n} dx^a$$

Bounds

$$|X^{\tau}_t| \leq \frac{(A|t - s|)^{|\tau|}}{\tau!}$$

Theorem (Tree multiplicative property)

$$X^{\tau}_t = \sum X^{\tau}_t X^{(1)}_{tu} X^{(2)}_{us} = X^{\Delta \tau}_{tu}$$
Example

\[ T_{ts}^\bullet = t - s, \quad T_{ts}^{[\tau_1\cdots\tau_n]} = \int_s^t T_{us}^{\tau_1} \cdots T_{us}^{\tau_n} du \]

By induction: \( T_{ts}^\tau = (t - s)^{|\tau|} (\tau!)^{-1} \)

Lemma (Tree Binomial)

For every \( \tau \in \mathcal{T} \) and \( a, b \geq 0 \) we have

\[
(a + b)^{|\tau|} = \sum_i \frac{\tau!}{\tau_i^{(1)}! \tau_i^{(2)}!} a^{|\tau_i^{(1)}|} b^{|\tau_i^{(2)}|}
\]  

(2)
Structure of solution to DDEs

Write $y_s^\tau = \phi^f(\tau)(y_s)/\sigma(\tau)$ so that

$$y_t - y_s = \sum_{\tau \in T_L} X_{ts}^\tau y_s^\tau$$

Lemma

For any $\tau \in T_L \cup \{\emptyset\}$ we have

$$y_t^\tau - y_s^\tau = \sum_{\sigma \in T_L, \rho \in T_L} c'(\sigma, \tau, \rho) X_{ts}^\rho y_s^\sigma$$

c' counting function of reduced coproduct: $\Delta'\sigma = \sum_{\tau, \rho} c'(\sigma, \tau, \rho) \tau \otimes \rho$. 

Integration of increments

Q: Given \( a \in \mathcal{C}_2 \) can we find \( f \in \mathcal{C}_1 \) such that \( f_t - f_s = a_{ts} \)?

\[
a_{ts} - a_{tu} - a_{us} = (f_t - f_s) - (f_t - f_u) - (f_u - f_s) = 0 \text{ (obstruction)}
\]

Increments: \( \mathcal{C}_n \subset C([0,1]^n, V) \), \( g \in \mathcal{C}_n \) iff \( g_{t_1 \ldots t_n} = 0 \) when \( t_i = t_{i+1} \)

Coboundary: \( \delta f_{ts} = f_t - f_s, \delta g_{tus} = g_{ts} - g_{tu} - g_{us}, \ldots, \delta^2 = 0 \)

\[
0 \to \mathbb{R} \to \mathcal{C}_1 \xrightarrow{\delta} \mathcal{C}_2 \xrightarrow{\delta} \mathcal{C}_3 \xrightarrow{\delta} \cdots
\]

Then \( \delta f = a \iff \delta a = 0 \).

Small 2-increments cannot be exact: \( a_{ts} = o(|t - s|) \Rightarrow a \neq \delta f \)

Unique decomposition: \( a = \delta f + o(|t - s|) \)?

Yes, if obstruction \( \delta a \) is small:

**Theorem**

If \( \delta a_{tus} = o(|t - s|) \), then \( \exists! f \in \mathcal{C}_1, r \in \mathcal{C}_2^{1+} \) such that

\[
a = \delta f + r, \quad \delta f = (1 - \Lambda \delta) a
\]
Examples

- Convergence of sums:

\[ S_{t0} = \sum_i a_{t_{i+1}t_i} = \sum_i (\delta f)_{t_{i+1}t_i} + \sum_i (r)_{t_{i+1}t_i} = (\delta f)_{t0} + \sum_i o(|t_{i+1} - t_i|) \to f_t - f_0 \]

- Young integrals: \( x \in C^\gamma, \gamma > 1/2 \) \( a_{ts} = \varphi(x_s)\delta x_{ts} \)

\[ \delta a_{tus} = \delta \varphi(x)_{tu} \delta x_{us} = o(|t - s|^{2\gamma}) \Rightarrow \delta f = (1 - \Lambda \delta)a =: \int \varphi(x)dx \]

