# Hydrodynamic Limits
# for the Boltzmann Equation

Claude Bardos,- François Golse-Diogo Arsenio

March 15, 2012

## Orders of magnitude, perfect gas

- For a monatomic gas at room temperature and atmospheric pressure, about $10^{20}$ gas molecules with radius $\simeq 10^{-8}$cm are to be found in any volume of $1$cm$^3$

- Excluded volume (i.e. the total volume occupied by the gas molecules if tightly packed): $10^{20} \times \frac{4\pi}{3} \times (10^{-8})^{-3} \simeq 5 \cdot 10^{-4}$cm$^3 \ll 1$cm$^3$

EXCLUDED VOLUME NEGLIGEABLE $\Rightarrow$ PERFECT GAS

- Equation of state for a perfect gas:

$p = k\rho\theta$, where $k$ =Boltzmann's constant$= 1.38 \cdot 10^{-23}$J/K

## Notion of mean-free path

•Roughly speaking, the average distance between two successive collisions for any given molecule in the gas

•Intuitively, the higher the gas density, the smaller the mean-free path; likewise, the bigger the molecules, the smaller the mean-free path; this suggests

$$\text{mean-free path} \approx \frac{1}{\mathcal{N} \times \mathcal{A}}$$

where $\mathcal{N}$ =number of gas molecules per unit volume and $\mathcal{A}$ =area of the section of any individual molecule

•For the same monatomic gas as before (at room temperature and atmospheric pressure), $\mathcal{N} = 10^{20}$ molecules/cm$^3$, while $\mathcal{A} = \pi \times (10^{-8})^2 \simeq 3 \cdot 10^{-16}$cm$^2$; hence the mean-free path is $\approx \frac{1}{3} \cdot 10^{-4}$cm $\ll 1$cm.

SMALL MEAN-FREE PATH REGIMES CAN OCCUR IN PERFECT GASES

•While keeping the same temperature, lower the pressure at $10^{-4}$atm; then $\mathcal{N} = 10^{16}$ molecules/cm$^3$ and the mean-free path becomes $\approx \frac{1}{3}$cm which is comparable to the size of the $1$cm$^3$ container

DEGREE OF RAREFACTION MEASURED BY KNUDSEN NUMBER

$$\mathrm{Kn} := \frac{\text{mean free path}}{\text{macroscopic length scale}}$$

## Kinetic vs. fluid regimes

•Kinetic regimes are characterized by $\boxed{\mathsf{Kn} = O(1)}$; since the gas is more rarefied, there are not enough collisions per unit of time for a local thermodynamic equilibrium to be reached. However, also because of rarefaction, correlations are weak $\Rightarrow$ state of the gas is adequately described by the distribution function:

$F \equiv F(t, x, v)$ single-particle phase-space density

The density (with respect to the Lebesgue measure $dxdv$) of particles which, at time $t$, are to be found at the position $x$ with velocity $v$.

## Macroscopic observables

•Macroscopic quantities (observables) are computed by averaging the corresponding quantity for a single particle w.r.t. the measure $F(t,x,v)dxdv$:

$$\text{density} = \int F(x,t,v)dv = \rho(x,t)$$

$$\text{momentum} = \int vF(t,x,v)dv = \rho(x,t)u(x,t),$$

$$\text{energy-temperature} = \int \tfrac{1}{2}|v|^2 F(t,x,v)dv = \rho(x,t)(\frac{1}{2}|u(x,t)|^2 + \frac{3}{2}\theta(x,t))$$

## The Boltzmann equation

•The number density $F$ is governed by the Boltzmann equation: in the absence of external force

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F)$$

where $\mathcal{B}(F, F)$ is the Boltzmann collision integral.

•Because of rarefaction, collisions other than BINARY are neglected.

