ON THE TRACE PROBLEM FOR SOLUTIONS OF THE VLASOV EQUATION

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Abstract - We study the trace problem for weak solutions of the Vlasov equation set in a domain. When the force field has Sobolev regularity, we prove the existence of a trace on the boundaries, which is defined thanks to a Green formula, and we show that the trace can be renormalized. We apply these results to prove existence and uniqueness of the Cauchy problem for the Vlasov equation with specular reflection at the boundary. We also give optimal trace theorems and solve the Cauchy problem with general Dirichlet conditions at the boundary.

1. Introduction and main results.

Let Ω be an open bounded or unbounded set of $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. We denote by $n(x)$ the unit outward normal vector at $x \in \partial\Omega$ and by $d\sigma_x$ the surface measure on $\partial\Omega$. We define the phases space $\mathcal{O} = \Omega \times \mathbb{R}^N$ and the domain $D = (0,T) \times \mathcal{O}$, with $T > 0$. We also define $\Sigma = \partial\Omega \times \mathbb{R}^N$, $\Sigma_\pm = \{(x,\xi) \in \Sigma,\, \pm\,\xi \cdot n(x) > 0\}$, $\Sigma_0 = \{(x,\xi) \in \Sigma,\, \xi \cdot n(x) = 0\}$, $\Gamma = (0,T) \times \Sigma$ and in the same way $\Gamma_\pm$ et Γ_0.

In this paper, we consider weak solutions $g = g(t,x,\xi) \in L^\infty(0,T; L^p_{loc}(\bar{\mathcal{O}}))$ of the Vlasov equation set in the domain D

$$\Lambda_E\, g = \frac{\partial g}{\partial t} + \xi \cdot \nabla_x g + E \cdot \nabla_\xi g = G \ \text{ in } D, \tag{1.1}$$

for a fixed vector field $E = E(t,x)$ and a fixed source term $G = G(t,x,\xi)$ which satisfy at least

$$E \in L^1\big(0,T; W^{1,p'}_{loc}(\bar\Omega)\big) \quad \text{and} \quad G \in L^1_{loc}([0,T]\times\bar{\mathcal O}). \tag{1.2}$$

Equation (1.1) must be understood in the distributional sense, which is

$$\iiint_D (g\,\Lambda_E\phi + G\,\phi)\, d\xi dx dt = 0, \tag{1.3}$$

for all test functions $\phi \in \mathcal{D}(D)$.

The main result is established in section 2 and state that a solution g of the Vlasov equation has a trace γg on Γ and for every $t\in[0,T]$ a trace $g(t,.)$ on $\{t\}\times\mathcal O$ in the sense of the Green formula. This problem of existence of a trace is fundamental for the Cauchy problem associated to (1.1) with boundary condition. Precisely, we prove the two following results.

Theorem 1. *We assume $p\in[1,\infty)$. Let $g\in L^\infty(0,T;L^p_{loc}(\bar{\mathcal O}))$ be a solution of equation (1.1)-(1.2). Then $g(t,.)$ is well defined for every $t\in[0,T]$ as a function of $L^p_{loc}(\bar{\mathcal O})$ and*

$$g \in C([0,T];L^1_{loc}(\mathcal O)), \tag{1.4}$$

and the trace of g on Γ is well defined, this is the unique function γg such that

$$\gamma\, g \in L^1_{loc}\big([0,T]\times\Sigma, (n(x)\cdot\xi)^2\, d\xi d\sigma_x dt\big), \tag{1.5}$$

which satisfies the Green formula

$$\begin{aligned}\int_{t_0}^{t_1}\iint_{\mathcal O} (g\,\Lambda_E\phi + G\,\phi)\, d\xi dx dt =\\ = \Big[\iint_{\mathcal O} g(\tau,.)\,\phi\, dx d\xi\Big]_{t_0}^{t_1} + \int_{t_0}^{t_1}\iint_\Sigma \gamma\, g\ \phi\, n(x)\cdot\xi\, d\xi d\sigma_x d\tau,\end{aligned} \tag{1.6}$$

for all t_0, $t_1\in[0,T]$ and for all test functions $\phi\in\mathcal D_0(\bar D)$, the space of functions $\phi\in\mathcal D(\bar D)$ such that $\phi=0$ on $(0,T)\times\Sigma_0$.

Theorem 2. *We assume $p=\infty$. Let $g\in L^\infty_{loc}(\bar D)$ be a solution of equation (1.1)-(1.2). Then $g(t,.)\in L^\infty_{loc}(\bar{\mathcal O})$ is well defined for all $t\in[0,T]$ and γg exists. They are uniquely defined by the Green formula (1.6) and they satisfy*

$$g\in C\big([0,T];L^a_{loc}(\bar{\mathcal O})\big)\ \forall a<\infty \ \text{ and } \ \gamma\, g \in L^\infty_{loc}([0,T]\times\Sigma,\, d\xi d\sigma_x dt). \tag{1.7}$$

Moreover, the Green formula (1.6) holds for all test functions $\phi \in \mathcal{D}(\bar{D})$.

The trace problem has been addressed in the case of free transport equation ($E = 0$) and neutronic equation ($E = 0$ and $\xi \in S^{N-1} = \{\xi \in \mathbb{R}^N, \ |\xi| = 1\}$) by V.I. Agoshkov [1], M. Cessenat [7], and by L. Arkeryd, C. Cercignani [2], M. Cannone, C. Cercignani [6], K. Hambdache [9] in connection with the investigation on Boltzmann equation. A. Heintz in [10] deals with the case of irregular domain. The case of E Lipschitz had been treated by C. Bardos [3] and S. Ukai [15].

The proofs we present here use new arguments for trace theorem, and in particular, they are not based on the characteristic method as precedent works are. On one hand, our approach is based on a multiplicator method, as the one introduced by P.-L. Lions and B. Perthame [11] to prove moments lemmas for transport equation, see also K. Hamdache [9] where similar multiplicator to ours is used, and on the other hand, it is based on a regularization method adapted from the one used by R.J. DiPerna et P.-L. Lions [8] in the framework of transport equation with coefficients of Sobolev regularity.

In these proofs, the trace γg is constructed as the strong limit of $g_k\big|_{\Gamma}$, where (g_k) is a sequence of smooth approximations of g, defined on $[0, T] \times \bar{\mathcal{O}}$. This implies that γg can be renormalized on the following sense

Corollary 1. *Under assumptions of Theorem 1 or 2, and for all functions $\beta \in W^{1,\infty}(\mathbb{R})$ we have $\Lambda_E \beta(g) = \beta'(g)\, G$ in $\mathcal{D}'(D)$, and the traces defined by Theorem 1 or 2 satisfy*

$$\gamma\, \beta(g) = \beta(\gamma\, g) \quad \text{and} \quad \beta(g)(t, .) = \beta(g(t, .)) \ \forall t \in [0, T]. \tag{1.8}$$

In section 3, we give some possible extensions of Theorem 1 and 2 and prove additional properties of the trace which are deduced from Corollary 1. We show that a stronger integrability assumption on E and G implies a stronger integrability of γg and we state a duality formula. We also show the strong and weak continuity of the trace γg with respect to g, E and G. We would like to emphasize that the renormalization property is very important. As in [8], this property allows us to prove uniqueness of the solution for some initial boundary value problem. It also makes possible to define the trace for a renormalized solution of the Vlasov equation (1.1), and we refer to [13] for an extension of the trace theory in this direction.

In section 4, we assume that E and G also satisfy

$$(1.9)\quad \frac{E(t,x)}{1+|x|} \in L^1(0,T;L^1(\Omega)) \cap L^1(0,T;L^\infty(\Omega)) \text{ and } G \in L^1(0,T;L^p(\mathcal{O})).$$

We study the initial boundary value problem for Vlasov equation (1.1) with initial data

$$(1.10)\qquad g(0,x,\xi) = g_0(x,\xi) \quad \text{in } \mathcal{O},$$

and specular reflection on the boundary

$$(1.11)\qquad \gamma_- g(t,x,\xi) = \gamma_+ g(t,x,R_x\xi) \;\text{ for a.e. }\; (t,x,\xi) \in \Gamma_-,$$

with $R_x\xi = \xi - 2\,(\xi \cdot n(x))\,n(x)$, and where we denote by $\gamma_+\, g$ (resp. $\gamma_-\, g$) the restriction of the trace $\gamma\, g$ to Γ_+ (resp. Γ_-).

We state in this framework the equivalent of existence and uniqueness results of R.J. DiPerna and P.L Lions [8].

Theorem 3. *Let $p \in [1,\infty]$. Assume $g_0 \in L^p(\mathcal{O})$, E and G such that (1.9) and (1.2) hold. Then there exists an unique solution g to (1.1) in $L^\infty(0,T;L^p(\mathcal{O}))$ satisfying (1.11), and corresponding to the initial datum g_0. Moreover, g satisfies*

$$(1.12)\qquad g \in C([0,T];L^p(\mathcal{O})) \;\text{ if }\; p < \infty.$$

Here, we focus our attention to the linear problem, where the force field E and the source G are fixed. But the present work is motivated by applications to non-linear problems which appear in plasma physic, in particular to the Vlasov-Poisson equation where precisely the force field E is only known to have Sobolev regularity. We refer to [13] for an application of the trace theory developed here to the initial boundary value problem for the Vlasov-Poisson-Boltzmann system. General references for mathematical results on the Vlasov-Poisson system are also given in [13].

Existence in Theorem 3 is obtained thanks to a penalty method, that we introduce, and which can be generalized to a lot of other situations. The penalty method is a classic tool which allows one to prove existence of a solution to a problem set in a domain from the existence of a family of solutions of problems set in the whole space, see C. Bardos and J. Rauch [4]. The idea of the penalty method is the following:

a) - We consider a family of fields $E_\varepsilon(x)$ which "tends to penalize Ω^c", forcing the particles to stay in the domain, and we consider the solution g_ε to

$$\Lambda_E\, g_\varepsilon + E_\varepsilon \cdot \nabla_\xi g_\varepsilon = G \;\text{ in } (0,T)\times\mathbb{R}^N\times\mathbb{R}^N, \tag{1.13}$$

corresponding to the initial datum (1.10) for which we get uniform bound in ε.

b) - First, we pass to the limit in the distributional sense in the interior of D, and up to the extraction of a sub-sequence, g_ε converges to a solution g to (1.1).

c) - We then multiply (1.13) by functions belonging to an appropriate class of functions and we pass to the limit in the whole space $(0,T)\times\mathbb{R}^N\times\mathbb{R}^N$. We show that g satisfy the reflection specular condition (1.11) in a weak sense, i.e. g satisfies (1.3) for all test functions ϕ in an appropriate class $\mathcal{RS}\subset\mathcal{D}_0(\bar D)$.

d) - We last use the trace Theorem 1 or 2 and the Green formula (1.6), and we get that γg satisfies the specular reflection condition (1.11).

Uniqueness in Theorem 3 is obtained by a very simply way using the renormalization property (1.8), the resolution of the backward problem of (1.1) and a duality formula.

