Spectral analysis of semigroups in Banach spaces and applications to PDEs

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works by K. Carrapatoso, I. Tristani

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Outline of the talk

1. Introduction

2. Examples of linear evolution PDE
   - Gallery of examples
   - Hypodissipativity result under weak positivity
   - Hypodissipativity result in large space

3. Nonlinear problems
   - Increasing the rate of convergence
   - Perturbation regime

4. Spectral theory in an abstract setting

5. Elements of proofs
   - The enlargement theorem
   - The spectral mapping theorem
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Spectral theory for general operator and its semigroup in general (large) Banach space, without regularity (≠ eventually norm continuous), without symmetry (≠ Hilbert space and self-adjoint op) and without (or with) positivity (Banach lattice)

- **Spectral map Theorem** $\iff \Sigma(e^{t\Lambda}) \simeq e^{t\Sigma(\Lambda)}$ and $\omega(\Lambda) = s(\Lambda)$
- **Weyl’s Theorem** $\iff$ (quantified) compact perturbation $\Sigma_{\text{ess}}(A + B) \simeq \Sigma_{\text{ess}}(B)$
- **Small perturbation** $\iff$ $\Sigma(\Lambda_\varepsilon) \simeq \Sigma(\Lambda)$ if $\Lambda_\varepsilon \to \Lambda$
- **Krein-Rutmann Theorem** $\iff$ $s(\Lambda) = \sup \Re \Sigma(\Lambda) \in \Sigma_d(\Lambda)$ when $S_\Lambda \geq 0$
- **Functional space extension (enlargement and shrinkage)** $\iff$ $\Sigma(L) \simeq \Sigma(L)$ when $L = L|_E$

**Structure:** operator which splits as

$$\Lambda = A + B, \quad A \prec B, \quad B \text{ dissipative}$$

**Examples:** Boltzmann, Fokker-Planck, Growth-Fragmentation operators and $W^{\sigma,p}(m)$ weighted Sobolev spaces
Applications / Motivations:

• (1) Convergence rate in large Banach space for linear dissipative and hypodissipative PDEs (ex: Fokker-Planck, growth-fragmentation)
• (2) Long time asymptotic for nonlinear PDEs via the spectral analysis of linearized PDEs (ex: Boltzmann, Landau, Keller-Segel) in natural $\varphi$ space
• (3) Existence, uniqueness and stability of equilibrium in “small perturbation regime” in large space (ex: inelastic Boltzmann, Wigner-Fokker-Planck, parabolic-parabolic Keller-Segel, neural network)

Is it new?

• Simple and quantified versions, unified theory (sectorial, KR, general) which holds for the “principal” part of the spectrum
• First enlargement result in an abstract framework by C. Mouhot (CMP06)
• Unusual splitting

$$\Lambda = \underbrace{A_0}_{\text{compact}} + \underbrace{B_0}_{\text{dissipative}} = \underbrace{A_\varepsilon}_{\text{smooth}} + \underbrace{A^c_\varepsilon}_{\text{dissipative}} + B_0$$

• The applications to these linear(ized) “kinetic” equations and to these nonlinear problems are clearly new
Old problems

- Fredholm, Hilbert, Weyl, Stone (Funct Analysis & sG Hilbert framework) ≤ 1932
- Hyle, Yosida, Phillips, Lumer, Dyson (sG Banach framework & dissipative operators) 1940-1960 and also Dunford, Schwartz
- Kato, Pazy, Voigt (analytic op., positive op.) 1960-1975
- Engel, Nagel, Gearhart, Metz, Diekmann, Prüss, Arendt, Greiner, Blake, Mokhtar-Kharoubi, Yao, ... 1975-

Spectral analysis of the linearized (in)homogeneous Boltzmann equation and convergence to the equilibrium

- Hilbert, Carleman, Grad, Ukai, Arkeryd, Esposito, Pulvirenti, Wennberg, Guo, Strain, ...

Spectral tide/spectral analysis in large space

- Bobylev (for linearized Boltzmann with Maxwell molecules, 1975), Gallay-Wayne (for harmonic Fokker-Planck, 2002)
Still active research field

• **Semigroup school (≥ 0, bio):** Arendt, Blake, Diekmann, Engel, Gearhart, Greiner, Metz, Mokhtar-Kharoubi, Nagel, Prüss, Webb, Yao, ...

