Landau equation for Coulomb potentials near Maxwellians and related problems

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Workshop interactions between PDEs & functional inequalities
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Results are picked up from

- Carrapatoso, M. *Landau equation for very soft and Coulomb potentials near Maxwellians*, submitted
- Kavian, M., *The Fokker-Planck equation with subcritical confinement force*, submitted
- M., *Semigroups in Banach spaces - factorization approach for spectral analysis and asymptotic estimates*, in progress

Generalize to a **weak dissipativity** framework some related previous works available in a dissipativity framework, in particular:

- Gualdani, M., Mouhot, *Factorization for non-symmetric operators and exponential H-Theorem*, arXiv 2010

which in turn formalize several reminiscent ideas from Bobylev 1975, Voigt 1980, Arkeryd 1988, Gallay-Wayne 2002, Mouhot 2006, ...
Outline of the talk

1. Introduction and main result
   - Hypodissipativity vs weak hypodissipativity
   - The Fokker-Planck equation with weak confinement
   - Landau equation with Coulomb potential near Maxwellians

2. Weak hypodissipativity in an abstract setting
   - From weak dissipativity to decay estimate
   - From decay estimate to weak dissipativity
   - Functional space extension (enlargement and shrinkage)
   - Spectral mapping theorem
   - Krein-Rutman theorem

3. About the proof for the Fokker-Planck equation
   - $F = \nabla V$
   - general forces

4. About the proof for the Landau equation
   - Estimates on the nonlinear problem and natural large space
   - Splitting trick, dissipativity and decay estimates on the linear operators
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Hypodissipative framework

With Mouhot, Gualdani and Scher, we have recently revisited the spectral theory of operators and semigroups in an hypodissipative and abstract general Banach framework, providing a set of results including:

- **Spectral mapping Theorem**
- **Weyl’s Theorem** about distribution of eigenvalues under compact perturbation
- **Stability of the spectrum Theorem** under small perturbation
- **Krein-Rutman Theorem**
- **Functional space extension (enlargement and shrinkage) Theorem**

These results were motivated by linear and nonlinear evolution PDEs to which they have been applied:

- Asymptotic behavior of linear PDEs in large space (Growth-Fragmentation, Kinetic Fokker-Planck, Run-and-Tumble in $W^{r,p}(m), -1 \leq r \leq 1 \leq p \leq \infty$)
- Optimal (= linearized) exponential decay estimates for nonlinear PDE (homogeneous (inelastic) Boltzmann, Parabolic-elliptic Keller-Segel)
- Existence, uniqueness and stability results in perturbative regime (inhomogeneous (inelastic) Boltzmann, Parabolic-parabolic Keller-Segel, kinetic FitzHugh-Nagumo and others neuronal networks)
Main feature and next step 1

- Our theory is suitable for semigroups $S_{\Lambda}$ which split as

$$S_{\Lambda}(t) = S_1(t) + S_2(t) = \sum_{\text{finite}} S_{\Lambda_j}(t)\Pi_j + O(e^{at})$$

with $S_1$ has asymptotically dominant finite dimensional range and $S_2$ is asymptotically negligible

with $\Lambda_j = \Lambda|_{\Pi_j}$, $\Sigma(\Lambda_j) = \{\lambda_j\}$, $\Re\lambda_j > a$, $\Pi_j$ commutes with $\Lambda$ and $\dim \Pi_j < \infty$

We establish a suitable **characterisation** of such semigroups **with spectral gap** $\simeq$ quantified principal spectral mapping theorem

- Next step 1 (open problems): take into account PDEs with boundary conditions such as the transport equation

$$\partial_t f + \partial_x f + a(x)f = 0, \quad f(t, 0) = \int_0^\infty a(y) f(t, y) \, dy, \quad a = \text{step function}$$

with applications to neurons network models as in Pakdaman-Perthame-Salort works, or kinetic PDEs

$$\partial_t f + \nu \cdot \nabla_x f = C(f) + \text{Maxwell boundary condition}$$

as in Guo and Briant-Guo works.
Reminder about dissipativity and hypodissipativity

For a semigroup $S_{\mathcal{B}}$ with generator $\mathcal{B}$ the following properties are “equivalent”:

1. $\mathcal{B}$ is dissipative:
   $$\langle f^*, \mathcal{B} f \rangle_X \leq a \|f\|_X^2,$$
   for any $f \in X_1^\mathcal{B}$ and $f^* \in X'$ dual element;

2. $S_{\mathcal{B}}$ satisfies the growth estimate
   $$\|S_{\mathcal{B}}(t)\|_{X \rightarrow X} \leq e^{at};$$

3. $\mathcal{B}$ is hypodissipative:
   $$\langle f^*, \mathcal{B} f \rangle_X \leq a \|f\|_X^2,$$
   for an equivalent norm $\|\cdot\|_X$ on $X$;

4. $S_{\mathcal{B}}$ satisfies the growth estimate
   $$\|S_{\mathcal{B}}(t)\|_{X \rightarrow X} \leq C e^{at}, \quad C \geq 1.$$
Reminder about dissipativity and hypodissipativity

For a semigroup $S_B$ with generator $B$ the following properties are “equivalent”:

1. $B$ is dissipative:
   $$\langle f^*, Bf \rangle_X \leq a \|f\|^2_X,$$
   for any $f \in X_1^B$ and $f^* \in X'$ dual element;

2. $S_B$ satisfies the growth estimate
   $$\|S_B(t)\|_{X \to X} \leq e^{at};$$

3. $B$ is hypodissipative:
   $$\langle f^*, Bf \rangle_X \leq a \|f\|^2_X,$$
   for an equivalent norm $\|\cdot\|_X$ on $X$;