- NCG & \( \Lambda \) map. \( L^2(\mathbb{R}), dg = [F, g], F^2 = c, \)

\[ (df)_{ts} = \frac{f_t - f_s}{t - s} \quad \int fdg = Tr_\omega(fdg) = \frac{1}{2c} Tr_\omega(Fdfdg) \]

so

\[ \Lambda(\delta f \delta g)_{-\infty,\infty} = -\frac{1}{2c} Tr_\omega(Fdfdg) \]
Rough integrals: $X^\bullet = \delta x$, $\delta X^{[\bullet]} = X^\bullet X^\bullet$, $X^\bullet \in C_2^\gamma$, $X^{[\bullet]} \in C_2^{2\gamma}$ ($\gamma > 1/3$)

$$\int \varphi(x)dx = (1 - \Lambda \delta) \left( \varphi(x)X^\bullet + \varphi'(x)X^{[\bullet]} \right),$$

$$\delta(\varphi(x)X^\bullet + \varphi'(x)X^{[\bullet]}) = (-\delta \varphi(x) + \varphi'(x)X^\bullet)X^\bullet - \delta \varphi'(x)X^{[\bullet]}$$

- Continuous map: $(\varphi, X^\bullet, X^{[\bullet]}) \mapsto \int \varphi(x)dx$
- Renormalized sums: $\sum_i (\varphi(x_{t_i})X_{t_{i+1}}^\bullet + \varphi'(x_{t_i})X_{t_{i+1}}^{[\bullet]}) \rightarrow \int \varphi(x)dx$
- $\int dx dx = X^{[\bullet]}$
- A finite number of iterated integrals determines all the other integrals.
Branched rough paths

The only data we need to build the family \( \{X^\tau\}_{\tau \in \mathcal{T}} \) is a family of maps \( \{I^a\}_{a \in \mathcal{L}} \) from \( \mathcal{C}_2 \) to \( \mathcal{C}_2 \) satisfying certain properties.

**Definition**

An integral is a linear map \( I : \mathcal{D}_I \rightarrow \mathcal{D}_I \) on an unital sub-algebra \( \mathcal{D}_I \subset \mathcal{C}_2^+ \) for which \( I(hf) = I(h)f, \forall h \in \mathcal{D}_I, f \in \mathcal{C}_1 \) and

\[
\delta I(h) = I(e)h + \sum_i I(h^{1,i})h^{2,i} \quad \text{when } h \in \mathcal{D}_I, \delta h = \sum_i h^{1,i}h^{2,i} \text{ and } h^{1,i} \in \mathcal{D}_I
\]

Then \( X^\bullet a = I^a(e), \quad X[^{\tau_1 \ldots \tau_k}] a = I^a(X^{\tau_1 \ldots \tau_k}), \quad X^{\tau_1 \ldots \tau_k} = X^{\tau_1} \circ \ldots \circ X^{\tau_k} \).

Tree multiplicative property still holds: \( \delta X^\tau = X^{\Delta' \tau} \).
Integrals are not necessarily Rota-Baxter maps: e.g. Itô–Z stochastic integral
\[
\int_0^t f_s dx_s \int_0^t g_s dx_s = \int_0^t f_s \int_0^s g_u dx_u dx_s + \int_0^t g_s \int_0^s f_u dx_u dx_s + \int_0^t f_s g_s ds
\]
\[
I(f)I(g) = I(I(f)g) + I(fI(g)) + J(fg), \quad J(f)I(g) = J(fI(g)) + I(gJ(f))
\]
Solution to \(dy = ydx, y_0 = 1\): \(y_t = \exp(x_t - t/2)\).
When they are Rota-Baxter we have shuffle relations:
\[
I^{a_1} (\cdots I^{a_n} (1)) \circ I^{b_1} (\cdots I^{b_m} (1)) = \sum_{\tau \in \text{Sh}(\vec{a}, \vec{b})} I^{c_1} (\cdots I^{c_{n+m}} (1))
\]
This relation reduces \(X^\tau\) for \(\tau \in \mathcal{T}_{\mathcal{L}}\) to a linear combination of \(\{X^\sigma\}_{\sigma \in \mathcal{T}^{\text{Chen}}_{\mathcal{L}}}\).
These are geometric rough-paths: the closure of smooth rough paths.
Growing a branched rough path

