On the long time asymptotic for the Growth-Fragmentation equation
a brief survey and a spectral analysis approach

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Outline of the talk

1. Introduction and brief survey
   - List of papers
   - Growth and fragmentation equation
   - Long time asymptotic
   - Main result: exponential rate of convergence
   - Several mathematical techniques

2. The spectral analysis approach
   - Spectral gap and semigroup decay
   - A Lyapunov condition
   - Some strong positivity conditions
   - the constant case

3. Open problems
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3 Open problems


Perthame, Ryzhik, *Exp. decay for the frag. or cell-division equation*, JDE (2005)


Caceres, Cañizo, M., *Rate of convergence to an asymptotic profile for the self-similar fragmentation and GF equations*, JMPA (2011), CAIM (2011)

Balagué, Cañizo, Gabriel, *Fine asymptotics of profiles and relaxation to equilibrium for growth-fragmentation equations with variable drift rates*, KRM (2013)


Bernard, Doumic, Gabriel, *Cyclic asymptotic behaviour of a population reproducing by fission into two equal parts*, arXiv 2016

And other papers by Banasiak, Bourgeron, Calvez, Doumic, Escobedo, Gabriel, Salvarani, ...
List of papers from the Probability community


And other papers by Berestycki, Bertoin, Miermont, Stephenson, Watson, ...
Growth and fragmentation equation

We will consider

$$\partial_t f = \Lambda f = D f + F f$$

on $f = f(t, x) \geq 0$ the number density of particles (or cells, polymers, organisms, individuals),
$t \geq 0$ is the time variable,
x $\in (0, \infty)$ is the size (or mass, age)

We take into account a fragmentation mechanism through

$$(F f)(x) := \int_x^\infty k(y, x)f(y)dy - K(x)f(x),$$

and possibly a growth mechanism by choosing $D = 0$ or

$$(D f)(x) := -\partial_x(\tau(x)f(x)) - \nu(x)f(x)$$
Fragmentation mechanism

The *fragmentation* operator writes

\[ \mathcal{F} := \mathcal{F}^+ - \mathcal{F}^-, \quad (\mathcal{F}^+ f)(x) := \int_x^\infty k(y, x) f(y) dy, \quad (\mathcal{F}^- f)(x) := K(x) f(x), \]

with fragmentation kernel \( k \) and total rate of fragmentation \( K \) related by

\[ K(x) = \int_0^x k(x, y) \frac{y}{x} dy. \]

Modeling the division (breakage) of a single *mother particle* of size \( x > 0 \) into two or more pieces (*daughter particles, offspring*) of size \( x_i > 0 \), conserving the mass

\[ \{x\} \xrightarrow{k} \{x_1\} + \ldots + \{x_i\} + \ldots, \quad x = \sum x_i. \]

We observe that

\[ (\mathcal{F}^* \phi)(x) = \int_0^x k(x, y) [\phi(y) - \frac{y}{x} \phi(x)] dy. \]

As a consequence, for \( \varphi_\alpha = x^\alpha \), there hold

\[ \mathcal{F}^* \varphi_0 > 0, \quad \mathcal{F}^* \varphi_1 = 0, \quad \mathcal{F}^* \varphi_2 < 0. \]

⇒ Mass is conserved and particles are produced.
Growth mechanism

The growth operator models the growth (for particles and cells) or the aging (for individuals) and the death. It writes

$$(Df)(x) := -\partial_x (\tau(x)f(x)) - \nu(x)f(x),$$

with drift speed (or growth rate) function $\tau : [0, \infty) \to \mathbb{R}$ and a damping rate $\nu : [0, \infty) \to [0, \infty)$. Schematically

$${\{x\} \xrightarrow{e^{-\nu(x)}} \{x + \tau(x) \, dx\}}.$$ 

Observing

$$(D^\ast \phi)(x) = \tau(x)\partial_x \phi(x) - \nu(x) \phi(x),$$

the only invariant is

$$\phi(x) = \phi_0 e^{\int_x^x \nu(y)/\tau(y) \, dy} \neq \varphi_1$$

except when $\tau(x)/\nu(x) = x$. For instance: $(Df)(x) := -\partial_x (xf(x)) - f(x)$

We will take $\tau(x) = 1$ (constant growth) or $\tau(x) = x$ (self-similar growth).
Well-posedness, conservation and steady state

The growth-fragmentation operator generates a positive and $C_0$-semigroup $S_\Lambda$ in $L^1$.