4. $S_B$ satisfies the growth estimate
   $$\|S_B(t)\|_{X \to X} \leq C e^{at}, \quad C \geq 1.$$

$(1) \Rightarrow (2)$: consequence of Gronwall lemma and the closed differential inequality

$$\frac{1}{2} \frac{d}{dt} \|f_t\|^2_X = \langle f_t^*, Bf_t \rangle \leq a \|f_t\|^2_X,$$

$f_t := S_B(t)f_0.$
Reminder about dissipativity and hypodissipativity

For a semigroup $S_B$ with generator $B$ the following properties are “equivalent”:

1. $B$ is dissipative:
   \[ \langle f^*, Bf \rangle_X \leq a \| f \|_X^2, \] for any $f \in X_1^B$ and $f^* \in X'$ dual element;

2. $S_B$ satisfies the growth estimate
   \[ \| S_B(t) \|_{X \to X} \leq e^{at}; \]

3. $B$ is hypodissipative:
   \[ \langle f^*, Bf \rangle_X \leq a \| f \|_X^2, \] for an equivalent norm $\| \cdot \|_X$ on $X$;

4. $S_B$ satisfies the growth estimate
   \[ \| S_B(t) \|_{X \to X} \leq C e^{at}, \quad C \geq 1. \]

(4) $\Rightarrow$ (3): one may choose the equivalent handy norm defined by
   \[ \| f \|_X^2 := \eta \| f \|_X^2 + \int_0^\infty \| S_B(\tau)f \|_X^2 e^{-b\tau} \, d\tau, \quad \eta > 0, \ b > a. \]
Next step 2: weakly hypodissipative framework

Possible extension to a weakly dissipative framework?
We do not assume the dissipativity inequality (1) but the weaker inequality
\[ \langle f^*, Bf \rangle_Y \leq a \|f\|_Z^2, \quad Y \subset Z, \quad a < 0. \]

▷ We cannot close a differential inequality with this only information.
However, assuming the additional (dissipativity) inequality
\[ \langle f^*, Bf \rangle_X \leq 0, \quad X \subset Y, \]
we may exploit these two inequalities together with an interpolation argument in order to get some rate of decay to 0 (as for the Allen-Cahn equation)

That corresponds to the (no spectral gap) situation:
\[ \Sigma_P(B) \cap \bar{\Delta}_0 = \emptyset, \quad \Sigma(B) \cap \bar{\Delta}_0 \neq \emptyset. \]

In this weakly dissipative framework, we will present:
• some (not all) abstract spectral analysis results
• some application to the Fokker-Planck equation with weak confinement force
• some application to the Landau equation for Coulomb potential near Maxwellians in the torus
The Fokker-Planck equation with weak confinement

Consider the Fokker-Planck equation

$$\partial_t f = \Lambda f = \Delta_v f + \text{div}_v(F f)$$

on $f = f(t, v) \in \mathbb{R}$, $t \geq 0$, $v \in \mathbb{R}^d$, with a weak confinement force field term $F$ such that

$$F(v) \approx v \langle v \rangle^{\gamma-2}, \quad \gamma \in (0, 1) \quad \text{(say =)}$$

and an initial datum

$$f(0) = f_0 \in W^{r,p}(m) \quad \text{(means } m f_0 \in W^{r,p}).$$

Here $p \in [1, \infty]$, $r = 0$ and $m$ is a polynomial weight

$$m = \langle v \rangle^k, \quad k > k^*(p, r, \gamma),$$

or an exponential weight

$$m = e^{\kappa \langle v \rangle^s}, \quad s \in (0, \gamma], \quad \kappa > 0.$$
Statement of the decay theorem

**Theorem 1.** (Kavian & M.)

There exists a unique “smooth”, positive and normalized steady state $f_\infty$. For any $f_0 \in L^p(m)$

$$
\| f(t) - \langle f_0 \rangle f_\infty \|_{L^p} \leq \Theta(t) \| f_0 - \langle f_0 \rangle f_\infty \|_{L^p(m)},
$$

with

$$
\Theta(t) = \frac{C}{\langle t \rangle^K}, \quad K = \frac{k - k^*(p)}{2 - \gamma} \quad \text{if} \quad m = \langle x \rangle^k
$$

$$
= Ce^{-\lambda t^\sigma}, \quad \sigma = \frac{s}{2 - \gamma} \quad \text{if} \quad m = e^{\kappa \langle x \rangle^s}.
$$

Improves (better rate and/or larger class of initial data) earlier results by Toscani, Villani, 2000 (based on log-Sobolev inequality) & Röckner, Wang, 2001 (based on weak Poincaré inequality).

Both works deal with a force field $F = \nabla V$ what is not necessary here.
Consider the Landau equation

\[ \partial_t f + \mathbf{v} \cdot \nabla_x f = Q(f, f) \]

\[ f(0, .) = f_0 \]

on density of the plasma \( f = f(t, x, \mathbf{v}) \geq 0 \), time \( t \geq 0 \), position \( x \in \mathbb{T}^3 \) (torus), velocity \( \mathbf{v} \in \mathbb{R}^3 \)

\( Q = \) the Landau (binary) collisions operator

\[ Q(g, f) = \partial_j \int_{\mathbb{R}^3} a_{ij}(\mathbf{v} - \mathbf{v}_*)(g_\ast \partial_j f - f \partial_j g_\ast) \, d\mathbf{v}_\ast \]

for the Coulomb potential cross section

\[ a_{ij}(z) = |z|^{\gamma + 2}(\delta_{ij} - \frac{z_i z_j}{|z|^2}), \quad \gamma = -3. \]
around the H-theorem

