Santiago de Chile - PREFALC June 30, 2018
March 26 - April 6, 2018

THE KELLER-SEGEL EQUATION

- STILL A DRAFT -

These notes are devoted to the analysis of the solutions to the Keller-Segel equation. It is a famous equation modelling physic and biology which is both quite simple in its formulation and has a rather rich mathematical bahaviour.

Contents

1. Introduction

1.1. **The Keller-Segel equation.** The Keller-Segel (KS), or Patlak-Keller-Segel, system for chemotaxis describes the collective motion of cells that are attracted by a chemical substance that they are able to emit. It writes

$$\partial_t f = \Delta f - \nabla(f\,\nabla c) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^d, \tag{1.1}$$

$$\tau\partial_t c = \Delta c - \alpha c + f \quad \text{in} \quad (0,\infty)\times\mathbb{R}^d, \tag{1.2}$$

for some real parameters $\alpha, \tau \geq 0$. Here $t \geq 0$ is the time variable, $x \in \mathbb{R}^d$ is the space variable, $f = f(t,x) \geq 0$ stands for the *mass density of cells* while $c = c(t,x) \in \mathbb{R}$ is the *chemo-attractant concentration.*

In the sequel, we will focus on the two dimensional case $d = 2$ without damping term ($\alpha = 0$) and in the quasi-static regime ($\tau = 0$). In other words, the Keller-Segel system (1.1)-(1.2) is a parabolic-elliptic system in which the equation (1.1) on the *mass density of cells* is unchanged and the *chemo-attractant concentration* c solves the (elliptic) Poisson equation

$$-\Delta c = f \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2. \tag{1.3}$$

The solution c to the Poisson equation is given by the representation formula

$$-c := \bar\kappa = \kappa * f, \quad \kappa := \frac{1}{2\pi}\log|z|.$$

In particular, we have

$$-\nabla c = \bar{\mathcal{K}} := \mathcal{K} * f, \quad \mathcal{K} := \nabla\kappa = \frac{1}{2\pi}\frac{z}{|z|^2},$$

and that is our definition of the *gradient of chemo-attractant concentration* in (1.1). In other words, the parabolic-elliptic Keller-Segel system (1.1)-(1.3) also writes as the Keller-Segel equation

$$\partial_t f = \Delta f + \nabla(\bar{\mathcal{K}}\,f) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2. \tag{1.4}$$

The evolution equation (1.4) is complemented with an initial condition

$$f(0,.) = f_0 \quad \text{in} \quad \mathbb{R}^2. \tag{1.5}$$

1.2. **Remarkable features.** We present now several remarkable identities satisfied (at least formally) by any solution f to the Keller-Segel equation (1.4).

• *Conservation of mass.* Because the RHS term is in divergence form, we (formally) have

$$\frac{d}{dt}\int f\,dx = \int_{\mathbb{R}^2} \mathrm{div}_x(\nabla f + f\,\bar{\mathcal{K}})\,dx = 0,$$

so that the mass is conserved

$$\int_{\mathbb{R}^2} f(t,x)\,dx = \int_{\mathbb{R}^2} f_0(x)\,dx =: M, \quad \forall\, t \geq 0. \tag{1.6}$$

• *Center of mass.* Similarly, we (formally) have

$$\frac{d}{dt}\int f\,x\,dx = -\int_{\mathbb{R}^2}(\nabla f + f\,\bar{\mathcal{K}})\,dx = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\;f(x)\,f(y)\,dx = 0,$$

because the function $(x,y)\mapsto\phi(x,y)=\mathcal{K}(x-y)f(x)f(y)$ in the last integral satisfies $\phi(y,x)=-\phi(x,y)$. Again, the center of mass is conserved

$$\int_{\mathbb{R}^2} f(t,x)\,x\,dx=\int_{\mathbb{R}^2} f_0(x)\,x\,dx,\quad \forall\,t\ge 0.$$

• *Energy.* We again (formally) compute

$$\begin{aligned}\frac{d}{dt}\int f\,|x|^2\,dx &= \int_{\mathbb{R}^2} f\,\Delta|x|^2\,dx-2\int_{\mathbb{R}^2} f\,\bar{\mathcal{K}}\cdot x\,dx\\ &= 4\int_{\mathbb{R}^2} f\,dx-\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\cdot(x-y)\,f(x)\,f(y)\,dx\\ &= 4M-\frac{M^2}{2\pi}=:C_1(M).\end{aligned}$$

We deduce the linear growth of the second moment

$$M_2(t):=\int_{\mathbb{R}^2} f(t,x)\,|x|^2\,dx=C_1(M)\,t+M_{2,0},\quad M_{2,0}:=\int_{\mathbb{R}^2} f_0(x)\,|x|^2\,dx, \tag{1.7}$$

for any $t\ge 0$.

• *Free energy.* We write the Keller-Segel equation in a "gradient flow" form

$$\begin{aligned}\partial_t f &= \mathrm{div}\,\big[\,\nabla f+f\nabla\kappa*f\big]\\ &= \mathrm{div}\,\big[f\nabla(\log f+\bar\kappa)\big],\end{aligned}$$

and we define the entropy functional $\mathcal{H}$ and the free energy functional $\mathcal{F}$ by

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(f):=\int_{\mathbb{R}^2} f\log f dx,\\ \mathcal{F} &= \mathcal{F}(f):=\mathcal{H}(f)+\frac12\int\!\!\int f(x)f(y)\,\kappa(x-y)\,dxdy.\end{aligned}$$

Denoting $\mathcal{F}(t)=\mathcal{F}(f(t))$, we compute

$$\begin{aligned}\frac{d}{dt}\,\mathcal{F}(t) &= \int\{\,1+\log f+\kappa*f\}\;\partial_t f\\ &= \int\{\log f+\kappa*f\}\;\mathrm{div}\,\big[f\nabla(\log f+\kappa*f)\big]\\ &= -\mathcal{D}_{\mathcal{F}}(t),\end{aligned}$$

where $\mathcal{D}_{\mathcal{F}}(t)=\mathcal{D}_{\mathcal{F}}(f(t))$ and $\mathcal{D}_{\mathcal{F}}$ stands for the dissipation of free energy defined by

$$\mathcal{D}_{\mathcal{F}}=\mathcal{D}_{\mathcal{F}}(f):=\int_{\mathbb{R}^2} f\,|\;\nabla(\log f+\bar\kappa)|^2\,dx.$$

We deduce the (formal) free energy-dissipation of the free energy identity

$$\mathcal{F}(t)+\int_0^t\mathcal{D}_{\mathcal{F}}(s)\,ds=\mathcal{F}_0:=\mathcal{F}(f_0),\quad\forall\,t\ge 0. \tag{1.8}$$

• *Positivity.* In order to justify the positivity assumption, we may argue as follows. We denote $\beta(s):=s_-$ and we (formally) compute

$$\beta'(f)\Delta f=\Delta\beta(f)-\beta''(f)|\nabla f|^2$$

and

$$\begin{aligned}\beta'(f)\,\mathrm{div}(\bar{\mathcal K}\,f) &= \beta'(f)f\,\mathrm{div}\,\bar{\mathcal K} + \beta'(f)\nabla f\cdot\bar{\mathcal K}\\ &= \beta(f)\,\mathrm{div}\,\bar{\mathcal K} + \nabla\beta(f)\cdot\bar{\mathcal K} \;=\; \mathrm{div}\,\big[\bar{\mathcal K}\,\beta(f)\big].\end{aligned}$$

We deduce

$$\frac{d}{dt}\int\beta(f) \;=\; -\int\beta''(f)|\nabla f|^2 + \int \mathrm{div}(\nabla\beta(f) + \bar{\mathcal K}\,\beta(f)) \le 0,$$

and finally

$$\int\beta(f(t,.)) \le \int\beta(f_0), \qquad t\ge 0.$$

Observing that $\beta(g)=0$ iff $g\ge 0$, we conclude that $f(t,.)\ge 0$ for any $t>0$, because $f_0\ge 0$.

Let us make some comments about the Keller-Segel equation and the already formally established properties of its solutions.
The most important feature of the parabolic-elliptic Keller-Segel equation is probably the fact that the critical mass $M_* := 8\pi$ is a threshold: solutions are global for subcritical initial mass $M\le M_*$ but there does not exist global nonnegative and mass preserving solution when $M>8\pi$. In other words, one can prove the existence and the uniqueness of (weak) solutions to the Keller-Segel equation on a time interval $(0,T^*)$ with $T^*=\infty$ if $M_*\le 8\pi$ and $T^*<\infty$ if $M_*>8\pi$.

On the one hand, one can easily figure out this last property by observing that identity (1.7) would imply that the second moment becomes negative at least after the finite time $T^{**} := -M_{2,0}/C_1(M)$ for a global and mass preserving solution with mass $M>8\pi$, what is in contradiction with the positivity of that solution after the same time T^{**}.

On the other hand, the RHS term in (1.3) is a well-defined as the divergence of a L^1 function whenever $D_{\mathcal F}\in L^1(0,T)$ because

$$\int_{\mathbb R^2} |f\,(\;\nabla(\log f + \bar\kappa)| \le M^{1/2}\mathcal D_{\mathcal F}^{1/2}, \tag{1.9}$$

thanks to the Cauchy-Schwarz, and that information is strong enough in order to build a solution. It turns out that one can get that bound on $\mathcal D_{\mathcal F}$ for any time $T>0$ when $M<M_*$ as a consequence of the logarithmic Hardy-Littlewood Sobolev inequality. On the contrary, in the critical case $M=8\pi$ and the supercritical case $M>8\pi$, the above argument using the logarithmic Hardy-Littlewood Sobolev inequality fails.

A next issue is about the time asymptotic of $f(t,.)$ as $t\nearrow T^*$. The supercritical case $M>8\pi$ and $T^*<\infty$ is still far to be completely understood but very much can be said about the cases $M<8\pi$ and $M=8\pi$.

Let us recall that to understand the behavior of the solution $f^\sharp$ to the heat equation

$$\partial_t f^\sharp = \Delta f^\sharp, \tag{1.10}$$

a classical trick is to perform the *self-similar change of variables*

$$g^\sharp(t,x) := e^{2t}\,f^\sharp(\frac{e^{2t}-1}{2}, e^t\,x)$$

and to observe that $g^\sharp$ is then a solution to the Fokker-Planck equation

$$\partial_t g^\sharp = \Delta g^\sharp + \mathrm{div}(x g^\sharp). \tag{1.11}$$

We immediately see that $G^\sharp(x) := (2\pi)^{-1}\exp(-|x|^2/2)$ is the unique positive and with mass one stationary solution to the Fokker-Planck equation. Because mass is conserved and the Fokker-Planck equation is still dissipative, it is likely that $g^\sharp(t) \to M(g_0)G^\sharp$ as $t \to \infty$ (a convergence result which can be indeed rigorously established). Such a convergence can be translated as an information about the self-similar behavior of the solutions to the heat equation, namely

$$f^\sharp(t,x) \;\sim\; R(t)^{-2} M(g_0) G^\sharp(R(t)^{-1}x) \quad \text{as } t \to \infty.$$

In the subcritical case $M < 8\pi$, we may obtain a very similar result by performing the same *self-similar change of variables*

$$g(t,x) := e^{2t}\, f(\frac{e^{2t}-1}{2}, e^t\, x) = e^{2t}\, f(\tau, w),$$

and by analyzing the resulting equation which writes

$$\partial_t g = \Delta g + \nabla((x + \bar{\mathcal{K}})\, g), \tag{1.12}$$

with now $\bar{\mathcal{K}} := \mathcal{K} * g$.

2. A priori estimates in the subcritical case $M < 8\pi$

We shall assume that

$$0 \le f_0 \in L^1_2(\mathbb{R}^2), \quad f_0 \log f_0 \in L^1(\mathbb{R}^2), \tag{2.1}$$

as well as

$$M := \langle f_0 \rangle = \int_{\mathbb{R}^2} f_0\, dx \in (0, 8\pi), \qquad \langle f_0\, x \rangle = 0,$$

excepted when it is explicitly mentioned the contrary. Here and below, for any weight function $\varpi : \mathbb{R}^2 \to \mathbb{R}_+$, we define the weighted Lebesgue space $L^p(\varpi)$ for $1 \le p \le \infty$ by

$$L^p(\varpi) := \{\, f \in L^1_{loc}(\mathbb{R}^2);\ \|\, f\|_{L^p(\varpi)} := \|\, f\, \varpi\|_{L^p} < \infty\},$$

as well as $L^1_+(\mathbb{R}^2)$ the cone of nonnegative functions of $L^1(\mathbb{R}^2)$. We use the shorthand L^p_k, $k \ge 0$, for the weighted Lebesgue space associated to the polynomial growth weight function $\varpi(x) := \langle x \rangle^k$, where $\langle x \rangle := (1 + |x|^2)^{1/2}$. We also introduce the notation (with no risk of confusion)

$$\langle g \rangle = \int_{\mathbb{R}^2} g\, dx, \quad \forall\, g \in L^1.$$

We recall and accept the following inequality.

Lemma 2.1 (logarithmic Hardy-Littlewood Sobolev inequality)**.** *For any function $0 \le f \in L^1(\mathbb{R}^2)$ with mass $M = \langle f \rangle \ge 0$, there holds*

$$\forall\, f \ge 0, \quad \int_{\mathbb{R}^2} f(x) \log f(x)\, dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)\, f(y)\, \log|x - y|\ \ dxdy \ge C_2(M), \tag{2.2}$$

with $C_2(M) := M\,(1 + \log\pi - \log M)$.

Using the logarithmic Hardy-Littlewood Sobolev inequality (2.2) and the subcritical mass hypothesis $M \in (0, 8\pi)$, we have

$$\begin{aligned}
\mathcal{F}(f) &= \mathcal{H}(f) + \frac{1}{4\pi}\int\int f(x)f(y)\log|x-y| \\
&= (1-\frac{M}{8\pi})\mathcal{H}(f) + \frac{M}{8\pi}\Big(\mathcal{H}(f) + \frac{2}{M}\int\int f(x)f(y)\log|x-y|\Big) \\
&\geq (1-\frac{M}{8\pi})\mathcal{H}(f) + \frac{M}{8\pi}C_2(M),
\end{aligned}$$

so that

$$\mathcal{H}(f) \leq C_3(M)\,\mathcal{F}(f) + C_4(M), \tag{2.3}$$

with $C_3(M) := 1/\big(1-\frac{M}{8\pi}\big)$, $C_4(M) := C_3(M)\,C_2(M)\,M/(8\pi)$.