•At the kinetic level of description, the size of particles is neglected everywhere but in the expression of the mean-free path: collisions are LOCAL and INSTANTANEOUS

$\Rightarrow$ $\mathcal{B}(F, F)$ operates only on the $v$-variable in $F$

## The collision integral -hard potential.

$$\mathcal{B}(F,F)(v)=\iint_{\mathbf{R}^3\times\mathbf{S}^2}\Big(F(v')F(v'_*)-F(v)F(v_*)\Big)\,b(v-v_*,\omega)d\omega dv_*$$

where the velocities $v'$ and $v'_*$ are defined in terms of $v$, $v_*$ and $\omega$ by

$$v' \equiv v'(v,v_*,\omega) = v-(v-v_*)\cdot\omega\omega$$
$$v'_* \equiv v'_*(v,v_*,\omega) = v_*+(v-v_*)\cdot\omega\omega$$

$(F_*, F'$ and $F'_*) = (F(v_*), F(v')$ and $F(v'_*)$

$$b(\omega, v-v^*) = |v-v^*|^{\beta}\widehat{b}(\omega.n)\,, n = \frac{(v-v^*)}{|v-v^*|}$$

Grad cutt off $\int_{\mathbf{S}^2}\widehat{b}(\omega.n)d\omega<\infty$ . hard spheres $b(\omega,v-v^*)=|(v-v^*)\cdot\omega|$

## Pre- to post-collision relations

•Given any velocity pair $(v, v_*) \in \mathbf{R}^6$, the pair $(v'(v, v_*, \omega), v'_*(v, v_*, \omega))$ runs through the set of solutions to the system of 4 equations

| | |
|---|---|
| $v' + v'_* = v + v_*$ | conservation of momentum |
| $\|v'\|^2 + \|v'_*\|^2 = \|v\|^2 + \|v_*\|^2$ | conservation of kinetic energy |

as $\omega$ runs through $\mathbf{S}^2$.

## Geometric interpretation of collision relations

The geometric interpretation of these formulas is as follows: in the reference frame of the center of mass of the particle pair, the velocity pair before and after collisions is made of two opposite vectors, $\pm\frac{1}{2}(v' - v'_*)$ and $\pm\frac{1}{2}(v - v_*)$.

Conservation of energy implies that $|v - v_*| = |v' - v'_*|$.

•Hence $v - v_*$ and $v' - v'_*$ are exchanged by some orthogonal symmetry, whose invariant plane is orthogonal to $\pm\omega$.

## Symmetries of the collision integral

•The collision integrand is invariant if one exchanges $v$ and $v_*$:

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F)\phi dv = \iiint (F'F'_* - FF_*)\phi b(v-v_*,\omega)d\omega dv_* dv$$
$$= \iiint (F'F'_* - FF_*)\frac{\phi+\phi_*}{2} b(v-v_*,\omega)d\omega dv_* dv$$

•The collision integrand is changed into its opposite if, given $\omega \in \mathbf{S}^2$, one exchanges $(v,v_*)$ and $(v',v'_*)$ (in the center of mass reference frame, this is a symmetry, and thus an involution).

•Further, $(v,v_*) \mapsto (v',v'_*)$ is an isometry of $\mathbf{R}^6$ (conservation of kinetic energy), so that $\boxed{dv dv_* = dv' dv'_*}$.

# Symmetries of the collision integral 2

**Theorem.** *Assume that* $F \in L^1(\mathbf{R}^3)$ *is rapidly decaying at infinity, i.e.*

$$F(v) = O(|v|^{-n}) \text{ as } |v| \to +\infty \text{ for all } n \geq 0$$

*while* $\phi \in C(\mathbf{R}^3)$ *has at most polynomial growth at infinity, i.e.*

$$\phi(v) = O(1 + |v|^m) \text{ as } |v| \to +\infty \text{ for some } m \geq 0$$

*Then, one has:*

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F)\phi dv = \iiint FF_* \frac{\phi + \phi_* - \phi' - \phi'_*}{2} |(v - v_*) \cdot \omega| d\omega dv_* dv$$

$$= \iiint_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2} (F'F'_* - FF_*) \frac{\phi + \phi_* - \phi' - \phi'_*}{4} |(v - v_*) \cdot \omega| d\omega dv_* dv$$

