Applications of controlled paths

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I will exhibit various applications of the idea of a "controlled path".

- Pre-history
- Rough path theory
- Averaging by oscillations
- Stochastic Burgers equation with derivative white noise perturbation
- NSE with random dispersion
- Korteweg–de Vries equation with distributional initial condition
Pre-historic controlled paths

If \( f : \mathbb{R} \to \mathbb{R} \) is \( \gamma \)-Hölder we know that a suitable generalization when \( \gamma \in (1, 2) \) is to require

\[
f_t - f_s = g_s(t - s) + O(|t - s|^{\gamma})
\]

for some given function \( g : \mathbb{R} \to \mathbb{R} \). Then we know also that

\[
f_t - f_s = \int_s^t g_r \, dr = \lim_{|\Pi_{s,t}| \to 0} \sum_{t_i \in \Pi_{s,t}} g_{t_i}(t_{i+1} - t_i)
\]
Young integral

Let $f, g$ two smooth function and consider the bilinear form

$$I(f, g)_t = \int_0^t f_t \, dg_r = \int_0^t f_r \partial_r g_r \, dr = f_t g_t - f_0 g_0 - \int_0^t g_r \partial_r f_r \, dr.$$ 

Then

$$I : C \times H^1 \to H^1 \text{ and } I : H^1 \times C \to C$$

The interpolation space $X_2 = [C, H^1]_{1/2}$ allows $I : X_2 \times X_2 \to X_2$. In practice it is enough to take $C^\gamma$ for $\gamma > 1/2$ and more generally, if $\gamma + \rho > 1$

$$I : C^\rho \times C^\gamma \to C^\gamma$$

Moreover $h = I(f, g)$ is the unique function which satisfy

$$h_t - h_s = f_s(g_t - g_s) + O(|t - s|^\gamma + \rho) \quad \text{or} \quad h_t - h_s = \lim_{|\Pi_{s,t}| \to 0} \sum_{t_i \in \Pi_{s,t}} f_i (g_{t_{i+1}} - g_{t_i})$$

**Remark:** This result say that $\partial_t g_t$ is a distribution for which the product $f_r \partial_r g_r$ is still a well-defined distribution.
Beyond Young: Controlled paths

Let $f \in C^\rho$ and $g \in C^\gamma$ and assume that the following equation

$$
\Phi_{s,t} - \Phi_{s,u} - \Phi_{u,t} = (f_s - f_u)(g_u - g_t), \quad i, j \in \{1, \ldots, d\}, 0 \leq s \leq u \leq t
$$

has a solution $\Phi(f, g) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^d \otimes \mathbb{R}^d$ such that $|\Phi(f, g)_{st}| \lesssim |t - s|^\rho + \gamma$, then if $\gamma + \rho + \theta > 1$, for any function $h$ such that

$$
h_t - h_s = h'_s(f_t - f_s) + O(|t - s|^\rho + \theta)
$$

with $h' \in C^\theta$ there exists a unique solution to the requirement

$$
z_t - z_s = h_s(g_t - g_s) + h'_s\Phi(f, g)_{s,t} + O(|t - s|^{\gamma + \rho + \theta})
$$

and moreover it holds that

$$
z_t - z_s = \lim_{|\Pi_{s,t}| \to 0} \sum_{t_i \in \Pi_{s,t}} h_{t_i}(g_{t_{i+1}} - g_{t_i}) + h'_{t_i}\Phi(f, g)_{t_i,t_{i+1}} = \int_s^t h_r dg_r
$$

Remark: The integration of controlled paths can be interpreted as a definition for the product of distributions.
Averaging along a Brownian motion

A. Davie has showed that if \( b : \mathbb{R}^d \to \mathbb{R} \) is a bounded function and \( B \) a \( d \)-dimensional Brownian motion. The average of \( b \) along the Brownian trajectory given by

\[
\sigma_t(y) = \int_0^t b(B_s + x) \, ds
\]

satisfy

\[
\mathbb{E}|\sigma_t(y) - \sigma_t(x)|^{2p} \leq C_p |x - y|^{2p} t^p
\]

From this it is possible to deduce that the ODE

\[
X_t = x + \int_0^t b(X_s) \, ds + B_t
\]

has a unique continuous solution for almost every sample path of \( B \).
Averaging along an fBm