Exercise 2.2. *Establish the convex inequality $uv \leq u\log u - u + e^v$, for any $u, v > 0$, and deduce*

$$\int_{|x-y|\ \leq 1} f(y)\log\frac{1}{|x-y|}\,dy \leq \frac{1}{\alpha}\mathcal{H}(f) + K(\alpha, M), \quad \forall\,\alpha \in (0,2).$$

Recover (2.3) *(possibly, for different constants). (Hint. One can take for instance $\alpha := 1 + M/(8\pi)$).*

We also recall the following classical functional inequality

Lemma 2.3 (positive part of the entropy)**.** *For any $0 \leq f \in L^1_2$, there holds*

$$\int f(\log f)_- \leq \frac{1}{2}M_2(f) + C(d).$$

In particular,

$$\mathcal{H}^+ := \mathcal{H}^+(f) = \int_{\mathbb{R}^2} f(\log f)_+ dx \leq \mathcal{H} + \frac{1}{2}\ M_2 + C_5(M), \tag{2.4}$$

Exercise 2.4. *Prove Lemma 2.3. (Hint. One may prove and use the estimate*

$$s\,(\log s)_- \leq \sqrt{s}\,\mathbf{1}_{0\leq s\leq e^{-a|x|^k}} + s\,a|x|^k\,\mathbf{1}_{e^{-a|x|^k}\leq s\leq 1}, \quad \forall\, s \geq 0,$$

with $k = 2$ and $a = 1/2$).

From (2.3) and Lemma 2.3, one concludes that (1.6), (1.7) and (1.8) provide a convenient family of a priori estimates in order to define weak solutions. More precisely, we get

$$\begin{aligned}
&\mathcal{H}^+(t) + \frac{1}{2}M_2(t) + C_3(M)\int_0^t \mathcal{D}_{\mathcal{F}}(s)\,ds \\
&\leq C_3(M)\Big\{\ \mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(s)\,ds\Big\} + M_2(t) + C_4(M) + C(d) \\
&\leq C_3(M)\,\mathcal{F}_0 + M_{2,0} + C_1(M)t + C_4(M) + C(d),
\end{aligned} \tag{2.5}$$

where the RHS term is finite under assumption (2.1) on f_0, since

$$\begin{aligned}
\mathcal{F}_0 &\leq \mathcal{H}_0 + \frac{1}{4\pi}\int\!\!\int f_0(x)\,f_0(y)\,(\log|x-y|)_+\ dxdy \\
&\leq \mathcal{H}_0 + \frac{1}{4\pi}\int\!\!\int f_0(x)\,f_0(y)\,|x-y|^2\ dxdy \leq \mathcal{H}_0 + \frac{1}{\pi}M\,M_{2,0},
\end{aligned} \tag{2.6}$$

with $\mathcal{H}_0 := \mathcal{H}(f_0)$. In other words, we have

$$\mathscr{A}_T(f) := \sup_{t\in[0,T]} \left\{ \ \mathcal{H}^+(f(t)) + M_2(f(t))\right\} \ + \int_0^T \mathcal{D}_{\mathcal{F}}(f(s))\,ds \le C(T), \tag{2.7}$$

for any $T \in (0, T^*)$. It is worth emphasizing that in the subcritical case $M < 8\pi$ we are considering, we have $T^* = +\infty$ and the constant $C(T)$ only depends on M, $M_{2,0}$, $\mathcal{H}_0$ and the final time T.

With the sole information (2.7) it is possible to directly define a notion of weak solution. We rather present additional and (apparently) stronger a priori estimates on the solutions that we may deduce from (2.7) and which will make the definition more classical and will be also useful during the further analysis of the equation (existence, uniqueness, large time behavior).

We start presenting some elementary functional (Gagliardo-Nirenberg-Sobolev type) inequalities which will be of main importance in the sequel.

Lemma 2.5. *For any $0 \le f \in L^1(\mathbb{R}^2)$ with finite mass M and finite Fisher information*

$$I = I(f) := \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f},$$

there hold

$$\forall\, p \in [1,\infty), \quad \|f\|_{L^p(\mathbb{R}^2)} \le C_p\, M^{1/p}\, I(f)^{1-1/p}, \tag{2.8}$$

$$\forall\, q \in [1,2), \quad \|\nabla f\|_{L^q(\mathbb{R}^2)} \le C_q\, M^{1/q-1/2}\, I(f)^{3/2-1/q}. \tag{2.9}$$

For any $0 \le f \in L^1(\mathbb{R}^2)$ with finite mass M, there holds

$$\forall\, p \in [2,\infty) \quad \|f\|_{L^{p+1}(\mathbb{R}^2)} \le C_p\, M^{1/(p+1)}\, \| \ \nabla (f^{p/2})\|_{L^2}^{2/(p+1)}. \tag{2.10}$$

Proof of Lemma 2.5. We start with (2.9). Let $q \in [1,2)$ and use the Hölder inequality:

$$\|\nabla f\|_{L^q}^q = \int \left|\frac{\nabla f}{\sqrt{f}}\right|^q f^{q/2} \le \left(\int \frac{|\nabla f|^2}{f}\right)^{q/2} \left(\int f^{q/(2-q)}\right)^{(2-q)/2} = I(f)^{q/2}\, \|f\|_{L^{q/(2-q)}}^{q/2}.$$

Denoting by $q^* = 2q/(2-q) \in [2,\infty)$ the Sobolev exponent associated to q in dimension 2, thanks to a standard interpolation (Holder) inequality and to the Sobolev inequality, we have

$$\begin{aligned} \|f\|_{L^{q/(2-q)}} &= \|f\|_{L^{q^*/2}} \le \|f\|_{L^1}^{1/(q^*-1)}\, \|f\|_{L^{q^*}}^{(q^*-2)/(q^*-1)} \\ &\le C_q\, \|f\|_{L^1}^{1/(q^*-1)}\, \|\nabla f\|_{L^q}^{(q^*-2)/(q^*-1)}. \end{aligned} \tag{2.11}$$

Gathering these two inequalities, it comes

$$\|\nabla f\|_{L^q} \le C_q\, I(f)^{1/2}\, \|f\|_{L^1}^{1/(2(q^*-1))}\, \|\nabla f\|_{L^q}^{(q^*-2)/(2(q^*-1))},$$

from which we deduce (2.9).

We now establish (2.8). For $p \in (1,\infty)$, we may write $p = q^*/2 = q/(2-q)$ with $q := 2p/(1+p) \in [1,2)$ and we may use (2.11) and (2.9) to get

$$\|f\|_{L^p} \le C_p\, \|f\|_{L^1}^{\frac{1}{q^*-1} + \frac{q^*-2}{q^*-1}\,(\frac{1}{q}-\frac{1}{2})}\, I(f)^{\frac{q^*-2}{q^*-1}\,(\frac{3}{2}-\frac{1}{q})},$$

from which one easily concludes.

We verify (2.10). From the Sobolev inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \| w^{2(1+1/p)}\|_{L^1(\mathbb{R}^2)} &= \| w^{1+1/p}\|^2_{L^2(\mathbb{R}^2)} \le \| \nabla(w^{1+1/p})\|^2_{L^1(\mathbb{R}^2)} \\ &\le (1+1/p)^2 \| w^{1/p}\|^2_{L^2} \| \nabla w\|^2_{L^2(\mathbb{R}^2)} \end{aligned} \tag{2.12}$$

and we conclude to (2.10) by taking $w := f^{p/2}$. □

We deduce a first key estimate on the solutions to the Keller-Segel equation as a consequence of (2.7) and Lemma 2.5.

Lemma 2.6. *For any solution f to the Keller-Segel equation* (1.3)-(1.5)-(2.1) *and any final time $T \in (0, T^*)$, there exists a constant $C := C(M, \mathscr{A}_T(f))$ such that*

$$\frac{1}{2}\int_0^T I(f(t))\,dt \le C. \tag{2.13}$$

In particular, in the subcritical case $M < 8\pi$ the constant C only depends on M, $\mathcal{H}_0$, $M_{2,0}$ and $T \in (0,\infty)$.

Proof of Lemma 2.6. On the one hand, we write

$$\begin{aligned} \mathcal{D}_{\mathcal{F}}(f) &= \int f\,|\nabla(\log f + \bar\kappa)|^2 \\ &\ge \int f\,|\nabla \log f|^2 + 2\int \nabla f\cdot\nabla\bar\kappa = I(f) - 2\int f^2. \end{aligned}$$

On the other hand, for any $A > 1$, using the Cauchy-Schwarz inequality and the inequality (2.8) for $p = 3$, we have

$$\begin{aligned} \int f^2\,\mathbf{1}_{f\ge A} &\le \Big(\int f\,\mathbf{1}_{f\ge A}\Big)^{1/2}\Big(\int f^3\Big)^{1/2} \\ &\le \Big(\int f\,\frac{(\log f)_+}{\log A}\Big)^{1/2}\Big(C_3^3\,M\,I(f)^2\Big)^{1/2}, \end{aligned}$$

from what we get for $A = A(M, \mathcal{H}^+(f))$ large enough, and more precisely taking A such that $\log A = 16\,\mathcal{H}^+(f)\,C_3^3\,M$, the bound

$$\int f^2\,\mathbf{1}_{f\ge A} \le C_3^{3/2}\,M^{1/2}\,\frac{\mathcal{H}^+(f)^{1/2}}{(\log A)^{1/2}}\,I(f) \le \frac{1}{4}\ I(f). \tag{2.14}$$

Together with the first estimate, we find

$$\begin{aligned} \frac{1}{2}I(f) &\le \mathcal{D}_{\mathcal{F}}(f) + 2\int f^2\,\mathbf{1}_{f\le A} \\ &\le \mathcal{D}_{\mathcal{F}}(f) + 2\,M\ \exp(16\,\mathcal{H}^+(f)\,C_3^3\,M), \end{aligned}$$

and we conclude thanks to (2.7). □

Remark 2.7. *As we have already mentioned, we are not able to use the logarithmic Hardy-Littlewood-Sobolev inequality* (2.2) *in the critical and supercritical cases. However, introducing the Maxwell function $\mathscr{M} := M\,(2\pi)^{-1}\exp(-|x|^2/2)$ of mass M and the relative entropy*

$$H(h|\mathscr{M}) := \int_{\mathbb{R}^2} (h\,\log(h/\mathscr{M}) - h + \mathscr{M})\,dx,$$

one may show that any solution f to the Keller-Segel equation (1.4) *formally satisfies*

$$
\begin{aligned}
\frac{d}{dt}\, H(f(t)|\mathscr{M}) &= -I(f(t)) + \int f(t)^2 + C_1/2 \\
&\le -I(f(t)) + MA + C_3^{3/2}\, M^{1/2}\, \frac{\mathcal{H}^+(f(t))^{1/2}}{(\log A)^{1/2}}\, I(f(t)) + C_1/2 \qquad (\forall\, A > 0) \\
&= -I(f(t)) + M \exp\big(4C_3^3\, M\, \mathcal{H}^+(f(t))\big) + C_1/2 \\
&= -I(f(t)) + M \exp\big\{C_6\, H(f(t)|\mathscr{M})\big\} + C_1/2,
\end{aligned}
$$

for a constant $C_6 = C_6(M)$ and where $C_1 = C_1(M)$ is defined in (1.7)*. In the above estimates, we have used* (2.14)*, we have made the choice $\log A := 4C_3^3\, M\, \mathcal{H}^+(f(t))$ and we have used a variant of inequality* (2.4)*. This differential inequality provides a local a priori estimate on the relative entropy which is the key estimate in order to get the same estimate as in subcritcal case (but thus only locally in time).*

As an immediate consequence of Lemmas 2.5 and 2.6, we have

Lemma 2.8. *For any $T \in (0, T^*)$, any solution f to the Keller-Segel equation* (1.3)-(1.5)-(2.1) *satisfies*

$$f \in L^{p/(p-1)}(0,T; L^p(\mathbb{R}^2)), \quad \forall\, p \in (1,\infty), \tag{2.15}$$

$$\bar{\mathcal{K}} \in L^{p/(p-1)}(0,T; L^{2p/(2-p)}(\mathbb{R}^2)), \quad \forall\, p \in (1,2), \tag{2.16}$$

$$\nabla_x \bar{\mathcal{K}} \in L^{p/(p-1)}(0,T; L^p(\mathbb{R}^2)), \quad \forall\, p \in (2,\infty). \tag{2.17}$$

Proof of Lemma 2.8. The bound (2.15) is a direct consequence of (2.13) and (2.8). The bound (2.16) then follows from the definition of K, the Hardy-Littlewood-Sobolev inequality

$$\Big\| \frac{1}{|z|} * f \Big\|_{L^{2r/(2-r)}(\mathbb{R}^2)} \le C_r \, \| f\|_{L^r(\mathbb{R}^2)}, \quad \forall\, r \in (1,2), \tag{2.18}$$

with $r = p$ and (2.15). Finally, from (2.13) and (2.9) we have

$$\nabla f \in L^{\frac{2q}{3q-2}}(0,T; L^q(\mathbb{R}^2)), \quad \forall\, q \in (1,2).$$

Applying the Hardy-Littlewood-Sobolev inequality (2.18) to $\nabla_x \bar{\mathcal{K}} = \mathcal{K} * (\nabla_x f)$ with $r = q$, we get

$$\nabla_x \bar{\mathcal{K}} \in L^{\frac{2q}{3q-2}}(0,T; L^{\frac{2q}{2-q}}(\mathbb{R}^2)), \quad \forall\, q \in (1,2),$$

which is nothing but (2.17). □

Exercise 2.9. *Prove that $f\bar{\mathcal{K}} \in L^{4/3}((0,T) \times \mathbb{R}^2)$. (Hint. Use the estimates $f \in L^{p'}(0,T; L^p)$, $\bar{\mathcal{K}} \in L^{q'}(0,T; L^{2q/(2-q)})$ together with the (generalized) Holder inequality and optimize the values of $p \in (1,\infty)$, $r \in (1,2)$).*

3. Existence theory

Definition 3.1. *For any initial datum f_0 satisfying* (2.1) *and any final time $T^* > 0$, we say that*

$$0 \le f \in C([0,T); \mathcal{D}'(\mathbb{R}^2)), \quad \forall\, T \in (0, T^*), \tag{3.1}$$

which satisfies the bound (2.7)*, is a weak solution to the Keller-Segel equation on the time interval* $(0,T^*)$ *associated to the initial condition* f_0 *whenever* f *satisfies the mass conservation identity* (1.6)*, the relaxed second moment (in)equation*

$$M_2(t) \le C_1(M)\, t + M_{2,0}, \quad \forall\, t \in (0,T^*), \tag{3.2}$$

and the relaxed free energy-dissipation of energy (in)equation

$$\mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(s)\, ds \le \mathcal{F}_0, \quad \forall\, t \in (0,T^*), \tag{3.3}$$

as well as the Keller-Segel equation (1.4)-(1.5) *in the distributional sense, namely*

$$\int_{\mathbb{R}^2} f_0(x)\, \varphi(0,x)\, dx = \int_0^{T^*}\!\!\int_{\mathbb{R}^2} f \left\{ (\nabla_x(\log f) + \ \bar{\mathcal{K}}) \cdot \nabla_x \varphi - \partial_t \varphi \right\} dxdt, \tag{3.4}$$

for any $\varphi \in C^2_c([0,T^*) \times \mathbb{R}^2)$.