In the last section, we show how the classical optimal weight theorems can be proved in the case of a field E with Sobolev regularity. This generalizes the already known results for the free transport equation, due to V.I. Agoshkov [1], M. Cessenat [7] and S. Ukai [15]. Let emphasize that without assumption on the geometry of the boundary, we can not hope in general, that

$$\gamma\, g \in L^1_{loc}([0,T]\times\Sigma, |n(x)\cdot\xi|\, d\xi d\sigma_x dt) \tag{1.14}$$

holds instead of (1.5). We refer to C. Bardos [3] where he builds a counter-example. Nevertheless, we prove

$$\gamma\, g \in L^p_{loc}([0,T]\times\Sigma, \tau_E(t,x,\xi)\, |n(x)\cdot\xi|\, d\xi d\sigma_x dt), \tag{1.15}$$

where $\tau_E(t,x,\xi)$ is the time of live in Ω of a particles submitted to the force field E, which at time t, has position $x \in \Omega$ and velocity ξ. This result is optimal since we are able to solve the Cauchy problem for the equation (1.1) with initial datum and Dirichlet condition in the incoming set $\gamma_- g = g_-$ on Γ_-, for every g_- satisfying (1.15).

Last, when $E = 0$ an elementary calculation leads to

$$\tau(t,x,\xi) \geq \min(2\,(R_0/R)\,|n(x)\cdot\xi|, T) \qquad \forall \xi \in B_R, \tag{1.16}$$

if Ω satisfies an uniform interior ball of radius R_0 condition, with equality in (1.16) when $\Omega = B_R$ and $|\xi| = R$, and $\tau(t,x,\xi) = T$ when Ω is an half-space. Thus, we have the two extremal situations: if Ω is an half-space, then γg satisfies (1.14); if Ω is a ball, then γg satisfies (1.5) and not better. We finish with two sufficient conditions when (1.14) holds.

2. Proofs of trace Theorems 1 and 2.

We begin with some notations. We assume that Ω satisfies the following regularity condition: Ω is locally on one side of $\partial\Omega$ and there exists a function $d = d_\Omega \in W^{2,\infty}(\mathbb{R}^N)$ such that for all x in an interior neighborhood of $\partial\Omega$ one has $d(x) = -\mathrm{dist}(x,\partial\Omega)$. (Such an assumption holds if for example $\partial\Omega$ is a C^2 manifold). We define in $\bar\Omega$ the gradient field $n(x) = \nabla_x d(x)$, which coincide with the unit outward normal vector to Ω on every point of $\partial\Omega$.

For a given real $R > 0$, we define $B_R = \{y \in R^N \ / \ |y| < R\}$, $\Omega_R = \Omega \cap B_R$, $\mathcal{O}_R = \Omega_R \times B_R$, $D_R = (0,T) \times \mathcal{O}_R$, $\Sigma_R = (\partial\Omega \cap B_R) \times B_R$ and $\Gamma_R = (0,T) \times \Sigma_R$. We also denote by $L^{a,b}_R$ the spaces $L^a(0,T;L^b(\mathcal{O}_R))$ or $L^a(0,T;L^b(\Omega_R))$, and $L^{a,b}_{loc}$ the spaces $L^a(0,T;L^b_{loc}(\bar{\mathcal{O}}))$ or $L^a(0,T;L^b_{loc}(\bar\Omega))$. We set $d\mu_i = |n(x)\cdot\xi|^i\, d\xi d\sigma_x dt$, with $i = 1$ or 2, the measures defined on Γ. We define $C_b(X)$ the space of continuous and bounded functions on X. Last, for a, $b \in \mathbb{R}$, we set $a \wedge b = \min(a,b)$ and for $a \in [1,\infty]$ we note $a' \in [1,\infty]$ the conjugate exponent of a, given by $1/a + 1/a' = 1$.

Proof of Theorem 1. We shall prove the Theorem in three steps.

First step: A priori bounds. Let assume $g \in W^{1,1}_{loc}(\bar D) \cap C(\bar D)$ in such a way that all the manipulations which will follow are allowed. Consider three functions, that we shall specify latter, $\psi = \psi(n(x)\cdot\xi) \in C^1(\mathbb{R})$ not decreasing,

$\psi(0) = 0$, $\varphi = \varphi(t,x,\xi) \in \mathcal{D}([0,T] \times \mathbb{R}^N \times \mathbb{R}^N)$ and $\beta \in C^1(\mathbb{R})$, and fix t_0, $t_1 \in [0,T]$. Using Stokes formula and equation (1.1) we get the following identity

$$
\begin{aligned}
&\int_{t_0}^{t_1} \iint_\Sigma \beta(g)\,\psi\,\varphi\; n(x)\cdot\xi\, d\xi d\sigma_x dt + \Big[\iint_{\mathcal{O}} \beta(g)\,\psi\,\varphi\, dx d\xi\Big]_{t_0}^{t_1} = \\
&\qquad = \int_{t_0}^{t_1} \iint_{\mathcal{O}} \Lambda_E \big(\beta(g(t,x,\xi))\,\varphi(t,x,\xi)\,\psi(n(x)\cdot\xi)\big)\, d\xi dx dt \\
&\qquad = \int_{t_0}^{t_1} \iint_{\mathcal{O}} \big\{\beta(g)\,\varphi\,\psi'(n(x)\cdot\xi)\,\big({}^t\xi\, D^2 d_\Omega\,\xi + E\cdot n(x)\big) \\
&\qquad\qquad\qquad + \beta(g)\,\psi\,\Lambda_E\,\varphi + \beta'(g)\,G\,\psi\,\varphi\big\}\, d\xi dx dt.
\end{aligned}
\tag{2.1}
$$

a) - Let fix $t_0 \in [0,T]$, a compact set K of $\mathcal{O}$, $\psi(z) = 1$ and $\beta = \beta_\varepsilon$, where (β_ε) is a sequence of smooth even and non negative functions, such that $\beta_\varepsilon(0) = 0$, $|\beta'_\varepsilon(y)| \le 1$ and $\beta_\varepsilon(y) \to |y|$, $\forall y \in \mathbb{R}$. One can then choose $\varphi \in \mathcal{D}(\mathcal{O})$ in such a way that $0 \le \varphi \le 1$ in D, $\varphi \equiv 1$ on K and we denote by $R > 0$ a real number satisfying $\operatorname{supp}\varphi \subset \mathcal{O}_R$. The identity (2.1) then implies that for all $t \in [0,T]$

$$
\begin{aligned}
\iint_{\mathcal{O}} \beta_\varepsilon(g(t_1,.))\,\varphi\, dx d\xi &= \iint_{\mathcal{O}} \beta_\varepsilon(g(t_0,.))\,\varphi\, dx d\xi \\
&\qquad + \int_{t_0}^{t_1} \iint_{\mathcal{O}} \big\{\beta_\varepsilon(g)\,\Lambda_E\,\varphi + \beta'_\varepsilon(g)\,G\,\varphi\big\}\, d\xi dx d\tau, \\
&\le \|\beta_\varepsilon(g(t_0,.))\|_{L^1_R} + \|G\|_{L^1_R} \\
&\qquad + C_{R,p}\,\|\nabla\varphi\|_{L^\infty(D)} \int_0^T \|\beta_\varepsilon(g)(t,.)\|_{L^p_R}\,(1 + \|E(t,.)\|_{L^{p'}_R})\, dt.
\end{aligned}
$$

One deduces, letting $\varepsilon \to 0$, a first a priori estimate

$$
\begin{aligned}
\sup_{t\in[0,T]} \|g(t,.)\|_{L^1(K)} &\le \|g(t_0,.)\|_{L^1_R} + \|G\|_{L^1_R} \\
&\qquad + C_R \int_0^T \|g(t,.)\|_{L^p_R}\,(1 + \|E(t,.)\|_{L^{p'}_R})\, dt.
\end{aligned}
\tag{2.2}
$$

b) - We now fix a compact set K of Σ, $\psi(z) = z$, $t_0 = 0$, $t_1 = T$ and β been unchanged. We chose $\varphi \in \mathcal{D}(\mathbb{R}^N \times \mathbb{R}^N)$ in such a way that $0 \le \varphi \le 1$ in $\mathcal{O}$, $\varphi \equiv 1$ on K, and we denote by $R > 0$ a real number satisfying $\operatorname{supp}\varphi \subset B_R \times B_R$. We deduce from the identity (2.1), in a same way that previously, a second a priori estimate

$$
\begin{aligned}
\|g\|_{L^1([0,T]\times K, d\mu_2)} \leq R \left(\|g(T,.)\|_{L^1} + \|g(0,.)\|_{L^1}\right) \\
+ C_R \int_0^T \left(1 + \|E(t,.)\|_{L_R^{p'}}\right) \|g(t,.)\|_{L_R^p}\, dt + R\, \|G\|_{L_R^1}.
\end{aligned} \tag{2.3}
$$

Second Step. Regularization. In this step we prove the following Lemma 1, which state that g can be approximated by a sequence g_k of regular functions, defined on $[0,T]\times\bar{\mathcal{O}}$, and solutions of (1.1) with an error term r_k which tends to 0 when k goes to ∞.

Given a sequence of mollifer ρ_k

$$
\rho_k(z) = k^N \rho(k\, z), \quad k \in \mathbb{N}^\star, \quad \rho \in \mathcal{D}_+(\mathbb{R}^N), \quad \operatorname{supp}\rho \subset B_1, \quad \int_{\mathbb{R}^N} \rho(z)\, dz = 1,
$$

we introduce the sequence of regularization functions $\tilde{g}_k = g \star_{x,k} \rho_k *_\xi \rho_k$ and $G_k = G \star_{x,k} \rho_k *_\xi \rho_k$, where $*$ denote the usual convolution and $\star_{x,k}$ denote the convolution-translation defined by

$$
(u \star_{x,k} h_k)(x) = [\tau_{2\, n(x)/k}(u * h_k)](x) = \int_{\mathbb{R}^N} u(y)\, h_k(x - \frac{2}{k}\, n(x) - y)\, dy, \tag{2.4}
$$

for all $u \in L^1_{loc}(\bar{\Omega})$ and $h_k \in L^1(\mathbb{R}^N)$ with $\operatorname{supp} h_k \subset B_{\frac{1}{k}}$.

Lemma 1. *For all $k \in \mathbb{N}^\star$, there exists a function $g_k \in C(\bar{D}) \cap W^{1,1}(0,T; W^{1,\infty}_{loc}(\bar{\mathcal{O}}))$ such that the sequence (g_k) satisfies*

$$
\begin{aligned}
&g_k \text{ is bounded in } L^\infty(0,T; L^p_{loc}(\bar{\mathcal{O}})), \\
&g_k \longrightarrow g \text{ in } L^a(0,T; L^p_{loc}(\bar{\mathcal{O}})) \quad \forall a < \infty
\end{aligned} \tag{2.5}
$$

and

$$
\Lambda_E g_k = G_k + r_k \quad \text{in} \quad \mathcal{D}'(D), \tag{2.6}
$$

where (r_k) converges to 0 in $L^1_{loc}([0,T]\times\bar{\mathcal{O}})$.

Proof. The proof is inspired from lemma II.1 of R. DiPerna and P.-L. Lions [8].

Noting $g = g(t,y,\eta)$, one multiplies equation (1.1) by the test function $\rho_k(x - \frac{2}{k}\, n(x) - y)\, \rho_k(\xi - \eta) \in \mathcal{D}(\Omega_y \times \mathbb{R}^N_\eta)$ for fixed $x \in \bar{\Omega}$ and $\xi \in \mathbb{R}^N$, and integrate on y, η. We get

$$\frac{\partial \tilde{g}_k}{\partial t} = G_k - (\xi \cdot \nabla_x g) \star_{x,k} \rho_k *_\xi \rho_k - (E \cdot \nabla_\xi g) \star_{x,k} \rho_k *_\xi \rho_k \in L^1(0,T; W^{1,\infty}_{loc}(\bar{\mathcal{O}})),$$

and in particular $\tilde{g}_k \in W^{1,1}(0,T; W^{1,\infty}_{loc}(\bar{\mathcal{O}}))$. Let define g_k to be the continuous representative of $\tilde{g}_k$ in the class of functions almost everywhere equal to $\tilde{g}_k$. Then g_k solves (2.6) with $r_k = r^1_k(g) + r^2_k(g)$, $r^1_k(g) = \xi \cdot \nabla_x \tilde{g}_k - (\xi \cdot \nabla_x g) \star_{x,k} \rho_k * \rho_k$ and $r^2_k(g) = E \cdot \nabla_\xi \tilde{g}_k - (E \cdot \nabla_\xi g) \star_{x,k} \rho_k * \rho_k$. We have to prove that $r^1_k(g)$ and $r^2_k(g)$ converges to 0 in L^1_{loc}. We shall prove the convergence of $r^1_k(g)$; the one of $r^2_k(g)$ is yet proved in [8].