## Collision invariants

•These are the functions $\phi \equiv \phi(v) \in C(\mathbf{R}^3)$ such that

$$\phi(v) + \phi(v_*) - \phi(v') - \phi(v'_*) = 0 \text{ for all } (v, v_*) \in \mathbf{R}^3 \text{ and } \omega \in \mathbf{S}^2$$

**Theorem.** *Any collision invariant is of the form*

$$\phi(v) = a + b_1 v_1 + b_2 v_2 + b_3 v_3 + c|v|^2 , \quad a, b_1, b_2, b_3, c \in \mathbf{R}$$

•If $\phi$ is any collision invariant and $F \in L^1(\mathbf{R}^3)$ is rapidly decaying, then

$$\int_{\mathbf{R}^3} \mathcal{B}(F, F)\phi dv = 0$$

Proof of the Theorem (Perthame).

Assume $\phi \geq 0$ with $(1 + |vl^2)\phi \in L^1(\mathbf{R}^3)$ with

$$\phi' \phi'_* = \phi \phi_* \text{ for a.e. } (v, v_*, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$$

Normalize with

$$\int_{\mathbf{R}^3} \phi(v) dv = 1 \qquad \int_{\mathbf{R}^3} v\phi(v) dv = 0$$

Fourier transform (with $\omega$ fixed) of $\phi\phi_*$ is continuous.

$$\widehat{\phi}(\xi)\widehat{\phi}(\xi_*) = \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \phi(v)\phi(v_*) e^{-i\xi v - i\xi_* v_*} dv dv_*$$

$$= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \phi(v')\phi(v'_*) e^{-i\xi v - i\xi_* v_*} dv dv_*$$

$$= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \phi(v)\phi(v_*) e^{-i\xi v' - i\xi_* v'_*} dv' dv'_*$$

$$= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \phi(v)\phi(v_*) e^{-i\xi v - i\xi_* v_*} e^{i((\xi - \xi_*)\cdot\omega)((v - v_*)\cdot\omega)} dv dv_*$$

Since the first term is independant of $\omega$ differentiating with respect to $\omega$ gives for $(\xi - \xi_*)\perp\omega$

$$0 = \iint_{\mathbf{R}^3\times\mathbf{R}^3} \phi(v)\phi(v_*)e^{-i\xi v - i\xi_* v_*}(v - v_*)\cdot\omega dv dv_*$$

for any $\xi \neq \xi_*$ and $\omega \in \mathbf{S}^2$ such that $\omega\perp(\xi_{-}\xi_*)$

$$\omega\perp(\xi - \xi_*) \Rightarrow (\nabla_\xi - \nabla_{\xi^*})\widehat{\phi}(\xi)\widehat{\phi}(\xi_*)\perp\omega$$
$$\Rightarrow (\nabla_\xi - \nabla_{\xi^*})\widehat{\phi}(\xi)\widehat{\phi}(\xi_*)\|(\xi - \xi_*)$$
$$\phi\widehat{(0)} = 1\,, \nabla_\xi\phi\widehat{(0)} = 0 \Rightarrow \nabla_\xi\phi(\xi)\|\xi \Rightarrow \phi(\xi) = \psi(|\xi|^2)$$
$$\xi\psi'(|\xi|^2)\psi(|\xi_*|^2) - \xi_*\psi(|\xi|^2)\psi'(|\xi_*|^2)\|(\xi - \xi^*)$$
$$\Rightarrow \psi'(|\xi|^2)\psi(|\xi_*|^2) = \psi(|\xi|^2)\psi'(|\xi_*|^2)$$
$$\Rightarrow \psi'(|\xi|^2) = \beta\psi(|\xi|^2)$$

## Local conservation laws

•In particular, if $F \equiv F(t,x,v)$ is a solution to the Boltzmann equation that is rapidly decaying in the $v$-variable

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F)dv = \int_{\mathbf{R}^3} v_k \mathcal{B}(F,F)dv = \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 \mathcal{B}(F,F)dv = 0$$

for $k = 1,2,3$.