It is worth emphasizing again that (2.7) is a consequence (the arguments are presented in Section 2) of (1.6), (1.7) and (3.3) in the subcritical case $M \in (0,8\pi)$ and that the RHS of (3.4) is well defined thanks to (1.9) and (2.7). As we will explain in Section 3.1 (see also Exercise 2.9), we can prove

$$\exists\, r \ge 1, \qquad f\ \bar{\mathcal{K}} \in L^r((0,T) \times B_R), \quad \forall\, R > 0,$$

so that equation (3.4) may be expressed in the simpler way

$$\int_{\mathbb{R}^2} f_0(x)\, \varphi(0,x)\, dx = \int_0^{T^*}\!\!\int_{\mathbb{R}^2} f \left\{ \ \bar{\mathcal{K}} \cdot \nabla_x \varphi - \partial_t \varphi - \Delta \varphi \right\} dxdt \tag{3.5}$$

for any $\varphi \in C^2_c([0,T^*) \times \mathbb{R}^2)$. We rather take this relation as the definition of the distributional formulation of the Keller-Segel equation (1.4)-(1.5) (instead of (3.4)).

This framework is well adapted for the existence theory.

Theorem 3.2. *For any initial datum* f_0 *satisfying* (2.1)*, there exists at least one weak and maximal solution on a time interval* $(0,T^*)$ *in the sense of Definition 3.1 to the Keller-Segel equation* (1.4)-(1.5) *with*

- $T^* = +\infty$*, when* $M \le 8\pi$*;*
- $T^* < +\infty$ *and* $H(f(t)|\mathscr{M}) \to \infty$ *as* $t \to T^*$*, when* $M > 8\pi$.

In the sequel we will only prove the existence result in the case when $M \in (0,8\pi)$ and we leave as an exercise the case $M \ge 8\pi$, for which the proof follows the same lines.

Exercise 3.3. *Use Remark 2.7 in order to prove the existence of at least local in time solutions to the Keller-Segel equations* (1.3)-(1.5)-(2.1) *in the critical and supercritical cases* $M \ge 8\pi$.

3.1. **A stability result.** Before passing to the proof of Theorem 3.2, we present a stability result which is the key argument in the cornerstone and last step of the existence result. We consider a sequence of functions (f_n) such that

$$0 \le f_n \in C([0,\infty); \mathcal{D}'(\mathbb{R}^2))$$

and f_n is a weak solution to the Keller-Segel equation in the sense of Definition 3.1. More precisely, we assume

$$M(f_n(t)) = M \in (0, 8\pi), \tag{3.6}$$

$$M_2(f_n(t)) \le M_2(f_n(0)) + tC_1(M), \tag{3.7}$$

$$\mathcal{F}(f_n(t)) + \int_0^t \mathcal{D}_{\mathcal{F}}(f_n(s))\, ds \le \mathcal{F}(f_n(0)), \tag{3.8}$$

for any $t \in (0, T)$ and the Keller-Segel equation

$$\partial_t f_n \;=\; \Delta f_n - \nabla(f_n \bar{\mathcal{K}}_n), \quad \bar{\mathcal{K}}_n := \mathcal{K} * f_n, \tag{3.9}$$

has to be understood in the distributional sense $\mathcal{D}'([0,\infty) \times \mathbb{R}^2)$. We finally assume that $f_n(0) \to f_0$ strongly in L^1, $M_2(f_n(0)) \to M_2(f_0)$ and $\mathcal{F}(f_n(0)) \to \mathcal{F}(f_0)$.

Proposition 3.4. *Under the above assumptions, there exists a subsequence still denoted as (f_n) such that $f_n \rightharpoonup f$ weakly in L^2 and f is a solution to the Keller-Segel equation associated to the initial datum f_0 in the sense of Definition 3.1.*

Because of the estimates established in section 2, we know that (f_n) satisfies (uniformly in n)

$$\sup_{[0,T]} \int_{\mathbb{R}^2} f_n \,(1 + |x|^2 + (\log f_n)_+)\, dx + \int_0^T \int_{\mathbb{R}^2} \frac{|\nabla f_n|^2}{f_n}\, dxdt \le C_T. \tag{3.10}$$

We then deduce that (up to the extraction of a subsequence)

$$f_n \rightharpoonup f \quad \text{weakly in} \quad L^2((0,T) \times \mathbb{R}^2), \tag{3.11}$$

because the Cauchy-Schwarz inequality and the Sobolev inequality imply

$$\int_{\mathbb{R}^2} f_n^2 dx \le C \Big(\int_{\mathbb{R}^2} |\nabla f_n| dx \Big)^2 \le C \int_{\mathbb{R}^2} f_n dx \times \int_{\mathbb{R}^2} \frac{|\nabla f_n|^2}{f_n}\, dx,$$

so that (f_n) is bounded in $L^2((0,T) \times \mathbb{R}^2)$ thanks to the Fisher information bound (3.10). The above estimate is also a particular case of the first point in Lemma 2.5 (with $p = 2$).

We aim to explain now why the following strong convergence result holds true, this one allows then to pass to the limit in the weak formulation of (1.4).

Lemma 3.5. *Under the above assumptions, there holds*

$$\bar{\mathcal{K}}_n \to \bar{\mathcal{K}} := \mathcal{K} * f \quad \textit{strongly in} \quad L^2((0,T) \times B_R) \;\; \forall\, R > 0. \tag{3.12}$$

Proof of Lemma 3.5. In order to get a more elementary and self-contained argument, we introduce a splitting of the kernel $\mathcal{K}$ which make possible to use the more classical Young inequality instead of the (very subtle) Hardy-Littlewood-Sobolev inequality (2.18).

Step 1. Using estimate (3.10) and repeating the proof of Lemma 2.8, we have

$$(f_n) \text{ is bounded in } \; L^{p'}(0,T; L^p(\mathbb{R}^2)) \quad \forall\, p \in [1, \infty),$$

and in particular

$$(f_n) \text{ is bounded in } \; L^3(0,T; L^{3/2}(\mathbb{R}^2)).$$

Introducing the splitting

$$\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_\infty, \quad \mathcal{K}_0 := \mathcal{K}\, \mathbf{1}_{B_1} \in L^{3/2}, \quad \mathcal{K}_\infty := \mathcal{K}\, \mathbf{1}_{B_1^c} \in L^{5/2},$$

and using twice the Young inequality

$$\| u * v\|_{L^r} \le \| u\|_{L^p} \| v\|_{L^q}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1,$$

with $(p,q) := (3/2,3/2)$ and $(p,q) := (3/2,5/2)$, we obtain

$$g \in L^3_t L^{3/2}_x \;\Rightarrow\; \mathcal{K} * g \in L^3_t (L^3_x + L^{15}_x),$$

or in other words

$$\| \mathcal{K} * g\|_{L^3((0,T)\times B_R)} \le C_R \| g\|_{L^3_t L^{3/2}_x}, \quad \forall R > 0.$$

We deduce then

$$(\mathcal{K} * f_n) \text{ is bounded in } L^3((0,T)\times B_R), \quad \forall R > 0. \tag{3.13}$$

Together with (3.11) and the (generalized) Holder inequality, we get

$$(f_n \, \mathcal{K} * f_n) \text{ is bounded in } L^{6/5}((0,T)\times B_R). \tag{3.14}$$

Step 2. We next observe that for any $\varphi \in \mathcal{D}(\mathbb{R}^2)$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^2} f_n(t,x)\, \varphi(x)\, dx = \int_{\mathbb{R}^2} f_n\, (\Delta\varphi - \bar{\mathcal{K}}_n \cdot \nabla\varphi)\, dx,$$

where the RHS term is bounded in $L^{6/5}(0,T)$ thanks to the first step. We deduce that $\langle f_n \varphi\rangle$ is bounded in $W^{1,6/5}(0,T) \subset C^{0,1/6}([0,T])$, thanks to Morrey's inequality, and

$$\langle f_n \varphi\rangle \to \langle f \varphi\rangle \quad \text{strongly in} \quad L^\infty(0,T),$$

thanks to the Ascoli Lemma. We immediately deduce that for any $\varphi \in \mathcal{D}(\mathbb{R}^2) \otimes \mathcal{D}(\mathbb{R}^2)$, the space of linear combinations of functions of separable variables $\phi(x,y) = \phi^1(x)\, \phi^2(y)$, we also have

$$\int_{\mathbb{R}^2} f_n(t,y)\, \varphi(x,y)\, dx \to \int_{\mathbb{R}^2} f(t,y)\, \varphi(x,y)\, dx \quad \text{in} \quad L^2((0,T)\times B_R), \tag{3.15}$$

for any $R > 0$.

Step 3. We fix now $\varphi \in L^2(B_R \times \mathbb{R}^2)$ and we recall the (Stone-Weierstrass) density result: there exists a sequence (φ_k) of functions of $\mathcal{D}(\mathbb{R}^2) \otimes \mathcal{D}(\mathbb{R}^2)$ such that

$$\varphi_k \to \varphi \quad \text{in} \quad L^2(B_R \times \mathbb{R}^2).$$

We write

$$\int f_n \varphi - \int f\varphi \;=\; \int f_n(\varphi - \varphi_k) + \int f_n \varphi_k - \int f\varphi_k + \int f(\varphi_k - \varphi).$$

We observe that by the Cauchy-Schwarz inequality

$$\begin{aligned}
\Big\| \int f(\varphi_k - \varphi)\Big\|^2_{L^2((0,T)\times B_R)} &\le \int_0^T \int_{B_R} \Big(\int_{\mathbb{R}^2} f(t,y)(\varphi_k(x,y) - \varphi(x,y))\, dy\Big)^2 dxdt \\
&\le \int_0^T \int_{B_R} \int_{\mathbb{R}^2} f^2(t,y)\, dydxdt \times \int_0^T \int_{B_R} \int_{\mathbb{R}^2} (\varphi_k(x,y) - \varphi(x,y))^2\, dydxdt \\
&\le T\, |B_R| \; \| f\|^2_{L^2} \| \varphi_k - \varphi\|^2_{L^2} \to 0,
\end{aligned}$$

and that we have a similar result (uniformly in n) for the first term. We then classically deduce that (3.15) also holds for such a function φ.

Step 4. We define

$$\varphi_\varepsilon(x,y) := K(x-y)\,\mathbf{1}_{\varepsilon<|x-y|<1/\varepsilon}$$

so that $\varphi_\varepsilon \in L^2(B_R\times\mathbb{R}^2)$ for any $\varepsilon\in(0,1)$. We write

$$\bar{\mathcal{K}}_n - \bar{\mathcal{K}} \;=\; \int f_n(k(x-y)-\varphi_\varepsilon(x,y)) + \int f_n\varphi_\varepsilon - \int f\varphi_\varepsilon + \int f(\varphi_\varepsilon(x,y)-k(x-y)).$$

We note $\Omega := (0,T)\times B_R$ and we define $\mathcal{K}_{0,\varepsilon} := \mathcal{K}\,\mathbf{1}_{B_\varepsilon}$, $\mathcal{K}_{\infty,\varepsilon} := \mathcal{K}\,\mathbf{1}_{B^c_{1/\varepsilon}}$. For the last term we have

$$\begin{aligned}\Big\| \int f(\varphi_\varepsilon(x,y)-k(x-y))\Big\|_{L^2(\Omega)} &\le \| k_{0,\varepsilon} * f\|_{L^2(\Omega)} + \| k_{\infty,\varepsilon}*f\|_{L^2(\Omega)}\\ &\le C\,(\|\mathcal{K}_{0,\varepsilon}\|_{3/2} + \|\mathcal{K}_{\infty,\varepsilon}\|_{5/2})\,\| f\|_{L^3_t(L^{3/2}_x)} \to 0,\end{aligned}$$

and we conclude to (3.12) in the same way as in the preceding step. □

Exercise 3.6. *Prove* (3.12) *without making use of Lemma 2.8 (which is based on the Hardy-Littlewood-Sobolev inequality) but by only using the Young inequality (and the splitting of $\mathcal{K}$) as we have done in the proof above.*

As announced, the convergence result of Lemma 3.5 together with (3.14) imply that

$$(\mathcal{K}*f_n)f_n \rightharpoonup (\mathcal{K}*f)f \text{ weakly in } \; L^{6/5}((0,T)\times B_R).$$

Using that convergence result and (3.11) there is no difficulty to pass to the limit in the distributional formulation of the Keller-Segel equation (3.5) satisfied by f_n. We deduce that f also satisfied the distributional formulation of the Keller-Segel equation, namely that (3.5) holds.

In order to conclude the proof of Proposition 3.4, we need a somehow stronger convergence result, that we present bellow. We split it into two pieces.