Let remark that if g is smooth, then of course, one has $\nabla_x (g \star_{x,k} \rho_k) = = \big(I - \dfrac{2}{k}\, D^2 d(x)\big)\, (\nabla_x g) \star_{x,k} \rho_k$ and therefore

$$r^1_k(g) \underset{k \to \infty}{\longrightarrow} 0 \quad \text{in} \quad L^p_{loc}(\bar{D}). \tag{2.7}$$

To deal with general $g \in L^{\infty,p}_{loc}$ we begin by proving an a priori estimate. One has

$$\begin{aligned} r^1_k = \iint & \xi \cdot \nabla_x g(t,y,\eta)\, \rho_k(x - \frac{2}{k}\, n(x) - y)\, \rho_k(\xi - \eta) \\ & \qquad - \eta \cdot \nabla_y \big(g(t,y,\eta)\, \rho_k(x - \frac{2}{k}\, n(x) - y)\, \rho_k(\xi - \eta)\big)\, dy d\xi \\ = \iint & g(t,y,\eta)\, \rho_k(\xi - \eta) \big\{ (\xi - \eta) \cdot \nabla \rho_k(x - \frac{2}{k}\, n(x) - y) \\ & \qquad - \frac{2}{k}\, \xi \cdot D^2 d(x)\, \nabla \rho_k(x - \frac{2}{k}\, n(x) - y) \big\}\, dy d\xi. \end{aligned}$$

Then, noting $(\nabla \rho)_k(z) = k^N\, \nabla \rho(k\, z)$, we get, for a constant C which only depends on p, R and $d(x)$, the bound

$$\begin{aligned} \|r^1_k\|^p_{L^p(D_R)} \le 2^p \int_0^T & \|g(t,.,.)\|^p_{L^p(\mathcal{O}_{R+1})}\, \|(\nabla \rho)_k\|^p_{L^1(\mathbb{R}^N_x)} \big\{ \|k\, \xi\, \rho_k(\xi)\|^p_{L^1(\mathbb{R}^N)} \\ & + 2\, \|\xi\, D^2 d(x)\|^p_{L^\infty(\mathcal{O}_{R+1})}\, \|\rho_k(\xi)\|^p_{L^1(\mathbb{R}^N_\xi)} \big\}\, dt \le C\, \|g\|^p_{L^p(D_{R+1})}. \end{aligned} \tag{2.8}$$

Then, for $g \in L^{\infty,p}_{loc}$ we argue by density: we consider a sequence g_ε of smooth functions, such that $g_\varepsilon \longrightarrow g$ in $L^p_{loc}(\bar{D})$ and we write $r^1_k(g) =$

$r_k^1(g_\varepsilon) + r_k^1(g - g_\varepsilon)$ which obviously converges to 0 in $L^p_{loc}(\bar{D})$ thanks to (2.7) and (2.8). $\square$

Third step. Passing to the limit. Thanks to (2.5), $g_k(t,.)$ converges to $g(t,.)$ in $L^p_{loc}(\bar{\mathcal{O}})$ for almost all $t \in [0,T]$, and we denote by t_0 such a time.

On the other hand, for all $k, \ell \in \mathbb{N}^\star$ the difference $g_k - g_\ell$ belongs to $W^{1,1}(0,T; W^{1,\infty}_{loc}(\bar{\mathcal{O}}))$ and solves

$$\Lambda_E(g_k - g_\ell) = G_k - G_\ell + r_k - r_\ell \quad \text{in} \quad \mathcal{D}'(D). \tag{2.9}$$

The estimate (2.2) applied to $g_k - g_\ell$ and Lemma 1 imply that for all compact sets $K \subset \mathcal{O}$ one has

$$\sup_{t\in[0,T]} \|(g_k - g_\ell)(t,.)\|_{L^1(K)} \underset{k,\ell\to+\infty}{\longrightarrow} 0. \tag{2.10}$$

We then deduce from (2.10) and the bound (2.5) that there exists for all $t \in [0,T]$ a function $\gamma_t g$ such that $g_k(t,.)$ converges to $\gamma_t g$ in $C([0,T]; L^1_{loc}(\mathcal{O}))$, and in particular

$$g(t,x,\xi) = \gamma_t g(x,\xi) \text{ for almost every } (t,x,\xi) \in D.$$

Moreover, for all $t \in [0,T]$ and $R > 0$ we have

$$\|\gamma_t g\|_{L^p_R} \leq \lim_{k\to\infty} \sup_{[0,T]} \|g_k(t,.)\|_{L^p_R} = \|g\|_{L^{\infty,p}_R}.$$

One has $g_k(t,.) = (\gamma_t\, g) \star_{x,k} \rho_k *_\xi \rho_k$ a.e. in $\mathcal{O}$ for all $k \in \mathbb{N}^\star$ and $t \in [0,T]$, and since the two functions are continuous, this holds everywhere in $\mathcal{O}$ and thus $g_k(t,.) \to \gamma_t g$ in $L^p_{loc}(\bar{\mathcal{O}})$ for all $t \in [0,T]$. In the sequel, we just note $\gamma_t\, g = g(t,.)$. From (2.10), we deduce $g \in C([0,T]; L^1_{loc}(\mathcal{O}))$.

Estimate (2.3) applied to $g_k - g_\ell$, Lemma 1 and the convergence (2.5) imply that for all compact subsets $K \subset \Sigma$

$$\iiint_{(0,T)\times K} |\gamma g_k - \gamma g_\ell| \, d\mu_2(t,x,\xi) \underset{k,\ell\to+\infty}{\longrightarrow} 0. \tag{2.11}$$

One deduces from (2.11) the existence of a function $\gamma g \in L^1_{loc}([0,T] \times \Sigma, d\mu_2)$ which is the limit of (γg_k).

Last, for a fixed $\phi \in \mathcal{D}_0(\bar{D})$ there is a constant C such that $|\phi(t,x,\xi)| \leq C\,|n(x)\cdot\xi|$ on Γ and therefore, the Green formula (1.6) is established, writing it first for g_k and passing next to the limit $k \to \infty$. Uniqueness of the trace follows from the Green formula. □

Proof of Theorem 2. The proof is really similar to the one of Theorem 1, and we describe it briefly. Let fix a compact subset K of $(0,T)\times\Sigma$, $\psi(z) = z$ and let chose $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^N\times\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on K, $\operatorname{supp}\varphi \subset (0,T)\times\bar{\mathcal{O}}_R$, $\beta(z) = |z|^\theta$ with $\theta \in [1,\infty)$. Using identity (2.1), one has a first estimate

$$\begin{aligned}(2.12)\quad \iiint_K |\gamma\, g|^\theta\, d\mu_2(t,x,\xi) \leq &(R\,\|\nabla\varphi\|_{L^\infty(D)}+1)\iiint_{D_R}|g|^\theta\,|E|\,d\xi dx dt\\ &+R^2\,\|D^2 d_\Omega\|_{L^\infty(\Omega)}\,\|g\|^\theta_{L^\theta_R} + R\,\|g\|^{\theta-1}_{L^\infty_R}\,\|G\|_{L^1_R}.\end{aligned}$$

For all compact sets $K\subset\mathcal{O}$, all $t_0\in[0,T]$ and all $a\in[1,\infty)$ we also prove that the following a priori estimate holds

$$\begin{aligned}(2.13)\quad \sup_{t\in[0,T]}\|g(t,.)\|^a_{L^a(K)} \leq &\|g(t_0,.)\|^a_{L^a_R}\\ &+\|\nabla\varphi\|_{L^\infty(D)}\int_0^T\iint_{\mathcal{O}_R}\left\{|g|^a\,(R+|E|)+a\,|g|^{a-1}\,|G|\right\}d\xi dx dt.\end{aligned}$$

Let then consider the sequence (g_k) of smooth approximations of g built in Theorem 1. This one satisfies

(2.14)

$$g_k \text{ is bounded in } L^\infty_{loc}(\bar{D}) \text{ and } g_k \underset{k\to\infty}{\longrightarrow} g \text{ in } L^a_{loc}([0,T]\times\bar{\mathcal{O}})\quad \forall a\in[1,\infty).$$

We conclude without difficulty thanks to the a priori estimates (2.12) and (2.13) and the convergence (2.14), in the same way that we have done in the proof of Theorem 1. □

Proof of Corollary 1. We just have to remark that sequences (g_k) and (G_k), defined in Lemma 1, obviously satisfy

$$\Lambda_E\beta(g_k) = \beta'(g_k)\,G_k + \beta'(g_k)\,r_k \quad\text{in}\quad \mathcal{D}'(D),$$
$$\beta(\gamma\, g_k) = \gamma\,\beta(g_k) \quad\text{and}\quad \beta(\gamma_t\, g_k) = \gamma_t\,\beta(g_k)\ \ \forall t\in[0,T],$$

and we pass to the limit $k \to \infty$ without difficulty, since g_k, $\gamma\, g_k$ and $\gamma_t\, g_k$ converge strongly. This proves (1.8). $\square$

3. Extensions and additional properties of the trace.

We begin with some remarks on Theorems 1 and 2.

Remark 1. (i) Theorems 1 and 2 can be generalized to a vector field $E = E(t, x, \xi)$ such that

$$E \in L^1(0, T; W^{1,p'}_{loc}(\bar{\mathcal{O}})) \quad \text{and} \quad div_\xi\, E \in L^1(0, T; L^\infty(\bar{\mathcal{O}})).$$

See [14].

(ii) A priori estimate (2.3) or (2.12) only use the bound of E in $L^1(0, T; L^{p'}_{loc}(\bar{\mathcal{O}}))$, but in general, we do not know how to give sense to the trace of g with only $E \in L^{1,p'}_{loc}$. Indeed, in the regularization step we use $E \in L^1(0, T; W^{1,p'}_{loc}(\bar{\mathcal{O}}))$. In particular, Theorem 1 does not apply to the Vlasov-Maxwell equation where the field $\mathcal{E}(t, x) + \xi \wedge \mathcal{B}(t, x)$ only belongs to $L^\infty([0, T]; L^2_{loc}(\bar{\mathcal{O}}))$.

(iii) Nevertheless, when $E \in L^{1,p'}_{loc}$ and $g \geq 0$, one can show existence of a measure trace γg, using the estimate (2.3) and regularizing in the only x variable. Furthermore, estimate (2.3) also allows one to show existence of a solution to the Vlasov equation (1.1) with Dirichlet condition or specular reflection condition on the boundary when $E \in L^{1,p'}_{loc}$ as we do in Theorem 4.