We recall that $\varphi = 1, \nu, |\nu|^2$ are collision invariants, meaning

$$\int_{\mathbb{R}^3} Q(f, f) \varphi \, d\nu = 0, \quad \forall f.$$ 

$\Rightarrow$ laws of conservation

$$\int_{\mathbb{R}^6} f \begin{pmatrix} 1 \\ \nu \\ |\nu|^2 \end{pmatrix} = \int_{\mathbb{R}^6} f_0 \begin{pmatrix} 1 \\ \nu \\ |\nu|^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

We also have the H-theorem, namely

$$\int_{\mathbb{R}^3} Q(f, f) \log f \begin{cases} \leq 0 \\ = 0 \Rightarrow f = \text{Maxwellian} \end{cases}$$

From both pieces of information, we expect

$$f(t, x, \nu) \underset{t \to \infty}{\longrightarrow} \mu(\nu) := \frac{1}{(2\pi)^{3/2}} e^{-|\nu|^2/2}.$$
Existence, uniqueness and stability in small perturbation regime

**Theorem 2. (Carrapatoso, M.)**

Take an “admissible” weight function $m$ such that

$$\langle v \rangle^{2+3/2} < m < e|v|^2.$$  

There exists $\varepsilon_0 > 0$ such that if

$$\|f_0 - \mu\|_{H^2_x L^2_v(m)} < \varepsilon_0,$$

there exists a unique global solution $f$ to the Landau Coulomb equation and

$$\|f(t) - \mu\|_{H^2_x L^2_v} \leq \Theta_m(t),$$

with

$$\Theta_m(t) \sim \begin{cases}  
t^{-(k-2-3/2)/|\gamma|} & \text{if } m = \langle v \rangle^k \\
e^{-\lambda t^{s/|\gamma|}} & \text{if } m = e^{\kappa |v|^s} \end{cases}$$
Comments on the main Theorem 2

• Improves (larger space) Guo and Strain’s results (CMP 2002, CPDE 2006, ARMA 2008) who proved a similar theorem in the higher order and strongly confinement Sobolev space $H^{8}_{x,v}(\mu^{-\theta})$, $\theta > 1/2$. Based on high order nonlinear (hypercoercivity) energy estimates.

• A corollary improves (faster rate) Desvillettes and Villani’s result (Invent. Math 2005) who proved polynomial rate of convergence for a priori suitably bounded solutions in a space inhomogeneous setting. Based on entropy and hypocoercivity methods.

• A corollary improves (faster rate) Carrapatoso, Desvillettes and He result (arXiv 2015) who proved polynomial and exponential rate for weak solutions in a space homogeneous setting. Based on an entropy method.

• Our proof mixes
  - Simple nonlinear estimates and trap argument in large space (no self-adjointness)
  - Decay and dissipativity estimates in appropriate norms for the linearized equation
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For a given Banach space $X$, we want to develop a spectral analysis theory for operators $\Lambda$ enjoying the splitting structure

$$\Lambda = A + B, \quad A \prec B, \quad B \text{ weakly hypodissipative.}$$

We will

• clarify the links between dissipativity and decay;
• present an extension of the decay estimate result;
• present a possible version of spectral mapping theorem;
• present a possible version of Krein-Rutman theorem.

• We do not present any version of Weyl’s theorem or perturbation theorem.

**Prop 1.**

Consider three “regular” Banach spaces $X \subset Y \subset Z$ and a generator $\Lambda$. Assume

$$\forall f \in Y_1^\Lambda, \quad \langle f_Y^*, \Lambda f \rangle_Y \lesssim -\|f\|^2_Z$$

$$\forall f \in X_1^\Lambda, \quad \langle f_X^*, \Lambda f \rangle_X \leq 0 \quad \text{(or $S_\Lambda$ is bounded $X$)}$$

$$\forall R > 0, \quad \varepsilon_R \|f\|^2_Y \leq \|f\|^2_Z + \theta_R \|f\|^2_X, \quad \varepsilon_R, \frac{\theta_R}{\varepsilon_R} \to 0.$$

There exists a decay function $\Theta$ such that

$$\|S_\Lambda(t)\|_{X \to Y} \leq \Theta(t) \to 0.$$

- We say that a Banach space $E$ is regular if $\varphi : E \to \mathbb{R}, f \mapsto \|f\|^2_E/2$ is $G$-differentiable and

  $$\{f^* \in E', \quad \langle f^*, f \rangle_E = \|f\|^2_E = \|f^*\|^2_{E'} \} = \{f^*_E\}, \quad f^*_E := D\varphi(f).$$

Hilbert spaces and $L^p$ spaces, $1 < p < \infty$, are regular spaces.

- We denote $E_s^\Lambda := \{f \in E, \quad \Lambda^s f \in E\}$ the abstract Sobolev spaces
Prop 1.

Consider three “regular” Banach spaces $X \subset Y \subset Z$ and a generator $\Lambda$. Assume

\[ \forall f \in Y_1^\Lambda, \quad \langle f_Y^*, \Lambda f \rangle_Y \lesssim -\|f\|_Z^2 \]

\[ \forall f \in X_1^\Lambda, \quad \langle f_X^*, \Lambda f \rangle_X \leq 0 \quad \text{(or } S_\Lambda \text{ is bounded } X) \]

\[ \forall R > 0, \quad \varepsilon_R \|f\|_Y^2 \leq \|f\|_Z^2 + \theta_R \|f\|_X^2, \quad \varepsilon_R, \frac{\theta_R}{\varepsilon_R} \to 0. \]

There exists a decay function $\Theta$ such that

\[ \|S_\Lambda(t)\|_{X \to Y} \leq \Theta(t) \to 0. \]

- We say that $m$ is an admissible if $m = \langle v \rangle^k$ or $m = e^{\kappa \langle v \rangle^s}$. We then write $m_0 \prec m_1$ or $m_1 \succ m_0$ or if $m_0/m_1 \to \infty$.