• **Schrodinger school / hypocoercivity and fluid mechanic:** Batty, Burq, Duyckaerts, Gallay, Helffer, Hérau, Lebeau, Nier, Sjöstrand, Wayne, ...

• **Probability school (as in Toulouse):** Bakry, Barthe, Bobkov, Cattiaux, Douc, Gozlan, Guillin, Fort, Ledoux, Roberto, Röckner, Wang, ...

• **Kinetic school (∼ Boltzmann):**
  ▶ Carlen, Carvalho, Toscani, Otto, Villani, ... *(log-Sobolev inequality)*
  ▶ Desvillettes, Villani, Mouhot, Baranger, Neuman, Strain, Dolbeault, Schmeiser, ... *(Poincaré inequality & hypocoercivity)*
  ▶ Guo school related to Ukai, Arkeryd, Esposito, Pulvirenti, Wennberg, ... *(existence in “small spaces” and “large spaces”)*
A list of related papers


- Gualdani, M., Mouhot, *Factorization for non-symmetric operators and exponential H-Theorem*, arXiv 2010


- Cañizo, Caceres, M., *Rate of convergence to the remarkable state for fragmentation and growth-fragmentation equations*, JMPA 2011 & CAIM 2011

- Egaña, M. *Uniqueness and long time asymptotic for the Keller-Segel equation - Part I. The parabolic-elliptic case*, arXiv 2013

- M., Mouhot, *Exponential stability of slowing decaying solutions to the kinetic Fokker-Planck equation*, in progress

- M., Scher *Spectral analysis of semigroups and growth-fragmentation eqs*, arXiv 2013

- Carrapatoso, *Exponential convergence ... homogeneous Landau equation*, arXiv 2013

- Tristani, *Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting*, arXiv 2013
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1) - Linear Boltzmann, e.g. $k(v, v_*) = \sigma(v, v_*) M(v_*)$, $\sigma(v_*, v) = \sigma(v, v_*)$,

$$\Lambda f = \int k(v, v_*) f(v_*) \, dv_* - \int k(v_*, v) \, dv_* \, f(v)$$

$$= : A f$$

$$= : B f$$

2) - Fokker-Planck, with $E(v) \approx v |v|^{\gamma - 2}$, $\gamma \geq 1$,

$$\Lambda = \Delta_v + \text{div}_v(E(v) \cdot) - M \chi_R + M \chi_R(v)$$

$$= : A$$

3) - Inhomogeneous/kinetic Fokker-Planck

$$\Lambda = \mathcal{T} + C - M \chi_R + M \chi_R(x, v)$$

$$= : A$$

with

$$\mathcal{T} := -v \cdot \nabla_x + F \cdot \nabla_x, \quad Cf := \Delta_v f + \text{div}_v(E(v) \, f)$$
4) - Growth fragmentation

$$\Lambda = \mathcal{F}^+ - \mathcal{F}^- + \mathcal{D} = \mathcal{F}^+\delta + \mathcal{F}^+\mathcal{c} - \mathcal{F}^- + \mathcal{D}$$

$$=: A \quad =: B$$

with

$$\mathcal{D}f = -\tau(x) \partial_x f - \nu f, \quad (\tau, \nu) = (1, 0) \text{ or } (x, 2)$$

$$\mathcal{F}^+(f) := \int_{-\infty}^{\infty} k(y, x) f(y) \, dy, \quad \mathcal{F}^- f := K(x) f$$

Mass conservation of $\mathcal{F}^+ - \mathcal{F}^-$ implies

$$K(x) = \int_0^x \frac{y}{x} k(x, y) \, dy$$

Self-similarity in $y/x$

$$k(x, y) = K(x) x^{-1} \theta(y/x), \quad \int_0^1 z \theta(z) \, dz = 1,$$

with

$$\theta \in \mathcal{D}(0, 1) \quad \text{or} \quad \theta(z) = 2 \delta_{z=1/2} \quad \text{or} \quad \theta(z) = \delta_{z=0} + \delta_{z=1}$$
Examples of operators - III