(iv) Theorem 1 is necessary local in ξ if we do not make moments assumption on g. One possibly global version of trace theorem is the following: let $g \in L^\infty\big(0, T; L^p(\mathcal{O}, (1 + |\xi|)\, dxd\xi)\big)$, $E \in L^1(0, T; W^{1,p'}(\bar{\mathcal{O}}))$ and $G \in L^1((0, T) \times \mathcal{O})$ then we have $\gamma g \in L^1(\Gamma; p_M(x, \xi)\, dxd\xi dt)$ $\forall M$, with $p_M(x, \xi) = |n(x) \cdot \xi|\, (|n(x) \cdot \xi| \wedge M)$.

(v) We can extend Theorems 1 and 2 for solution g to the Vlasov-Fokker-Planck equation

$$\frac{\partial}{\partial t} g + \xi \cdot \nabla_x g + E \cdot \nabla_\xi g - \sigma\, \Delta_\xi g = G \;\text{ in }\; \mathcal{D}'(D), \quad \sigma \in \mathbb{R}.$$

See [14].

(vi) We obtain similar trace results in the case of the stationary Vlasov equation.

(vii) We can make other assumptions concerning the integrability in time of g, E and G in Theorems 1 and 2. One has for instance

Proposition 1. Let $E \in L^{a'}(0,T;W^{1,p'}_{loc}(\bar\Omega))$, $G \in L^1_{loc}(\bar D)$ and $g \in L^a(0,T;L^p_{loc}(\bar{\mathcal{O}}))$ a solution to (1.1), with $a,p \in [1,\infty)$. Then g satisfies $g \in C([0,T];L^1_{loc}(\mathcal{O}))$, $g(t,.) \in L^1_{loc}(\bar{\mathcal{O}}\backslash\Sigma_0)$ for all $t \in (0,T)$ and there exists a trace function $\gamma g \in L^1_{loc}((0,T)\times\Sigma, d\mu_2)$ such that the Green formula (1.6) holds for all $\phi \in \mathcal{D}(\bar D)$ which vanishes in a neighborhood of $[0,T]\times\Sigma_0 \cup \{0\}\times\Sigma\cup\{T\}\times\Sigma$.

The proof of Proposition 1 is a variant of the proof of Theorem 1. One can show the following a priori bound: for every compact set K of $\mathcal{O}$ and for all $\varepsilon, R > 0$ there is a constant C such that

$$\sup_{[0,T]}\|g(t,.)\|_{L^1(K)} + \sup_{[\varepsilon,T-\varepsilon]}\|g(t,.)\,n(x)\cdot\xi\|_{L^1(\mathcal{O}_R)}$$
$$+\int_\varepsilon^{T-\varepsilon}\iint_{\Sigma_R}|\gamma g|\,d\mu_2(t,x,\xi) \le C\left\{(1+\|E\|_{L^{a',p'}_{R+1}})\,\|g\|_{L^{a,p}_{R+1}} + \|G\|_{L^1_{R+1}}\right\}.$$

Then, one can conclude using the sequence of regularized functions (g_k).

We give now some additional results concerning the properties of the trace. We begin with a stronger integrability result on γg and an improvement of Corollary 1.

Proposition 2. *We make assumption of Theorem 1, we assume that additionally*

$$\text{(3.1)}\qquad E \in L^1\big(0,T;L^q_{loc}(\bar\Omega)\big),\ G \in L^1\big(0,T;L^s_{loc}(\bar{\mathcal{O}})\big)\ \textit{with}\ \frac{1}{s} = \frac{1}{q}+\frac{1}{p} \le 1,$$

and we set $r = p\,(1-1/q)$. Then, we have

$$\text{(3.2)}\qquad |\gamma\, g|^r \in L^1_{loc}\big([0,T]\times\Sigma, (n(x)\cdot\xi)^2\,d\xi d\sigma_x dt\big),$$

and Corollary 1 holds with every $\beta \in W^{1,\infty}_{loc}$ such that $|\beta(y)| \le C\,(1+|y|^r)$ $\forall y \in \mathbb{R}$. Last, if $r \ge 2$, then the Green formula (1.6) holds for all test functions $\phi \in \mathcal{D}(\bar D)$.

Proof. We consider a sequence (β_ε) of smooth non negative, even and bounded functions, such that $-r\,|y|^{r-1} \le \beta_\varepsilon(y) \le r\,|y|^{r-1}$ and $\beta_\varepsilon(y) \to |y|^r$

$\forall y \in \mathbb{R}$. From Corollary 1, we can write the Green formula (1.6) for $\beta_\varepsilon(g)$ and a test function $\varphi = \xi \cdot n(x)\,\chi$, with $\chi \in \mathcal{D}([0,T]\times\bar{\mathcal{O}})$ such that $0 \le \chi \le 1$, $\chi = 1$ on $[0,T]\times\bar{\mathcal{O}}_R$ and $\operatorname{supp}\chi \subset [0,T]\times\mathcal{O}_{R+1}$. Using the Holder inequality, we easily obtain the estimate

$$\int_0^T\!\!\iint_{\Sigma_R} \beta_\varepsilon(g)\,(n(x)\cdot\xi)^2\,d\xi d\sigma_x dt \le R\iint_{\mathcal{O}_{R+1}} \big\{\beta_\varepsilon(g(0,.) + \beta_\varepsilon(g(T,.)\big\}\,dxd\xi$$

$$+\int_0^T\!\!\iint_{\mathcal{O}_{R+1}} \big\{\beta_\varepsilon(g)\,\big(\big|{}^t\xi\,D^2 d_\Omega\,\xi\big| + |E| + |\xi|\,|\Lambda_E\chi|\big) + |\xi|\,|\beta'_\varepsilon(g)|\,|G|\big\}\,d\xi dxdt$$

$$\le R\,\big(\|g(0,.)\|_{L^r_{R+1}} + \|g(T,.)\|_{L^r_{R+1}}\big)$$

$$+\,C_R\int_0^T \big(1+\|E(t,.)\|_{L^q_{R+1}}\big)\,\|g(t,.)\|^r_{L^p_{R+1}}\,dt + R\,\|g\|^{r-1}_{L^{\infty,p}_{R+1}}\,\|G\|_{L^{1,s}_{R+1}}.$$

Letting $\varepsilon \to 0$ we get (3.2). Let now $T \in C^1(\mathbb{R})$ be a not decreasing and odd function such that $T(z) = z$ if $0 \le z \le 1$ and $T(z) = 2$ if $z \ge 3$, and let define $T_\ell(z) = \ell\,T(z/\ell)$ for $\ell \in \mathbb{N}^\star$. For a given β satisfying the above assumptions we can use Corollary 1 with $T_\ell \circ \beta$ and then (1.8) holds with $T_\ell \circ \beta$ instead of β. Then, thanks to (3.2), we can let ℓ goes to ∞ and we obtain that Corollary 1 still holds with such a β. Of course, when $r \ge 2$ the embedding $L^r_{loc}(\Gamma, d\mu_2) \subset L^{r/2}_{loc}(\Gamma, d\mu_1)$ permits us to write (1.6) for all $\phi \in \mathcal{D}(\bar{D})$. □

The next duality formula is important, it will be used in the sequel in order to prove uniqueness of the solution to the Cauchy problem with boundary conditions.

Proposition 3 (Duality formula). Let p_1, p_2, $q \in [1,\infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{q} \le 1$. We define $\frac{1}{s_1} = \frac{1}{p_1} + \frac{1}{q}$, $\frac{1}{s_1} = \frac{1}{p_2} + \frac{1}{q}$, $r_i = p_i\,(1-\frac{1}{q})$ if $p_i < \infty$, $r_i = \infty$ if $p_i = \infty$ and $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Let g_1 and g_2 be two solutions of Vlasov equations

$$\Lambda_E g_1 = G_1 \text{ and } \Lambda_E g_2 = G_2 \text{ in } \mathcal{D}'(D),$$

with $E \in L^1\big(0,T; W^{1,p'}_{loc}(\bar\Omega)\cap L^q_{loc}(\bar\Omega)\big)$, $p = p_1 \wedge p_2$, $g_i \in L^\infty(0,T; L^{p_i}_{loc}(\bar{\mathcal{O}}))$, $G_i \in L^1(0,T; L^{s_i}_{loc}(\bar{\mathcal{O}}))$ for $i = 1$ and 2. Then the following Duality formula holds

$$\iiint_\Gamma \gamma\,g_1\;\gamma\,g_2\,\chi\,n(x)\cdot\xi\,d\xi d\sigma_x dt + \Big[\iint_{\mathcal{O}} g_1\,g_2\,\chi\,dxd\xi\Big]_0^T = \tag{3.3}$$
$$= \iiint_D \big(g_1\,G_2\,\chi + G_1\,g_2\,\chi + g_1\,g_2\,\Lambda_E\chi\big)\,d\xi dxdt,$$

for all test functions $\chi \in \mathcal{D}_0(\bar{\mathcal{O}})$, and if $r \ge 2$, for all test functions $\chi \in \mathcal{D}(\bar{\mathcal{O}})$.

Proof. We consider the regularized functions $g_{i,k} = g_i \star_{x,k} \rho_k *_\xi \rho_k$, $i = 1, 2$ introduced in Lemma 1 and T_ℓ defined in the proof of Proposition 2. We have

$$\Lambda_E T_\ell(g_{1,k}\, g_{1,k}) = T'_\ell(g_{1,k}\, g_{1,k}) \big(g_{1,k}\, G_{2,k} + g_{2,k}\, G_{1,k}\big) \ \text{ in } \ \mathcal{D}'(D), \tag{3.4}$$

with $G_{i,k} \to G_i$ in $L^1(0,T; L^{s_i}_{loc})$, $i = 1, 2$, and then the Green formula (1.6) writes

(3.5)

$$\Big[\iint_{\mathcal{O}} T_\ell(g_{1,k}\, g_{2,k})\, \chi\, dx d\xi \Big]_0^T + \iiint_\Gamma T_\ell(\gamma g_{1,k}\, \gamma g_{2,k})\, \chi\, n(x)\cdot\xi\, d\xi d\sigma_x dt =$$
$$= \iiint_D \big\{ T'_\ell(g_{1,k}\, g_{1,k}) \big(g_{1,k}\, G_{2,k} + g_{2,k}\, G_{1,k}\big)\, \chi + T_\ell(g_{1,k}\, g_{2,k})\, \Lambda_E \chi \big\}\, d\xi dx dt.$$

We first pass to the limit $k \to \infty$ using the fact that $\gamma g_{i,k}$ converges to γg_i *a.e.* in $[0,T] \times \Sigma$ and $g_{i,k}(t,.)$ converges to $g_i(t,.)$ *a.e.* in $\mathcal{O}$ for every $t \in [0,T]$ and for $i = 1$ and 2. We then get (3.3) letting ℓ go to ∞ and using the bound (1.5), (1.7) or (3.2). □

In the next proposition we prove weak and strong continuity of the traces $\gamma\, g$ and $\gamma_t\, g$ with respect to g, E and G.

Proposition 4. Let $p \in [1,\infty]$ and (g_ε), (E_ε) and (G_ε) be three sequences of functions which satisfy assumptions of Theorem 1 or 2.

1) Assume that $g_\varepsilon \rightharpoonup g$ in $L^{\infty,p}_{loc}$, $E_\varepsilon \longrightarrow E$ in $L^{1,p'}_{loc}$ and $G_\varepsilon \rightharpoonup G$ in L^1_{loc}, and moreover that there is a function $\beta_0 \in W^{1,\infty}_{loc}(\mathbb{R})$ such that

$$\begin{cases} \beta_0 \text{ is stricly superlinear at the infinity, i.e. } \beta_0(z)/|z| \underset{|z|\to\infty}{\longrightarrow} \infty \\ \text{and } \beta_0(g_\varepsilon) \text{ is bounded in } L^{\infty,p}_{loc}. \end{cases} \tag{3.6}$$

Then g solves Vlasov equation (1.1), g has a trace $\gamma g \in L^1_{loc}([0,T] \times \Sigma, d\mu_2)$ and a trace $g(t,.) \in L^p_{loc}(\bar{\mathcal{O}})$ on $\{t\} \times \mathcal{O}$ for all $t \in [0,T]$ in the sense of Green formula (1.6), which are the weak limits of γg_ε and $g_\varepsilon(t,.)$ respectively. Moreover, $g \in C([0,T]; L^p_{loc}(\bar{\mathcal{O}})\, weak)$.

