Regularization by oscillations

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Ascona, May 26th, 2011
I would like to discuss three different regularization phenomena due to the presence of noise which however share similar structural properties.

▶ Davie’s theorem for SDEs
▶ Stochastic Burgers equation
▶ Schrödinger equation with random dispersion
Davie’s phenomenon

Consider this integral equation in $\mathbb{R}^d$:

$$x_t = \int_0^t b(s, x_s)ds + w_t, \quad 0 \leq t \leq 1$$

where $w \in C([0, 1]; \mathbb{R}^d)$ is a path picked according Wiener measure and $b$ is a generic bounded vectorfield.

A. M. Davie showed that there exist a full measure set $\Gamma \subset C([0, 1]; \mathbb{R}^d)$ such that every $w \in \Gamma$ admits only one solution $x \in C([0, 1]; \mathbb{R}^d)$ to the integral equation.

Easy consequence: any discrete scheme will converge to this solution (in particular non-adapted schemes).

Related work: [Veretennikov, Krylov-Röckner, Flandoli-Priola-G.]
Smoothing effect of typical brownian paths

Let \( x_t = u_t + \omega_t \):

\[
    u_t = \int_0^t b(s, w_s + u_s) \, ds, \quad 0 \leq t \leq 1
\]

Key fact

\[
    \mathbb{E} \left[ \left( \int_s^t (b(r, W_r + x) - b(r, W_r + y)) \, dr \right)^p \right] \leq C_p |x - y|^p |t - s|^{p/2}
\]

⇒ The random field \((t, s, x) \mapsto \int_s^t b(r, W_r + x) \, dr\) is almost surely almost \textbf{Lipshitz} in the space variable.

⇒ Via an approximation argument this results in uniqueness for the ODE.
A simple argument, or the Itô trick

Consider the Fourier transform of local time of a Brownian motion. By Itô formula

\[ Y(t, \xi) = \int_0^t e^{i\xi \cdot W_s} \, ds = 2\frac{e^{i\xi \cdot W_t} - e^{i\xi \cdot W_0}}{|\xi|^2} - \frac{2i\xi}{|\xi|^2} \cdot \int_0^t e^{i\xi \cdot W_s} \, dW_s \]

And for large \( \xi \in \mathbb{R}^d \)

\[ \mathbb{E} [|Y(t, \xi)|^p] \leq C_p |\xi|^{-p} |t|^{p/2} \]

so there is a gain of one power of \( \xi \), with respect to a trivial estimate.

Alternative approach using Gaussian computations

\[ \mathbb{E} [|Y(t, \xi)|^2] = \int_0^t \int_0^t \mathbb{E}[e^{i\xi \cdot (B_r - B_s)}] \, ds \, dr \]

\[ = 2 \int_0^t \int_0^r e^{-|\xi|^2 (r-s)/2} \, ds \, dr \leq \frac{t}{|\xi|} \]

\[ \Rightarrow \text{Extensions to fBM} \quad [\text{ongoing work with R. Catellier}] \]
Stochastic Burgers equation

Here the stochastic Burgers equation on $\mathbb{T} = [-\pi, \pi]$

$$d u_t = \frac{1}{2} \partial_{\xi}^2 u_t(\xi) dt + \frac{1}{2} \partial_{\xi} (u_t(\xi))^2 dt + \partial_{\xi} dW_t$$

where $W_t(\xi) = \sum_{k \in \mathbb{Z}_0} e_k(\xi) \beta_t^k$ with $\mathbb{Z}_0 = \mathbb{Z}\{0\}$, $e_k(\xi) = e^{ik\xi} / \sqrt{2\pi}$ and $(\beta_t^k)_{t \geq 0, k \in \mathbb{Z}_0}$ are complex Brownian motions with $(\beta_t^k)^* = \beta_t^{-k}$ and covariance $\mathbb{E}[\beta_t^k \beta_t^q] = \delta_{q+k=0} t$.

The solution $u$ would like to be the derivative of the solution of the Kardar–Parisi–Zhang equation

$$d h_t = \frac{1}{2} \partial_{\xi}^2 h_t(\xi) dt + \frac{1}{2} (\partial_{\xi} h_t(\xi))^2 dt + dW_t. \quad (1)$$

which is believed to capture the macroscopic behavior of a large class of surface growth phenomena.
Problems with the weak formulation

For sufficiently smooth test functions $\varphi : \mathbb{T} \to \mathbb{R}$ look for solutions of

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\partial^2_\xi \varphi) \, ds + \int_0^t \langle \partial_\xi \varphi, B(u_s) \rangle \, ds + W_t(\partial_\xi \varphi)$$

where $B(u_s)(\xi) = (u_s(\xi))^2$.

- We would like to start the equation from initial condition $u_0$ which is space white noise, this is expected to be an invariant measure.
- The linearized equation

$$X_t(\varphi) = u_0(\varphi) + \int_0^t X_s(\partial^2_\xi \varphi) \, ds + W_t(\partial_\xi \varphi)$$

has trajectories which look like white noise in space.