5) - Linearized Boltzmann

\[ \Lambda h = Q(h, M) + Q(M, h) \]
\[ = Q^+(h, M) + Q^+(M, h) - L(h) M - L(M) h \]
\[ = \underbrace{Q^+_\delta, \ast [h]}_{=: A h} + \underbrace{Q^+_\delta, \ast, c [h] - L(M) h}_{=: B h} \]

6) - Inhomogeneous linearized Boltzmann (in the torus)

\[ \Lambda h = Q^+_\delta, \ast [h] + Q^+_\delta, \ast, c [h] - L(M) h + T h, \quad T := -v \cdot \nabla_x \]

7) - other operators: homogeneous/inhomogeneous linearized inelastic Boltzmann, homogeneous linearized Landau, Fokker-Planck with fractional diffusion, linearized Keller-Segel (parabolic-elliptic), homogeneous Boltzmann for hard potential without angular cut-off
The Growth-Fragmentation equation (as an application of the KR theorem)

**Th 1.** (M., Scher)

Assume that for \( \gamma \geq 0, x_0 \geq 0, 0 < K_0 \leq K_1 < \infty \):

\[
K_0 x^\gamma 1_{x \geq x_0} \leq K(x) \leq K_1 x^\gamma.
\]

There exists a (unique) \((\lambda, f_\infty)\) with \(\lambda \in \mathbb{R}\) and \(f_\infty\) is the unique solution to

\[
\mathcal{F} f_\infty + D f_\infty = \lambda f_\infty, \quad f_\infty \geq 0, \quad \langle f_\infty, 1 \rangle = 1.
\]

There exists \(a < \lambda, C > 0\) such that \(\forall f_0 \in L^1_\alpha, \alpha > 1\)

\[
\|f e^{\lambda t} f_0 - e^{\lambda t} \Pi_0 f_0\|_{L^1_\alpha} \leq C e^{at} \|f_0 - e^{\lambda t} \Pi_0 f_0\|_{L^1_\alpha},
\]

where \(\Pi_0\) is the projector on the eigenspace \(\text{Vect}(f_\infty)\).

The Fokker-Planck equation (as a consequence of extension or KR theorems)

Consider

\[ \partial_t f = \Lambda f = \Delta_v f + \text{div}_v (F f) \]

with a (friction) force field \( F \) such that

\[ F \cdot x \geq |x|^{\gamma}, \quad \text{div} F \leq C_F |x|^{\gamma - 2}, \quad \forall x \in B_R^c \]

**Th 2.** Gualdani-M.-Mouhot; M.-Mouhot; Ndao

There exists a unique positive and unit mass stationary solution \( f_\infty \), and for any \( \sigma \in \{-1, 0, 1\} \), \( p \in [1, \infty] \), any \( m = \langle v \rangle^k \), \( k > k^*(p, \sigma, \gamma) \) or \( m = e^{\kappa \langle v \rangle^s} \), \( s \in [2 - \gamma, \gamma] \), \( \gamma \geq 1, \ \kappa < 1/\gamma \) if \( s = \gamma \), any \( a \in (a^*_\sigma(p, m), 0) \), there exists \( C = C(a, p, \sigma, \gamma, m) \) such that for any \( f_0 \in W^{\sigma, p}(m) \)

\[ \| e^{t\Lambda} f_0 - \langle f_0 \rangle f_\infty \|_{W^{\sigma, p}(m)} \leq C e^{\alpha t} \| f_0 - \langle f_0 \rangle f_\infty \|_{W^{\sigma, p}(m)}. \]

- Generalizes similar results known in \( L^2(f_\infty^{-1/2}) \)
- The same result holds for the kinetic Fokker-Planck in the torus and in \( \mathbb{R}_x^d \) with confinement potential
- Provides decay in Wasserstein distance (see also Bolley-Gentil-Guillin (2012))
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Conditionally (up to time uniform strong estimate) exponential $H$-Theorem

- $(f_t)_{t \geq 0}$ solution to the inhomogeneous Boltzmann equation for hard spheres interactions in the torus with strong estimate

$$\sup_{t \geq 0} (\|f_t\|_{H^k} + \|f_t\|_{L^1(1+|v|^s)}) \leq C_{s,k} < \infty.$$