2) Assume that g_ε is bounded in $L^{\infty,p}_{loc}$, $g_\varepsilon \longrightarrow g$ in $L^{a,p}_{loc}$, $\forall a < \infty$, $E_\varepsilon \rightharpoonup E$ in $L^{1,p'}_{loc}$, with $E \in L^1(0,T; W^{1,p'}_{loc}(\bar\Omega))$ and $G_\varepsilon \rightharpoonup G$ in L^1_{loc}, then $\gamma\, g_\varepsilon \longrightarrow \gamma\, g$ in $L^1_{loc}([0,T] \times \Sigma, d\mu_2)$ and $g_\varepsilon(t,.) \to g(t,.)$ in $L^1_{loc}(\bar{\mathcal{O}})$ for all $t \in [0,T]$.

Remark 2. Result 1) shows that a solution $g \in L^{\infty,p}_{loc}$ to the Vlasov equation (1.1) with $E \in L^{1,p'}_{loc}$, $G \in L^1_{loc}$ has a trace $\gamma g \in L^1_{loc}([0,T] \times \Sigma, d\mu_2)$ if g is obtained as the weak limit of a sequence $g_\varepsilon \in L^{\infty,p}_{loc}$ of solutions to the Vlasov equation (1.1) with $E_\varepsilon \in L^1(0,T; W^{1,p'}_{loc}(\bar{\mathcal{O}}))$, $G_\varepsilon \in L^1_{loc}$. But, in general, we can not say that the trace function γg constructed by this way satisfies $\beta(\gamma g) = \gamma \beta(g)$ for every $\beta \in W^{1,\infty}(\mathbb{R})$. Compare to Remark 1. ii) and 1. iii).

We start with the statement and the proof of a technical lemma that we shall use in the sequel.

Lemma 2. *Under assumption (3.6), there exists $\beta \in W^{1,\infty}_{loc}(\mathbb{R})$ strictly superlinear at the infinity such that $\beta(g_\varepsilon)$ is bounded in $L^{\infty,p}_{loc}$, $\beta(G_\varepsilon)$ and $\beta'(g_\varepsilon)\, G_\varepsilon$ are bounded in L^1_{loc}.*

Proof of Lemma 2. First remark that it is not a restriction to assume that moreover β_0 is even, convex, not decreasing on $\mathbb{R}_+$ and satisfies $\beta_0(z) \geq 1+|z|$ $\forall z \in \mathbb{R}$. We can also assume, thanks to Dunford-Pettis lemma, that $\beta_0(G_\varepsilon)$ is bounded in $L^1_{loc}(\bar{D})$. In order to construct β let define b_k as the infimum of positive reals such that

$$\frac{\beta_0(z)}{z} \geq k \quad \text{if} \quad z \geq b_k.$$

We then define by induction on $k \in \mathbb{N}$ the even function β by

$$\begin{cases} \beta(z) = 1 \text{ if } z \in [0,1], \quad \beta(z) = z \text{ if } z \in [1, a_2], \text{ with } a_2 = \max(1, b_2), \\ \beta(z) = k\,(z - a_k) + \beta(a_k) \text{ if } z \in [a_k, a_{k+1}], \\ \qquad\qquad\qquad\qquad \text{with } a_{k+1} = \max(b_{k+1}, \dfrac{2k\, a_k - \beta(a_k)}{k-1}), \ k \geq 2. \end{cases}$$

By construction of the a_k, β satisfies $\beta'(z)\, z \leq 2\, \beta(z)$ and $\beta(z) \leq k\, z \leq \beta_0(z)$ in each segment $[a_k, a_{k+1}]$ and β is strictly superlinear, since $\beta'(z) \geq k$, $\forall z \geq a_k$. Therefore, $\beta(g_\varepsilon)$ is bounded in $L^{\infty,p}_{loc}$ and $\beta(G_\varepsilon)$ is bounded in L^1_{loc}. Last, we have $\big|\beta'(g_\varepsilon)\, G_\varepsilon\big| \leq \beta'(|g_\varepsilon|)\, |g_\varepsilon| + \beta'(|G_\varepsilon|)\, |G_\varepsilon| \leq 2(\beta(g_\varepsilon) + \beta(G_\varepsilon))$. □

Proof of Proposition 4. We begin with 1). We claim that the sequence $(g_\varepsilon(t,.))$ is weakly compact in $L^p_{loc}(\bar{\mathcal{O}})$ and the sequence (γg_ε) is weakly compact in $L^1_{loc}([0,T] \times \Sigma, d\mu_2)$. Indeed, we observe that thanks to Proposition 2 and Lemma 2, we can write $\Lambda_E \beta(g_\varepsilon) = \beta'(g_\varepsilon)\, G_\varepsilon$ in $\mathcal{D}'(D)$ and then that, uniformly in ε, $\beta(\gamma\, g_\varepsilon)$ is bounded in $L^1_{loc}([0,T]\times\Sigma, d\mu_2)$ and $\beta(g_\varepsilon(t,.))$ is bounded in $L^1_{loc}(\mathcal{O})$ for all $t\in[0,T]$. We conclude with the help of Dunford-Pettis Lemma. Now, for fixed $t_0, t_1\in[0,T]$, there is a sub-sequence, denoted by ε', such that $\gamma g_{\varepsilon'}$ and $g_{\varepsilon'}(t_i,.)$, $i=0$ or 1, converge; and we note γg and $\gamma_{t_i} g$ the resulting limits. We write then the Green formula (1.6) for the sequence $g_{\varepsilon'}$, we pass to the limit $\varepsilon'\to 0$ and we get that γg is the trace of g on Γ and that is the all sequence γg_ε which converges. The same holds for $g(t,.) := \gamma_t g$. The continuity $t\mapsto g(t,.)$ in $L^p_{loc}(\bar{\mathcal{O}})$ weak is a consequence of the bound $\|\beta(g)\|_{L^{\infty,p}_R} \le \liminf_{\varepsilon\to 0}\|\beta(g_\varepsilon)\|_{L^{\infty,p}_R}$ and of the continuity in the distributional sense following from (1.6).

In order to prove 2), we first remark that, passing to the limit in the Green formula (1.6), we obtain $\gamma\, g_\varepsilon\ (n(x)\cdot\xi)^2 \rightharpoonup \gamma\, g\ (n(x)\cdot\xi)^2$ in the sense of the weak$\star$ topology $\sigma\big(M^1([0,T]\times\Sigma_R), C([0,T]\times\Sigma_R)\big)$, for all $R>0$. We then choose a strictly convex function $\beta\in C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$, $\beta(z)\le|z|$. Since we have $\beta(g_\varepsilon)\longrightarrow\beta(g)$ in $L^p_{loc}(D)$ with $\beta(g_\varepsilon)$ bounded in $L^\infty(D)$, $E_\varepsilon\rightharpoonup E$ in $L^{1,p'}_{loc}(D)$ and $\beta'(g_\varepsilon)\, G_\varepsilon\rightharpoonup\beta'(g)\, G$ in $L^1_{loc}(D)$ we also obtain that $\beta(\gamma\, g_\varepsilon)=\gamma\,\beta(g_\varepsilon)\rightharpoonup\gamma\,\beta(g)=\beta(\gamma\, g)$, which implies that $\gamma\, g_\varepsilon$ converges to $\gamma\, g$ strongly, see H. Brézis [5]. In the same way we prove that $g_\varepsilon(t,.)$ converges to $g(t,.)$ in $L^1_{loc}(\bar{\mathcal{O}})$. $\square$

4. Vlasov equation with specular reflection on the boundary.

We show in this section existence and uniqueness of the solution to the Cauchy problem for the Vlasov equation (1.1), with initial datum (1.10) and specular reflection at the boundary (1.11).

Existence is proved thanks to the penalty method that we have described in the introduction. Such a method had been used by P.L. Lions and A.S. Snitzmann [12] to prove existence of a solution to an E.D.O. set in an open set, with "reflection" when the trajectory touches the boundary. To our knowledge, it is the first time that a penalty method is used in the framework of kinetic equation.

In order to define the penalty term let introduce some notations. We extend $d=d_\Omega$ as a function $d\in W^{2,\infty}(\mathbb{R}^N)$ such that in an exterior neighborhood $\mathcal{V}$ of $\partial\Omega$ one has $d(x)=\mathrm{dist}(x,\partial\Omega)$ and $d(x)\ge d_0>0$ outside of $\mathcal{V}$.

One defines $\delta(x) = d(x)\,\mathbf{1}_{\{x\in\Omega^c\}}$, and thus $\delta(x) = \mathrm{dist}(x,\bar\Omega)$ in $\mathcal{V}$. In a neighborhood $\mathcal{W}$ of $\partial\Omega$ the vector field $n(x) = \nabla_x d(x)$ does not vanish, and we can define on $\mathcal{W}$ the field Π_x of projector operators on the hyperplane which is orthogonal to $n(x)$, in such a way that we have $\xi = (n(x)\cdot\xi)\,n(x) + \Pi_x\,\xi$ and $n(x)\cdot\Pi_x\,\xi = 0$, for all $\xi\in\mathbb{R}^N$, and we extend it arbitrarily outside of $\mathcal{W}$. Last, we define de vector field $E_\varepsilon(x) = -\nabla_x \dfrac{\delta^2(x)}{2\,\varepsilon} = -\dfrac{\delta(x)}{\varepsilon}\,n(x)$. For a given function H defined on D or on $(0,T)\times\mathbb{R}^N\times\mathbb{R}^N$, we note $\tilde H$, or just H when there is no ambiguity, the function defined on $(0,T)\times\mathbb{R}^N\times\mathbb{R}^N$ by $\tilde H = H$ on D, $\tilde H = 0$ on $(0,T)\times\Omega^c\times\mathbb{R}^N$.

We begin with a first theorem which implies the existence result in Theorem 3.