•Therefore, one has the local conservation laws:

$$\begin{aligned}
\partial_t \int_{\mathbf{R}^3} F dv + \operatorname{div}_x \int_{\mathbf{R}^3} vF dv &= 0\,, \quad \text{(mass)}\\
\partial_t \int_{\mathbf{R}^3} vF dv + \operatorname{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} F dv &= 0\,, \quad \text{(momentum)}\\
\partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 F dv + \operatorname{div}_x \int_{\mathbf{R}^3} v\tfrac{1}{2}|v|^2 F dv &= 0\,, \quad \text{(energy)}
\end{aligned}$$

## Boltzmann's H Theorem

•Assume that $0 < F \in L^1(\mathbf{R}^3)$ is rapidly decaying and such that $\ln F$ has polynomial growth at infinity. Then

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F) \ln F dv =$$
$$-\frac{1}{4}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) |(v-v_*)\cdot\omega| d\omega dv dv_* \leq 0$$

•The following conditions are equivalent:

$$\int_{\mathbf{R}^3} \mathcal{B}(F,F) \ln F dv = 0 \;\Leftrightarrow\; \mathcal{B}(F,F) = 0 \text{ a.e. } \;\Leftrightarrow\; F \text{ is a Maxwellian}$$

i.e. $F(v)$ is of the form

$$F(v) = M_{\rho,u,\theta}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}} \text{ for some } \rho,\theta > 0 \text{ and } u \in \mathbf{R}^3$$

## Implications of conservation laws + H Theorem

•If $F \equiv F(t,x,v) > 0$ is a solution to the Boltzmann equation that is rapidly decaying and such that $\ln F$ has polynomial growth in the $v$-variable, then

$$\begin{aligned}
\partial_t \int_{\mathbf{R}^3} F dv + \mathrm{div}_x \int_{\mathbf{R}^3} vF dv = 0\,, &\quad \text{(mass)}\\
\partial_t \int_{\mathbf{R}^3} vF dv + \mathrm{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} F dv = 0\,, &\quad \text{(momentum)}\\
\partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 F dv + \mathrm{div}_x \int_{\mathbf{R}^3} v\tfrac{1}{2}|v|^2 F dv = 0\,, &\quad \text{(energy)}\\
\partial_t \int_{\mathbf{R}^3} F \ln F dv + \mathrm{div}_x \int_{\mathbf{R}^3} vF \ln F dv \le 0\,, &\quad \text{(entropy)}
\end{aligned}$$

The last differential inequality bearing on the entropy density is reminiscent of the Lax-Friedrichs entropy condition that selects admissible solutions of hyperbolic systems of conservation laws.

## Dimensionless form of the Boltzmann equation

•Choose macroscopic scales of time $T$ and length $L$, and a reference temperature $\Theta$; this defines 2 velocity scales:

$$V = \frac{L}{T} \text{ (macroscopic velocity)} \ , \qquad \text{and } c = \sqrt{\Theta} \text{ (thermal speed)}$$

Finally, set $\mathcal{N}$ to be the total number of particles.

•Define dimensionless time, position, and velocity variables by

$$\hat{t} = \frac{t}{T}\,, \quad \hat{x} = \frac{x}{L}\,, \quad \hat{v} = \frac{v}{c}$$

and a dimensionless number density

$$\hat{F}(\hat{t}, \hat{x}, \hat{v}) = \frac{L^3 c^3}{\mathcal{N}} F(t, x, v)$$

## Dimensionless form of the Boltzmann equation 2

- One finds that

$$\frac{L}{cT}\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \frac{\mathcal{N}r^2}{L^2}\iint(\hat{F}'\hat{F}'_* - \hat{F}\hat{F}_*)|(\hat{v}-\hat{v}_*)\cdot\omega|d\omega d\hat{v}_*$$