- Desvillettes, Villani proved [Invent. Math. 2005]: for any $s \geq s_0$, $k \geq k_0$

$$\forall t \geq 0 \int_{T \times \mathbb{R}^d} f_t \log \frac{f_t}{G_1(v)} \, dvdx \leq C_{s,k} (1 + t)^{-\tau_{s,k}}$$

with $C_{s,k} < \infty$, $\tau_{s,k} \to \infty$ when $s, k \to \infty$, $G_1 :=$ Maxwell function

**Th 3. Gualdani-M.-Mouhot**

$\exists s_1, k_1$ s.t. for any $a > \lambda_2$ exists $C_a$

$$\forall t \geq 0 \int_{T \times \mathbb{R}^d} f_t \log \frac{f_t}{G_1(v)} \, dvdx \leq C_a e^{\frac{a}{2} t},$$

with $\lambda_2 < 0$ ($2^{nd}$ eigenvalue of the linearized Boltzmann eq. in $L^2(G_1^{-1})$).
Global existence and uniqueness for weakly inhomogeneous initial data for the elastic and inelastic inhomogeneous Boltzmann equation for hard spheres interactions in the torus

**Th 4. Gualdani-M.-Mouhot; Tristani**

For any $F_0 \in L^1_3(\mathbb{R}^d)$ there exists $e_0 \in (0, 1)$ and $\varepsilon_0 > 0$ such that if $f_0 \in W^{k,1}_t(T^d; L^1_3(\mathbb{R}^d))$ satisfies $\|f_0 - F_0\| \leq \varepsilon_0$ and if $e \in [e_0, 1]$ then

- there exists a unique global mild solution $f(t, x, v)$ starting from $f_0$;
- $f(t) \to G_1$ when $t \to \infty$ (with rate) when $e = 1$;
- $f(t) \to \tilde{G}_e$ when $t \to \infty$ (with rate) when $e < 1$ (diffuse forcing).

- The case $e \sim 1$ is proved thanks to a small perturbation argument in a large space because $\tilde{G}_e(v) \geq e^{-|v|^{3/2}} \notin L^2(G_1^{-1/2})$.

- The case $e = 1$ has been treated by non constructive arguments by Arkeryd-Esposito-Pulvirenti (CMP 1987), Wennberg (Nonlinear Anal. 1993) and for the space homogeneous analogous by Arkeryd (ARMA 1988), Wennberg (Adv. MAS 1992)

- Extend to a larger class of initial data similar results due to Ukai, Guo, Strain and collaborators
More results about constructive exponential rate of convergence

For
- homogeneous Boltzmann eq for hard spheres (Mouhot 2006)
- homogeneous weakly inelastic Boltzmann eq for hard spheres (M-Mouhot 2009)
- homogeneous Landau eq for hard potential (Carrapatoso 2013)
- parabolic-elliptic Keller-Segel eq (Egaña-M 2013)
- homogeneous Boltzmann eq for hard potential (Tristani, soon on arXiv)

In all these cases, we prove that under minimal assumptions on the initial datum $f_0$ (bounded mass, energy, entropy, ...) the associated solution $f(t)$ satisfies

$$f(t) \to G \text{ when } t \to \infty \text{ (with exponential rate)}$$

where $G$ is the unique associated equilibrium/self-similar profile

We know (except for the inelastic Boltzmann eq) that the associated linearized operator $\mathcal{L}$ is self-adjoint and has a spectral gap in the very small space $L^2(G^{-1/2}_1)$ in which a general solution does not belong (even for large time).