Lemma 3.7. *Under the above assumptions, there holds*

$$f_n \to f \quad \textit{strongly in } \; L^p((0,T)\times B_R),\;\; \forall\, p\in[1,2),\;\; \forall\, R>0. \tag{3.16}$$

Proof of Lemma 3.7. Step 1. We argue similarly as in the proof of Lemma 3.5. We introduce a sequence of mollifiers (ρ_ε), that is $\rho_\varepsilon(x) := \varepsilon^{-2}\rho(\varepsilon^{-1}x)$ with $0\le\rho\in\mathcal{D}(\mathbb{R}^2)$, $\langle\rho\rangle = 1$. We observe that

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^2} f_n(t,y)\,\rho_\varepsilon(x-y)\,dx = \int_{\mathbb{R}^2} f_n\,(\Delta\rho_\varepsilon - \bar{\mathcal{K}}_n\cdot\nabla\rho_\varepsilon)\,dy,$$

where the RHS term is bounded in $L^{6/5}((0,T)\times B_R)$ uniformly in n for any fixed $\varepsilon>0$, thanks to the first step in Lemma 3.5. We also clearly have

$$\nabla_x\int_{\mathbb{R}^2} f_n(t,y)\,\rho_\varepsilon(x-y)\,dx = -\int_{\mathbb{R}^2} f_n\nabla_y\rho_\varepsilon(x-y)\,dy,$$

where again the RHS term is bounded in $L^{6/5}((0,T)\times B_R)$ uniformly in n for any fixed $\varepsilon>0$, thanks to (3.11). In other words, $f_n*\rho_\varepsilon$ is bounded in $W^{1,6/5}((0,T)\times B_R)$. Thanks to the Rellich-Kondrachov Theorem, we get that (up to the extraction of a subsequence) $(f_n*\rho_\varepsilon)_n$ is strongly convergent in $L^{6/5}((0,T)\times B_R)$. Thanks to (3.11) and for any fixed $\varepsilon>0$, we then get

$$f_n*\rho_\varepsilon \to f*\rho_\varepsilon \text{ strongly in } L^1((0,T)\times B_R) \text{ as } n\to\infty.$$

Step 2. Now, we observe that

$$\begin{aligned}\int_{(0,T)\times\mathbb{R}^2} |g - g*\rho_\varepsilon|\,dxdt &= \int_{(0,T)\times\mathbb{R}^2} \Big|\int_{\mathbb{R}^2} (g(t,x)-g(t,x-y))\rho_\varepsilon(y)\,dy\Big|\,dxdt \\ &= \int_{(0,T)\times\mathbb{R}^2} \Big|\int_{\mathbb{R}^2}\int_0^1 \nabla_x g(t,z_s)\cdot y\rho_\varepsilon(y)\,dsdy\Big|\,dxdt,\end{aligned}$$

with $z_s := x + sy$. As a consequence, we have

$$\begin{aligned}\int_{(0,T)\times\mathbb{R}^2} |g - g*\rho_\varepsilon|\,dxdt &\le \varepsilon\int_{(0,T)\times\mathbb{R}^2} \Big|\int_{\mathbb{R}^2}\int_0^1 |\nabla_x g(t,z_s)|\frac{1}{\varepsilon^2}\frac{|y|}{\varepsilon}\rho\big(\frac{y}{\varepsilon}\big)\,dsdydxdt \\ &\le \varepsilon\int_{(0,T)\times\mathbb{R}^2} |\nabla_x g(t,z)|dtdz\int_{\mathbb{R}^2}|z|\rho(z)\,dy \\ &\le \varepsilon C\int_0^T M^{1/2} I(g(t,.))^{1/2}dt.\end{aligned}$$

We conclude that $f_n \to f$ in $L^1((0,T)\times B_R)$ by writing

$$f_n - f = (f_n - f_n*\rho_\varepsilon) + (f_n*\rho - f*\rho) + (f*\rho_\varepsilon - f)$$

and using the previous convergence and estimates. Together with (3.11), we conclude to (3.16). □

Lemma 3.8. *Under the above assumptions, there holds $f_n, f \in C([0,T]; L^1\text{-weak})$ in the following sense*

$$\int g(t)\varphi\,dx \to \int g(t)\varphi\,dx \quad as \quad t\to s, \tag{3.17}$$

for $g = f_n, f$ and any $\varphi\in L^\infty_{-1}$, as well as $f_n\to f$ for the convergence in $C([0,T]; L^1_1\text{-weak})$ in the following sense

$$\int f_n(t)\varphi\,dx \to \int f(t)\varphi\,dx \quad as \quad n\to\infty \tag{3.18}$$

uniformly in $t\in[0,T]$ and for any $\varphi\in L^\infty_{-1}$.

Proof of Lemma 3.8. We only prove (3.18) since the proof of (3.17) is similar (and even simpler). Th eproof is split into three steps.

Step 1. We prove $f_n \to f$ for the $C([0,T];(C_0)'\text{-weak})$ topology. Coming back to Step 2 in the proof of Lemma 3.5, we have

$$\int_{\mathbb{R}^2} f_n(t,x)\varphi(x)\,dx \text{ is bounded in } C^{0,1/6}([0,T])$$

for any $\varphi\in\mathcal{D}(\mathbb{R}^2)$. Since there exists a countable subset $\mathcal{U}\subset\mathcal{D}(\mathbb{R}^2)$ which is dense in $C_0(\mathbb{R}^2)$, the space of continuous and vanish at the infinity functions, the Ascoli theorem and a Cantor's diagonalisation argument tell us that there exists a subsequence $(f_{n'})$ such such that $(\langle f_{n'}(t)\varphi\rangle)$ uniformly converges in $C([0,T])$ for any $\varphi\in\mathcal{U}$ and then for any $\varphi\in C_0(\mathbb{R}^2)$. Denoting $\ell_\varphi(t)$ its limit, we have $|\ell_\varphi(t)| \le M\|\,\varphi\|_\infty$ for any $\varphi\in C_0(\mathbb{R}^2)$, and we may identify it as a Borel measure $\tilde f(t)\in (C_0(\mathbb{R}^2))'$. In other words, there exists $\tilde f\in C([0,T];(C_0)'\text{-weak})$ such that $f_{n'}\to\tilde f$ for the $C([0,T];(C_0)'\text{-weak})$ topology, and thus also in $\mathcal{D}'((0,T)\times\mathbb{R}^2)$. Since the convergence $f_n\to f$ also holds in $\mathcal{D}'((0,T)\times\mathbb{R}^2)$, we have $f=\tilde f$ a.e., and from now on, we we make the identification with the continuous representent $f = \tilde{f}$. The all sequence (f_n) convergences by the standard "uniqueness of the limit" argument.

Step 2. We prove that f satsifies the same mass, moment and entropy bounds as the sequence (f_n). Let us define $\varphi_R(x) := \varphi(x/R)$, for $\varphi \in \mathcal{D}(\mathbb{R}^2)$, $\mathbf{1}_{B_1} \le \varphi \le 1$. From (3.7), we have

$$\int f_n(t)|x|^2\varphi_R\,dx \le M_2(f_n(0)) + tC_1(M).$$

Passing to the limit first as $n \to \infty$ and next as $R \to \infty$, we deduce that f satisfies the second moment inequality (3.2).

Denoting $\varphi_R^c := 1 - \varphi_R$, we compute

$$\begin{aligned}\Big|\int f(t) - M\Big| &= \Big|\int (f - f_n)\varphi_R + \int (f - f_n)\varphi_R^c\Big| \\ &\le \Big|\int (f - f_n)\varphi_R\Big| + \frac{1}{R^2}\int (f(t) + f_n(t))|x|^2\,\mathbf{1}_{|x|\ge R},\end{aligned}$$

and the RHS term is as small as we wish by taking R large enough (and using the second moment estimates) and then n large enough (and using the convergence $f_n(t) \rightharpoonup f(t)$ weakly in $(C_0)'$). We deduce that f satisfies the mass conservation identity (1.6). With a similar argument, we get

$$\int f_n(t)|x| \to \int f(t)|x| \quad \text{as} \quad n \to \infty, \tag{3.19}$$

for any fixed $t \in [0, T]$.

We observe next that

$$g \log g = j(g/q)q + g\log q,$$

with $q = e^{-|x|}$ and $j(s) = s\log s - s + 1 = \sup_{\sigma\in\mathbb{R}}(s\sigma - e^\sigma + 1)$ for any $s > 0$. For a sequence (g_n) such that $g_n \rightharpoonup g$ in $(C_0)'$, we deduce that

$$\begin{aligned}\int (\sigma g/q - e^\sigma + 1)q &= \lim \int (\sigma g_n/q - e^\sigma + 1)q \\ &\le \liminf \int g_n \log g_n - \int g_n \log q,\end{aligned}$$

for any $\sigma \in C_0(\mathbb{R}^2)$. Taking the supremum over $\sigma \in C_0$, we obtain

$$\int g\log g - \int g\log q \le \liminf \int g_n \log g_n - \int g_n \log q.$$

Together with (3.19), we get

$$\int f(t)\log f(t) \le \liminf \int f_n(t)\log f_n(t),$$

for any $t \in [0, T]$.

Step 3. We establish (3.18). For any fixed $\varphi \in L^\infty_{-1}$, we may introdce a sequence (φ_ε) of $C_0(\mathbb{R}^2)$ such that $\varphi_\varepsilon \to \varphi$ a.e. and $\|\varphi_\varepsilon\|_{L^\infty_{-1}} \le \|\varphi\|_{L^\infty_{-1}}$. We then split

$$\Big|\int f_n(t)\varphi - f(t)\varphi\Big| \le \Big|\int (f(t) - f_n(t))\varphi_\varepsilon\Big| + \int (f_n(t) + f(t))|\varphi - \varphi_\varepsilon|$$

and we use the inequality

$$0 \le g \le \mathbf{1}_{B_R} A + \mathbf{1}_{B_R} g \, \mathbf{1}_{g \ge A} \ + \mathbf{1}_{B_R^c} g \tag{3.20}$$

in order to control the last term. □

We leave as an exercise the next convergence result.

Exercise 3.9. *Consider a sequence (g_n) such that $g_n \to g$ in the $C([0,T);L^1\text{-weak})$ sense and $\varphi \in L^\infty_{-1}$, then*

$$g_n * \varphi \to g * \varphi \ \text{ in } C([0,T);L^1_{-4}), \ \text{ bounded in } \ L^\infty(0,T;L^\infty_{-1})$$

Consider moreover a sequence (a_n) such that $a_n \to a$ in the $C([0,T);L^1_{-4})$ sense, bounded in $L^\infty(0,T;L^\infty_{-1})$, then

$$\int g_n \, a_n \, dx \to \int g \, a \, dx \ \text{ in } C([0,T]). \tag{3.21}$$

(Hint. For (3.21)*, one may use the Egorov lemma).*

We are now ready to prove relaxed free energy-dissipation of energy inequality (3.3) for the limit function. We define

$$\kappa_\varepsilon(z) := \kappa(z) \, \mathbf{1}_{|z| \le \varepsilon}, \quad \kappa_\varepsilon^c(z) := \kappa(z) \, \mathbf{1}_{|z| \ge \varepsilon}, \quad \forall \, \varepsilon > 0.$$

On the one hand, we have $\kappa_\varepsilon^c \in L^\infty_{-1}$ and we easilly deduce

$$\int f_n(t)(f_n(t) * \kappa_\varepsilon^c) \, dx \to \int f(t)(f(t) * \kappa_\varepsilon^c) \, dx, \quad \forall \, t \in [0,T],$$

by using the two results stated in Exercise 3.9
On the other hand, using the convexity inequality $u\,v \le u \log v + e^v$, $\forall \, u, v > 0$, we have

$$\int_{|x-y|<\varepsilon} g(y) \log \frac{1}{|x-y|} \, dy \ \ \le \ \ \int_{|x-y| \le \varepsilon} \Big\{ \frac{1}{|x-y|)} + g(y) \log g(y) \Big\} \, dy.$$

We deduce

$$\int g(g * \kappa_\varepsilon) \, dx \ \ \le \ \ 2\varepsilon \int g + \frac{1}{2\pi} \int g(y) \log g(y) \int \mathbf{1}_{|x-y|<\varepsilon} g(x) \to 0$$

as $\varepsilon \to 0$, uniformly in g bounded in $L^1_2 \cap L^1 \log L^1$, where we may use (3.20) to estimate the second term. In other words, we have

$$\int f_n(t)(f_n(t) * \kappa_\varepsilon) \, dx, \ \int f(t)(f(t) * \kappa_\varepsilon) \, dx \to 0$$

as $\varepsilon \to 0$, uniformly in $n \ge 1$. Together with the above convergence on the term associated to κ_ε^c, we get

$$\int f_n(t)(f_n(t) * \kappa) \, dx \to \int f(t)(f(t) * \kappa) \, dx, \quad \forall \, t \in [0,T].$$

Using the lsc of $\mathcal{H}$ established in Lemma 3.8, we conclude to

$$\mathcal{F}(f(t)) \le \liminf \mathcal{F}(f_n(t)), \quad \forall \, t \in [0,T].$$

Finally, we observe that

$$\int_0^T \mathcal{D}_\mathcal{F}(f_n) \, dt = \| \ 2\nabla\sqrt{f_n} - \mathcal{K}_n f_n \|^2_{L^2((0,T)\times\mathbb{R}^2)}$$

and that

$$2\nabla\sqrt{f_n} - \mathcal{K}_n f_n \rightharpoonup 2\nabla\sqrt{f} - \mathcal{K} f,$$

as $n \to \infty$ in the distributional sense $\mathcal{D}'((0,T) \times \mathbb{R}^2)$, thanks to (3.12) and (3.16). From the lsc of the $L^2((0,T) \times \mathbb{R}^2)$ norm, we deduce that

$$\int_0^T \mathcal{D}_\mathcal{F}(f)\, dt \le \liminf \int_0^T \mathcal{D}_\mathcal{F}(f_n)\, dt.$$

We summarize some previous result in the following theorem.

Lemma 3.10. *The entropy functional $\mathcal{H}$, the free energy functional $\mathcal{F}$ and the dissipation of free energy functional $\mathcal{D}_\mathcal{F}$ are lower semi continuous for the convergence $g_n \rightharpoonup g$ in the weak sense $\sigma(L^1, L^\infty)$ together with $(\mathcal{H}(g_n))$ and $(M_2(g_n))$ are bounded sequences.*

We conclude to (3.3) by passing to the limit in (3.8) thanks to the above pieces of information, and more precisely combining (3.18) and Lemma 3.10.

3.2. **Strategies of proof.** In the rest of the section we will explain how to establish the existence of a solution to the Keller-Segel equation (1.4). The general idea consists in introducing a truncated nonlinear problem for which we get the existence in several quite standard steps. We then remove the truncation and we use the previous stability/weak compactness argument in order to conclude.

For the truncated nonlinear problem we may proceed along the following strategies:

(1) A semigroup approach taking advantage that the Keller-Segel equation is a (nonlinear) perturbation of the heat equation. Using a Banach fixed point theorem, we may obtain a local in time solution $f \in C([0,T); L^2_k)$ for any $f_0 \in L^2_k$, $\mathcal{K} \in L^\infty$. We then need to prove that $f \ge 0$, f is mass conservative, $T = \infty$, f is smooth enough in order to justify the computation of the free energy and that f is a weak solution to the (truncated) Keller-Segel equation.

(2) A Hille-Yosida approach or almost equivalently a semigroup approach in a functional space with more regularity. The advantage is that all the additional properties are simpler to established, except the global existence.

(3) A variational approach at the level of an associated linear problem and next to perform a Banach fixed point theorem for the nonlinear problem. The advantage of this approach is that we easily prove the global existence but the construction is a bit more abstract than with the semigroup approach.

In the sequel, we mix these strategies in order to get a rather direct and almost self-contained proof.

3.3. **Linear problem in H^1_k.** We consider the linear problem

$$\frac{\partial}{\partial t} f = \Delta f + \nabla \cdot (E\, f) \quad \text{in } (0,\infty) \times \mathbb{R}^d \tag{3.22}$$

$$f(0,x) = f_0(x) \quad \text{in } \mathbb{R}^d, \tag{3.23}$$

with

$$E = E(t,x) \in L^\infty(0,T; W^{1,\infty})$$

and

$$0 \le f_0 \in H^1_k := \{f \in L^2_k, \nabla f \in L^2_k\}, \quad k > 3.$$

We prove the existence and uniqueness of a solution to (3.22)-(3.23) in several steps. The precise result will be stated at the end of the section.