$\Rightarrow$ The nonlinear term $B(u_s)$ is not defined.
Smoothing

Write \( u_t = X_t + v_t \), then

\[
\nu_t(\phi) = \int_0^t \nu_s(\partial^2_{\xi} \phi) ds + \int_0^t \langle \partial_{\xi} \phi, B(X_s + v_s) \rangle ds
\]

The covariance of the OU process is

\[
\mathbb{E}[X_t(e_k)X_s(e_m)] = \delta_{m+k=0}e^{-m^2|t-s|/2}
\]

**Key fact**

The quantity

\[
\int_0^t \langle \partial_{\xi} \phi, B(X_s) \rangle ds
\]

is well defined due to the rapid space-time decorrelation of the OU process.
Call "good" a process $y$ such that

$$y_t(\varphi) = y_0(\varphi) + \int_0^t v_s(\partial^2_\xi \varphi) ds + A_t(\varphi) + W_t(\partial_\xi \varphi)$$

where

- $A_t(\varphi)$ is a zero-quadratic variation process
- $y_t$ is space-time white noise at all times
- The reversed process $\hat{y}_t = y_{T-t}$ has the same properties with drift $\hat{A} = -A$. 

Forward/backward Itô trick

Adding Itô formula for the finite quadratic variation process $y$

$$h(y_t) = h(y_0) + \int_0^t L^0 h(y_s) \, ds + \int_0^t Dh(y_s) \, dA_s + M^+_t$$

(here $L^0$ is the OU generator) with Itô formula for the backward process

$$h(y_{T-t}) = h(y_T) + \int_T^{T-t} L^0 h(y_{T-s}) \, ds - \int_T^{T-t} Dh(y_{T-s}) \, dA_{T-s} + M^-_t$$

gives

$$M^+_t - M^-_{T-t} + M^-_T = \int_0^t 2L^0 h(y_s) \, ds$$
Easy to find an $H$ such that $2L^0H = \partial_\xi B$ which allows to replace the Burgers drift

$$\int_0^t \partial_\xi B(y_s)ds$$

with a sum of forward and backward martingales such that

$$\langle M^\pm(\varphi) \rangle_T = \int_0^T \mathcal{E}(\langle \varphi, H(\cdot) \rangle)(y_s)ds$$

where

$$\mathcal{E}(h)(x) = \sum_{q \in \mathbb{Z}_0} q^2(D_qh)(x)^2.$$

The function $\mathcal{E}(\langle \varphi, H(\cdot) \rangle)(y_s)$ is now well defined for $y_s$ sampled according white noise and we can estimate it.
Formulation of the equation

Let $B_\varepsilon(x) = B(\rho_\varepsilon * x)$ a regularization of the non-linearity.

By previous arguments we have that for good processes $y$ this limit exists

$$
\lim_{\varepsilon \to 0} \int_0^t \langle \varphi, \partial_\xi B_\varepsilon(y_s) \rangle ds = B_t(\varphi)
$$

and we can use it to define the drift in the Burgers equation.

A solution $u$ of the Burgers equation is a good process such that

$$
u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\partial_\xi^2 \varphi) ds + B_t(\varphi) + W_t(\partial_\xi \varphi)
$$

The Itô trick provides compactness estimates for Galerkin approximation. Uniqueness is open (in this approach).

The process $B_t(\varphi)$ is only $3/2$–Hölder in time.
Schrödinger equation with random dispersion

Consider the (Stratonovich-) stochastic Schrödinger equation

\[ d\phi_t = i\Delta\phi_t \circ dB_t + |\phi_t|^2\phi_t dt \]

for \( \phi : [0, T] \times \mathbb{T} \to \mathbb{C} \). [Debussche–De Bouard]

Let

\[ U_t = e^{i\Delta B_t} \]

so that distributionally

\[ dU_t = i\Delta U_t \circ dB_t \]

and set \( \psi_t = U_t^{-1}\phi_t \). Then

\[ \psi_t = \psi_0 + \int_0^t U_s^{-1}(|U_s\psi_s|^2U_s\psi_s)ds. \]
Regularization

Define
\[ X_t(\theta) = \int_0^t U_s^{-1}(|U_s\theta|^2U_s\theta)\,ds \]

It turns out that this random map has the following pathwise regularity
\[ \|X_t(\theta) - X_s(\theta)\|_{L^2(\mathbb{T})} \lesssim |t - s|^{\gamma} \|\theta\|^3_{L^2(\mathbb{T})} \]
for some \( \gamma > 1/2 \).

Then take any \( \theta \in C^\gamma([0, T], L^2(\mathbb{T})) \) and consider
\[ \lim_{\Delta t \to 0} \sum_i [X_{t_{i+1}}(\theta_{t_i}) - X_{t_i}(\theta_{t_i})] = \int_0^t X_{ds}(\theta_s). \]

By Young theory we have the estimate
\[ \left\| \int_0^t X_{ds}(\theta_s) \right\|_{C^\gamma([0, T], L^2(\mathbb{T}))} \lesssim \|\theta\|^3_{C^\gamma([0, T], L^2(\mathbb{T}))}. \]
Existence and uniqueness of global solutions

The Schrödinger equation can be reformulated as a Young equation

\[ \psi_t = \psi_0 + \int_0^t X_{ds}(\psi_s) \]

giving rise to local solutions

\[ \psi \in C^\gamma([0, T_\ast], L^2(\mathbb{T})). \]

The \( L^2 \) conservation law allows to obtain global solutions.

Standard arguments for Young equations allows to prove convergence of the Euler scheme

\[ \psi_{t_{i+1}} = \psi_{t_i} + [X_{t_{i+1}}(\psi_{t_i}) - X_{t_i}(\psi_{t_i})] \]

Similar phenomena for the deterministic KdV equation

\[ \partial_t u_t = \partial^3_{\xi} u_t + \partial_{\xi} u_t^2. \]

with initial conditions in \( H^\alpha(\mathbb{T}) \) for \( \alpha > -1/2 \).

[see paper on CPAA]
Thanks