▷ we start by “enlarge” the space in which $\mathcal{L}$ has a spectral gap and then we (classically) prove a nonlinear stability result

▷ for the weakly inelastic Boltzmann eq we additionally use perturbation argument
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Main issue

For a given operator $\Lambda$ in a Banach space $X$, we want to prove

\[(1)\; \Sigma(\Lambda) \cap \Delta_a = \{\xi_1\}, \quad \xi_1 = 0\]

with $\Sigma(\Lambda) =$ spectrum, $\Delta_\alpha := \{z \in \mathbb{C}, \Re z > \alpha\}$

\[(2)\; \Pi_{\Lambda, \xi_1} = \text{finite rank projection, i.e. } \xi_1 \in \Sigma_d(\Lambda)\]

\[(3)\; \|S_\Lambda(I - \Pi_{\Lambda, \xi_1})\|_{X \to X} \leq C_a e^{at}, \quad a < \Re \xi_1\]

Definition: We say that $L - a$ is hypodissipative iff $\|e^{tL}\|_{X \to X} \leq C e^{at}$. 
Th 1. (M., Scher)

(0) \( \Lambda = A + B \), where \( A \) is \( B^{\zeta'} \)-bounded with \( 0 \leq \zeta' < 1 \),

(1) \( \| S_B \ast (AS_B)^{(\ast \ell)} \|_{X \rightarrow X} \leq C_\ell e^{at}, \ \forall \ a > a^*, \ \forall \ \ell \geq 0 \),

(2) \( \| S_B \ast (AS_B)^{(\ast n)} \|_{X \rightarrow D(\Lambda^{\zeta})} \leq C_n e^{at}, \ \forall \ a > a^*, \ \text{with} \ \zeta > \zeta' \),

(3) \( \Sigma(\Lambda) \cap (\Delta_{a^{**}} \setminus \Delta_{a^*}) = \emptyset, \ a^* < a^{**} \),

is equivalent to

(4) there exists a projector \( \Pi \) which commutes with \( \Lambda \) such that

\[ \Lambda_1 := \Lambda|_{X_1} \in \mathcal{B}(X_1), \ X_1 := R\Pi, \ \Sigma(\Lambda_1) \subset \Delta_{a^*} \]

\[ \| S_{\Lambda}(t) (I - \Pi) \|_{X \rightarrow X} \leq C_a e^{at}, \ \forall \ a > a^* \]

In particular

\[ \Sigma(e^{t \Lambda}) \cap \Delta_{e^{at}} = e^{t \Sigma(\Lambda) \cap \Delta_a} \quad \forall \ t \geq 0, \ a > a^* \]

and

\[ \max(s(\Lambda), a^*) = \max(\omega(\Lambda), a^*) \]
Weyl’s theorem - characterization

Th 2. (M., Scher)
(0) $\Lambda = A + B$, where $A$ is $B^{\zeta'}$-bounded with $0 \leq \zeta' < 1$,
(1) $\| S_B \ast (AS_B)^{(\ell)} \|_{X \rightarrow X} \leq C_\ell e^{at}, \ \forall \ a > a^*, \ \forall \ \ell \geq 0$, 
(2) $\| S_B \ast (AS_B)^{(n)} \|_{X \rightarrow X_\zeta} \leq C_n e^{at}, \ \forall \ a > a^*, \ \text{with} \ \zeta > \zeta'$,
(3) $\int_0^\infty \| (A S_B)^{(n+1)} \|_{X \rightarrow Y} e^{-at} \, dt < \infty, \ \forall \ a > a^*, \ \text{with} \ Y \subset \subset X$, 

is equivalent to

(4) there exist $\xi_1, \ldots, \xi_J \in \tilde{\Delta}_a$, there exist $\Pi_1, \ldots, \Pi_J$ some finite rank projectors, there exists $T_j \in B(R \Pi_j)$ such that $\Lambda \Pi_j = \Pi_j \Lambda = T_j \Pi_j$, $\Sigma(T_j) = \{\xi_j\}$, in particular

$$\Sigma(\Lambda) \cap \tilde{\Delta}_a = \{\xi_1, \ldots, \xi_J\} \subset \Sigma_d(\Sigma)$$

and there exists a constant $C_a$ such that

$$\| S_\Lambda(t) - \sum_{j=1}^J e^{tT_j} \Pi_j \|_{X \rightarrow X} \leq C_a e^{at}, \ \forall \ a > a^*$$
Th 3. (M. & Mouhot; Tristani)

Assume

(0) $\Lambda_\varepsilon = A_\varepsilon + B_\varepsilon$ in $X_i$, $X_{-1} \subset X_0 = X \subset X_1$, $A_\varepsilon \prec B_\varepsilon$,