- For $X = L^p(m_1)$, $Y = L^p(m_0)$, $Z = L^p(m_0 \langle v \rangle^{\alpha/p})$, with $\alpha < 0$ and $m_1 \succ m_0$, we get

\[ \Theta(t) \sim \begin{cases} 
    t^{-(k_1-k_0)/|\alpha|} & \text{if } m_i = \langle v \rangle^{k_i} \\
    e^{-\lambda t^{s/|\alpha|}} & \text{if } m_1 = e^{\kappa |v|^s}
\end{cases} \]
Proof of Proposition 1

We define \( f_t := S_\Lambda(t)f_0, \) \( f_0 \in X, \) and we compute

\[
\frac{d}{dt}\|f_t\|^2_X \leq 0 \quad \Rightarrow \quad \|f_t\|_X \leq C\|f_0\|_X, \quad C \geq 1,
\]

\[
\frac{d}{dt}\|f_t\|^2_Y \precsim -\|f_t\|^2_Z
\]

\[
\precsim -\varepsilon R\|f_t\|^2_Y + \theta R\|f_0\|^2_X,
\]

and from Gronwall lemma

\[
\|f_t\|^2_Y \precsim e^{-\varepsilon R t}\|f_0\|^2_Y + \frac{\theta R}{\varepsilon R}\|f_0\|^2_X
\]

\[
\precsim \Theta(t)^2\|f_0\|^2_X,
\]

with

\[
\Theta(t)^2 := \inf_{R > 0} \left( e^{-\varepsilon R t} + \frac{\theta R}{\varepsilon R} \right).
\]
Prop 2. Consider three “regular” Banach spaces $X \subset Y \subset Z$ and a generator $L$. Assume

- $\|S_L(t)\|_{X \to Z} \leq \Theta(t)$, with $\Theta \in L^2(\mathbb{R}_+)$ a decay function (i.e. which tends to 0)
- $L = A + B$, $A \prec B$, with

\[
\forall f \in X^B_1, \quad \langle f^*, Bf \rangle_X \lesssim -\|f\|^2_Y \\
\forall f \in X^A_1, \quad \langle f^*, Af \rangle_X \lesssim \|f\|^2_Z.
\]

Then, $L$ is weakly hypodissipative

\[
\langle \langle f^*, Lf \rangle \rangle_X \lesssim -\|f\|^2_Y
\]

for the duality product $\langle \langle, \rangle \rangle_X$ associated to the norm defined by

\[
\|f\|^2 := \eta \|f\|^2_X + \int_0^\infty \|S_L(\tau)f\|^2_Z d\tau,
\]

for $\eta > 0$ small enough. That norm is equivalent to the initial norm in $X$. 
Proof of Proposition 2

We observe that $\| \cdot \| \sim \| \cdot \|_X$ because $\Theta \in L^2(\mathbb{R}_+)$. 

We set $f_t := S_{\mathcal{L}}(t)f_0$ and we compute 

$$
\frac{d}{dt} \| f_t \|^2 = 2\eta \langle f_t^*, \mathcal{L}f_t \rangle_X + \int_0^\infty \frac{d}{d\tau} \| S_{\mathcal{L}}(\tau + t)f_0 \|^2_Z \ d\tau 
$$

$$
= 2\eta \langle f_t^*, Bf_t \rangle_X + \eta \langle f_t^*, Af_t \rangle_X - \| f_t \|^2_Z
$$

$$
\leq -2\eta C_1 \| f_t \|^2_Y + (\eta C_2 - 1) \| f_t \|^2_Z
$$

$$
\geq -\| f_t \|^2_Y
$$

as well as 

$$
\frac{d}{dt} \| f_t \|^2 \sim \langle \langle f_t^*, \mathcal{L}f_t \rangle \rangle_X
$$
Prop 3. Consider a decay function $\Theta$ such that
\[
\Theta^{-1}(t) \lesssim \Theta^{-1}(t - s)\Theta^{-1}(s) \quad \text{for any } 0 < s < t.
\]
We consider two sets of Banach spaces $X_1 \subset X_0$ and $Y_1 \subset Y_0$ and a generator $\Lambda$. We assume

- $\|S_\Lambda(t)\|_{X_1 \to X_0 \Theta^{-1}} \in L^\infty$
- $\Lambda = A + B$, $A \prec B$, with

\[
\forall \ell, \quad \|S_B \ast (AS_B)^{(*\ell)}\|_{Y_1 \to Y_0 \Theta^{-1}} \in L^\infty
\]
\[
\exists n, \quad \|(AS_B)^{(*n)}\|_{Y_1 \to X_1 \Theta^{-1}} \in L^1 \quad \text{if } X_0 \subset Y_0 \quad \text{(enlargement)}
\]
\[
\exists n, \quad \|(S_B A)^{(*n)}\|_{X_0 \to Y_1 \Theta^{-1}} \in L^1 \quad \text{if } Y_1 \subset X_1 \quad \text{(shrinkage)}
\]

Then,
\[
\|S_\Lambda(t)\|_{Y_1 \to Y_0 \Theta^{-1}} \in L^\infty.
\]
Proof of Proposition 3