Fix $\gamma \in (0, 1]$, consider $q_\gamma : \mathcal{F} \to \mathbb{R}_+$ on forests as $q_\gamma(\tau) = 1$ for $|\tau| \leq 1/\gamma$ and

\[
q_\gamma(\tau) = 1, \text{ if } |\tau| \leq 1/\gamma \quad q_\gamma(\tau) = \frac{1}{2^\gamma |\tau|} \sum q_\gamma(\tau^{(1)})q_\gamma(\tau^{(2)}) \text{ otherwise}
\]

$q_\gamma(\tau_1 \cdots \tau_n) = q_\gamma(\tau_1) \cdots q_\gamma(\tau_n)$.

**Theorem**

*Given a partial homomorphism $X : A_n \mathcal{T}_L \to \mathcal{C}_2$ satisfying the multiplicative property*

\[
|X_{ts}^\tau| \leq BA^{\gamma|\tau|} q_\gamma(\tau) |t - s|^\gamma |\tau|, \quad \tau \in \mathcal{T}_L^n \tag{4}
\]

*with $\gamma(n + 1) > 1$, then $\exists! X : A \mathcal{T}_L \to \mathcal{C}_2$ such that eq. (4) holds $\forall \tau \in \mathcal{T}_L$.*

Construction via the equation: $X^\tau = \Lambda(X^{\Delta', \tau})$. 
Speed of growth

Conjecture

\[ q_\gamma(\tau) \asymp C(\tau!)^{-\gamma} \]

True for linear Chen trees \( \mathcal{T}_{\text{Chen}} \):

\[
\sum_{k=0}^{n} \frac{a^{\gamma k} b^{\gamma (n-k)}}{(k!)^{\gamma} (n!)^{\gamma}} \leq c_\gamma \frac{(a + b)^{\gamma n}}{(n!)^{\gamma}}, \quad \gamma \in (0, 1], \ a, b \geq 0
\]

Variant of Lyons' neo-classical inequality

\[
\sum_{k=0}^{n} \frac{a^{\gamma k} b^{\gamma (n-k)}}{(\gamma k)! [\gamma (n - k) n]!} \leq c_\gamma \frac{(a + b)^{\gamma n}}{(\gamma n)!}
\]

"neo-classical tree inequality"?

\[
\sum \frac{a^{\gamma |\tau^{(1)}|} b^{\gamma |\tau^{(2)}|}}{(|\tau^{(1)}|)! [\gamma (\tau^{(2)!})] \gamma} \leq c_\gamma \frac{(a + b)^{\gamma |\tau|}}{(|\tau|)! \gamma}
\]

OK for \( \gamma = 1 \): tree binomial formula.
Controlled paths

Definition

Let $n$ the largest integer such that $n \gamma \leq 1$. For any $\kappa \in (1/(n + 1), \gamma]$ a path $y$ is a $\kappa$-weakly controlled by $X$ if

$$
\delta y = \sum_{\tau \in \mathcal{F}^{n-1}_L} X^\tau y^\tau + y^{\#}, \quad \delta y^\tau = \sum_{\sigma \in \mathcal{F}^{n-1}_L} \sum_{\rho} c'(\sigma, \tau, \rho) X^\rho y^\sigma + y^{\tau,\#}, \quad \tau \in \mathcal{F}^{n-1}_L
$$

with $y^\tau \in C|\tau|_2^{-\kappa}$, $y^{\#\tau} \in C^{(n-|\tau|)_2}$. Then we write $y \in \mathcal{Q}_\kappa(X; V)$.