(1) $\|S_{B_\varepsilon} (A_\varepsilon S_{B_\varepsilon})^{(\ell)}\|_{X_i \rightarrow X_i} \leq C_\ell e^{a t}$, $\forall a > a^*$, $\forall \ell \geq 0$, $i = 0, \pm 1$,

(2) $\|S_{B_\varepsilon} (A_\varepsilon S_{B_\varepsilon})^{(n)}\|_{X_i \rightarrow X_{i+1}} \leq C_n e^{a t}$, $\forall a > a^*$, $i = 0, -1$,

(3) $X_{i+1} \subset D(B_\varepsilon|X_i)$, $D(A_\varepsilon|X_i)$ for $i = -1, 0$ and

$$\|A_\varepsilon - A_0\|_{X_i \rightarrow X_{i-1}} + \|B_\varepsilon - B_0\|_{X_i \rightarrow X_{i-1}} \leq \eta_1(\varepsilon) \rightarrow 0, \ i = 0, 1,$$

(4) the limit operator satisfies (in both spaces $X_0$ and $X_1$)

$$\Sigma(\Lambda_0) \cap \Delta_a = \{\xi_1, \ldots, \xi_k\} \subset \Sigma_d(\Lambda_0).$$

Then

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_a = \{\xi_{1,1}^\varepsilon, \ldots, \xi_{1,d_1^\varepsilon}^\varepsilon, \ldots, \xi_{k,1}^\varepsilon, \ldots, \xi_{k,d_k^\varepsilon}^\varepsilon\} \subset \Sigma_d(\Lambda_\varepsilon),$$

$|\xi_j - \xi_j^{\varepsilon,j'}| \leq \eta(\varepsilon) \rightarrow 0 \ \forall \ 1 \leq j \leq k, \forall \ 1 \leq j' \leq d_j$;

$$\dim R(\Pi_{\Lambda_\varepsilon,\xi_{j,1}^\varepsilon} + \ldots + \Pi_{\Lambda_\varepsilon,\xi_{j,d_j^\varepsilon}}) = \dim R(\Pi_{\Lambda_0,\xi_j});$$
**Th 4.** (M. & Scher) Consider a semigroup generator \( \Lambda \) on a “Banach lattice of functions” \( X \),

1. \( \Lambda \) such as in Weyl’s Theorem for some \( a^* \in \mathbb{R} \);
2. \( \exists b > a^* \) and \( \psi \in D(\Lambda^*) \cap X'_+ \setminus \{0\} \) such that \( \Lambda^* \psi \geq b \psi \);
3. \( S_\Lambda \) is positive (and \( \Lambda \) satisfies Kato’s inequalities);
4. \( -\Lambda \) satisfies a strong maximum principle.

Defining \( \lambda := s(\Lambda) \), there holds

\[
a^* < \lambda = \omega(\Lambda) \quad \text{and} \quad \lambda \in \Sigma_d(\Lambda),
\]

and there exists \( 0 < f_\infty \in D(\Lambda) \) and \( 0 < \phi \in D(\Lambda^*) \) such that

\[
\Lambda f_\infty = \lambda f_\infty, \quad \Lambda^* \phi = \lambda \phi, \quad R \Pi_{\Lambda, \lambda} = \text{Vect}(f_\infty),
\]

and then

\[
\Pi_{\Lambda, \lambda} f = \langle f, \phi \rangle f_\infty \quad \forall f \in X.
\]

Moreover, there exist \( \alpha \in (a^*, \lambda) \) and \( C > 0 \) such that for any \( f_0 \in X \)

\[
\| S_\Lambda(t)f_0 - e^{\lambda t} \Pi_{\Lambda, \lambda} f_0 \|_X \leq C e^{\alpha t} \| f_0 - \Pi_{\Lambda, \lambda} f_0 \|_X \quad \forall \ t \geq 0.
\]
Change (enlargement and shrinkage) of the functional space of the spectral analysis and semigroup decay

**Th 5.** (Moutot 06, Gualdani, M. & Mouhot) Assume

\[ \mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{L} = A + B, \quad A = \mathcal{A}|_E, \quad B = \mathcal{B}|_E, \quad E \subset \mathcal{E} \]