From now, we exclude singular fragmentation kernels at the origine (⇒ shattering phenomenon = lost of mass) of the discussion.

Questions:

- Is $S_\Lambda^*$ Markov? = conservation law?
- ∃ of invariant measure for $S_\Lambda^*$? = ∃ of steady state for (GF) equation?

First answers:

- In general, no trivial conservation law (except $\varphi_1$ when $D = 0$).
- In general, no explicit steady state.
- For the pure fragmentation equation ($D = 0$) no steady state. But for a total rate $K(x) = x^\gamma$, $\gamma > 0$, we may change variables into "self-similar" variables (which adds a new growth operator $\mathcal{D}f := -\partial_x(xf) - f$) such that the new equation have a steady state (a self-similar profile for the initial equation). We denote this model as the (SSF) equation.
A complete answer thanks to Krein-Rutman (Perron-Frobenius) theory

Krein-Rutmann theory says

$$\exists (\lambda, G, \phi), \quad \lambda \in \mathbb{R}, \ G > 0, \ \phi > 0, \ (\Lambda - \lambda)G = 0, \ (\Lambda^* - \lambda)\phi = 0.$$  

- Finite dimensional approximation and compactness argument
- Semigroup and compactness argument

Up to a change of unknown, we may then assume

$$\exists (G, \phi), \ G > 0, \ \phi > 0, \ \Lambda G = 0, \ \Lambda^* \phi = 0.$$  

As a consequence, any solution $f$ to the GF equation satisfies

$$\frac{d}{dt} \int f \phi = 0, \quad \frac{d}{dt} \int j\left(\frac{f}{G}\right)G\phi = -D_j(f) \leq 0$$

for any $j : \mathbb{R} \rightarrow \mathbb{R}$ convex function, with

$$D_j(f) := \iint k_\ast G_\ast \phi(j(u) - j(u_\ast) - j'(u_\ast)(u - u_\ast)) \, dx \, dx_\ast, \quad u := \frac{f}{G}.$$
Main result: exponential rate of convergence

Assume (for simplicity)

\[
k(x, y) = K(x) \varphi(y/x)/x, \quad \int_0^1 z \varphi(dz) = 1;
\]

\[
K(x) \sim x^\gamma, \quad \gamma \geq 0;
\]

\((GF)\) \quad \tau(x) = 1 \quad \text{and} \quad \varphi \text{ smooth and positive or } \varphi = \delta_{1/2};

\((SSF)\) \quad \tau(x) = x \quad \text{and} \quad \varphi \text{ smooth and positive}.

**Theorem**

There exist \(a < 0, \ C \geq 1\) and a weight function \(m : [0, \infty) \to [1, \infty)\) such that

\[
\|S_\Lambda(t)f_0 - \Pi_{\Lambda,0}f_0\|_{L^1_m} \leq C e^{at} \|f_0 - \Pi_{\Lambda,0}f_0\|_{L^1_m}
\]

for any \(f_0 \in L^1_m\), with \(\Pi_{\Lambda,0}f := G \langle f, \varphi \rangle\).

Furthermore, \(a, C\) are constructive when \(K = \text{cst}\) as well as in the (SSF) case.
Several mathematical techniques for convergence and rate of convergence

- compactness argument + Lyapunov/dissipation of entropy
  - Escobedo, M., Rodriguez (2005); Michel, M., Perthame (2005), Bernard, Doumic, Gabriel (arXiv 2016)

- ad hoc $W_1$ distance when $K \sim \text{cst}$
  - Perthame, Ryzhik (2005), Laurençot, Perthame (2009)

- dissipation of entropy-entropy inequality

- spectral analysis of semigroup
  - M., Scher (2016)

- direct Laplace / Mellin analysis
  - Doumic, Escobedo (2016); Bertoin, Watson (arXiv 2017)

* with constructive constants
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3. Open problems
Spectral gap and semigroup decay

In the spirit of Perron-Frobenius (∼1907) and Krein-Rutman (1948) theory

Positive semigroup theory by “german school” (Arendt, Engel, Grabosch, Greiner, Groh, Nagel, Voigt, ... ∼ 80's)

The Harris-Meyn-Tweedie-Down theory about Markov chain and semigroup (1956–90's) (revisited by Hairer-Mattingly in 2011)

From positivity and conservation, we know that $S_\Lambda$ is a SG of contractions

$$\Rightarrow \Sigma(\Lambda) \cap \Delta_0 = \emptyset, \quad \Delta_a := \{ z \in \mathbb{C}, \, \Re e z > a \}. $$

We also know that $0 \in \Sigma(\Lambda)$ and $G \in E_0 :=$ eigenspace associated to the eigenvalue 0.