Enlargement result. We iterate the Duhamel formula

\[ S_\Lambda = S_B + S_\Lambda \ast (A S_B) \]

to get a “stopped Dyson-Phillips series” (the D-P series corresponds to \( n = \infty \))

\[ S_\Lambda = \sum_{\ell=0}^{n-1} S_B \ast (A S_B)^{(*\ell)} + S_\Lambda \ast (A S_B)^{(*n)} =: S_1 + S_2. \]

From the assumptions, we immediately have

\[ \| S_\Lambda \|_{Y_1 \to Y_0 \Theta^{-1}} \leq \| S_1 \|_{Y_1 \to Y_0 \Theta^{-1}} + \| S_\Lambda \Theta^{-1} \|_{X_1 \to X_0} \ast \| (A S_B)^{(*n)} \Theta^{-1} \|_{Y_1 \to X_1} \in L^\infty \]

Shrinkage result. We argue similarly starting with the iterated the Duhamel formula / stopped Dyson-Phillips series

\[ S_\Lambda = \sum_{\ell=0}^{n-1} S_B \ast (A S_B)^{(*\ell)} + (S_B A)^{(*n)} \ast S_\Lambda. \]
Prop 4. (rough version) We consider two Banach spaces $X \subset Y$ and a generator $\Lambda$. We assume $X^{1}_{\Lambda} \subset Y$ is compact and $\Theta(t) \approx e^{-\lambda t^{1/(1+j)}}$

- $\Sigma_{P}(\Lambda) \cap \tilde{\Delta}_{0} = \emptyset$, with $\Delta_{0} := \{z \in \mathbb{C}; \Re z > 0\}$
- $\Lambda = \mathcal{A} + \mathcal{B}$, with $\mathcal{A} \in \mathcal{B}(Y, X)$, $\zeta \in (0, 1]$ and

(a1) $\forall \ell$, $\|S_{\mathcal{B}} \ast (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}\|_{X \rightarrow Y} \Theta^{-1} \in L^{\infty}$

(a2) $\forall \ell$, $\sup_{z \in \tilde{\Delta}_{0}} \|(R_{\mathcal{B}}(z))^{\ell}\|_{X \rightarrow Y} \leq C (\ell!)^{j}$

(a3) $\forall \ell$, $\sup_{z \in \tilde{\Delta}_{0}} \|R_{\mathcal{B}}(z)\|_{Y \rightarrow X^{\Lambda}_{\zeta}} \leq C (\ell!)^{j}$

Then,

$\|S_{\Lambda}(t)\|_{X \rightarrow Y} \Theta^{-1} \in L^{\infty}$. 
Proof of Proposition 4

We start again with the stopped Dyson-Phillips series

\[ S_\Lambda = \sum_{\ell=0}^{N-1} S_B * (A S_B)^{(*\ell)} + S_\Lambda * (A S_B)^{(*N)} = S_1 + S_2 \]

The first \( N - 1 \) terms are fine. For the last one, we use the inverse Laplace formula

\[ S_2(t)f = \frac{i}{2\pi} \int_{\uparrow_0} e^{zt} R_\Lambda(z)(AR_B(z))^N f \, dz \]

\[ \approx \frac{1}{t^k} \int_{\uparrow_0} e^{zt} \frac{d^k \Phi}{dz^k} \, dz f \]

\[ \lesssim \frac{C^k}{t^k} k! \int_{\uparrow_0} \sup_{|\alpha| \leq k} \left\{ \|R_\Lambda^{1+\alpha_1}(z)\|_{X \to Y} \right\}_{\in L^\infty(\uparrow_0)} \left\{ \|AR_B^{1+\alpha_1} ... AR_B^{1+\alpha_N}(z)\|_{X \to X} \right\}_{\in L^1(\uparrow_0)} dz \|f\|_X, \]

where \( \uparrow_0 := \{ z = 0 + iy, y \in \mathbb{R} \} \) and because

\[ \frac{d^k \Phi}{dz^k} \approx \sum_{|\alpha| \leq k} \alpha! R_\Lambda^{1+\alpha_0} AR_B^{1+\alpha_1} ... AR_B^{1+\alpha_N} \]
Key estimates

• Using (a2), (a3), the compact embedding $X^1_{\Lambda} \subset Y$ and the fact that there is not punctual spectrum in $\bar{\Delta}_0$, we get

$$
\sup_{z \in \bar{\Delta}_0} \| R_{\Lambda}(z)^{\ell} \|_{X \to Y} \leq C (\ell!)^j
$$

• $A \in \mathcal{B}(Y, X)$ and the resolvent identity

$$
R_B(z) = \frac{1}{z} (R_B(z)B - I) \in \mathcal{B}(X_1, X)
$$

imply

$$
\| AR_B(z) \|_{X \to X} \leq C / |z| \quad \forall z \in \bar{\Delta}_0.
$$

Together with (a2) (where we assume that $\zeta = 1$ in order to make the proof simpler) we get

$$
\| AR_B(z)^{\ell_1} AR_B(z)^{\ell_2} \|_{X \to X} \leq C (\ell_1!)^j (\ell_2!)^j \langle z \rangle^{-1}
$$

• Choosing $N = 4$ and gathering the two estimates, we get

$$
\| \frac{d^k \Phi}{dz^k} (z) \|_{X \to Y} \leq C^k (k!)^j \langle z \rangle^{-2} \in L^1(\uparrow_0).
$$
Coming back to the term $S_2$, we have

\[
S_2(t) \lesssim C^k k^{(1+j)k} t^{-k}.
\]

\[
\lesssim e^{-\lambda t^{1/(1+j)}} = \Theta(t),
\]

by choosing appropriately $k = k(t)$.
Prop 5.