(i) \((B - a)\) is hypodissipative on \(E\), \((B - a)\) is hypodissipative on \(\mathcal{E}\);
(ii) \(A \in \mathcal{B}(E), \quad \mathcal{A} \in \mathcal{B}(\mathcal{E})\);
(iii) there is \(n \geq 1\) and \(C_a > 0\) such that

\[
\| (\mathcal{A}S_B)^{(n)}(t) \|_{\mathcal{E} \to E} \leq C_a e^{at}.
\]

Then the following for \((X, \Lambda) = (E, L), \ (\mathcal{E}, \mathcal{L})\) are equivalent:

\[
\exists \xi_j \in \Delta_a \text{ and finite rank projector } \Pi_{j,\Lambda} \in \mathcal{B}(X), \ 1 \leq j \leq k, \text{ which commute with } \Lambda \text{ and satisfy } \Sigma(\Lambda|_{\Pi_{j,\Lambda}}) = \{\xi_j\}, \text{ so that}
\]

\[
\forall \ t \geq 0, \quad \left\| S_\Lambda(t) - \sum_{j=1}^{k} S(t) \Pi_{j,\Lambda} \right\|_{X \to X} \leq C_{\Lambda,a} e^{at}
\]
Discussion / perspective

- In Theorem 1, 2, 3, 4, one can take $n = 1$ in the simplest situations (most of space homogeneous equations), but one need to take $n = 2$ for the equal mitosis equation or for the space inhomogeneous Boltzmann equation.

- In Theorem 5, one need to take $n > d/4$ for the space homogeneous Fokker-Planck equation in order to extend the spectral analysis from $L^2$ (well-known) to $L^1$.

- Beyond the “dissipative case”?
  - Example of the Fokker-Planck equation when $\gamma \in (0, 1)$ and relation with “weak Poincaré inequality” by Röckner-Wang.
  - Links with semi-uniform stability by Lebeau & co-authors, Burq, Liu-R, Bátkal-E-P-S, Batty-D, ...
  - Applications to Boltzmann and Landau equation associated to “soft potential”.

- Inhomogeneous linearized Landau, linearized Keller-Segel (parabolic-parabolic), neural network, Fokker-Planck in the subcritical case $\gamma \in (0, 1)$. 

S. Mischler (CEREMADE & IUF)  
Semigroups spectral analysis  
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Proof of the enlargement theorem

We split the semigroup into invariant linear sub-manifolds (eigenspaces)

\[ S_L = \Pi S_L + (I - \Pi) S_L (I - \Pi) \]

and write the (iterated) Duhamel formula or “stopped” Dyson-Phillips series (the Dyson-Phillips series corresponds to the choice \( n = \infty \))

\[ S_L = \sum_{\ell=0}^{n-1} S_B * (AS_B)^{(*\ell)} + S_L * (AS_B)^{(n)} \]

or

\[ + \ (AS_B)^{(n)} * S_L. \]

These two identities together

\[ S_L = \Pi S_L + (I - \Pi) \left\{ \sum_{\ell=0}^{n-1} S_B * (AS_B)^{(*\ell)} \right\} (I - \Pi) \]

\[ + \ \{(I - \Pi) S_L\} * (AS_B)^{(n)}(I - \Pi) \]

or

\[ + \ (I - \Pi)(AS_B)^{(n)} * \{ S_L(I - \Pi) \} \]
Sketch of the proof of the spectral mapping theorem

We introduce the resolvent

\[ R_\Lambda(z) = (\Lambda - z)^{-1} = -\int_0^\infty S_\Lambda(t) e^{-zt} \, dt. \]

Using the inverse Laplace formula for \( b > \omega(\Lambda) \geq s(\Lambda) = \sup \Re \Sigma(\Lambda) \) and the fact that \( \Pi^\perp R_\Lambda(z) \) is analytic in \( \Delta_{a^*} \), \( \Pi^\perp := I - \Pi \), we get

\[
S_\Lambda(t)\Pi^\perp = \frac{i}{2\pi} \int_{b-i\infty}^{b+i\infty} e^{zt} \Pi^\perp R_\Lambda(z) \, dz \\
= \lim_{M \to \infty} \frac{i}{2\pi} \int_{a-iM}^{a+iM} e^{zt} \Pi^\perp R_\Lambda(z) \, dz
\]