Theorem 4 (Existence). *Let $p\in[1,\infty]$, $g_0\in L^p(\mathcal{O})$, $E\in L^1(0,T;L^{p'}_{loc}(\Omega))$, $G\in L^1(0,T;L^p(\mathcal{O}))$. Then, there exists a solution g to (1.1) in $L^\infty(0,T;L^p(\mathcal{O}))$ satisfying (1.11), and corresponding to the initial datum g_0.*

Proof. First step. We assume in this first step that $E\in L^1(0,T;W^{1,p'}_{loc}(\Omega))$ and $p\in(1,\infty]$.
We shall deduce the existence from the existence result of DiPerna-Lions [8].

a) - Proposition II.1 [8] state that there exists a solution $g_\varepsilon\in L^\infty(0,T;L^p(\mathbb{R}^N\times\mathbb{R}^N))$ to

$$\frac{\partial}{\partial t}g_\varepsilon + \xi\cdot\nabla_x g_\varepsilon + (E_\varepsilon+E)\cdot\nabla_\xi g_\varepsilon = G \ \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^N\times\mathbb{R}^N), \tag{4.1}$$

corresponding to the initial datum g_0, and satisfying the uniform bound

$$\sup_{[0,T]}\|g_\varepsilon(t,.)\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)} \le C(T,\|g_0\|_{L^p(\mathcal{O})},\|G\|_{L^1((0,T);L^p(\mathcal{O}))}). \tag{4.2}$$

Therefore, up to the extraction of a subsequence, g_ε converges to a function g in $L^\infty((0,T);L^p(\mathbb{R}^N\times\mathbb{R}^N))$ weak.

b) - We remark that $E_\varepsilon\equiv 0$ in the domain D. Thus, passing to the limit in the sense of $\mathcal{D}'([0,T)\times\mathcal{O})$ in equation (4.1), we obtain that g solves (1.1), corresponding to the initial datum g_0, i.e. g satisfies

$$\iint_{\mathcal{O}} g_0\,\phi(0,.)\,dxd\xi + \iiint_D (g\,\Lambda_E\phi + G\,\phi)\,d\xi dxd\tau = 0, \tag{4.3}$$

for all $\phi \in \mathcal{D}([0,T) \times \mathcal{O})$. We still have to show that g satisfies (1.11).

c) - Given three functions $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^N)$, $\psi \in \mathcal{D}([0,+\infty))$, $\psi(0)=0$, and $\Psi \in \mathcal{D}(\mathbb{R}^{N-1})$, we set

$$\Phi_\varepsilon = \varphi\,\psi_\varepsilon\,\Psi = \varphi(t,x)\,\psi((n(x)\cdot\xi)^2 + \frac{\delta^2(x)}{\varepsilon})\,\Psi(\Pi_x\,\xi),$$

and we define the class $\mathcal{RS} \subset \mathcal{D}_0((0,T)\times\mathbb{R}^N)$ as the space of functions which write $\Phi(t,x,\xi) = \varphi(t,x)\,\psi((n(x)\cdot\xi)^2)\,\Psi(\Pi_x\,\xi)$. We choose Φ_ε as a test function in (4.1), and we get

$$\begin{aligned} &-\int_0^T\!\!\iint_{\mathbb{R}^N\times\mathbb{R}^N} g_\varepsilon \Big\{\psi_\varepsilon\,\Psi\,(\frac{\partial}{\partial t}\varphi + \xi\cdot\nabla_x\varphi) + \varphi\,\Psi\,\big(\xi\cdot\nabla_x\psi_\varepsilon + (E_\varepsilon+E)\cdot\nabla_\xi\psi_\varepsilon\big) \\ &+ \varphi\,\psi_\varepsilon\,\big(\xi\cdot\nabla_x\Psi + (E_\varepsilon+E)\cdot\nabla_\xi\Psi\big)\Big\}\,dxd\xi dt = \int_0^T\!\!\iint_{\mathbb{R}^N\times\mathbb{R}^N} G\,\varphi\,\psi_\varepsilon\,\Psi\,dxd\xi dt. \end{aligned}$$

We pass to the limit $\varepsilon \to 0$ noting that

$$\xi\cdot\nabla_x\psi_\varepsilon + E_\varepsilon\cdot\nabla_\xi\psi_\varepsilon = 2\,\xi\cdot\nabla_x\big[(n(x)\cdot\xi)^2\big]\,\psi'((n(x)\cdot\xi)^2 + \frac{\delta^2}{\varepsilon}),$$

$$E_\varepsilon\cdot\nabla_\xi\Psi = -\frac{\delta(x)}{\varepsilon}\,n(x)\cdot\nabla_\xi\Psi(\Pi_x\,\xi) = -\frac{\delta(x)}{\varepsilon}\,n(x)\cdot\big[\Pi_x\,\nabla_\eta\Psi\,(\Pi_x\,\xi)\big] = 0,$$

and

$$g_\varepsilon\,\psi_\varepsilon = g_\varepsilon\,\tilde\psi + g_\varepsilon\,\mathbf{1}_{x\notin\Omega}\,\psi_\varepsilon \rightharpoonup g\,\tilde\psi \quad L^p_{loc}\text{ weak},$$

since $\psi_\varepsilon \longrightarrow 0$ a.e. $x \in \Omega^c$ and $\|\psi_\varepsilon\|_\infty \le \|\psi\|_\infty$, and thus

$$\begin{aligned} &-\iiint_D g\,\Big\{\psi\,\Psi\,(\frac{\partial}{\partial t}\varphi + \xi\cdot\nabla_x\varphi) + \varphi\,\Psi\,(\xi\cdot\nabla_x\psi + E\cdot\nabla_\xi\psi) \\ &\qquad + \varphi\,\psi\,(\xi\cdot\nabla_x\Psi + E\cdot\nabla_\xi\Psi)\Big\}\,dxd\xi dt = \iiint_D G\,\Phi\,dxd\xi dt, \end{aligned}$$

or, in other words,

$$\iiint_D (g\,\Lambda_E\Phi + G\,\Phi)\,dxd\xi dt = 0, \quad \forall\Phi \in \mathcal{RS}. \tag{4.4}$$

This last equation is a weak formulation of the specular reflection condition.

d) - Indeed, Theorem 1 or 2 imply that the trace γg is well defined, and, thanks to Green formula (1.6), γg satisfies

$$\iiint_\Gamma \gamma\,g(t,x,\xi)\,\Phi(t,x,\xi)\,n(x)\cdot\xi\,d\sigma(x)d\xi dt = 0, \quad \forall\Phi \in \mathcal{RS}. \tag{4.5}$$

Therefore, for almost every $(t,x) \in (0,T) \times \partial\Omega$, for all $\bar\psi$ odd, $|\bar\psi(z)| \le C\, z^2$ and for all Ψ we have shown that

$$\int_{\xi' \in \Pi_x(\mathbb{R}^N)} \int_{\xi'' \in \mathbb{R}_+} \big[\gamma g(t,x,\xi' + \xi''\, n(x)) \\ - \gamma g(t,x,\xi' - \xi''\, n(x))\big]\, \Psi(\xi')\, \bar\psi(\xi'')\, d\xi' d\xi'' = 0,$$

which is equivalent to say that $\gamma g(t,x,\xi) = \gamma g(t,x,R_x\,\xi)$ for almost every $(t,x,\xi) \in \Gamma$.

Second step. We deal now with the general case $E \in L^1(0,T;L^{p'}_{loc}(\Omega))$ and $p \ge 1$. Let consider a sequence of approximations $E_\ell \in L^1(0,T;W^{1,p'}_{loc}(\Omega))$ such that $E_\ell \to E$ in $L^1(0,T;L^{p'}_{loc}(\bar{\mathcal{O}}))$. To deal with the case $p = 1$ we introduce an additional approximation: we define the sequence $g_0^\ell = T_\ell\, g_0$ and $G_\ell = T_\ell\, G$, where T_ℓ is defined just like in the proof of Proposition 2, in such a way that $g_0^\ell \to g_0$ in L^p and $G_\ell \to G$ in $L^{1,p}$. We note g_ℓ the solution of the Vlasov equation (1.1)-(1.10)-(1.11) corresponding to the field E_ℓ, the source G_ℓ and initial datum g_0^ℓ, constructed thanks to the first step.

When $p > 1$, the sequence g_ℓ satisfies the a priori bound (4.2), and thus, up to the extraction of a subsequence, g_ℓ converges to a function g in $L^\infty(0,T;L^p(\mathbb{R}^N \times \mathbb{R}^N))$ weak, which solves (1.1)-(1.10). But, in order to prove that g satisfies (1.11), and in order to deal with the case $p = 1$, we shall need an a priori estimate a little stronger than (4.2).

Thanks to Dunford-Pettis Lemma and Lemma 2, there is a convex, even and superlinear function β such that $\beta(g_0) \in L^p(\mathcal{O})$, $\beta(G) \in L^1(0,T;L^p(\mathcal{O}))$ and $\beta'(z)\,z \le 2\,\beta(z)$ for all $z \ge 0$. With the notations of Lemma 2, we define the function β_k, which is even and increases linearly at the infinity, by $\beta_k(z) = \beta(z)$ if $z \in [0,a_k]$ and $\beta_k(z) = k\,(z - a_k) + \beta(a_k)$ if $z \ge a_k$. Proposition 2 implies

$$\Lambda_{E_\ell}\beta_k(g_\ell) = \beta_k'(g_\ell)\, G_\ell \quad \text{in } \ \mathcal{D}'(D), \tag{4.6}$$

and thanks to the Gronwall lemma we get the estimate

$$\sup_{[0,T]} \|\beta_k(g_\ell)\|_{L^p} \le C(\|\beta_k(g_0^\ell)\|_{L^p}, \|\beta_k(G_\ell)\|_{L^1(L^p)}, T),$$

from which we deduce using Lebesgue Theorem and Beppo-Levy theorem

$$\sup_{[0,T]} \|\beta(g_\ell)\|_{L^p} \le C(\|\beta(g_0)\|_{L^p}, \|\beta(G)\|_{L^1(L^p)}, T). \tag{4.7}$$

Therefore, we are able to pass to the limit $k \to \infty$ in (4.6), and we obtain

$$\Lambda_{E_\ell} \beta(g_\ell) = \beta'(g_\ell)\, G_\ell \quad \text{in} \;\; \mathcal{D}'(D). \tag{4.8}$$

When $p = 1$, estimate (4.7) and Dunford-Pettis lemma show that, up to the extraction of a subsequence, g_ℓ converges to a function g in $L^\infty(0,T; L^1(\mathbb{R}^N \times \mathbb{R}^N))$ weak, which solves (1.1)-(1.10).

In order to prove that the specular reflection condition (1.11) holds, we use Proposition 4, part 1) which says that g has a trace γg and that $\gamma g_\ell \rightharpoonup \gamma g$ in $L^1([0,T] \times \Sigma, d\mu_2)$. Passing to the limit in (4.5) written for g_ℓ, we get that (4.5) also holds for g. This proves that g satisfies (1.11). □

Proof of Theorem 3. Existence part is stated in Theorem 4. For the uniqueness result we shall argue by duality. Thanks to Theorem 1 or 2, the trace γg is well defined, $\gamma g \in L^1_{loc}([0,T] \times \Sigma, d\mu_2)$, and therefore the boundary condition makes sense. Let us consider two solutions g_1 and g_2 of (1.1)-(1.10)-(1.11), and let us set $f = g_2 - g_1$. For a fixed $\beta \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, such that $\beta(0) = 0$, $\beta(s) > 0$ $\forall s \in \mathbb{R}^\star$, the function $\beta(f) \in L^\infty(D)$ satisfies

$$\begin{gathered} \Lambda_E\, \beta(f) = 0 \;\; \text{in} \;\; \mathcal{D}'(D), \quad \beta(f)(0,x,\xi) = 0 \\ \text{and} \;\; \beta(f)(t,x,\xi) = \beta(f)(t,x,R_x\,\xi) \;\; \text{a.e. on} \;\; \Gamma. \end{gathered}$$