Consider a semigroup generator $\Lambda$ on a Banach lattice $X$, and assume

1. $\Lambda$ such as the spectral mapping Theorem holds (for $\|f\|_Y = \langle |f|, \phi \rangle$);
2. $\phi \in D(\Lambda^*)$, $\phi \succ 0$ such that $\Lambda^*\phi = 0$;
3. $S_\Lambda$ is positive (and $\Lambda$ satisfies Kato’s inequalities);
4. $-\Lambda$ satisfies a strong maximum principle.

There exists $0 < f_\infty \in D(\Lambda)$ such that

$$\Lambda f_\infty = 0, \quad \Sigma_P(\Lambda) \cap \tilde{\Delta}_0 = \{0\}, \quad \Sigma_P(\Lambda_1) \cap \tilde{\Delta}_0 = \emptyset$$

with $\Lambda_1 := \Lambda|_{X_1}$, $X_1 = R(I - \Pi_0) = (I - \Pi_0)X$,

$$\Pi_0 f = \langle f, \phi \rangle f_\infty \quad \forall f \in X.$$ 

Moreover the decay function $\Theta$ defined in the spectral mapping Theorem :

$$\|S_\Lambda(t)(I - \Pi_0)f_0\|_Y \lesssim \Theta(t) \|(I - \Pi_0)f_0\|_X \quad \forall t \geq 0, \forall f_0 \in X.$$
Outline of the talk

1. Introduction and main result
   - Hypodissipativity vs weak hypodissipativity
   - The Fokker-Planck equation with weak confinement
   - Landau equation with Coulomb potential near Maxwellians

2. Weak hypodissipativity in an abstract setting
   - From weak dissipativity to decay estimate
   - From decay estimate to weak dissipativity
   - Functional space extension (enlargement and shrinkage)
   - Spectral mapping theorem
   - Krein-Rutman theorem

3. About the proof for the Fokker-Planck equation
   - $F = \nabla V$
   - general forces

4. About the proof for the Landau equation
   - Estimates on the nonlinear problem and natural large space
   - Splitting trick, dissipativity and decay estimates on the linear operators
Elements of proof of Theorem 1 - The case: $F = \nabla V$, $V = |v|^{\gamma}/\gamma$

- Weak Poincaré inequality

$$\langle \Lambda f, f \rangle_{E_0} \lesssim -\|f\|^2_{E_*}, \quad \forall f \in E_0, \ 〈f〉 = 0,$$

with $E_0 := L^2(f_{\infty}^{-1/2})$, $f_{\infty} := e^{-V}$, and $E_* := L^2(⟨v⟩^{-1}f_{\infty}^{-1/2})$.

- By the generalized relative entropy inequality

$$\forall f, \forall p \geq 1, \quad \langle \Lambda f, (f/f_{\infty})^{p-1} \rangle \leq 0,$$

and passing to the limit as $p \to \infty$, we deduce the semigroup (of contractions) estimate

$$\|f_t\|_{E_1} \leq \|f_0\|_{E_1}, \quad E_1 := L^\infty(f_{\infty}^{-1}).$$

- For any $f_0 \in E_1$, $〈f_0〉 = 0$, both inequalities and an interpolation argument imply (as in Prop 1)

$$\|f_t\|_{E_0} \leq \Theta(t)\|f_0\|_{E_1}, \quad \Theta(t) \simeq e^{-t^2/\gamma}.$$
We introduce the splitting \( \Lambda = A + B \), with \( A \) a multiplication operator

\[
Af = M\chi_{R}(v)f, \quad \chi_{R}(v) = \chi(v/R), \quad 0 \leq \chi \leq 1, \quad \chi \in \mathcal{D}(\mathbb{R}^{d})
\]

\( \triangleright \) \( A \in \mathcal{B}(X_{0}, X_{1}), X_{i} = W^{r,p}(m_{i}), m_{1} \geq m_{0} \)

\( \triangleright \) \( B \) is not \( a \)-dissipative in \( X = W^{r,p}(m) \) with \( a < 0 \). However, it is weakly dissipative. For \( p \in (1, \infty) \), and \( M, R > 0 \) large enough, we have

\[
\langle f^{*}, Bf \rangle_{L^{p}} \lesssim -\|f\|^{2}_{L^{p}(m(v)(\gamma-2+s)/p)}, \quad s := 0 \text{ for polynomial weight}
\]

That is a consequence of the identity

\[
\int (\Lambda f)^{p-1} m^{p} = (1 - p) \int |\nabla(fm)|^{2}(fm)^{p-1} + \int (fm)^{p} \psi
\]

\[
\psi = \left( \frac{2}{p} - 1 \right) \frac{\Delta m}{m} + 2(1 - \frac{1}{p}) \frac{|\nabla m|^{2}}{m^{2}} + (1 - \frac{1}{p}) \text{div}F - F \cdot \frac{\nabla m}{m}
\]

\[
\sim -F \cdot \frac{\nabla m}{m} \sim -(v)^{s+\gamma-2}
\]
• the estimate
\[ \| S_B \ast (AS_B)^{(*\ell)} \|_{X_1 \to X_0} \leq \Theta(t) \]
follows from Proposition 1.