Similarly as for the (iterated) Duhamel formula, we have

\[
R_\Lambda = \sum_{\ell=0}^{N-1} (-1)^\ell R_\beta(\mathcal{A}R_\beta)^\ell + (-1)^N R_\Lambda(\mathcal{A}R_\beta)^N
\]
These two identities together

\[
S_{\mathcal{L}}(t) \Pi \downarrow = \Pi \downarrow \sum_{\ell=0}^{N-1} (-1)^\ell \frac{i}{2\pi} \int_{a-i\infty}^{a+i\infty} e^{zt} R_B(z)(AR_B(z))^\ell \, dz \\
+ (-1)^N \Pi \downarrow \frac{i}{2\pi} \int_{a-i\infty}^{a+i\infty} e^{zt} R_A(z)(AR_B(z))^N \, dz
\]

\[
= \sum_{\ell=0}^{N-1} \Pi \downarrow S_B \ast (AS_B)^{(*)\ell} \\
+ (-1)^N \frac{i}{2\pi} \int_{a-i\infty}^{a+i\infty} e^{zt} \Pi \downarrow R_A(z)(AR_B(z))^N \, dz
\]

and we have to explain why the last term is of order \(O(e^{at})\). We clearly have

\[
\sup_{z=a+iy, \, y \in [-M,M]} \| \Pi \downarrow R_A(z)(AR_B(z))^N \| \leq C_M
\]

and it is then enough to get the bound

\[
\| R_A(z)(AR_B(z))^N \| \leq C/|y|^2, \quad \forall z = a + iy, \, |y| \geq M, \, a > a_\
\]

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The key estimate

We assume (in order to make the proof simpler) that $\zeta = 1$, namely
\[
\|(AS_B)^{(\ast n)}\|_{X \to X_1} = O(e^{at}) \quad \forall \ t \geq 0,
\]
with $X_1 := D(\Lambda) = D(B)$, which implies
\[
\|(AR_B(z))^n\|_{X \to X_1} \leq C_a \quad \forall \ z = a + iy, \ a > a_\ast.
\]
We also assume (for the same reason) that $\zeta' = 0$, so that
\[
\mathcal{A} \in \mathcal{L}(X) \quad \text{and} \quad R_B(z) = \frac{1}{z} (R_B(z)B - I) \in \mathcal{L}(X_1, X)
\]
imply
\[
\|AR_B(z)\|_{X_1 \to X} \leq C_a/|z| \quad \forall \ z = a + iy, \ a > a_\ast.
\]
The two estimates together imply
\[
(\ast) \quad \|(AR_B(z))^{n+1}\|_{X \to X} \leq C_a/|z| \quad \forall \ z = a + iy, \ a > a_\ast.
\]

• In order to deal with the general case $0 \leq \zeta' < \zeta \leq 1$ one has to use some additional interpolation arguments
We write

\[ R_\Lambda (1 - \mathcal{V}) = \mathcal{U} \]

with

\[ \mathcal{U} := \sum_{\ell=0}^{n} (-1)^\ell R_B (\mathcal{A} R_B)^\ell, \quad \mathcal{V} := (-1)^{n+1} (\mathcal{A} R_B)^{n+1} \]

For \( M \) large enough

\[ (*) \quad \| \mathcal{V}(z) \| \leq 1/2 \quad \forall z = a + iy, \ |y| \geq M, \]

and we may write the Neuman series

\[ R_\Lambda (z) = \underbrace{\mathcal{U}(z)}_{\text{bounded}} \sum_{j=0}^{\infty} \underbrace{\mathcal{V}(z)^j}_{\text{bounded}} \]

For \( N = 2(n + 1) \), we finally get from \((*)\) and \((***)\)

\[ \| R_\Lambda (z) (\mathcal{A} R_B(z))^N \| \leq C / \langle y \rangle^2, \quad \forall z = a + iy, \ |y| \geq M \]