- The pre-factor multiplying the collision integral is

$$L \times \frac{\mathcal{N}r^2}{L^3} = \frac{L}{\pi \times \text{mean free path}} = \frac{1}{\pi \mathsf{Kn}}$$

- The pre-factor multiplying the time derivative is

$$\frac{\frac{1}{T}\times L}{c} = \mathsf{St}\,, \qquad \text{(kinetic Strouhal number)}$$

$$\mathsf{St}\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \frac{1}{\pi\mathsf{Kn}}\iint_{\mathbf{R}^3\times\mathbf{S}^2}(\hat{F}'\hat{F}'_* - \hat{F}\hat{F}_*)|(\hat{v}-\hat{v}_*)\cdot\omega|d\omega d\hat{v}_*$$

## Compressible Euler scaling

•This scaling limit corresponds to $\boxed{\mathrm{St} = 1}$ and $\boxed{\pi\mathrm{Kn} =: \epsilon \ll 1}$, leading to the singular perturbation problem

$$\boxed{\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon) = \frac{1}{\epsilon}\iint (F'_\epsilon F'_{\epsilon *} - F_\epsilon F_{\epsilon *})|(v - v_*) \cdot \omega| d\omega d\hat{v}_*}$$

•One expects that, as $\epsilon \to 0$, $F_\epsilon \to F$ and $\mathcal{B}(F_\epsilon, F_\epsilon) \to \mathcal{B}(F, F) = 0$; hence $F(t, x, \cdot)$ is a Maxwellian for all $(t, x)$, i.e.

$$\boxed{F(t, x, v) = M_{\rho(t,x),u(t,x),\theta(t,x)}(v) = \frac{\rho(t, x)}{(2\pi\theta(t, x))^{3/2}} e^{-\frac{|v-u(t,x)|^2}{2\theta(t,x)}}}$$

In other words, $F$ is a local equilibrium.

•<u>Problem:</u> to find the governing equations for $\rho(t, x)$, $u(t, x)$ and $\theta(t, x)$.

## Formal Euler limit by the moment method

•Assume that $F_\epsilon$ is rapidly decaying and such that $\ln F_\epsilon$ has polynomial growth for large $v$'s; assume further that $F_\epsilon \to F$, and that the decay properties above are uniform in this limit.

•H Theorem implies that $F$ is a local Maxwellian $M_{\rho,u,\theta}$:

$$\int_0^{+\infty}\iint \mathcal{B}(F_\epsilon, F_\epsilon) \ln F_\epsilon dvdxdt = \epsilon \iint F_\epsilon \ln F_\epsilon\Big|_{t=0} dxdv - \epsilon \varliminf_{t\to+\infty} \iint F_\epsilon \ln F_\epsilon\Big|_t dxdv \to 0$$

as $\epsilon \to 0$; hence

$$\int_0^{+\infty}\iint \mathcal{B}(F, F) \ln F dvdxdt = 0$$

•Passing to the limit in the local conservation laws + the entropy differential inequality leads to the system of conservation laws for $(\rho, u, \theta)$

$$\partial_t \int_{\mathbf{R}^3} M_{\rho,u,\theta} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v M_{\rho,u,\theta} dv = 0$$
$$\partial_t \int_{\mathbf{R}^3} v M_{\rho,u,\theta} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v^{\otimes 2} M_{\rho,u,\theta} dv = 0$$
$$\partial_t \int_{\mathbf{R}^3} \tfrac{1}{2}|v|^2 M_{\rho,u,\theta} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v \tfrac{1}{2}|v|^2 M_{\rho,u,\theta} dv = 0$$

as well as the differential inequality

$$\partial_t \int_{\mathbf{R}^3} M_{\rho,u,\theta} \ln M_{\rho,u,\theta} dv + \mathrm{div}_x \int_{\mathbf{R}^3} v M_{\rho,u,\theta} \ln M_{\rho,u,\theta} dv \le 0$$