Let us now consider $\phi \in \mathcal{D}(D)$ and $\Phi \in L^\infty(0,T; L^1(\mathcal{O})) \cap L^\infty(D)$ a solution to the backward problem

$$\Lambda_E \Phi = \phi \;\; \text{in} \;\; \mathcal{D}'(D), \;\; \Phi|_{t=T} = 0 \text{ and } \Phi(t,x,\xi) = \Phi(t,x,R_x\xi) \text{ on } \Gamma,$$

given by Theorem 4. Last consider $\mathcal{X}_R = \chi_R(|\xi|)\,\chi_R(|x|)$ a smooth troncature function, with $\chi_R = \chi(\frac{\cdot}{R})$, $R \geq 1$, $\chi \in \mathcal{D}_+(\mathbb{R}_+)$, $\operatorname{supp}\chi \in [0,2)$, $\chi \equiv 1$ on $[0,1]$. We use the duality formula (3.3) with the functions $\beta(f)$, Φ, and the test function $\mathcal{X}_R \in \mathcal{D}(\bar{\mathcal{O}})$, and we obtain

$$\begin{aligned} \iiint_D \beta(f)\,\phi\,\mathcal{X}_R\, d\xi dx dt = &\iiint_\Gamma \gamma\,\Phi\,\gamma\beta(f)\,\mathcal{X}_R\, n(x)\cdot\xi\, d\xi d\sigma_x dt \\ &- \iiint_D \beta(f)\,\Phi\left(\chi_R\,\xi\cdot\nabla_x\,\chi_R + \chi_R\,E\cdot\nabla_\xi\chi_R\right) d\xi dx dt. \end{aligned}$$

The first right hand term vanishes thanks to the specular reflection condition, and the last term is bounded by

$$
(4.9)\qquad \begin{aligned} &\|\beta\|_{L^\infty(\mathbb{R})} \iiint_D \Phi \, \big(\mathbf{1}_{\{|\xi|\le 2\,R\}} \, \frac{|\xi|}{R} \, \mathbf{1}_{\{R\le |x| \le 2\,R\}} \\ &\qquad\qquad + \mathbf{1}_{\{|x|\le 2\,R\}} \, \frac{|E|}{R} \, \mathbf{1}_{\{R\le |\xi|\le 2\,R\}}\big) \, dx d\xi dt \underset{R\to\infty}{\longrightarrow} 0, \end{aligned}
$$

thanks to Lebesgue theorem. We deduce that $\iiint_D \beta(f)\,\phi\,d\xi dx dt = 0$ for all $\phi \in \mathcal{D}(D)$, which implies $g_2 \equiv g_1$.

For the continuity result, first remark that thanks to Theorem 1 we already have

$$
(4.10)\qquad g \in C([0,T]; L^1_{loc}(\bar{\mathcal{O}})).
$$

Moreover, for all $\beta \in W^{1,\infty}(\mathbb{R})$ and for all $\zeta \in C_b(\bar{\mathcal{O}})$, radial in the ξ variable (i.e. $\zeta(x,\xi') = \zeta(x,\xi)$ if $|\xi'| = |\xi|$), the following identity is satisfied

$$
(4.11)\qquad \frac{d}{dt} \iint_{\mathcal{O}} \beta(g)\,\zeta\,d\xi dx = \iint_{\mathcal{O}} \big(\beta'(g)\,G\,\zeta + \beta(g)\,\Lambda_E \zeta\big)\,d\xi dx.
$$

In order to establish (4.11), we just have to write the Green formula (1.6) with $\beta(g)$ and $\zeta\,\mathcal{X}_R$, and to pass to the limit $R \to \infty$ using an estimate like (4.9). Taking $\zeta = 1$, we get

$$
\iint_{\mathcal{O}} \beta(g(t,.))\,d\xi dx = \iint_{\mathcal{O}} \beta(g_0)\,d\xi dx + \int_0^t \iint_{\mathcal{O}} \beta'(g)\,G\,d\xi dx \in C([0,T]).
$$

When $p > 1$ we let increase β to $|.|^p$, and thus $\|g(t,.)\|_{L^p} \in C([0,T])$, which is enough to conclude in view of (4.10). When $p = 1$, we let increase β to a stricly superlinear fonction β_0, constructed as in the second step of Theorem 4, and we deduce

$$
(4.12)\qquad \sup_{t\in[0,T]} \iint_{\mathcal{O}} \beta_0(g(t,x,\xi))\,d\xi dx < \infty.
$$

We write (4.11) with $\beta(z) = |z| \wedge M$ and $\zeta(\xi) = \zeta_R(|\xi|)$, $\zeta_R = \zeta(\frac{\cdot}{R})$, $R \ge 1$, $\zeta \in C_b(\mathbb{R}_+)$, $\zeta \equiv 0$ on $[0,1]$ and $\zeta \equiv 1$ on $[2,\infty)$, and we obtain

$$
\frac{d}{dt} \iint_{\mathcal{O}} |g| \wedge M\,\zeta_R\,d\xi dx = \iint_{\mathcal{O}} \big(\mathbf{1}_{|g|\le M}\,G\zeta_R + |g| \wedge M\,\Lambda_E \zeta_R\big)\,d\xi dx.
$$

Using again the estimate made in (4.9), we deduce

$$\sup_{t\in[0,T]} \iint_{\mathcal{O}} |g(t,.)| \wedge M\, \mathbf{1}_{|(x,\xi)|\geq R}\, d\xi dx \underset{R\to\infty}{\longrightarrow} 0. \tag{4.13}$$

Therefore, from (4.10), (4.12) and (4.13) we deduce that $g \in C([0,T]; L^1(\mathcal{O}))$.
□

5. Trace theorem with optimal weight and resolution of the Dirichlet problem.

We begin with an existence result for the Vlasov equation (1.1) which is an extension of the result of C. Bardos [3] to the case of a force field with Sobolev regularity. Our proof follows the method of R. DiPerna and P.-L. Lions [8].

Lemma 3. *Assume $E \in L^1(0,T; W^{1,1}_{loc}(\bar{\Omega}))$. For given $\Phi \in C_b(\bar{D})$, $\phi_- \in C_b(\Gamma_-)$ (resp. $\phi_+ \in C_b(\Gamma_+)$) and $\phi_0 \in C_b(\bar{\mathcal{O}})$ (resp. $\phi_T \in C_b(\bar{\mathcal{O}})$), there exists a solution $\phi \in L^\infty(D) \cap C([0,T]; L^1_{loc}(\bar{\mathcal{O}}))$ to*

$$(5.1)\ \begin{cases} \Lambda_E\phi = \Phi \ \ \textit{in } \mathcal{D}'(D), \\ \gamma_-\phi = \phi_-, \\ \phi(0) = \phi_0, \end{cases} \qquad \left(\textit{resp.}\ (5.1')\ \begin{cases} \Lambda_E\phi = \Phi \ \ \textit{in } \mathcal{D}'(D), \\ \gamma_+\phi = \phi_+, \\ \phi(T) = \phi_T \end{cases}\right).$$

Furthermore, the solution ϕ of (5.1) satisfies the estimate

$$\|\phi\|_{L^\infty(D)} \leq \mathit{Max}(\|\phi_0\|_{L^\infty(\mathcal{O})}, \|\phi_-\|_{L^\infty(\Gamma_-)}, T\,\|\Phi\|_{L^\infty(D)}), \tag{5.2}$$

and if $\phi_0 \geq 0$, $\phi_- \geq 0$ and $\Phi \geq 0$, then $\phi \geq 0$ in D.
We also have that the solution ϕ of (5.1′) satisfies

$$\|\phi\|_{L^\infty(D)} \leq \mathit{Max}(\|\phi_T\|_{L^\infty(\mathcal{O})}, \|\phi_+\|_{L^\infty(\Gamma_+)}, T\,\|\Phi\|_{L^\infty(D)}), \tag{5.2'}$$

and if $\phi_T \geq 0$, $\phi_+ \geq 0$ and $\Phi \leq 0$, then $\phi \geq 0$ in D.

Proof. We only deal with problem (5.1), since the proof in the case of problem (5.1′) can be performed in the same way. For a smooth field E, C. Bardos in [3] solves the problem using a characteristic and semi-group method. The solution, that he constructs, satisfies the bound (5.2) and the positivity property.

When $E \in L^1(0,T;W^{1,1}_{loc}(\bar{\Omega}))$, we consider a sequence $E_\nu \in C^1([0,T]\times\bar{\Omega})$, such that $E_\nu \to E$ in $L^1(0,T;L^1_{loc}(\bar{\Omega}))$, and we note ϕ_ν the corresponding solution to (5.1); for which estimate (5.2) holds uniformly in ν. Up to the extraction of a subsequence, ϕ_ν converges to a function ϕ in $L^\infty(D)$, which satisfies (5.2). Proposition 4 implies that ϕ solves the Dirichlet problem. Continuity follows from Theorem 2, and the positivity of ϕ is deduced from the positivity of ϕ_ν. □

Lemma 4. (Uniqueness). *Let a, $p \in [1,\infty]$. Let $E \in L^{a'}(0,T;W^{1,p'}_{loc}(\bar{\Omega}))$ satisfying (1.9) and $g \in L^a(0,T;L^p_{loc}(\bar{\mathcal{O}}))$ a solution of*

$$(5.3)\quad \begin{cases} \Lambda_E g = 0 \quad \textit{in } \mathcal{D}'(D), \\ \gamma_- g = 0, \\ g(0) = 0, \end{cases} \qquad \left(\textit{resp. } (5.3')\quad \begin{cases} \Lambda_E g = 0 \quad \textit{in } \mathcal{D}'(D), \\ \gamma_+ g = 0, \\ g(T) = 0 \end{cases}\right),$$

then $g \equiv 0$.

Proof. Again, we only treat the case of equation (5.3). One fixes $\beta \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ such that $\beta(0) = 0$ and $\beta(s) > 0$ if $s \neq 0$, in such a way that $\beta(g) \in L^\infty(D)$ is still a solution of (5.3). For all $\varphi \in \mathcal{D}(D)$, we solve thanks to lemma 3 the backward problem

$$\begin{cases} \Lambda_E \phi = \varphi \quad \text{in } \mathcal{D}'(D), \\ \gamma_+ \phi = 0, \\ \phi(T) = 0. \end{cases}$$

We take $\mathcal{X}_R \in \mathcal{D}(\bar{\mathcal{O}})$ as in the proof of Theorem 3, and using the duality formula (3.3), we get

$$\iiint_D \beta(g)\, \varphi\, \mathcal{X}_R \, dx d\xi dt + \iiint_D \beta(g)\, \phi\, \Lambda_E \mathcal{X}_R \, dx d\xi dt = 0.$$

We let R tend to ∞ and we obtain $\iiint_D \beta(g)\, \varphi \, dx d\xi dt = 0$, $\forall \varphi \in \mathcal{D}(D)$. We deduce that $\beta(g) = 0$ and thus $g = 0$. □

We note τ_E^- and τ_E^+ the solutions of

$$(5.4)\quad \begin{cases} \Lambda_E \tau_E^- = 1 \ \text{ in } \mathcal{D}'(D), \\ \tau_E^-\big|_{\Gamma_-} = 0, \\ \tau_E^-(0) = 0, \end{cases} \qquad \text{and} \qquad (5.4')\quad \begin{cases} \Lambda_E \tau_E^+ = -1 \ \text{ in } \mathcal{D}'(D), \\ \tau_E^+\big|_{\Gamma_+} = 0, \\ \tau_E^+(T) = 0. \end{cases}$$

Thanks to lemma 3, one has $0 \leq \tau_E^-, \tau_E^+ \leq T$. We set $\tau_E = \tau_E^+ + \tau_E^-$, and then $\tau_E(t,x,\xi)$ is the "time of life in D" of a particle which at time t has position x and velocity ξ.