Lemma (Stability)

Let $\varphi \in C^n_b(\mathbb{R}^k, \mathbb{R})$ and $y \in \mathcal{Q}_\kappa(X; \mathbb{R}^k)$, then $z_t = \varphi(y_t)$ is a weakly controlled path, $z \in \mathcal{Q}_\kappa(X; \mathbb{R})$ where its coefficients are given by

$$
z^\tau = \sum_{m=1}^{n-1} \sum_{|\vec{b}|=m} \frac{\varphi_{\vec{b}}(y)}{m!} \sum_{\tau_1, \ldots, \tau_m \in \mathcal{F}^{n-1}_L \atop \tau_1 \cdots \tau_m = \tau} y^{\tau_1, b_1} \cdots y^{\tau_m, b_m}, \quad \tau \in \mathcal{F}^{n-1}_L
$$
Integration of controlled paths

Theorem

The integral maps \( \{I^a\}_{a \in \Lambda} \) can be extended to maps \( I^a : \mathcal{Q}_\kappa(X) \to \delta \mathcal{Q}_\kappa(X) \)

\[
y \in \mathcal{Q}_\kappa(X) \mapsto \delta z = I^a(y) = X^{\bullet a}z^{\bullet a} + \sum_{\tau \in \mathcal{J}^n_\Lambda} X^\tau z^\tau + z^b,
\]

where \( z^b \in \mathcal{C}_2^{\kappa(n+1)} \), \( z^{\bullet a} = y \), \( z^{[\tau]}_a = y^\tau \) and zero otherwise.

Remark

If \( y \in \mathcal{Q}_\kappa(X; \mathbb{R}^n \otimes \mathbb{R}^d) \) then \( \{J^b(\cdot) = \sum_{a \in \Lambda} I^a(y^{ab}_\cdot)\}_{b \in \Lambda_1} \) defines a family of integrals with an associated branched rough path \( Y \) indexed by \( \mathcal{T}_\Lambda_1 \). An explicit recursion is

\[
Y^{\bullet b} = \sum_{a \in \Lambda} I^a(y^{ab}), \quad Y^{[\tau^1 \cdots \tau^k]}_b = \sum_{a \in \Lambda} I^a(y^{ab} Y^{\tau^1} \circ \cdots \circ Y^{\tau_k}), \quad b \in \Lambda_1
\]
Example

\[ \delta y = X \cdot y' + X \cdot y' + X \cdot y'' + X \cdot y' + X \cdot y' + X \cdot y' + X \cdots y'' + X \cdot y' + y' \]

\[ \delta y' = X \cdot (y' + 2y'' + y') + X \cdot (y' + y') + X \cdot (y' + y' + 3y''') + y''' \]

\[ \delta y'' = X \cdot (y'' + 2y' + y') + y'' \]

\[ \delta y''' = X \cdot (y''' + y''') + y''' \]

\[ \delta y' = y', \quad \delta y'' = y'', \quad \delta y''' = y''', \quad \delta y'''' = y'''' \]

\[ \delta z = \delta I(y) = X \cdot y + X \cdot y' + X \cdot y' + X \cdot y' + X \cdot y' + X \cdot y' + X \cdot y' + X \cdot y' + X \cdot y' + z' \]

\[ = X \cdot z' + X \cdot z' + X \cdot z' + X \cdot z' + z' \]

with

\[ z' = \Lambda \left[ X \cdot y' + X \cdot y' + X \cdot y' + X \cdot y' + X \cdot y' + X \cdot y' + X \cdot y' + X \cdot y' + X \cdot y' \right] \]
Rough differential equations

Take vectorfields \( \{f_a \in C^n_b(\mathbb{R}^k; \mathbb{R}^k)\}_{a \in \mathcal{L}} \) and integral maps \( \{I^a\}_{a \in \mathcal{L}} \) and consider the rough differential equation

\[
\delta y = I^a(f_a(y)), \quad y_0 = \eta \in \mathbb{R}^k
\]

in the time interval \([0, T]\).