• We introduce the heat semigroup

$$S_t\psi(x) = (\gamma_t * \psi)(x) = \int_{\mathbb{R}^2} \gamma_t(x-y)\,\psi(y)\,dy, \quad \gamma_s(z) := \frac{1}{2\pi s} e^{-\frac{|z|^2}{2s}}.$$

Observing that

$$\|\,\gamma_t\|_{L^1_\ell} \le C, \quad \|\,\nabla\gamma_t\|_{L^1_\ell} \le \frac{C}{\sqrt{t}}, \quad \forall\, t>0,$$

for any $\ell \ge 0$, we deduce that

$$\|\,S_t\|_{H^s_\ell \to H^s_\ell} \le C,\;\; s=0,1, \quad \|\,S_t\|_{L^2_\ell \to H^1_\ell} \le \frac{C}{\sqrt{t}}, \quad \forall\, t>0.$$

For $g \in L^\infty(0,T;H^1_k)$, we define the function

$$h_t := S_t f_0 + \int_0^t S_{t-s}\mathrm{div}(E_s g_s)\,ds.$$

We clearly have

$$\mathrm{div}(E_s g_s) = (\mathrm{div} E_s)\,g_s + E_s\cdot\nabla g_s \in L^\infty(0,T;L^2_k),$$

and then $h_t \in C([0,T];H^1_k)$. For two functions $g^1, g^2 \in C([0,T];H^1_k)$ and the two associated $h^1, h^2 \in C([0,T];H^1_k)$, we write

$$h_t = \int_0^t S_{t-s}\mathrm{div}(E_s g_s)\,ds,$$

with $g = g^2 - g^1$ and $h = h^2 - h^1$. We compute

$$\begin{aligned} \|h_t\|_{L^2_k} &\le \int_0^t C\|\mathrm{div}(E_s g_s)\|_{L^2_k}\,ds \\ &\le \int_0^t C\|\,E_s\|_{W^{1,\infty}}\,\|\,g_s\|_{H^1_k}\,ds \le CT\,\|\,g_s\|_{L^\infty(0,T;H^1_k)} \end{aligned}$$

and similarly

$$\begin{aligned} \|\nabla h_t\|_{L^2_k} &\le \int_0^t \frac{C}{t-s}\|\mathrm{div}(E_s g_s)\|_{L^2_k}\,ds \\ &\le C\sqrt{T}\,\|\,g_s\|_{L^\infty(0,T;H^1_k)}. \end{aligned}$$

For $T_* > 0$ small enough the mapping $g \mapsto h$ is a contraction in $C([0,T_*];H^1_k)$, so that there exists a unique $f \in C([0,T_*];H^1_k)$ satisfying

$$f_t = S_t f_0 + \int_0^t S_{t-s}\mathrm{div}(E_s f_s)\,ds, \quad \forall\, t\in(0,T_*). \tag{3.24}$$

Repeating the same argument, we construct a global solution $f \in C([0,T];H^1_k)$ for any $T>0$.

• As an equation in L^2_k, it is clear that (3.24) also write

$$f_t = S_t f_0 + \int_0^t \nabla S_{t-s}(E_s f_s)\,ds, \quad \forall\, t\in(0,T). \tag{3.25}$$

• Moreover, for any $\varphi_t \in C^2_c([0,T) \times \mathbb{R}^2)$ and introducing the notation $G_s := \mathrm{div}(E_s f_s)$, we have

$$\langle \varphi_t, f_t \rangle = \langle S_t \varphi_t, g_0 \rangle + \int_0^t \langle S_{t-s}\varphi_t, G_s \rangle \, ds \in C^1([0,T])$$

because $S^*_t = S_t$, and then

$$\begin{aligned} \frac{d}{dt} \langle f_t, \varphi_t \rangle &:= \langle S^*_t(\Delta\varphi_t + \varphi'_t), f_0 \rangle + \int_0^t \langle S_{t-s}(\Delta\varphi_t + \varphi'_t), G_s \rangle \, ds + \langle G_t, \varphi_t \rangle \\ &= \langle \Delta\varphi_t + \varphi'_t, f_t \rangle + \langle \varphi_t, G_t \rangle. \end{aligned}$$

Integrating in time that last identity, we get that f is a weak solution in the following sense

$$\Big[\int f_s \varphi_s \Big]_0^t = \int_0^t \int \Big\{ (\Delta\varphi_s + \partial_s \varphi_s) f_s + \varphi_s \mathrm{div}(E_s f_s) \Big\} \, ds,$$

or equivalently

$$\frac{\partial}{\partial t} f = \Delta f + \nabla \cdot (E\, f) \quad \text{in} \quad \mathcal{D}'([0,T) \times \mathbb{R}^2).$$

• On the one hand, performing one integration by part and using a straightforward density argument, we get that f satisfies the equation in the variational sense

$$\Big[\int f_s \varphi_s \Big]_0^t = \int_0^t \int \Big\{ \partial_s \varphi_s f_s - \nabla\varphi_s \cdot (\nabla f_s + E_s f_s) \Big\} \, ds, \tag{3.26}$$

for any $\varphi \in C([0,T]; L^2_k) \cap L^2(0,T; H^1_k) \cap H^1(0,T; H^{-1})$. It is worth emphasizing that for any $\varphi \in C^1_c((0,T); H^1)$, we have

$$\begin{aligned} \Big| \int_0^T \langle f_t, \partial_t \varphi_t \rangle dt \Big| &= \Big| \int_0^T \int \nabla\varphi_t \cdot (\nabla f_t + E_t f_t) dt \Big| \\ &\le \| \nabla f + E f \|_{L^2(0,T;H^1)} \| \varphi \|_{L^2(0,T;H^1)} \le C \| \varphi \|_{L^2(0,T;H^1)}, \end{aligned}$$

which exactly means that $f \in X_T := H^1(0,T; H^{-1})$, so that f belongs to the same space as the test functions.

• Given a mollifier (ρ_ε) and introducing the regularized sequence $f_\varepsilon = \rho_\varepsilon * f$ and $G_\varepsilon := G * \rho_\varepsilon$, we have

$$\partial_t f_\varepsilon = \Delta f_\varepsilon + G_\varepsilon \in L^2,$$

so that $f_\varepsilon \in H^1_t H^2_x$. We fix $\beta : \mathbb{R} \to \mathbb{R}$ a C^2 function such that $\beta'' \in L^\infty$ and $\beta(0) = 0$ so that $\beta(f_\varepsilon) \in W^{1,1}_t W^{2,1}_x$. From the chain rule, we have

$$\partial_t f_\varepsilon = -\beta''(f_\varepsilon)|\nabla f_\varepsilon|^2 + \Delta\beta(f_\varepsilon) + \beta'(f_\varepsilon) G_\varepsilon.$$

Passing in the limit as $\varepsilon \to 0$, we get that f satisfies the equation in the renormalized sense, namely

$$\partial_t \beta(f) = -\beta''(f)|\nabla f|^2 + \Delta\beta(f) + \beta'(f)\mathrm{div}(Ef),$$

in the distributional sense $\mathcal{D}'([0,T) \times \mathbb{R}^2)$. In particular, for any $\varphi \in C^2_c(\mathbb{R}^2)$, we have

$$\frac{d}{dt} \int \beta(f)\varphi + \int \beta''(f)|\nabla f|^2 \varphi = \int \beta(f)\Delta\varphi + \beta'(f)\mathrm{div}(Ef)\varphi. \tag{3.27}$$

We assume that β is convex, globally Lipschitz and non negative and we take $\varphi(x) = \varphi_R(x) = \varphi(x/R)$ with $\mathbf{1}_{B_1} \le \varphi \in \mathcal{D}(\mathbb{R}^2)$. We obtain

$$\frac{d}{dt}\int \beta(f)\varphi_R \le \int \Big\{ \beta(f)[\Delta\varphi_R - E\cdot\nabla\varphi_R] + [\beta'(f)f - \beta(f)](\mathrm{div}E)\varphi_R \Big\}$$

and then

$$\frac{d}{dt}\int \beta(f) \le \int [\beta'(f)f - \beta(f)](\mathrm{div}E),$$

by passing to the limit $R \to \infty$ and observing that $\varphi_R \to 1$, $\nabla\varphi_R, \Delta\varphi_R \to 0$ as well as $\beta(f), \beta'(f)f \in L^\infty(0,T;L^1)$. Using a density argument, we may take $\beta(s) = s_-$ so that $\beta'(s)s - \beta(s) = 0$ and then

$$\frac{d}{dt}\int f_- \le 0.$$

We conclude that $f \ge 0$. We may also choose $\beta(s) = s_-^2$ and we get the same conclusion $f \ge 0$ (without using an additional density argument). Taking $\beta(f) = f$ in (3.27) and arguing similarly, we deduce that f is mass conserving.

As a conclusion, we have established the following result.

Proposition 3.11. *Under the above assumptions on E and f_0, there exists a unique global nonnegative and mass conservative solution $f \in X_T$, $\forall T > 0$, to the linear equation* (3.22)-(3.23) *in the mild sense* (3.25)*, in the variational sense* (3.26) *and in the renormalized sense* (3.27)*.*

3.4. **Linear problem in $L^2 \cap L^1_2$.** We consider now the same linear problem (3.22)-(3.23) but with

$$E = E(t,x) \in L^\infty(0,T\times\mathbb{R}^2)$$

and assuming

$$0 \le f_0 \in L^2 \cap L^1_2.$$

We first observe that

$$\begin{aligned}\frac{d}{dt}\int f^2 &= -2\int |\nabla f|^2 + 2\int \nabla f\cdot Ef \\ &\le -\int |\nabla f|^2 + \|E\|_{L^\infty_{tx}}\int f^2,\end{aligned}$$

by taking $\beta(s) = s^2$ in equation (3.27). As a consequence, we deduce a first a priori estimate

$$\sup_{[0,T]}\int f^2 + \int_0^T\int |\nabla f|^2 \le e^{\|E\|_{L^\infty_{tx}}T}\int f_0^2, \tag{3.28}$$

thanks to the Gronwall Lemma. Similarly, we have

$$\begin{aligned}\frac{d}{dt}\int f\langle x\rangle^2 &= 4\int f + \int 2x\cdot Ef \\ &\le (4 + \|E\|_{L^\infty_{tx}})\int f\langle x\rangle^2,\end{aligned}$$

and the second a priori estimate

$$\sup_{[0,T]}\int f\langle x\rangle^2 \le e^{(4+\|E\|_{L^\infty_{tx}})T}\int f_0\langle x\rangle^2. \tag{3.29}$$

We build the solution to (3.22)-(3.23) thanks to an approximation argument. We consider a sequence (f_{0k}) of H^1_k such that $f_{0k} \to f_0$ in $L^1_2 \cap L^2$ and a sequence (E_k) of $L^\infty_t W^{1,\infty}_x$ such that $E_k \to E$ in L^∞_{tx}. We denote by f_k the corresponding solution exhibited in the previous section. For $n \ge m$, we define $f_{n,m} = f_m - f_n$, $E_{n,m} := E_m - E_n$ and we compute

$$\begin{aligned}\frac{d}{dt} \int f^2_{n,m} &= -2\int |\nabla f_{n,m}|^2 - 2\int \nabla f_{n,m} \cdot [\, E_{n,m} f_m + f_{n,m} E_n)] \\ &\le \int |E_{n,m} f_m + f_{n,m} E_n|^2 \\ &\le \|E_{n,m}\|^2_{L^\infty_{tx}} \|f_m\|^2_{L^\infty_t L^2_x} + \|E_n\|^2_{L^\infty_{tx}} \|f_{n,m}\|^2_{L^\infty_t L^2_x}.\end{aligned}$$

We immediately deduce $\|f_n - f_m\|_{L^\infty_t L^2_x} \to 0$ thanks to the Gronwall Lemma and the apriori estimate (3.29). In other words, (f_n) is a Cauchy sequence in $C([0,T); L^2)$ and it is bounded in $L^\infty_t L^1_{2x} \cap L^2_t H^1_x$. We deduce that there exists

$$f \in Y_T := C([0,T); L^2) \cap L^\infty_t L^1_{2x} \cap L^2_t H^1_x \cap H^1_t H^{-1}_x,$$

such that $f_n \to f$ strongly in $C([0,T); L^2)$ and weakly in $L^\infty_t L^1_{2x} \cap L^2_t H^1_x$. With this information, we may pass to the limit in the mild formulation (3.25), in the variational formulation (3.26) and in the renormalized formulation (3.27).

As a conclusion, we have established the following second well posedness result.

Proposition 3.12. *Under the above assumptions on E and f_0, there exists a unique global nonnegative and mass conservative solution $f \in Y_T$, $\forall T > 0$, to the linear equation* (3.22)-(3.23) *in the mild sense* (3.25), *in the variational formulation* (3.26) *and in the renormalized sense* (3.27).

Remark 3.13. *It is worth mentioning that (a slightly modified version of) Proposition 3.12 (for $f_0 \in L^2_k \subset L^1_2$) can be obtained more directly by using J.-L. Lions theory of variational solutions.*

3.5. **Truncated nonlinear problem in $L^2 \cap L^1_2$.** For $\varepsilon \in (0,1)$, we define the truncated logarithmic function

$$\begin{aligned}\log_\varepsilon(r) &:= r/\varepsilon + \log\varepsilon - 1 \quad \text{if } r \in (0,\varepsilon); \\ &:= \log r \quad \text{if } r \in (\varepsilon, 1/\varepsilon); \\ &:= -\log\varepsilon \quad \text{if } r > 1/\varepsilon,\end{aligned}$$

so that in particular

$$(\log_\varepsilon)'(r) = \frac{1}{\varepsilon} \wedge \frac{1}{r}\ \mathbf{1}_{r<1/\varepsilon}.$$

Abusing notations, we define the truncated potential κ_ε on $\mathbb{R}^2$ by $\kappa_\varepsilon(z) = \kappa_\varepsilon(|z|)$ with

$$\kappa_\varepsilon(r) := -\frac{1}{2\pi} \log_\varepsilon r \quad \forall r \in (0,\infty).$$

The associated force field is finally define by

$$a(x) := \nabla\kappa_\varepsilon(x) = \frac{x}{|x|}\ \kappa'_\varepsilon(|x|) \in L^\infty \cap L^1.$$

We define the functional set

$$Z_T := \{\ f \in Y_T;\ f \ge 0,\ \langle f\rangle = \langle f_0\rangle,\ \|f\|_{Y_T} \le C\},$$

with

$$\|f\|_{Y_T} := \max \{ \|f\|_{L^\infty_t L^2_x}, \|f\|_{L^\infty_t L^1_2}, \|\nabla_x f\|_{L^2_{tx}} \}.$$