Theorem 5 (Optimal weight). *Let $p \in [1,\infty)$, $a \in [p,\infty]$ and $b \in [1,\infty]$, with $a \geq (p-1)\,b'$ if $p > 1$. Let $E \in L^{a'}(0,T;W^{1,p'}_{loc}(\bar\Omega))$, $G \in L^b(0,T;L^p(\mathcal{O}))$ and $g \in L^a(0,T;L^p(\mathcal{O}))$ a solution to (1.1). Then the trace γg satisfies*

$$\gamma\, g \in L^p\big([0,T]\times\Sigma, |n(x)\cdot\xi|\,\tau_E(t,x,\xi)\,d\xi d\sigma_x dt\big).$$

Proof. One fixes $\beta_M(z) = (|z|\wedge M)^p$. With the notations of Theorem 1, we have

$$\Lambda_E(\beta_M(g_k)\,\tau_E^+) = \beta_M'(g_k)\,(G_k + r_k)\,\tau_E^+ - \beta_M(g_k) \quad \text{in } \mathcal{D}'(D).$$

Theorem 2 and Green formula (1.6) imply

$$-\iint_{\mathcal{O}} \beta_M(g_k(0))\,\tau_E^+(0)\,d\xi dx + \iiint_{\Gamma_-} \gamma\beta_M(g_k)\,\tau_E^+\,n(x)\cdot\xi\,d\xi d\sigma_x dt = \\ = \iiint_D \big\{\beta_M'(g_k)\,(G_k + r_k)\,\tau_E^+ - \beta_M(g_k)\big\}\,d\xi dx dt.$$

We pass to the limit $k \to \infty$ and we get

$$\iint_{\mathcal{O}} \beta_M(g(0))\,\tau_E^+(0)\,d\xi dx + \iiint_{\Gamma_-} \beta_M(\gamma g)\,\tau_E^+\,|n(x)\cdot\xi|\,d\xi d\sigma_x dt = \\ -\iiint_D \big\{\beta_M'(g)\,G\,\tau_E^+ - \beta_M(g)\big\}\,d\xi dx dt \leq C_T\,\|g\|_{L^{a,p}}^{p-1}\,\big\{\|g\|_{L^{a,p}} + \|G\|_{L^{b,p}}\big\}.$$

It is then enough to pass for second time to the limit $M \to \infty$ in order to obtain

$$\gamma_-\, g \in L^p(\Gamma_-, |n(x)\cdot\xi|\,\tau_E(t,x,\xi)\,d\xi d\sigma_x dt).$$

In a very same way, we prove $\gamma_+ g \in L^p(\Gamma_+, |n(x)\cdot\xi|\,\tau_E(t,x,\xi)\,d\xi d\sigma_x dt)$. $\square$

Theorem 6 (Dirichlet problem). *Let $E \in L^1(0,T;W^{1,p'}_{loc}(\bar\Omega))$ satisfy (1.9). For given $g_- \in L^p(\Gamma_-, |n(x)\cdot\xi|\,\tau_E(t,x,\xi)\,d\xi d\sigma_x dt)$, $g_0 \in L^p(\mathcal{O})$ and $G \in L^p(D)$, there exists an unique solution $g \in L^p(D)$ of (1.1) such that $\gamma_- g = g_-$ and $g(0) = g_0$.*

Proof. We consider some sequences (g_-^ε), (g_0^ε) and (G^ε) of approximations of g_-, g_0 and G respectively, such that for all $\varepsilon > 0$ one has $g_-^\varepsilon \in C_c(\Gamma_-)$, $g_0^\varepsilon \in C_c(\mathcal{O})$ and $G^\varepsilon \in C_c(D)$, in such a way that thanks to lemma 3, there is a sequence of solutions $g^\varepsilon \in L^\infty(\bar D)$ to

$$\begin{cases} \Lambda_E g^\varepsilon = G^\varepsilon \ \text{ in } \ \mathcal{D}'(D), \\ \quad g^\varepsilon = g_-^\varepsilon \ \text{ on } \ \Gamma_-, \\ g^\varepsilon(0) = g_0^\varepsilon \ \text{ in } \ \mathcal{O}. \end{cases} \tag{5.5}$$

For $p > 1$ we fix $\beta(y) = |y|^p$, and for $p = 1$ we take β given by Lemma 2 and such that

$$\iint_{\mathcal{O}} \beta(g_0^\varepsilon)\,dxd\xi, \quad \iiint_{\Gamma_-} \beta(g_-^\varepsilon)\,\tau_E\,d\mu_1, \quad \iiint_D \beta(G^\varepsilon)\,dxd\xi dt \le C < \infty.$$

The function $\beta(g^\varepsilon)\tau_E^+$ belongs to $L^\infty(D)$ and satisfies

$$\Lambda_E\left(\beta(g^\varepsilon)\,\tau_E^+\right) = \beta'(g^\varepsilon)\,G^\varepsilon\,\tau_E^+ - \beta(g^\varepsilon).$$

Using the Green formula (1.6) with $\mathcal{X}_R \in \mathcal{D}(\bar D)$ defined in Theorem 3, we have

$$\iiint_\Gamma \beta(g^\varepsilon)\,\tau_E^+\,\mathcal{X}_R\,n(x)\cdot\xi\,d\xi d\sigma_x dt + \Big[\iint_{\mathcal{O}} \beta(g^\varepsilon)\,\tau_E^+\,\mathcal{X}_R\,d\xi dx\Big]_0^T =$$
$$= \iiint_D \big(\beta'(g^\varepsilon)\,G^\varepsilon\,\tau_E^+ - \beta(g^\varepsilon)\big)\,\mathcal{X}_R\,d\xi dx dt.$$

Recalling that $\tau_E^+ = 0$ on Γ_+ and in $t = T$, $0 \le \tau_E^+ \le T$, and letting $R \to \infty$, we deduce

$$\iiint_D \beta(g^\varepsilon)\,d\xi dx dt = \iiint_{\Gamma_-} \beta(g^\varepsilon)\,\tau_E^+\,|n(x)\cdot\xi|\,d\xi d\sigma_x dt$$
$$+ \iint_{\mathcal{O}} \beta(g_0^\varepsilon)\,\tau_E^+\,d\xi dx + \iiint_D \beta'(g^\varepsilon)\,G^\varepsilon\,\tau_E^+\,d\xi dx dt.$$

For $p = 1$ we use the inequality $\beta'(y)\, z \le \dfrac{\beta(y)}{2T} + \dfrac{\beta(4T\, z)}{2T}$ and for $p > 1$ the inequality $\beta'(y)\, z \le \dfrac{\beta(y)}{2T} + C_T\, \beta(z)$, and we obtain the following a priori estimate on g^ε

$$\iiint_D \beta(g^\varepsilon)\, d\xi dx dt \le \iiint_{\Gamma_-} \beta(g^\varepsilon_-)\ \tau_E^+ |n(x)\cdot\xi|\, d\xi d\sigma_x dt \\ + C_T \left\{ \iint_{\mathcal{O}} \beta(g_0^\varepsilon)\, d\xi dx + \iiint_D \beta(G^\varepsilon)\, d\xi dx dt \right\}.$$

Therefore, up to the extraction of a sub-sequence, g^ε converges weakly to a function $g \in L^p(D)$. We pass to the limit $\varepsilon \to 0$ in equation (5.5) using Proposition 4. This ends the existence proof. Uniqueness follows from lemma 4. □

Last, we present two situations where the weight (1.14) can be obtained.

Proposition 5. *We assume in addition to the assumptions made in Proposition 2 that* $\gamma_+ g \in L^r_{loc}(\Gamma_+, d\mu_1)$ *(resp.* $\gamma_- g \in L^r_{loc}(\Gamma_-, d\mu_1)$*). Therefore, we have*

$$\gamma_- g \in L^r_{loc}(\Gamma_-, d\mu_1) \quad (\textit{resp. } \gamma_+ g \in L^r_{loc}(\Gamma_+, d\mu_1)\,).$$

Proposition 6. *Let* $S \subset \partial\Omega$*. We assume in addition to the assumptions made in Proposition 2 that in a neighborhood* ω *of* S*,* $\omega \subset \Omega$ *and* $S \subset\subset \partial\omega \cap \partial\Omega$ *we have* $(\star)$ $\forall (t,x) \in (0,T)\times\omega$ $D^2 d_\Omega(x) \le 0$ *and* $E(t,x)\cdot n(x) \le 0$*, (this is for instance the case for* $S = \partial\Omega$ *if* $E = 0$*, and* Ω *is the exterior of a closed convex set), then*

$$\gamma\, g \in L^r_{loc}([0,T]\times S \times \mathbb{R}^N, d\mu_1).$$

Idea of the proof of Propositions 5 and 6. Once again, we just establish formally two a priori estimates that easily permit to conclude with the help of the regularization procedure presented in Theorem 1. We begin with Proposition 6. For a given $R > 0$, we fix $\varphi \in \mathcal{D}(\mathbb{R}^N \times \mathbb{R}^N)$ in such a way that $0 \le \varphi \le 1$ in $\mathcal{O}$, $\varphi \equiv 1$ on $S \times B_R$ and $\operatorname{supp}\varphi \subset B_{R+1}\times B_{R+1}$. We write the identity (2.1) with $\beta = |.|^r$, $t_0 = 0$, $t_1 = T$, φ and $\psi = \psi_\sigma$, where

$\psi'_\sigma(y) = \dfrac{1}{\sigma}\rho(\dfrac{y}{\sigma})$, $\rho \in \mathcal{D}_+(\mathbb{R})$, $\displaystyle\int_{\mathbb{R}} \rho(y)\,dy = 2$, in such a way that $\psi_\sigma(y)$ tends to the sign function $\mathrm{sign}(y)$ when $\sigma \to 0$. We remark that the term multiplied to ψ' in (2.1) is non positive thanks to assumption $(\star)$, and therefore, letting σ go to 0 we get

$$\Big(\int_0^T \iint_{S\times B_R} |g|^r\, d\mu_1(t,x,\xi)\Big)^{1/r} \le C\,(1 + \|E\|^{1/r}_{L^{1,q}_{R+1}})\|g\|_{L^{\infty,p}_{R+1}} + \|G\|^{1/r}_{L^{1,s}_{R+1}}.$$

For Proposition 5, we just treat the case $\gamma_+\, g \in L^r_{loc}(\Gamma_+, (n(x)\cdot\xi)\,d\xi d\sigma_x dt)$. We write the identity (2.1) with $\beta = |.|^r$, $t_0 = 0$, $t_1 = T$, $\psi = 1$ and $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^N\times\mathbb{R}^N)$, with $0 \le \varphi \le 1$ in D, $\varphi \equiv 1$ on $K^-_R = [1/R, T-1/R]\times(\Sigma_-\cap\Sigma_R)$ and $\mathrm{supp}\,\varphi \subset (0,T)\times B_{R+1}\times B_{R+1}$, for a given $R > 1/(2T)$ and we get

$$\iiint_{\Gamma_-} |\gamma_-\, g|^r\,\varphi(t,x,\xi)\,d\mu_1(t,x,\xi) = -\iiint_D \big(|g|^p\,\Lambda_E\,\varphi$$
$$+ \mathrm{sign}\, g\,|g|^{p-1}\,G\,\varphi\big)\,d\xi dx dt + \iiint_{\Gamma_+} |\gamma_+\, g|^p\,\varphi(t,x,\xi)\,(n(x)\cdot\xi)\,d\xi d\sigma_x dt.$$

We deduce

$$\|\gamma_-\, g\|_{L^r(K^-_R, d\mu_1)} \le C_R(1+\|E\|^{1/r}_{L^{1,q}_{R+1}})\|g\|_{L^{\infty,p}_{R+1}} + \|G\|_{L^{1,s}_{R+1}} + \|\gamma_+\, g\|_{L^r(\Gamma^+_{R+1}, d\mu_1)},$$

with the notation $\Gamma^+_{R+1} = \Gamma_+\cap([0,T]\times\Sigma_{R+1})$. □

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