**Theorem**

The rough differential equation (6) has a global solution \( y \in \mathcal{Q}_\gamma(X; \mathbb{R}^k) \) for any initial condition \( \eta \in \mathbb{R}^k \). If the vectorfields are \( C^{n+1}_b \) the solution is unique and has Lipshitz dependence on data.
The KdV equation

1d periodic KdV equation:

\[ \partial_t u(t, \xi) + \partial^3_\xi u(t, \xi) + \frac{1}{2} \partial_\xi u(t, \xi)^2 = 0, \quad u(0, \xi) = u_0(\xi), \quad (t, \xi) \in \mathbb{R} \times \mathbb{T} \]

where initial condition \( u_0 \in H^\alpha(\mathbb{T}), \mathbb{T} = [-\pi, \pi] \). Linear part: Airy group \( U(t) \) (isometries on \( H^\alpha \)). Go to Fourier variables and let \( v_t = U(-t)u_t \):

\[ v_t(k) = v_0(k) + \frac{ik}{2} \sum_{k_1} \int_0^t e^{-i3kk_1k_2s} v_s(k_1)v_s(k_2) \, ds, \quad t \in [0, T], k \in \mathbb{Z}_* \]

where \( k_2 = k - k_1 \) and \( v_0(k) = u_0(k) \). Restrict to \( v_0(0) = 0 \). It has the form

\[ v_t = v_s + \int_s^t \dot{X}_\sigma(v_\sigma, v_\sigma) \, d\sigma, \quad t, s \in [0, T]. \]

where \( \dot{X}_\sigma(\varphi, \varphi) = \frac{ik}{2} \sum_{k_1} e^{-i3kk_1k_2\sigma} \varphi(k_1)\varphi(k_2) \).
The KdV equation

Expansion

\[ \delta v_{ts} = X^\bullet (v^x2) + X\dot{1}(v^x3) + X\dot{2}(v^x4) + X\dot{3}(v^x4) + r \]  

(7)

with multi-linear operators \( X^\tau \):

\[ X_{ts}^\bullet (\varphi_1, \varphi_2) = \int_s^t \dot{X}_\sigma (\varphi_1, \varphi_2) d\sigma; \]

\[ X_{ts}^{[\tau_1]} (\varphi_1, \ldots, \varphi_{m+1}) = \int_s^t \dot{X}_\sigma (X_{\sigma s}^{\tau_1}(\varphi_1, \ldots, \varphi_m), \varphi_{m+1}) d\sigma \]

and

\[ X_{ts}^{[\tau_1 \tau_2]} (\varphi_1, \ldots, \varphi_{m+n}) = \int_s^t \dot{X}_\sigma (X_{\sigma s}^{\tau_1}(\varphi_1, \ldots, \varphi_m), X_{\sigma s}^{\tau_2}(\varphi_{m+1}, \ldots, \varphi_{m+n})) d\sigma. \]

Eq.7 is a rough equation which can be solved with fixed-point:

\[ \delta v = (1 - \Lambda \delta)[X^\bullet (v^x2) + X\dot{1}(v^x3)] \]
Shadows of the conservation law

Lemma

\[
\langle \varphi_1, \dot{X}_s(\varphi_2, \varphi_3) \rangle + \langle \varphi_2, \dot{X}_s(\varphi_1, \varphi_3) \rangle + \langle \varphi_3, \dot{X}_s(\varphi_2, \varphi_1) \rangle = 0, \quad s \in [0, T]
\]

\[
\langle \varphi, X_{ts}(\varphi, \varphi) \rangle = 0 \quad 2\langle \varphi, X_{ts}^2(\varphi, \varphi, \varphi) \rangle + \langle X_{ts}(\varphi, \varphi), X_{ts}(\varphi, \varphi) \rangle = 0
\]

\[
[\delta \langle v, v \rangle]_{ts} = 2\langle X_{ts}(v_s, v_s) + X_{ts}^2(v_s, v_s, v_s) + v_{ts}^b, v_s \rangle
\]

\[
+ \langle X_{ts}(v_s, v_s), X_{ts}(v_s, v_s) \rangle + 2\langle X_{ts}(v_s, v_s), v_{ts}^\# \rangle + \langle v_{ts}^\#, v_{ts}^\# \rangle
\]

\[
= 2\langle v_{ts}^b, v_s \rangle + 2\langle X_{ts}(v_s, v_s), v_{ts}^\# \rangle + \langle v_{ts}^\#, v_{ts}^\# \rangle = O(|t - s|^{3\gamma})
\]