For $g \in Z_T$, we observe that

$$\| a * g\|_{L^\infty_{tx}} \le \| a\|_{L^\infty} \|g\|_{L^\infty_t L^1_x} \le \| a\|_{L^\infty} \langle f_0 \rangle,$$

and we then may define $h = \Phi[g] \in Y_T$ as the solution to

$$\frac{\partial}{\partial t} h = \Delta h + \nabla \cdot ((a * g)\, h) \quad \text{in } (0,\infty) \times \mathbb{R}^2 \tag{3.30}$$

$$f(0,x) = f_0(x) \quad \text{in } \mathbb{R}^2, \tag{3.31}$$

exhibited in Section 3.4 for $0 \le f_0 \in L^1_2 \cap L^2$. For any $T > 0$, we choose

$$C := \exp((4 + \| a\|_{L^\infty} M) T) \| f_0\|_{L^1_2 \cap L^2},$$

so that $h \in Z_T$ thanks to the two estimates (3.28) and (3.29) and the above bound on $a * g$. Next, for two functions $g_1, g_2 \in Z_T$ and the two associated solutions $h_1, h_2 \in Z_T$, we define $h := h_2 - h_1$, $g := g_2 - g_1$, and we compute

$$\begin{aligned} \frac{d}{dt} \int h^2 &= -2 \int |\nabla h|^2 - 2 \int \nabla h \cdot [\, (a * g) h_2 + h (a * g_1)] \\ &\le \| a * g\|^2_{L^\infty_{tx}} \| h_2\|^2_{L^\infty_t L^2_x} + \| a * g_1\|^2_{L^\infty_{tx}} \| h\|^2_{L^\infty_t L^2_x} \\ &\le \| a\|^2_{L^2} \|g\|^2_{L^\infty_t L^2_x} \| h_2\|^2_{L^\infty_t L^2_x} + \| a\|^2_{L^2} \|g_1\|^2_{L^\infty_t L^2_x} \| h\|^2_{L^\infty_t L^2_x}, \\ &\le 2A \|g\|^2_{L^\infty_t L^2_x} + 2A \| h\|^2_{L^\infty_t L^2_x}, \end{aligned}$$

with $A = \| a\|^2_{L^2} C^2/2$. Together with the Gronwall lemma, we deduce

$$\| h\|_{L^\infty_t L^2} \le (e^{AT} - 1) \|g\|_{L^\infty_t L^2},$$

so that the mapping $\Phi : Z_\tau \to Z_\tau$, $g \mapsto h$ is a contraction for the $L^\infty(0,\tau; L^2_x)$ norm when $\tau > 0$ small enough. We deduce that Φ admits a unique fixed point $f = \Phi[f] \in Y_\tau$, which is thus a global nonnegative and mass conservative solution to the linear equation (3.22)-(3.23) in the mild sense (3.25), in the variational sense (3.26) and in the renormalized sense (3.27) on the time intervalle $(0,\tau)$. We get that f is a solution on the time intervalle $(0,T)$ by an iterative process and observing that exactly the same argument makes possible to build a solution on $[m\tau, (m+1)\tau]$ as long as $(m+1)\tau \le T$, because the value of τ only depends on the constant A which is finite on any given intervalle of time $[0,T]$. Because T is arbitrary, we have proven the existence of a global solution (here without restriction on the mass).

3.6. **The compactness argument.** For the smooth function $\beta(s) := (s+\alpha)\log(s+\alpha)$, $\alpha > 0$, the renormalized formulation of the equation writes

$$\begin{aligned} \partial_t \int \beta(f) &= -\int \beta''(f) |\nabla f|^2 + \beta''(f) \nabla f \cdot \bar{\mathcal{K}}_\varepsilon f, \\ &= -\int \frac{|\nabla f|^2}{\alpha + f} + \frac{\nabla f}{\sqrt{\alpha + f}} \cdot \frac{\bar{\mathcal{K}}_\varepsilon f}{\sqrt{\alpha + f}} \\ &\le -\frac{1}{2} \int \frac{|\nabla f|^2}{\alpha + f} + \frac{1}{2} \int \frac{\bar{\mathcal{K}}^2_\varepsilon f^2}{\alpha + f} \\ &\le -\frac{1}{2} \int \frac{|\nabla f|^2}{\alpha + f} + \frac{1}{2} \| \mathcal{K}_\varepsilon\|^2_{L^\infty} \langle f_0 \rangle^3. \end{aligned}$$

Passing to the limit $\alpha \to 0$ in the last inequality and using the dominated convergence and monotonous convergence theorems, we deduce the estimate

$$\sup_{[0,T]} \int f \log f + \frac{1}{2} \int_0^T \int \frac{|\nabla f|^2}{f} \le \int f_0 \log f_0 + CT.$$

Using that information, we may then pass to the limit in the second equality in the previous series of equations, and we get

$$\begin{aligned} \partial_t \int f \log f &= -\int \frac{|\nabla f|^2}{f} + \frac{\nabla f}{\sqrt{f}} \cdot \bar{\mathcal{K}}_\varepsilon \sqrt{f} \\ &= -\int |\sqrt{f}\, \nabla \log f|^2 + (\sqrt{f}\, \nabla \log f) \cdot \bar{\mathcal{K}}_\varepsilon \sqrt{f}. \end{aligned}$$

We observe that $\bar\kappa_\varepsilon = \kappa_\varepsilon * f_\varepsilon \in Y_T$. On the one hand, we have

$$\frac{d}{dt} \int f \bar\kappa_\varepsilon = \int (\partial_t f) \bar\kappa_\varepsilon + \int f \partial_t \bar\kappa_\varepsilon = 2 \int f \partial_t \bar\kappa_\varepsilon$$

for a smooth function f and thus also for $f \in Y_T$. On the other hand, from the variational formulation (3.26), we have

$$\frac{d}{dt} \int f \bar\kappa_\varepsilon = \int f \partial_t \bar\kappa_\varepsilon - \int \nabla \bar\kappa_\varepsilon \cdot (\nabla f + \bar{\mathcal{K}}_\varepsilon f).$$

Both together, we deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int f \bar\kappa_\varepsilon &= -\int \nabla \bar\kappa_\varepsilon \cdot (\nabla f + \bar{\mathcal{K}}_\varepsilon f) \\ &= -\int \sqrt{f} \bar{\mathcal{K}}_\varepsilon \cdot (\sqrt{f} \nabla \log f + \sqrt{f} \bar{\mathcal{K}}_\varepsilon). \end{aligned}$$

For the free energy associated to truncated Keller-Segel equation (3.30)-(3.31), we finally have

$$\begin{aligned} &\frac{d}{dt} \left\{ \int f \log f + \frac{1}{2} \int f \bar\kappa_\varepsilon \right\} \\ &\quad = -\int |\sqrt{f}\, \nabla \log f|^2 + 2(\sqrt{f}\, \nabla \log f) \cdot \bar{\mathcal{K}}_\varepsilon \sqrt{f} + |\sqrt{f} \bar{\mathcal{K}}_\varepsilon|^2 \\ &\quad = -\int |\sqrt{f}\, \nabla \log f + \sqrt{f} \bar{\mathcal{K}}_\varepsilon|^2. \end{aligned}$$

We may repeat the proof of the estimates obtained in the case of the (untruncated) Keller-Segel model in Section 1.2 and Section 2 and then readily adapt the proof of the stability result Proposition 3.4 in order to get that (f_ε) converges (up to the extraction of a subsequence) to a solution f to the Keller-Segel equation associated to the initial datum f_0 in the sense of Definition 3.1.

4. Uniqueness result

We accept the following regularity result.

Theorem 4.1. *For any initial datum f_0 satisfying* (2.1)*, the associated solution f is smooth for positive time, namely $f \in C^\infty((0,T^*) \times \mathbb{R}^2)$, and satisfies the identity* (1.8) *on* $(0,T^*)$.

About the proof of Theorem 4.1. We just mention the formal computation which leads to the L^2 estimate. Multiplying the Keller-Segel by f and intergarting in the position variable, we get

$$\frac{d}{dt}\int f^2 = -2\int |\nabla f|^2 + \int f^3$$

and we have to explain how we may bound the last term. For any $A > 1$, we write

$$\begin{aligned}\int f^3 &\lesssim \int (f\wedge A)^3 + \int (f-A)_+^3 \\ &\lesssim A^2 \int f + \|(f-A)_+\|_{L^1} \,\|\, \nabla (f-A)_+\|_{L^2}^2 \\ &\lesssim A^2 M + \frac{\mathcal{H}_+(f)}{\log A} \,\|\, \nabla f\|_{L^2}^2,\end{aligned}$$

where we have used the Gagliardo-Nirenberg-Sobolev inequality (2.10) (with $p = 2$) in the second line. All together, we deduce

$$\frac{d}{dt}\int f^2 = -\int |\nabla f|^2 + A^2 M,$$

by choosing A large enough. □

We establish that the previous framework (for existence of weak solutions) is also well adapted for the well-posedness issue.

Theorem 4.2. *For any initial datum f_0 satisfying* (2.1)*, there exists at most one weak solution in the sense of Definition 3.1 to the Keller-Segel equation* (1.4)-(1.5)*.*

We split the proof into two steps. We recall that from Theorem 4.1 we already know that $\|f\|_{L^2} \in C^1(0,T)$ and $\|f\|_{L^p} \in L^\infty(t_0,T)$ for any $0 < t_0 < T < T^*$ and any $p \in [1,\infty]$.

Step 1. We establish our new main estimate, namely that any weak solution satisfies

$$t^{1/4}\| \ (t,.)\|_{L^{4/3}} \to 0 \quad \text{as } t \to 0. \tag{4.1}$$

First, from (1.4) and the regularity of the solution, we have

$$\frac{d}{dt}\,\|f\|_{L^2}^2 + 2\|\nabla_x f\|_{L^2}^2 = \|f\|_{L^3}^3 \quad \text{on} \quad (0,T).$$

As in the element of proof of Theorem 4.1, we deduce that

$$\frac{d}{dt}\,\|f\|_{L^2}^2 + \|\nabla_x f\|_{L^2}^2 \le A^2 M \quad \text{on} \quad (0,T),$$

for A large enough. Thanks to the Nash inequality

$$\|f\|_{L^2}^2 \le C\,M\,\|\nabla f\|_{L^2}, \tag{4.2}$$

we thus obtain

$$\frac{d}{dt}\,\|f\|_{L^2}^2 + c_M\|f\|_{L^2}^4 \le A^2 M \quad \text{on} \quad (0,T).$$

It is a classical trick of ordinary differential inequality to deduce that there exists a constant K (which only depends on c_M, A^2M and T) so that

$$t\,\|f(t,.)\|_{L^2}^2 \le K \qquad \forall\, t \in (0,T). \tag{4.3}$$

We now prove (4.1) from (4.3), the conservation of mass and an interpolation argument (Hölder inequality). On the one hand, introducing the notation $\widetilde{\log}_+ f := 2 + (\log f)_+$, we use the Hölder inequality in order to get

$$\begin{aligned} \int f^{4/3} &= \int f^{2/3} \, (\widetilde{\log}_+ f)^{2/3} \, f^{2/3} \, (\widetilde{\log}_+ f)^{-2/3} \\ &\le \Big(\int f \, \widetilde{\log}_+ f \Big)^{2/3} \Big(\int f^2 \, (\widetilde{\log}_+ f)^{-2} \Big)^{1/3}, \end{aligned}$$

or, in other words, and using a similar estimate as (2.4), we have

$$\|f\|_{L^{4/3}} \le C(\mathcal{H}(f), M_2(f)) \Big(\int f^2 \, (\widetilde{\log}_+ f)^{-2} \Big)^{1/4}. \tag{4.4}$$

On the other hand, we observe that for any $R \in (0, \infty)$

$$\begin{aligned} t \int f^2 \, (\widetilde{\log}_+ f)^{-2} &\le t \int_{f \le R} f^2 \, (\widetilde{\log}_+ f)^{-2} + t \int_{f \ge R} f^2 \, (\widetilde{\log}_+ f)^{-2} \\ &\le t \, \frac{R}{(\widetilde{\log}_+ R)^2} \int_{f \le R} f + \frac{t}{(\widetilde{\log}_+ R)^2} \int_{f \ge R} f^2 \\ &\le t \, \frac{MR}{(\widetilde{\log}_+ R)^2} + \frac{K}{(\widetilde{\log}_+ R)^2} \le \frac{M+K}{(\widetilde{\log}_+ 1/t)^2} \to 0, \end{aligned} \tag{4.5}$$

where we have used that $s \mapsto s/(\widetilde{\log}_+ s)^2$ is an increasing function in the second line, then the mass conservation and estimate (4.3) in the third line, and we have chosen $R := t^{-1}$ in order to get the last inequality. We conclude to (4.1) by gathering (4.4) and (4.5).

Step 3. Contraction argument. We consider two weak solutions f_1 and f_2 to the Keller-Segel equation (1.4) with same initial datum f_0, that we write in the mild form

$$f_i(t) = e^{t\Delta} f_0 + \int_0^t e^{(t-s)\Delta} \nabla (\bar{\mathcal{K}}_i(s) \, f_i(s)) \, ds. \quad \bar{\mathcal{K}}_i = \mathcal{K} * f_i.$$

Here $e^{t\Delta}$ stands for the heat semigroup defined in $\mathbb{R}^2$ by $e^{t\Delta} f := \gamma_t * f$, $\gamma_t(x) := (2\pi t)^{-1} \exp(-|x|^2/(2t))$. The difference $F := f_2 - f_1$ satisfies

$$F(t) = \int_0^t \nabla \cdot e^{(t-s)\Delta} (V_2(s) \, F(s)) \, ds + \int_0^t \nabla \cdot e^{(t-s)\Delta} (W(s) \, f_1(s)) \, ds = I_1 + I_2,$$

with $W := \bar{\mathcal{K}}_2 - \bar{\mathcal{K}}_1$. For any $t > 0$, we define

$$Z_i(t) := \sup_{0 < s \le t} s^{1/4} \, \|f_i(s)\|_{L^{4/3}}, \quad \Delta(t) := \sup_{0 < s \le t} s^{1/4} \, \|F(s)\|_{L^{4/3}}.$$

We then compute

$$\begin{aligned}
J_1 &:= t^{1/4}\,\|I_1(t)\|_{L^{4/3}} \\
&\le t^{1/4}\int_0^t \|\nabla\cdot e^{(t-s)\Delta}(\bar{\mathcal{K}}_2(s)\,F(s))\|_{L^{4/3}}\,ds \\
&\le t^{1/4}\int_0^t \frac{C}{(t-s)^{3/4}}\,\|\bar{\mathcal{K}}_2(s)\,F(s)\|_{L^1}\,ds \\
&\le t^{1/4}\int_0^t \frac{C}{(t-s)^{3/4}}\,\|\bar{\mathcal{K}}_2(s)\|_{L^4}\,\|F(s)\|_{L^{4/3}}\,ds \\
&\le t^{1/4}\int_0^t \frac{C}{(t-s)^{3/4}}\,\|f_2(s)\|_{L^{4/3}}\,\|F(s)\|_{L^{4/3}}\,ds \\
&\le \int_0^t \frac{C}{(t-s)^{3/4}}\,\frac{t^{1/4}}{s^{1/2}}\,ds\,Z_2(t)\,\Delta(t) \\
&= \int_0^1 \frac{C}{(1-u)^{3/4}}\,\frac{du}{u^{1/2}}\,Z_2(t)\,\Delta(t),
\end{aligned}$$

where we have used the regularizing effect of the heat equation

$$\|\nabla(e^{t\Delta}g)\|_{L^{4/3}} \ \le \|\nabla\gamma_t\|_{L^{4/3}}\,\|g\|_{L^1} \le \frac{C}{t^{3/4}}\,\|g\|_{L^1},$$

at the third line, the Hölder inequality at the fourth line and the critical Hardy-Littlewood-Sobolev inequality (2.18) with $r = 4/3$ (and thus $2r/(2-r) = 4$) at the fifth line. Similarly, we have

$$J_2 := t^{1/4}\,\|I_2(t)\|_{L^{4/3}} \le \int_0^1 \frac{C}{(1-u)^{3/4}}\,\frac{du}{u^{1/2}}\,\Delta(t)\,Z_1(t).$$