•The following formulas for the moments of a Maxwellian

$$\int M_{\rho,u,\theta} dv = \rho\,, \quad \int v M_{\rho,u,\theta} dv = \rho u\,,$$
$$\int v^{\otimes 2} M_{\rho,u,\theta} dv = \rho(u^{\otimes 2} + \theta I)\,, \quad \int \tfrac{1}{2}|v|^2 M_{\rho,u,\theta} dv = \tfrac{1}{2}\rho(|u|^2 + 3\theta)$$
$$\int v \tfrac{1}{2}|v|^2 M_{\rho,u,\theta} dv = \tfrac{1}{2}\rho u(|u|^2 + 5\theta)$$

and for its entropy and entropy flux

$$\int M_{\rho,u,\theta} \ln M_{\rho,u,\theta} dv = \rho \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right) - \tfrac{3}{2}\rho$$
$$\int v M_{\rho,u,\theta} \ln M_{\rho,u,\theta} dv = \rho u \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right) - \tfrac{3}{2}\rho u$$

show that $(\rho, u, \theta)$ is an admissible solution of Euler's system.

## Compressible / Incompressible Navier -Stokes equation and scaling

*The derivation of the compressble Euler equation is independent of the cross section! Properties of the cross section appear in the compressible Euler which is an higher order approximation with $\epsilon$ being the Knudsen number*

$$
\begin{aligned}
\partial_t \rho + \mathrm{div}_x(\rho u) &= 0 \\
\partial_t (\rho u) + \mathrm{div}_x(\rho (u)^{\otimes 2}) + \nabla_x(\rho\theta) &= \epsilon\, \mathrm{div}_x(\mu D(u)) \\
\partial_t \left(\rho(\tfrac{1}{2}|u|^2 + \tfrac{3}{2}\theta)\right) + \mathrm{div}_x \left(\rho u(\tfrac{1}{2}|u|^2 + \tfrac{5}{2}\theta)\right) &= \epsilon\, \mathrm{div}_x(\kappa \nabla_x \theta) \\
&\quad + \epsilon\, \mathrm{div}_x(\mu D(u)\cdot u)
\end{aligned}
$$

With $m$ the Mach number : Small fluctuations

$$\rho = 1 + m\tilde{\rho}, u = m\tilde{u}\,, \quad \theta = 1 + m\tilde{\theta}$$

and change of scale in time (adapted to $\tilde{u}$) $t \to t/m$

$$\mathrm{div}_x\,\tilde{u} = 0 \quad \tilde{u} + \tilde{\rho} = 0$$

$$\partial_t\tilde{u} + \mathrm{div}_x(\tilde{u} \otimes \tilde{u}) + \nabla_x p = \frac{\epsilon}{m}\mu\Delta_x\tilde{u}$$

$$\frac{5}{2}(\partial_t\tilde{\theta} + \mathrm{div}_x(\tilde{u}\tilde{\theta})) = \frac{\epsilon}{m}\kappa\Delta_x\tilde{\theta}$$

Hence the Von Karman relation

$$\mathrm{Reynolds} \simeq \frac{\mathrm{Mach}}{\mathrm{Knudsen}}$$

## The incompressible Navier-Stokes scaling

•Consider the dimensionless Boltzmann equation in the incompressible Navier-Stokes scaling, i.e. with $\boxed{\mathrm{St} = \pi\mathrm{Kn} = \epsilon \ll 1}$:

$$\epsilon\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)$$

•Start with an initial data that is a perturbation of some uniform Maxwellian (say, the centered reduced Gaussian $M = M_{1,0,1}$) with Mach number $\boxed{\mathrm{Ma} = O(\epsilon)}$:

$$F_\epsilon^{in} = M_{1,0,1} + \epsilon f_\epsilon^{in}$$

•Example 1: pick $u^{in} \in L^2(\mathbf{R}^3)$ a divergence-free vector field; then the distribution function

$$F_\epsilon^{in}(x,v) = M_{1,\epsilon u^{in}(x),1}(v)$$

is of the type above.