• the estimate
\[ \|(AS_B)^{(*n)}\|_{B(L^1(m_1), H^1(m_2))} \leq \Theta(t) \]
follows from (1) and the use a “Nash + regularity” trick for small time. More precisely, introducing
\[ F(t, h) := \| h \|^2_{L^1(m)} + t^\bullet \| h \|^2_{L^2(m)} + t^\bullet \| \nabla v h \|^2_{L^2(m)} \]
we are able to prove (for convenient exponents \( \bullet > 1 \))
\[ \frac{d}{dt} F(t, S_B(t)h) \leq 0 \quad \text{and then} \quad \| S_B(t)h \|^2_{H^1(m)} \leq \frac{1}{t^\bullet} \| h \|^2_{L^1(m)} \]

• In the case \( F = \nabla V \), we conclude thanks to Prop 3 (enlargement argument)

• For the general case, we use the Krein-Rutman theory. The Fokker-Planck semigroup is obviously mass conservative and positive and the Fokker-Planck operator satisfies the strong maximum principle. The last point in order to apply Proposition 5 is to verify that assumption (a2) in Proposition 4 is satisfied.
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Strategy of proof of Theorem 2

The method consists in introducing the variation function \( g = f - \mu \) and the corresponding Landau equation

\[
\partial_t g = \bar{L} g + Q(g, g),
\]

\[
\bar{L} = -\nu \cdot \nabla_x + \mathcal{L}, \quad \mathcal{L} = Q(\cdot, \mu) + Q(\mu, \cdot)
\]

• As a starting point, we use the known weak dissipativity estimate

\[
(Lg, g)_{L^2(\mu^{-1/2})} \lesssim -\|\Pi g\|^2_{H^1_\ast} \approx H^1(\mu^{1/2}(v)(\gamma + 2)/2),
\]

\( \Pi := I - \Pi_0, \ \Pi_0 := \text{projector on } N(\mathcal{L}), \)

• in order to prove the weak hypodissipativity estimate

\[
(\bar{L}g, g)_{H^1_{x,v}(\mu^{-1/2})} \lesssim -\|\bar{\Pi} g\|^2_{H^1_{x}H^1_{v}} \approx H^1_{x,v}(\mu^{1/2}(v)(\gamma + 2)/2),
\]

\( \bar{\Pi} := I - \bar{\Pi}_0, \ \bar{\Pi}_0 := \text{projector on } N(\bar{L}), \)

• and next factorization and semigroup tricks in order to get similar information in the space \( \mathcal{X} := H^2_x L^2(m) \).
Estimate on nonlinear operator

A classical result asserts that for any weight functions \( m, m_1 \succ \langle v \rangle^{2+3/2} \) and \( m_0 \succ \langle v \rangle^2 \)

\[
\langle Q(f, g), h \rangle_{L^2(m)} \lesssim \left( \| f \|_{L^2(m)} \| g \|_{H^1_*(m_1)} + \| f \|_{H^1(m_0)} \| g \|_{L^2(m)} \right) \| h \|_{H^1_*(m)}
\]

with

\[
\| f \|_{H^1_*(m)}^2 := \| f \|_{L^2(m, \langle v \rangle^{(\gamma+\sigma)/2})}^2 + \| \tilde{\nabla} f \|_{L^2(m, \langle v \rangle^{\gamma/2})}^2,
\]

and

\[
\tilde{\nabla}_v f = P_v \nabla_v f + \langle v \rangle (I - P_v) \nabla_v f, \quad P_v \xi = \left( \xi \cdot \frac{v}{|v|} \right) \frac{v}{|v|}.
\]

As a consequence, we have

**Prop 6.**

for \( m \succ \langle v \rangle^{2+3/2} \), defining \( X := H^2_x L^2_v(m) \), \( Y := H^2_x H^1_v,*(m) \), \( Z := H^2_x H^{-1}_v,*(m) \), we have

\[
\langle Q(f, g), h \rangle_X \lesssim \left( \| f \|_X \| g \|_Y + \| f \|_Y \| g \|_X \right) \| h \|_Y
\]

\[
\| Q(f, g) \|_Z \lesssim \left( \| f \|_X \| g \|_Y + \| f \|_Y \| g \|_X \right).
\]
Nonlinear a priori estimate

A introduce the equivalent norm on $R\bar{\Pi}$

$$
\|g\|_X^2 := \eta \|g\|_X^2 + \int_0^\infty \|S_{\bar{L}}(\tau)g\|_{X_0}^2 d\tau,
$$

with $X_0 := H^2_x L^2_v$, $Y_0 := H^2_x H^1_v$, $Z_0 := H^2_x H^{-1}_v$ (without weight!)

We consider a solution $g$ to the Landau equation

$$
\frac{d}{dt}g = \bar{L}g + Q(g, g)
$$

and we compute

$$
\frac{1}{2} \frac{d}{dt} \|g\|_X^2 = \langle \langle \bar{L}g, g \rangle \rangle_X + \eta \langle Q(g, g), g \rangle_X
$$

$$
+ \int_0^\infty \langle S_{\bar{L}}(\tau)Q(g, g), S_L(\tau)g \rangle_{X_0} d\tau =: T_1 + T_2 + T_3.
$$

From Propositions 1, 2, 3, we expect to have

$$
T_1 \lesssim -\|g\|_Y^2.
$$

Thanks to the choice of the norm and Proposition 6, we have

$$
T_2 \leq C \|g\|_X \|g\|_Y^2.
$$
Nonlinear a priori estimate (continuation)

For the last term, thanks to Proposition 6, we have

\[
T_3 = \int_0^\infty \langle S_L(\tau)Q(g,g), S_L(\tau)g \rangle \chi_0 \, d\tau
\]

\[
\lesssim \int_0^\infty \|S_L(\tau)Q(g,g)\|_{Z_0} \|S_L(\tau)g\|_{Y_0} \, d\tau
\]

\[
\lesssim \|Q(g,g)\|_{Z} \|g\|_{Y} \int_0^\infty \Theta(\tau)^2 \, d\tau \lesssim \|g\|_{X} \|g\|_{Y}^2,
\]

under the condition that

\[
t \mapsto \|S(\tau)\|_{Y \to Y_0}, \|S(\tau)\|_{Z \to Z_0} \in L^2(\mathbb{R}_+).
\]