Gathering the preceding estimates on J_1 and J_2 and using the asymptotic estimate (4.1), we conclude to

$$\Delta(t) \ \le \ \int_0^1 \frac{C}{(1-u)^{3/4}}\,\frac{du}{u^{1/2}}\,(Z_1(t)+Z_2(t))\,\Delta(t) \le \frac{1}{2}\Delta(t),$$

for $t\in(0,T)$, $T>0$ small enough. That in turn implies $\Delta(t)\equiv 0$ on $[0,T)$. □

Exercise 4.3. *Prove the interpolation inequality* $\|f\|_{L^2} \le \|f\|_{L^1}^{1/4}\,\|f\|_{L^3}^{3/4}$. *Deduce the Nash inequality* (4.2). *(Hint. Use the Gagliardo-Nirenberg-Sobolev inequality* (2.12) *with* $p=1$*).*

5. Self-similar solutions

In this section, we consider the long time behavior of solution for subcritical mass $M\in(0,8\pi)$ issue. For that purpose it is convenient to work with self-similar variables. We introduce the rescaled functions g and u defined by

(5.1) $$g(t,x) := R(t)^{-2} f(\log R(t), R(t)^{-1}x), \quad u(t,x) := c(\log R(t), R(t)^{-1}x),$$

with $R(t) := (1+2t)^{1/2}$. The function g can be equivalently defined as the solution to the rescaled parabolic-elliptic KS system

(5.2) $$\begin{aligned}
\partial_t g &= \Delta g + \nabla(gx - g\,\nabla u) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2, \\
u &= -\kappa * g \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2,
\end{aligned}$$

associated to the same initial datum $g(0,.) = f_0$.

The derivation of the (self-similar) rescaled Keller-Segel equation (5.2) from the original equation (1.4) and the change of variables (5.1) is a elementary differential calculus exercise that we briefly draw bellow. We compute successively

$$\begin{aligned}\partial_t g &= 2e^{2t} f(\tau,w) + e^{4t}\ (\partial_\tau f)(\tau,w) + e^{3t} x\cdot(\nabla_w f)(\tau,w)\\ &= e^{2t}(\mathrm{div}_w(wf))(\tau,w) + e^{4t}\ (\partial_\tau f)(\tau,w),\end{aligned}$$

as well as

$$\Delta_x g = e^{4t}\ (\Delta_w f)(\tau,w)$$

and finally

$$\begin{aligned}\mathrm{div}_x(xg) &= 2g + x\cdot\nabla_x g\\ &= 2e^{2t} f(\tau,w) + e^{3t} x\cdot\nabla_w f(\tau,w)\\ &= e^{2t}(\mathrm{div}_w(wf))(\tau,w).\end{aligned}$$

For the last term, we compute

$$\begin{aligned}\mathcal{K}*g &\simeq \int \frac{x-y}{|x-y|^2} g(t,y)\,dy \simeq e^{2t}\int \frac{x-y}{|x-y|^2} f(\tau,e^t y)\,dy\\ &\simeq \int \frac{x-e^{-t}v}{|x-e^{-t}v|^2} f(\tau,v)\,dv \simeq e^t \int \frac{w-v}{|w-v|^2} f(\tau,v)\,dv\\ &= e^t(\mathcal{K}*f)(\tau,w),\end{aligned}$$

and $(\mathrm{div}_x\mathcal{K})*g = g$, so that

$$\begin{aligned}\mathrm{div}_x[(\mathcal{K}*g)g] &= g^2 + (\mathcal{K}*g)\cdot\nabla_x g\\ &= e^{4t} f^2(\tau,w) + e^{4t}((\mathcal{K}*f)\cdot\nabla_w f)(\tau,w)\\ &= e^{4t}\mathrm{div}_w[(\mathcal{K}*f)f](\tau,w).\end{aligned}$$

We conclude to (5.2) by putting all theses identities together.

For a solution g to the rescaled equation equation (5.2), we may adapt straightforwardly the computations on the evolution of the moments we made for the solutions of the original equation (1.4). We easily have

$$\begin{aligned}\int_{\mathbb{R}^2} g(t)\,dx = \int_{\mathbb{R}^2} f_0\,dx = M,\\ \int_{\mathbb{R}^2} g(t)\,x\,dx = \int_{\mathbb{R}^2} f_0\,x\,dx = 0,\end{aligned}$$

because

$$\begin{aligned}\frac{d}{dt}\int g\,x &= -\int g\,x + \int g(\nabla(-\Delta g))\\ &= -\int g\,x + \frac{1}{2}\int \mathrm{div}(|\nabla g|^2) = -\int g\,x,\end{aligned}$$

and

$$\int_{\mathbb{R}^2} g(t)\,|x|^2\,dx = \frac{M^2}{2}(1-e^{-2t}) + e^{-2t}\int_{\mathbb{R}^2} f_0\,|x|^2\,dx,$$

because

$$\begin{aligned}\frac{d}{dt}\int g\,|x|^2 &= -2\int g\,|x|^2 + 2\int x\,g(x)\int \frac{(x-y)}{|x-y|^2} g(y)\,dydx \\ &= -2\int g\,|x|^2 + \iint (x-y)\,g(x)\,\frac{(x-y)}{|x-y|^2} g(y)\,dydx \\ &= -2\int g\,|x|^2 + M^2.\end{aligned}$$

Observing that the rescaled Keller-Segel equation (5.2) may be written in the following gradient flow form

$$\partial_t g = \mathrm{div}[\, g\nabla(\log g + \frac{|x|^2}{2} + \kappa * g)],$$

we then naturally introduce the rescaled free energy

$$\mathcal{E}(g) := \int g\log g\,dx - M + \int g\,\frac{|x|^2}{2}\,dx + \frac{1}{2}\int \kappa(x-y)\,g(x)\,g(y)\,dxdy,$$

and the dissipation of rescaled free energy

$$\begin{aligned}\mathcal{D}_{\mathcal{E}}(g) &:= \int g\,|\nabla(\log g + \frac{|x|^2}{2} + \kappa * g)|^2\,dx, \\ &= -\int (\log g + \frac{|x|^2}{2} + \kappa * g)\mathrm{div}\big[\, g\nabla(\log g + \frac{|x|^2}{2} + \kappa * g)\big].\end{aligned}$$

We easily deduce the decay of the rescaled free energy

$$\frac{d}{dt}\mathcal{E}(g) = -\mathcal{D}_{\mathcal{E}}(g) \le 0. \tag{5.3}$$

Our first mathematical result is about the stationary issues.

Proposition 5.1. *For any $M \in (0, 8\pi)$, let us define the set $D := \{\ 0 \le g \in L^1\log L^1 \cap L^1_2,\ \langle g\rangle = M\}$. There exists a unique solution G in D to the three following equivalent assertions*

- *G minimises the rescaled free energy: $\mathcal{E}(G) = \min\{\ \mathcal{E}(g),\ g \ge 0,\ \langle g\rangle = M\}$;*
- *G is a zero of the dissipation of rescaled free energy: $\mathcal{D}_{\mathcal{E}}(G) = 0$;*
- *G is a stationary state of the rescaled Keller-Segel equation:*

$$\Delta G + \nabla(Gx + G\,\mathcal{K} * G) = 0. \tag{5.4}$$

Furthermore, G is smooth, radially symmetric, positive and

$$G(x) = \theta\, e^{-\frac{|x|^2}{2} - (\kappa * G)(x)} \in L^1(e^{\alpha|x|^2}), \quad \forall\, \alpha \in (0, 1/2). \tag{5.5}$$

Proof of Proposition 5.1. The proof consistes in establishing the existence of a solution to the first minimizing problem, by establishing several implications and concluding by explaining why the stationary equation (5.4) has at most one solution.

Step 1. As the limit of a minimizing sequence (g_n), there exists a solution $G^* \in D$ to the minimizing problem. We use the logarithmic LHS inequality in order to get $\mathcal{H}_+(g_n) + M_2(g_n) \lesssim \mathcal{E}(g_1) + C(M)$, the Dunford-Pettis Lemma which implies that $g_{n_k} \rightharpoonup G^*$ in L^1-weak and finally the weak lsc of the functional $\mathcal{E}$ which is a straightforward consequence of the lsc of the functional $\mathcal{F}$ established in Lemma 3.10.

Step 2. Consider G^* a solution to the minimizing problem. We consider the solution g_s to the rescaled KS equation starting from $g_0 = G^*$ and we get

$$0 \le \int_0^T \mathcal{D}_\mathcal{E}(g_s)\, ds = \mathcal{E}(G^*) - \mathcal{E}(g_T) \le 0.$$

We deduce that $\mathcal{D}_\mathcal{E}(g_s) = 0$ for any $s > 0$ and thus $\mathcal{D}_\mathcal{E}(G^*) = 0$ (at least when G^* is smooth enough, say $\mathcal{D}_\mathcal{E}(G^*) < \infty$). In that case we deduce $I(G^*) < \infty$ as well as $G^*, \mathcal{K} * G^* \in L^q$ for any $q \in (1,\infty)$. In the general case when we do not assume a priori $\mathcal{D}_\mathcal{E}(G^*) < \infty$ we have however $\mathcal{D}_\mathcal{E}(g_s) = 0$ for any $s > 0$ and the following steps will imply that g_s can be identifies as the unique steady state G to rescaled KS equation. We then also get $G^* = G$ by passing to the limit as $s \to 0$.

Step 3. Consider G such that $\mathcal{D}_\mathcal{E}(G) = 0$. It then satisfies

$$2\nabla\sqrt{G} \ + \sqrt{G}\mathcal{K} * G + \sqrt{G}x = 0,$$

and $\sqrt{G} \in W^{1,q}_{loc}$ for any $q \in (1,\infty)$, in particular $G \in C^0(\mathbb{R}^2)$. Defining $\mathcal{O} := \{x \in \mathbb{R}^2;\ G(x) > 0\}$, we have $G \in W^{1,q}_{loc}(\mathcal{O})$ and then

$$\nabla(\log G \ + \kappa * G + |x|^2/2) = 0 \text{ on } \ \mathcal{O}.$$

It means that G is given by the expression (5.5) on $\mathcal{O}$. By a continuity argument, we must have $\mathcal{O} = \mathbb{R}^2$.

Step 4. The above equations also writes

$$\nabla G + G(\mathcal{K} * G + x) = 0,$$

from what we immediately deduce that G is a stationary state of the rescaled Keller-Segel equation. By a bootstrap argument, we deduce that G is a smooth function.

Step 5. We accept that using a Schwarz symmetrization argument (or a moving plane argument), we may deduce of the PDE that G is radially symmetric and next, writing the associated non local ODE, that G is unique. □

Proposition 5.2. *We have*

$$\sup_{t\ge 1} \|g_t\|_{W^{2,p}\cap L^1_k} \le C.$$

Proof of Proposition 5.2. With $\Phi(x) = |x|^k$, $k \ge 2$, we easily compute

$$\begin{aligned} \frac{d}{dt}\int f\,|x|^k &= -k\int f\,|x|^k + \frac12 \iint (\Phi'(x) - \Phi'(y))\, f(x)\,\frac{(x-y)}{|x-y|^2} f(y)\,dydx \\ &\le -k\int f\,|x|^k + C_k\, M_{k-2}\, M, \end{aligned}$$

from what we deduce

$$\sup_{t\ge 0}\ M_k(t) \le \max(C_k', M_k(0)).$$

Next, we observe that

$$\sup_{t\ge 0}\{\mathcal{H}_+(g_t) + M_2(g_t)\} \ \le C = C(M, \mathcal{E}(g_0)),$$

thanks to the logarithmic LHS inequality. We may then proceed in the same way as for the solution f in the original variables, but we get here uniform in time regularity estimates instead of local in time regularity estimates. □

Proposition 5.3. *We have $g(t) \to G$ in L^p as $t \to \infty$, for any $p \in [1,\infty]$.*

Proof of Proposition 5.3. We use a mix of the La Salle principle and the dissipation of entropy method. We take $t_n \to \infty$ and we write the free energy identity (5.3) in time integrated form

$$\mathcal{E}(g_{t_n+T}) + \int_0^T \mathcal{D}_{\mathcal{E}}(g(t_n+s))\,ds = \mathcal{E}(g_{t_n}).$$

in particular the mapping $t \mapsto \mathcal{E}(g(t))$ is decreasing and thus $\mathcal{E}(g(t_n)), \mathcal{E}(g_{t_n+T}) \to \bar{\mathcal{E}}$ (a constant which satisfies $\bar{\mathcal{E}} \geq \mathcal{E}(G)$). On the other hand, because of Lemma 3.8, and after extraction of a subsequence, we have $g(t_{n'}+s) \to \bar{g}(s)$ in the sense of $C([0,T]; L^p(\mathbb{R}^2) - \text{weak})$. Passing to the limit in the above free energy equation, we get

$$\int_0^T \mathcal{D}_{\mathcal{E}}(\bar{g}(s))\,ds = 0,$$

so that $\bar{g}(s) = G$ for any $s \in [0,T]$. In particular, $g(t_{n'}) \to \bar{g}(0) = G$ and we conclude (by the usual contradiction argument) that the all sequence $g(t_n)$ converges to the same limit for the strong norm sense L^p thanks to Proposition 5.2. □

6. Linear stability of the self-similar profile in L^2

We introduce the linearized operator

$$\mathcal{L}f := \Delta f + \operatorname{div}(fx + f\,\mathcal{K} * G + G\,\mathcal{K} * f),$$

that we want to analyse in the suitable Hilbert space

$$\mathcal{H}_0 := \{\, f \in L^2(G^{-1});\ \langle f \rangle = 0\}.$$

To that purpose, we introduce the bilinear form

$$\langle f, g \rangle := \int f g\, G^{-1} dx + \int\int f(x) g(y) \kappa(x-y)\,dxdy,$$

as well as the two related quadratic forms

$$\begin{aligned} Q_1[f] &:= \langle f, f \rangle, \\ Q_2[f] &:= \langle -\mathcal{L}f, f \rangle. \end{aligned}$$

Let us comment some basic facts. First, we may observe that for any $f \in \mathcal{H}_0 \cap \mathcal{D}(\mathbb{R}^2)$, we have

$$Q_1[f] = \lim_{\varepsilon \to 0} \frac{2}{\varepsilon^2}\,(\mathcal{E}(G+\varepsilon f) - \mathcal{E}(G)) \geq 0 \tag{6.1}$$

and

$$Q_2[f] = \lim_{\varepsilon \to 0} \frac{2}{\varepsilon^2}\,(\mathcal{D}_{\mathcal{E}}(G+\varepsilon f) - \mathcal{D}_{\mathcal{E}}(G)) \geq 0,$$

so that if g is a solution to the rescaled Keller-Segel equation that we write as $g = G + \varepsilon f$ with given initial datum $f_0 \in \mathcal{H}_0$, we may pas to the limit $\varepsilon \to 0$ in the rescaled free energy equation (5.3), and we get

$$\frac{d}{dt}\,Q_1[f] = -Q_2[f] \leq 0.$$

It explains why these quadratic forms are introduced.