•Example 2: If in addition $\theta^{in} \in L^2 \cap L^\infty(\mathbf{R}^3)$, the distribution function

$$F_\epsilon^{in}(x,v) = M_{1-\epsilon\theta^{in}(x), \frac{\epsilon u^{in}(x)}{1-\epsilon\theta^{in}(x)}, \frac{1}{1-\epsilon\theta^{in}(x)}}(v)$$

is also of the type above. (Pick $0 < \epsilon < \frac{1}{\|\theta^{in}\|_{L^\infty}}$, then $1 - \epsilon\theta^{in} > 0$ a.e.).

## FORMAL INCOMPRESSIBLE LIMITS

$$\epsilon\,\partial_t F_\epsilon + v \cdot \nabla F_\epsilon = \frac{1}{\epsilon^q}\mathcal{B}(F_\epsilon, F_\epsilon) \quad q \geq 1 \tag{1}$$

$$F_\epsilon = M G_\epsilon = M(1 + \epsilon g_\epsilon)\,. \tag{2}$$

## The linearized collision operator

•Viscosity and heat diffusion given linearization at a Maxwellian $M = M_{1,0,1}$ (the centered, reduced Gaussian) of Boltzmann's collision integral

$$\begin{aligned}\mathcal{L}_M\phi &= -2M^{-1}\mathcal{B}(M, M\phi)\\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2}(\phi+\phi_*-\phi'-\phi'_*)|(v-v_*)\cdot\omega|d\omega M_* dv_*\end{aligned}$$

The operator $\mathcal{L}_M$ takes the form

$$(\mathcal{L}_M\phi)(v) = \lambda_M(|v|)\phi(v) - (\mathcal{K}_M\phi)(v)$$

where $\lambda(|v|)$ is the collision frequency, while $\mathcal{K}_M$ is an integral operator

$$\lambda(|v|) = 2\pi\int_{\mathbf{R}^3}|v-v_*|M_* dv_*\,,\quad \mathcal{K}_M\phi = \mathcal{K}_{1,M} - \mathcal{K}_{2,M}$$

with

$$\frac{1}{c}(1+|v|) \leq \lambda(|v|) \leq c(1+|v|)$$

$$\mathcal{K}_{1,M}\phi = 2\iint_{\mathbf{R}^3\times\mathbf{S}^2} \phi'|(v-v_*)\cdot\omega|d\omega M_* dv_*$$
$$\mathcal{K}_{2,M}\phi = 2\pi\int_{\mathbf{R}^3} \phi_*|v-v_*|M_* dv_*$$

**Theorem.** *The operator $\mathcal{L}_M$ is a nonnegative, unbounded self-adjoint Fredholm operator on $L^2(Mdv)$ (scalar product denoted $\langle .,.\rangle$) with domain $L^2(\lambda(|v|)^2 Mdv)$. Further, its nullspace is the set of collision invariants, i.e.*

$$\ker \mathcal{L}_M = \text{span}\{1, v_1, v_2, v_3, |v|^2\}\,.$$

*Moreover, there exists $c_0 > 0$ such that, for each $\phi \in L^2(\lambda(|v|)Mdv)$:*

$$\phi \perp \ker \mathcal{L}_M \Rightarrow \langle \phi, \phi \rangle = \int_{\mathbf{R}^3} \phi \mathcal{L}_M \phi M dv \geq c_0 \int_{\mathbf{R}^3} \phi^2 \lambda(|v|) M dv\,.$$

•Fredholm's alternative: Consider the (integral) equation $\mathcal{L}_M \phi = \psi$. Either

- $\boxed{\psi \perp \ker \mathcal{L}_M}$ $\Rightarrow$ there exists a unique solution $\phi_0 \perp \ker \mathcal{L}_M$ (denoted by $\phi_0 = c L_M^{-1} \psi$); all solutions are of the form $\phi_0 + n$ with $n \in \ker \mathcal{L}_M$;

- otherwise, there exists no solution $\phi$ to the above equation.