Let $\mathcal{F}L^\alpha$ the set of distribution $b : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$N_\alpha(b) = \int_{\mathbb{R}^d} (1 + |\xi|)^\alpha |\hat{b}(\xi)| d\xi < +\infty.$$ 

Then it is possible to show that if $(w_t)_{t \geq 0}$ is the sample path of a $d$-dim. fractional Brownian motion and $x \in Q^w \subset C(\mathbb{R}; \mathbb{R}^d)$ is controlled by $w$ in the sense that

$$x_t - x_s = w_t - w_s + O(|t - s|^\rho)$$

for some $\rho > 1/2$, for all $b \in \mathcal{F}L^\alpha$ with $\alpha > 1 - 1/2H$ the integral

$$\lim_{n \to \infty} \int_0^t b_n(x_s) \, ds =: \int_0^t b(x_s) \, ds$$

is well defined for any sequence of smooth function $(b_n)_{n \geq 1}$ such that $N_\alpha(b - b_n) \to 0$ and independent of the sequence. Moreover the map $t \mapsto \int_0^t b(x_s) \, ds$ is $C^\gamma$ for some $\gamma > 1/2$.

[joint work with R. Catellier]
Regularization by oscillations

If $\alpha > 2 - 1/2H$ the map

$$y \mapsto \int_0^t b(x_s + y)\,ds$$

is Lipshitz:

$$\left| \int_s^t b(x_r + y)\,dr - \int_s^t b(x_r + z)\,dr \right| \lesssim_{x,w} N_\alpha(b) |y - z| |t - s|^\gamma.$$  

The previous results allows to study the ODE in $\mathbb{R}^d$

$$x_t = x_0 + \int_0^t b(x_s)\,ds + w_t$$

where $b \in FL^\alpha$.

- Existence in $Q^w_\gamma$ for $\alpha > 1 - 1/2H$
- Uniqueness in $Q^w_\gamma$ for $\alpha > 2 - 1/2H +$ Lipshitz flow.
- If $b$ is not random we can have uniqueness for $\alpha > 1 - 1/2H$. 
Here the stochastic Burgers equation on $\mathbb{T} = [-\pi, \pi]$

$$du_t = \frac{1}{2} \partial^2_{\xi} u_t(\xi)dt + \frac{1}{2} \partial_{\xi} (u_t(\xi))^2 dt + \partial_{\xi} dW_t$$

where $dW_t$ is space-time white noise.

The solution $u$ would like to be the derivative of the solution of the Kardar–Parisi–Zhang equation

$$dh_t = \frac{1}{2} \partial^2_{\xi} h_t(\xi)dt + \frac{1}{2} (\partial_{\xi} h_t(\xi))^2 dt + dW_t.$$

which captures the macroscopic behavior of a large class of surface growth phenomena.
Problems with the weak formulation

For sufficiently smooth test functions $\varphi : T \to \mathbb{R}$ look for solutions of

$$u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\partial^2_\xi \varphi) \, ds + \int_0^t \langle \partial_\xi \varphi, B(u_s) \rangle \, ds + W_t(\partial_\xi \varphi)$$

where $B(u_s)(\xi) = (u_s(\xi))^2$.

- We would like to start the equation from initial condition $u_0$ which is space white noise, this is expected to be an invariant measure.
- The linearized equation

$$X_t(\varphi) = u_0(\varphi) + \int_0^t X_s(\partial^2_\xi \varphi) \, ds + W_t(\partial_\xi \varphi)$$

has trajectories which looks like white noise in space.

$\Rightarrow$ The nonlinear term $B(u_s)$ is not defined.
"Lazy" smoothing estimation

Here a controlled process \( y \) is such that

\[
y_t(\varphi) = y_0(\varphi) + \int_0^t v_s(\partial^2_\xi \varphi) \, ds + A_t(\varphi) + W_t(\partial_\xi \varphi)
\]

where

- \( A_t(\varphi) \) is a zero-quadratic variation process
- \( y_t \) is space-time white noise at all times
- The reversed process \( \hat{y}_t = y_{T-t} \) has the same properties with drift \( \hat{A} = -A \).
Formulation of the equation

Let \( B_\varepsilon(x) = B(\rho_\varepsilon * x) \) a regularization of the non-linearity.

Can show that for a controlled path \( y \) this limit exists:

\[
\lim_{\varepsilon \to 0} \int_0^t \langle \phi, \partial_\xi B_\varepsilon(y_s) \rangle \, ds = B_t(\phi)
\]

(independently of regularization) and we can use it to define the drift in the Burgers equation.