Spectral analysis issues:

$\triangleright $ $E_0 = \text{Vect}(G) \,$? $\Sigma(\Lambda) \cap \Delta_a = \{ 0 \}$ for some $a < 0$?

Semigroup issue:

$\triangleright $ deduce the corresponding semigroup decay ($\sim$ quantified spectral mapping theorem)?
Λ satisfies a “Lyapunov condition”

**Proposition**

There exist $m : [0, \infty) \to [1, \infty)$, $m(x) \to \infty$ as $x \to \infty$, and $a < 0$, $M \geq 0$, such that

$$\Lambda^* m \leq am + M$$

For the GF operator, assuming

$$K_0 x^\gamma \leq K(x) \leq K_1 x^\gamma, \quad \forall x \geq x_1,$$

and taking $\beta > \beta^*$ such that

$$\varphi_{\beta^*} = K_0 / K_1 \in (0, 1], \quad \varphi_{\beta} := \int_0^1 z^\beta \varphi(dz),$$

we may choose $m(x) = e^{-K(x)}1_{x \leq x_1} + x^\beta 1_{x \geq x_1}$.

For the SSF operator, we may choose $m(x) = x^\alpha 1_{x \leq 1} + \eta x^\beta 1_{x \geq 1}$, whatever are $\alpha < 1 < \beta$ and $\eta > 0$ small enough.

Key point: for ”large” particles fragmentation dominates growth while the inverse holds for ”small” particles.
Strong maximum principle and uniqueness of the steady state

**Lemma (strong MP).**

For any solution to $\Lambda f = 0$, $f \geq 0$, there holds $f \equiv 0$ or $f > 0$.

**Proof:** When $f \not\equiv 0$, we have

$$\tau(x)\partial_x f + (K(x) + \mu)f \geq \mathcal{F}^+ f \geq 0, \not\equiv 0,$$

and we spread out positivity.

**Corollary (uniqueness).**

$E_0 = \text{Vect}(G)$

**Proof:** Consider $f$ another steady state. We may reduce to the case when $f$ is nonnegative and has unit mass. Then $g := f - G$ satisfies $\Lambda g = 0$. In particular,

$$\Lambda g_+ \geq (\text{sign}_+ g)\Lambda g = 0 \quad \text{and} \quad \int (\Lambda g_+)\phi = \int g_+(\Lambda^*\phi) = 0,$$

so that $\Lambda g_+ = 0$. From the strong MP, we deduce $g_+ = 0$ or $g_+ > 0$. In the second case, we get $g > 0$ and then

$$1 = \langle |f| \rangle \geq \langle f \rangle > \langle G \rangle = 1 \quad \text{absurd!}$$

In a similar way, we have $g_- = 0$ and we conclude with $f - G = g = 0$. 
Lemma (Strong Kato’s inequality).

The case of saturation in Kato’s inequality

\[ \Lambda |f| = \Re(\text{sign} f) \Lambda f \]

implies \( \exists u \in \mathbb{C} \) such that \( f = u |f| \).

Fails in the case \( \tau(x) = x \) and \( \varphi = \delta_{1/2} \)!

Corollary (about the spectrum on the imaginary axis).

There is no other eigenvalue on \( i\mathbb{R} \): \( \Sigma(\Lambda) \cap i\mathbb{R} = \{0\} \) and 0 is (algebraically) simple.

Proof: If \( \Lambda f = \mu f \) with \( \Re \mu = 0 \), we write

\[ 0 = (\Re \mu) |f| = \Re(\text{sign} f) \Lambda f \leq \Lambda |f| \]

and then \( \Lambda |f| = 0 \) by integration. We may apply the strong Kato’s inequality to get \( f = u |f| \) and then \( \Lambda f = 0 \). That implies \( \mu = 0 \).
In $X := L^1_m$, we split
\[ \Lambda = A + B, \quad A := F_R^+, \]
and we proved (using in particular Proposition 1) that for some $a^* < 0$
\[ t \mapsto S_B \ast (A S_B)^{(*k)}(t) e^{-a^* t} \in L^\infty(\mathbb{R}_+; B(X)), \quad \forall k \geq 0, \]
\[ t \mapsto (A S_B)^{(*2)}(t) e^{-a^* t} \in L^\infty(\mathbb{R}_+; B(X, Y)), \quad Y \subset D(\Lambda^{1/2}), \ Y \subset\subset X. \]