We conclude with

\[
\frac{d}{dt} \|g\|_{X}^2 \lesssim \|g\|_{Y}^2 (1 - C \|g\|_{X})
\]

We deduce

▷ a priori uniform estimate for \( \|g_0\|_{X}^2 \) small, and then classically existence and uniqueness

▷ considering two weight functions \( m \succ \tilde{m} \), the above a priori estimate implies

\[
\frac{d}{dt} \|g\|_{X}^2 \lesssim -\|g\|_{Y}^2, \quad \frac{d}{dt} \|g\|_{X}^2 \lesssim 0,
\]

and we get decay estimate just repeating the proof of Proposition 1.
Splitting of the operator

We introduce the splitting $\bar{L} = A + B$

$$A g := Q(g, \mu) + M_{\chi_R} g = (a_{ij} \ast g) \partial_{ij} \mu - (c \ast g) \mu + M_{\chi_R} g,$$

$$B g := Q(\mu, g) - M_{\chi_R} g - \nu \cdot \nabla_x g = (a_{ij} \ast \mu) \partial_{ij} g - (c \ast \mu) g - M_{\chi_R} g - \nu \cdot \nabla_x g,$$

with

$$b_i(z) = \partial_j a_{ij}(z) = -2 |z|^\gamma z_i, \quad c(z) = \partial_j a_{ij}(z) = -8\pi \delta_0$$

We show

- Weak dissipativity of $B$ in many spaces (twisting trick, duality trick)

$$\langle B f, f \rangle_{H^2_x L^2(m)} \lesssim - \|f \|_{H^2_x H^1_*, \nu(m)}^2$$

$$\langle B f, f \rangle_{H^2_x H^1_\nu(m)} \lesssim - \|f \|_{H^2_x H^1(m, \nu(\gamma+2)/2)}^2$$

$$\langle B^* f, f \rangle_{H^2_x H^1_\nu(m)} \lesssim -...$$

- Decay estimate of $S_B$ in many spaces by Proposition 1.

- Regularization property of $S_B$ in many spaces by using “Hormander-Hérau-Villani” hypoelliptic trick. More precisely, introducing

$$F(t, h) := \|h\|_{L^2(m)}^2 + t^\bullet \|\nabla h\|_{L^2(m)}^2 + t^\bullet (\nabla h, \nabla h)_{L^2(m)} + t^\bullet \|\nabla h\|_{L^2(m)}^2$$

we get (for convenient exponents $\bullet \geq 1$)

$$\frac{d}{dt} F(t, S_B(t)h) \leq 0, \quad \forall t \in [0, 1].$$
and factorization trick

- \( A \in B(H^\alpha_x H^\beta_v(m_0), H^\alpha_x H^\beta_v(m_1)) \) for any weight functions \( m_1 \succeq m_0 \).
- In the space of self-adjointness \( L^2(\mu^{-1/2}) \) we have the nice dissipativity estimate

\[
\langle Lg, g \rangle_{L^2(\mu^{-1/2})} \lesssim -\| \Pi g \|^2_{H^1_x(\mu^{-1/2})}
\]

from which we deduce thanks to the twisting hypocoercivity Nier-Hérau-Villini trick

\[
\langle \langle \tilde{L}g, g \rangle \rangle_{H^1_{x,v}(\mu^{-1/2})} \lesssim -\| \tilde{\Pi} g \|^2_{H^1_x H^1_{x,v}(\mu^{-1/2})}
\]

We deduce

- \( S_{\tilde{\mathcal{L}}} \) is bounded in many spaces because \( S_{\tilde{\mathcal{L}}} \) is bounded in one space and \( \tilde{\mathcal{L}} \) splits in a suitable way (Proposition 3 of extension).
- \( S_{\tilde{\mathcal{L}}} \) is fast decaying in one space \( B(H^1_{x,v}(\mu^{-3/2}, H^1_{x,v}(\mu^{-3/2})) \) because it is bounded in \( H^1_{x,v}(\mu^{-3/2}) \) and weakly dissipative in \( H^1_{x,v}(\mu^{-1/2}) \) (Proposition 1).
- \( S_{\tilde{\mathcal{L}}} \) is decaying in many space because \( S_{\tilde{\mathcal{L}}} \) is decaying in one space and \( \tilde{\mathcal{L}} \) splits in a suitable way (Proposition 3 of extension).
As a conclusion, we are able to prove

- On the one hand,
  \[ \| S_L \|_{x \to x_0} \leq \Theta(t), \]
  and \( L = A + B \) with
  \[ \langle f, Bf \rangle_x \lesssim -\| f \|_Y^2, \quad \langle f, Af \rangle_x \lesssim -\| f \|_{x_0}^2 \]
  in order to use Proposition 2 and define the weak dissipative norm \( \| \cdot \|_x \) for \( \tilde{L} \)
- On the other hand,
  \[ t \mapsto \| S_L \|_{Y \to Y_0}, \| S_L \|_{Z \to Z_0} \in L^2(\mathbb{R}_+) \]

\( \triangleright \) That are the needed properties in order to get the a priori nonlinear estimate!
Open problems:

• Suitable spectral analysis theory in an abstract setting and a weakly dissipative framework?

• What about the Boltzmann equation without Grad’s cut-off (~ fractional diffusion in the velocity variable)?
  ▷ Work in progress by Hérau, Tonon, Tristani, ...

• What about the grazing collisions limit (from Boltzmann to Landau)?