A solution \( u \) of the Burgers equation is a good process such that

\[
u_t(\varphi) = u_0(\varphi) + \int_0^t u_s(\partial^2_\xi \varphi) \, ds + B_t(\varphi) + W_t(\partial_\xi \varphi)
\]

The controlled path approach provides compactness estimates for Galerkin approximation. Uniqueness seems difficult in this approach.

The process \( B_t(\varphi) \) is only \( 3/2 \)– Hölder in time.
Schrödinger equation with random dispersion

Consider the (Stratonovich-) stochastic Schrödinger equation

\[ d\phi_t = i\Delta \phi_t \circ dB_t + |\phi_t|^2 \phi_t dt \]

for \( \phi : [0, T] \times T \to \mathbb{C} \).

Let \( U_t = e^{i\Delta B_t} \) so that

\[ dU_t = i\Delta U_t \circ dB_t \]

then

\[ \phi_t = U_t(\phi_0 + \int_0^t U_s^{-1}(|\phi_s|^2 \phi_s) ds). \]
Formulation as a controlled path problem

The path \( \phi \) is controlled if

\[
\phi_t = U_t \psi_t
\]

with \( \psi_t \in C^\rho([R_+; L^2(T))] \) for some \( \rho > 1/2 \). Then it is possible to show that

\[
t \mapsto \int_0^t U_s^{-1}(|\phi_s|^2 \phi_s)ds
\]

exists, coincide with the following limit

\[
\lim_{n \to \infty} \int_0^t U_s^{-1}(|P_n \phi_s|^2 P_n \phi_s)ds
\]

(\( P_n \) is the projector on the Fourier modes \( |k| \leq n \)) and is \( \gamma \)-Hölder in time for some \( \gamma > 1/2 \) and locally Lipshitz in \( \phi \) in the controlled path norm.

By standard fixed-point argument we get a (unique) local solution to the NSE and the \( L^2 \) conservation law allows to extend it to a global one.
1d periodic KdV equation

\[
\begin{cases}
    \partial_t u(t, \xi) + \partial^3_\xi u(t, \xi) + \frac{1}{2} \partial_\xi u(t, \xi)^2 = 0 \\
    u(0, \xi) = u_0(\xi)
\end{cases} \quad (t, \xi) \in \mathbb{R} \times \mathbb{T}
\]

with initial condition \( u_0 \in H^\alpha(\mathbb{T}), \, \mathbb{T} = [-\pi, \pi] \).

We look for solutions for any \( \alpha > -1/2 \).

Airy group

\[ \mathcal{F}(U_t \varphi)(k) = e^{-ik^3 t} \hat{\varphi}(k), \quad k \in \mathbb{Z}. \]

Mild form

\[ u_t = U_t(u_0 + \int_0^t U_{-s}(\partial_\xi u_s^2)ds) \]
Formally the series expansion of the solution looks like

\[ u_t = U_{t-s}(u_s) + \int_s^t U_{r-s} \partial_\xi(U_r u_s)^2 dr + \cdots \]

A computation shows that the bilinear operator

\[ X_{s,t}^1(\varphi_1, \varphi_2) = \int_s^t U_{r-s} \partial_\xi \left[ (U_r \varphi_1)(\partial_\xi(U_r \varphi_2)) \right] dr \]

is bounded from \( H^\alpha \times H^\alpha \) to \( H^\alpha \) for \( \alpha > -1/2 \) and that the norm is of order \( |t-s|\gamma \) for some \( \gamma > 1/3 \).
Uniqueness of weak solutions

Using rough paths theory we can prove that the nonlinear term is defined of every controlled path:

Let $N(\varphi)(t, \xi) = \partial_\xi (\varphi(t, \xi)^2)/2$ for smooth functions $\varphi$. Any path $u$ in $H^\alpha$ such that

$$u_t = U_{t-s}u_s + U_t \int_s^t U_{t-r} (\partial_\xi (U_{r-s}v_s)^2) dr + U_t O(|t-s|^{2\gamma})$$

for some $v \in C^\gamma(\mathbb{R}, H^\alpha)$ enjoy the property that

$$N(P_N u) \rightarrow N(u)$$

as space-time distribution. The non-linear term is then well-defined.

There exists a unique local controlled solutions to the distributional equation

$$\partial_t u + \partial^3_\xi u + N(u) = 0$$