From Voigt’s power compact version of Weyl’s theorem, we have
\[ Y \subset\subset X \quad \text{implies} \quad \Sigma(\Lambda) \cap \Delta_{a^*} \quad \text{is discrete.} \]

Similarly, we have
\[ Y \subset D(\Lambda^{1/2}) \quad \text{implies} \quad \Sigma(\Lambda) \cap \Delta_{a^*} \quad \text{is bounded, and thus finite!} \]

Conclusion: $\exists \ a < 0$ such that $\Sigma(\Lambda) \cap \Delta_a = \{0\}$ and $\Pi := \langle \cdot, \phi \rangle G$. 
More details about the spectral gap via Weyl’s theorem

**Lemma (spectral gap).**

There is $a < 0$ such that $\Sigma(\Lambda) \cap \Delta_a = \{0\}$.

**Proof.** For an generator $L$ we define the resolvent operator

$$R_L(z) = (L - z)^{-1} = -\int_0^\infty S_L(t) e^{-zt} dt.$$  

From $\Lambda = A + B$, we get

$$R_\Lambda = R_B - R_\Lambda AR_B = R_B - R_B AR_B + R_\Lambda (AR_B)^2$$

from what we deduce

$$R_\Lambda(z)(1 - (AR_B(z))^2) = R_B(z) - R_B(z)AR_B(z).$$

- From

$$\|AR_B(z)f_0\|_Y^2 \leq \int_0^\infty \|AS_B(t)f_0\|_Y^2 e^{-2at} dt \leq C_a \|f_0\|_X^2, \quad \forall f_0 \in X, \; z \in \Delta_a,$$

we get the estimate

$$AR_B(z) : X \to Y \text{ as } O(1), \quad \forall z \in \Delta_a, \quad a < 0.$$
End of the proof of the spectral gap

- On the one hand, together with the interpolation estimate

\[
\begin{align*}
R_B(z) : X_1 &\to X \text{ as } \mathcal{O}(\langle z \rangle^{-1}) \\
R_B(z) : X &\to X \text{ as } \mathcal{O}(1)
\end{align*}
\]

imply

\[
R_B(z) : X_{1/2} \to X \text{ as } \mathcal{O}(\langle z \rangle^{-1/2}),
\]

and observing that \( Y \subset X_{1/2} \), we deduce

\[
(AR_B(z))^2 = A R_B(z)(AR_B(z)) : X \to X \text{ as } \mathcal{O}(\langle z \rangle^{-1/2}).
\]

In particular, \( I - (AR_B(z))^2 \) is invertible in \( \Delta_a \cap B(0, M)^c \) for \( M > 1 \) large.

- On the other hand, because \( Y \subset X \) with compact embedding, the operator \( I - (AR_B(z))^2 \) is an analytic and compact perturbation of the identity, and the Ribarič-Vidav-Voigt’s version of Weyl’s theorem implies that

\[
\Sigma(\Lambda) \cap \Delta_a = \Sigma_d(\Lambda) \cap \Delta_a = \text{discrete set}.
\]

- Both information together, we have

\[
\Sigma(\Lambda) \cap \Delta_a = \Sigma_d(\Lambda) \cap \Delta_a = \text{finite set}.
\]

We conclude by using that \( \Sigma(\Lambda) \cap \bar{\Delta}_0 = \{0\} \).
Lemma (semigroup decay in $L^1_m$).

Defining $\Pi g := G\langle g, \phi \rangle$, there holds

$$\|S_\Lambda(t)(I - \Pi)\|_{X \to X} \lesssim e^{at}, \quad \forall \ t \geq 0, \ \forall \ a > a^*.$$  

Proof. We set $\Pi^\perp = I - \Pi$ and we write

$$S_\Lambda(t)\Pi^\perp = \Pi^\perp\{S_B + \ldots + S_B \ast (AS_B)^{(*n-1)} + S_\Lambda \ast (AS_B)^{*n}\}$$

$$\approx \Pi^\perp\{S_B + \ldots + S_B \ast (AS_B)^{(*n-1)}\} + \int_{t_a} \Pi^\perp R_\Lambda(z)(AR_B)^n e^{zt} \, dz.$$  

Because $\|\Pi^\perp R_\Lambda(z)\|$ is uniformly bounded on $\tilde{\Delta}_a$, and $\|(AR_B)^n(z)\| \lesssim \langle z \rangle^{-3/2}$, we obtain that each term is of order $O(e^{at})$. 