On the other hand, we may also write the linearized rescaled Keller-Segel operator in gradient flow form

$$\begin{aligned}\mathcal{L}f &= \Delta f + \mathrm{div}(-f\nabla \log G + G\,\mathcal{K} * f)\\ &= \mathrm{div}[\ G\nabla(f/G + \kappa * f)],\end{aligned}$$

the bilinear form as

$$\langle f, g\rangle = \int f\,(g/G + \kappa * g),$$

and then compute directly

$$\begin{aligned}\langle -\mathcal{L}f, f\rangle &= -\int \mathrm{div}[\ G\nabla(f/G + \kappa * f)]\,(f/G + \kappa * f)\\ &= \int G|\nabla(f/G + \kappa * f)|^2 =: Q_2[f].\end{aligned}$$

We start with a technical Lemma that we will use in the sequel in order to get some fundamental properties about Q_1 and Q_2.

Lemma 6.1. *The function*

$$F := \partial_M G$$

is a solution to the linearized problem

$$\mathcal{L}F = 0, \quad \langle F\rangle = 1, \quad F(0) \neq 0.$$

Proof of Lemma 6.1. We obtain the two first equations by just differentiating the equation (5.4) and the mass condition $\langle G\rangle = M$. On the other hand, we have

$$0 = \langle -\mathcal{L}F, F\rangle = Q_2[F],$$

so that

$$\nabla(F/G + \kappa * F) = 0.$$

Taking the divergence of that equation and recalling that $\Delta\kappa = \delta$, the function $Z := F/G$ satisfies

$$\Delta Z + GZ = 0.$$

Because G and then F and Z are radially symmetric and smooth, we obtain

$$Z'' + \frac{1}{r}Z + GZ' = 0, \quad Z'(0) = 0.$$

We now claim that if $w \in C^2(\mathbb{R}_+)$ satisfies

$$w'' + \frac{1}{r}w' + Gw = 0, \quad w(0) = w'(0) = 0, \tag{6.2}$$

then $w \equiv 0$. Indeed, we may introduce the quantity

$$U := \frac{1}{2}\,(w')^2 + \frac{1}{2}\,Gw^2$$

which satisfies

$$U' = w'(-\frac{1}{r}w' - Gw) + Gww' + \frac{1}{2}\,G'w^2 \le \frac{1}{2}(G')_+ w^2 \le C\,U,$$

because $G' = -(r + \mathcal{K} * G)G \le 0$ on $[R, \infty)$ and $(G')_+ \le CG$ on $[0, R]$. We deduce $U \equiv 0$ from the Gronwall lemma, and thus $w \equiv 0$.

The mass condition $\langle F\rangle = 1$ implies $F \not\equiv 0$, and thus $F(0) \neq 0$, from the above ODE argument. □

As a first consequence of the proof, we have :

Lemma 6.2. *The bilinear form $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathcal{H}_0$ and the associated norm is equivalent to the usual norm.*

Proof of Lemma 6.2. It is clear that $Q_1[f] \lesssim \| f\|^2_{\mathcal{H}_0}$. Once proved that $\langle \cdot, \cdot \rangle$ defines a norm, we get the equivalence between the two norms by using the open map theorem. Since we already know that $Q_1 \ge 0$, we only have to prove that Q_1 is not degenerated. Assume now that $Q_1[f] = 0$, so that $Q_1[h] \ge Q_1[f]$ for any $h \in \mathcal{H}_0$. The associated Euler equation reads

$$\int_{\mathbb{R}^2} (f G^{-1} + f * \kappa)\, h = 0, \quad \forall\, h \in \mathcal{H}_0.$$

Choosing $0 \le \chi \in C_c(\mathbb{R}^2)$ such that $\langle \chi \rangle = 1$ and choosing $h := g - \langle g \rangle \chi$ with arbitrary $g \in C_c(\mathbb{R}^2)$ in the above Euler equation, we deduce that

$$f G^{-1} + f * \kappa = C := \int [f G^{-1} + f * \kappa]\; \chi.$$

As a consequence, the function $z := f/G$ satisfies

$$-\Delta z = z G.$$

From the last equation and using symmetrization techniques, we may establish that z is radially symmetric, and we accept this fact. Recalling that the function F introduced in the previous Lemma satisfies $F(0) \ne 0$, we may define $\lambda := z(0)G(0)/F(0)$, in such a way that the function $w := z - \lambda F/G$ satisfies (6.2). We have seen that this implies $w \equiv 0$ and then $f = \lambda F$. Together with the mass conditions on f and F, we have

$$0 = \langle f \rangle = \lambda \langle F \rangle = \lambda,$$

so that $f = 0$. $\square$

Lemma 6.3. *The operator $\mathcal{L}$ is self-adjoint for the scalar product $\langle \cdot, \cdot \rangle$, it is dissipative and $N(\mathcal{L}) = \{0\}$.*

Proof of Lemma 6.3. We already know that $\langle \mathcal{L} f, f \rangle \le 0$ for any $f \in \mathcal{H}_0$, namely that $\mathcal{L}$ is coercive. We easily compute

$$\begin{aligned} \langle \mathcal{L} f, g \rangle &= \int \mathrm{div}[\, G\, \nabla (f/G + \kappa * f)]\; (g/G + \kappa * g) \\ &= -\int G\, \nabla (f/G + \kappa * f) \cdot \nabla (g/G + \kappa * g) \\ &= \langle f, \mathcal{L} g \rangle \end{aligned}$$

so that $\mathcal{L}^* = \mathcal{L}$ for this scalar product. Finally, we have seen in the proof of Lemma 6.1 that $\mathcal{L} f = 0$ for some $f \in \mathcal{H}_0$ implies that the function $z := f/G$ satisfies

$$\Delta z + G z = 0.$$

We conclude that $f = 0$ exactly in the same way as in the end of the proof of the preceding Lemma. $\square$

Let us sum up and emphasizing the elementary structure that we have used several times in the first results of this section. The following equivalence holds true

$$\mathcal{L} f = 0 \Leftrightarrow Q_2[f] = 0 \Leftrightarrow f/G + \kappa * f = C \Leftrightarrow -\Delta z = z G,$$

where $z := f/G$, and where for proving the reverse sense in the first equivalence we use the Cauchy-Schwarz inequality

$$\langle -\mathcal{L} f, g\rangle \le \langle -\mathcal{L} f, f\rangle^{1/2} \langle -\mathcal{L} g, g\rangle^{1/2}.$$

In order to go further, we state a Poincaré inequality.

Lemma 6.4. *The stationary state G satisfies the Poincaré inequality, in the sense that there exists $\lambda > 0$ such that*

$$\int |\nabla h|^2 G \ge \lambda \int |h|^2 \langle x\rangle^2 G, \quad \forall\, h, \quad \langle hG\rangle = 0. \tag{6.3}$$

Proof of Lemma 6.4. The proof is a variation around the proof in the gaussian case and it is only alluded. Put a complete proof ?

• We may check that G satisfies the so-called Lyapunov condition

$$\mathcal{L}^* e^{\alpha |x|^2} \lesssim -e^{\alpha |x|^2} + b \mathbf{1}_{B_R},$$

for any $\alpha \in (0,1)$ and for some $b, R > 0$.

• The following is true but probably not necessary if we use the more recent proof of the Poncaré inequality. Thanks to the convexity inequality $s\,t \le s \log s + e^t \; \forall\, s,t > 0$, we have

$$\begin{aligned} |-\log G + \omega - |x|^2/2| &\le \int_{|x-y|\le 1} (-\log|x-y|)\, G(y)\, dy + \int_{|x-y|\ge 1} \log|x-y|\, G(y)\, dy \\ &\le \int_{|x-y|\le 1} \Big\{ \frac{1}{|x-y|)} + G(y) \log G(y) \Big\}\, dy \\ &\quad + \int_{|x-y|\ge 1} (\log\langle x\rangle + \log\langle y\rangle)\, G(y)\, dy \\ &\le 2\pi + \int G \log G + C(M)\, \log\langle x\rangle, \end{aligned}$$

so that $G_-\, e^{-\alpha|x|^2} \le G \le G_+\, e^{-\beta |x|^2}$ for any $\beta < 1/2 < \alpha$. That implies that the local Poincaré inequality

$$\int_{B_R} h^2 G \lesssim \int_{B_R} |\nabla h|^2 G + \Big(\int_{B_R} hG \Big)^2, \quad \forall\, h,$$

holds true.

• The Lyapunov condition and the local Poincaré inequality together classically imply the (strong) Poincaré inequality (6.3). □

Corollary 6.5. *The spectrum of $\mathcal{L}$ is discrete and thus*

$$\exists \lambda_1 < 0, \quad \langle \mathcal{L} f, f\rangle \le 2\lambda_1 Q_1[f], \quad \forall\, f \in \mathcal{H}_0. \tag{6.4}$$

Finally, we deduce

$$Q_1[e^{\mathcal{L} t}\, f_0] \le e^{-2\lambda_1 t} Q_1[f_0], \quad \forall\, f_0 \in \mathcal{H}_0,$$

and thus

$$\|e^{\mathcal{L} t}\, f_0\|_{L^2(G^{-1})} \le C e^{\lambda t} \|f_0\|_{L^2(G^{-1})}, \quad \forall\, f_0 \in \mathcal{H}_0.$$

Elements of proof. That result is a very classical consequence (and variant) of Lemma 6.4. The main point is that because of the previous estimate the operator $\mathcal{L}$ has compact resolvent so that there exists a decreasing sequence (λ_k) of eigenvalues with finite dimensional associated eigenspace such that

$$\Sigma(\mathcal{L}) = \{\lambda_n,\ n \geq 1\} \subset] - \infty, \lambda_1],$$

and $\lambda_1 < 0$ because of Lemma 6.3. By decomposing any function f in the Hilbert space of eigenvectors we deduce (6.4). The end of the proof is straightforward from (6.4). □

7. Bibliographic discussion

We refer to [5] for the origibal existence proof.
The uniqueness of the weak solution is established in [16].
The smoothness of the solution is proved in [5, 16] as well as the entropy identity.
The uniform estimates on g and the convergence to the stationary profile can be found in [5, 16].
We refer to [5, 3, 10, 20] for the proof of the uniqueness of G.
The analysis of the linearized operator in $L^2(G^{-1})$ is presented in [11, 12]
The proof of the convergence to G for general initial datum is established in [16].

References

[1] Beckner, W. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. of Math. (2) 138*, 1 (1993), 213–242.

[2] Ben-Artzi, M. Global solutions of two-dimensional Navier-Stokes and Euler equations. *Arch. Rational Mech. Anal. 128*, 4 (1994), 329–358.

[3] Biler, P., Karch, G., Laurençot, P., and Nadzieja, T. The 8π-problem for radially symmetric solutions of a chemotaxis model in the plane. *Math. Methods Appl. Sci. 29*, 13 (2006), 1563–1583.

[4] Blanchet, A., Carrillo, J. A., and Masmoudi, N. Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$. *Comm. Pure Appl. Math. 61* (2008), 1449–1481.

[5] Blanchet, A., Dolbeault, J., and Perthame, B. Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. *Electron. J. Differential Equations* (2006), No. 44, 32 pp. (electronic).

[6] Brezis, H. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[7] Brezis, H. Remarks on the preceding paper by M. Ben-Artzi: "Global solutions of two-dimensional Navier-Stokes and Euler equations" [Arch. Rational Mech. Anal. **128** (1994), no. 4, 329–358; MR1308857 (96h:35148)]. *Arch. Rational Mech. Anal. 128*, 4 (1994), 359–360.

[8] Caglioti, E., Lions, P.-L., Marchioro, C., and Pulvirenti, M. A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. *Comm. Math. Phys. 143*, 3 (1992), 501–525.

[9] Caglioti, E., Lions, P.-L., Marchioro, C., and Pulvirenti, M. A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description. II. *Comm. Math. Phys. 174*, 2 (1995), 229–260.

[10] Calvez, V., and Carrillo, J. A. Refined asymptotics for the subcritical Keller-Segel system and related functional inequalities. *Proc. Amer. Math. Soc. 140*, 10 (2012), 3515–3530.

[11] Campos, J., and Dolbeault, J. A functional framework for the Keller-Segel system: logarithmic Hardy-Littlewood-Sobolev and related spectral gap inequalities. *C. R. Math. Acad. Sci. Paris 350*, 21-22 (2012), 949–954.

[12] Campos, J. F., and Dolbeault, J. Asymptotic estimates for the parabolic-elliptic Keller-Segel model in the plane. *Comm. Partial Differential Equations 39*, 5 (2014), 806-841.
[13] Carlen, E., and Loss, M. Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on S^n. *Geom. Funct. Anal. 2*, 1 (1992), 90–104.
[14] Carrillo, J. A., Lisini, S., and Mainini, E. Uniqueness for Keller-Segel-type chemotaxis models. *Discrete Contin. Dyn. Syst. 34*, 4 (2014), 1319–1338.
[15] DiPerna, R. J., and Lions, P.-L. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math. 98*, 3 (1989), 511–547.
[16] Egaña, G. and Mischler, S. Uniqueness and long time asymptotic for the Keller-Segel equation: the parabolic-elliptic case. *Arch. Ration. Mech. Anal.*, 220 (2016), no. 3, 1159–1194
[17] Fournier, N., Hauray, M., and Mischler, S. Propagation of chaos for the 2d viscous vortex model. *Journal of the European Mathematical Society (JEMS)*, 16 (2014), no. 7, 1423–1466
[18] Gajewski, H., and Zacharias, K. Global behaviour of a reaction-diffusion system modelling chemotaxis. *Math. Nachr. 195* (1998), 77–114.
[19] Lieb, E. H., and Loss, M. *Analysis*, vol. 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.
[20] Naito, Y. Symmetry results for semilinear elliptic equations in $\mathbb{R}^2$. In *Proceedings of the Third World Congress of Nonlinear Analysts, Part 6 (Catania, 2000)* (2001), vol. 47, pp. 3661–3670.