Theorem (Integral conservation law)

If \( v \) is a solution of KdV then \( |v_t|^2 = |v_0|^2 \) for any \( t \).
The NS equation

The $d$-dimensional NS equation (or the Burgers’ equation) have the abstract form

$$u_t = S_t u_0 + \int_0^t S_{t-s} B(u_s, u_s) \, ds.$$  \hspace{1cm} (8)

$S$ bounded semi-group on $\mathcal{B}$, $B$ symmetric bilinear operator. Define $d(\tau)$-multilinear operator by

$$X_{ts}^\bullet(\varphi \times 2) = \int_s^t S_{t-u} B(S_{u-s} \varphi, S_{u-s} \varphi) \, du$$

$$X_{ts}^{[\tau^1]}(\varphi \times (d(\tau^1) + 1)) = \int_s^t S_{t-u} B(X_{us}^{\tau^1} (\varphi \times d(\tau^1)), \varphi) \, du$$

and

$$X_{ts}^{[\tau^1 \tau^2]}(\varphi \times (d(\tau^1) + d(\tau^2))) = \int_s^t S_{t-u} B(X_{us}^{\tau^1} (\varphi \times d(\tau^1)), X_{us}^{\tau^2} (\varphi \times d(\tau^2))) \, du$$

where $d(\tau)$ is an appropriate degree function.
The $X$ operators allow bounds in $\mathcal{B}$ of the form

$$\left| X^{\tau}(\varphi \times d(\tau)) \right|_\mathcal{B} \leq C \frac{|t - s|^{\epsilon |\tau|}}{(|\tau|!)^\epsilon} |\varphi|_\mathcal{B}^{d(\tau)}$$

where $\epsilon \geq 0$ is a constant depending on the particular Banach space $\mathcal{B}$ we choose.

We have the (norm convergent) series representation

$$u_t = S_t u_0 + \sum_{\tau \in \mathcal{J}_B} X^\tau_{t0}(u_0 \times d(\tau))$$  \hspace{1cm} (9)$$

which gives local solutions to NS.

Regularity: $|u(k)| \leq Ce^{-|k|\sqrt{t}}$ by controlling growth of the terms in the series.
A cochain complex \( (\hat{C}_*, \hat{\delta}) \) adapted to the study of convolution integrals.

Coboundary \( \hat{\delta} h = \delta h - ah - ha \) with \( a_{ts} = S_{t-s} - \text{Id} \) the 2-increment associated to the semi-group (parallel transport).

Associated integration theory (\( \tilde{\Lambda} \)-map as inverse to \( \hat{\delta} \)).

Algebraic relations, e.g.:

\[
\hat{\delta} X^\bullet(\varphi \times^3) = X^\bullet (X^\bullet(\varphi \times^2), \varphi)
\]

Applications to stochastic partial differential equations (SPDEs):

\[
u_t = S_t u_0 + \int_0^t S_{t-s} dw_s f(u_s)
\]
Perspectives & open problems

- Rough integrals as renormalized integrals
- Growth of $X$ and generalized B-series:
  \[ \sum_{\tau} c_{\tau} \frac{a^{|\tau|}}{(\tau!)^\varepsilon} \]
- Birkhoff decomposition for PDEs (cf. ERGE)
- Scaling in PDEs (RG):
  - Blowup of solutions via series methods (cf. Sinai for cNS)
  - Long-time asymptotics
- Nonperturbative solutions of DSE
- Hochschild cohomology for $(\mathcal{C}, \delta)$