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2nd strong positivity condition and dissipation of entropy-entropy inequality

We may prove (with constructive constants in the (SSF) case)

\[ A_1 e^{-\kappa_1 \Lambda(x)} \leq G(x) \leq A_0 e^{-\kappa_0 \Lambda(x)}, \quad \kappa_i, A_i > 0, \]

with \( \Lambda(x) = x^\gamma \) in the (SSF) case and \( \Lambda(x) = x^{\gamma+1} \) in the (GF) case. We also recall that \( \phi(x) = x \) in the (SSF) case and in the (GF) case, we may prove

\[ C_\alpha (1 + x)^\alpha \leq \phi(x) \leq C (1 + x), \quad \forall \alpha \in (0, 1). \]

We recall

\[
H_2(f \mid G) := \int (u - 1)^2 G \phi = \| f - G \|_{L^2_\Omega}^2, \quad u := \frac{f}{G},
\]

\[
-D_2(f \mid G) := -\frac{d}{dt} H_2(f \mid G) = \iint k_* G_* \phi (u - u_*)^2 dxdx_*,
\]

**Proposition** In both (GF) and (SSF) cases, when \( \gamma \in (0, 2) \) and \( \phi \) is smooth

\[ \exists a < 0, \quad D_2(f \mid G) \geq (-a) H_2(f \mid G). \]

**Corollary** In both (GF) and (SSF) cases, when \( \gamma \in (0, 2) \) and \( \phi \) is smooth

\[ \| S_\Lambda(t)f_0 - \Pi f_0 \|_{L^2_\Omega}^2 \leq e^{at} \| f_0 - \Pi f_0 \|_{L^2_\Omega}^2, \quad \forall t \geq 0. \]
For a normalized eigenvalue-eigenfunction \((\xi, f)\) with \(\xi \in \Delta_a \cap \Sigma(\Lambda) \subset B(0, R)\), we have
\[
\langle |f|, \phi \rangle = 1, \quad \|f\|_{L^1} \leq C, \quad \|f\|_{W^{1,\infty}(\delta, \delta^{-1})} \leq C.
\]
When \(\phi\) is smooth, we deduce
\[
D_1(f|G) := \Re\langle \Lambda|f| - (\Lambda f)\text{sign} f, \phi \rangle \geq \kappa := -a^{**} > 0,
\]
with constructive constant when \(\phi(x) = x\).
As a consequence:
\[
\Re \xi \langle |f|, \phi \rangle = \Re\langle \xi f \text{sign} f, \phi \rangle = \Re\langle (\Lambda f) \text{sign} f, \phi \rangle \leq \langle \Lambda|f|, \phi \rangle + a^{**}
\]
and then \(\Re \xi \leq a^{**}\). As a consequence, \(\Delta_{a^{**}} \cap \Sigma(\Lambda) = \{0\}\).

**Corollary**

Constructive rate of convergence for (SSF) case when \(\gamma > 0\) and \(\phi\) is smooth.
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Open problems:

- Constructive exponential rate of convergence in the case $\gamma \geq 0$
  > Generalize M. & Scher’s spectral analysis approach to the more general framework of Balagué, Cañizo, Doumic, Gabriel?
  > Make all the constants constructive in the upper and lower bound of $G$?
  > Restriction on $\gamma$? Prove first $D(f \mid G) \geq \|f - G\|_*^2$ for a weaker norm?
  > Make all the constants constructive in the positivity and regularity estimates on an eigenvector $f$ when the associated eigenvalue $\xi \in \Delta_a$?

- Meyn-Tweedie approach: Is is true
  \[
  \forall C, R > 0, \exists T, \kappa > 0, (S_\Lambda(T)f_0)(x) \geq \kappa, \forall x \in (0, R)
  \]
  for any $f_0 \geq 0, \langle f_0, \phi \rangle = 1, \|f_0\|_{L^1_m} \leq C$?

- Beyond spectral gap
  > polynomial rate of convergence when $\gamma < 0$? (subgeometric framework)
  > rate of convergence for the ergodic behavior in the critical case (SSF) equation with $\varphi = \delta_{1/2}$?