•Basic examples: Consider the vector field $B$ and the tensor field $A$ defined by

$$A(v) = v^{\otimes 2} - \tfrac{1}{3}|v|^2 I\,, \quad B(v) = \tfrac{1}{2} v(|v|^2 - 5)$$

Notice that $A \perp \ker \mathcal{L}_M$, $B \perp \ker \mathcal{L}_M$ and $A \perp B$; there exist $\mathcal{L}_M^{-1} A \perp \ker \mathcal{L}_M$ and $\mathcal{L}_M^{-1} B \perp \ker \mathcal{L}_M$

• Rotational invariance of $\mathcal{B}$ Let $R \in O_3(\mathbf{R})$; it acts on functions $f$ on $\mathbf{R}^3$, on vector fields $U$ on $\mathbf{R}^3$, and on 2-contravariant tensors fields $S$ on $\mathbf{R}^3$ as follows:

$$f_R(v) = f(R^T v)\,, \quad U_R(v) = RU(R^T v)\,, \quad S_R(v) = RS(R^T v)R^T$$

• The Boltzmann collision integral is rotationally invariant:

$$\mathcal{B}(F_R, F_R) = \mathcal{B}(F, F)_R\,, \text{ therefore } \mathcal{L}_{M_{1,0,1}}\phi_R = (\mathcal{L}_{M_{1,0,1}}\phi)_R$$

since $M_{1,0,1}$ is a radial function.

• One has $A_R = A$ and $B_R = B$; hence $(\mathcal{L}_M^{-1} A)_R = \mathcal{L}_M^{-1} A$ and $(\mathcal{L}_M^{-1} B)_R = \mathcal{L}_M^{-1} B$. Therefore, there exist $\alpha \equiv \alpha(|v|)$ and $\beta \equiv \beta(|v|)$ s.t.

$$\mathcal{L}_M^{-1} A(v) = \alpha(|v|)A(v)\,, \quad \mathcal{L}_M^{-1} B(v) = \beta(|v|)B(v)$$

$$\langle f, g \rangle = \int_{\mathbf{R}^3} f(v)g(v)M(v)dv = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbf{R}^3} f(v)g(v)e^{-\frac{|vl^2}{2}}dv$$

**Theorem.** *Let $F_\epsilon(t,x,v) = MG_\epsilon = M(1+\epsilon g_\epsilon)$ be a sequence of non-negative solutions to the scaled kinetic equation (1) and (2) with “good, reasonable" convergence properties then: Then the limiting $g$ has the form*

$$g = v \cdot u + (\frac{1}{2}|v|^2 - \frac{5}{2})\theta\,,$$

*where the velocity $u$ is divergence free and the density and temperature fluctuations, $\rho$ and $\theta$, satisfy the Boussinesq relation*

$$\nabla u = 0\,, \qquad \nabla_x(\rho + \theta) = 0\,.$$

*Moreover, the functions $\rho$, $u$ and $\theta$ are weak solutions of the equations*

$$\partial_t u + u\nabla u + \nabla p = \mu\Delta u\,, \quad \frac{5}{2}(\partial_t\theta + u\cdot\nabla\theta) = \kappa\Delta\theta\,, \quad \textit{if } q = 1\,;$$

$$\partial_t u + u\nabla u + \nabla p = 0\,, \quad \partial_t\theta + u\cdot\nabla\theta = 0\,, \quad \textit{if } q > 1\,;$$

*With $\mu$ and $\kappa$ given by the formulas:*

$$\nu = \tfrac{1}{10}\int \alpha(|v|)A : AMdv\,, \quad \kappa = \tfrac{1}{3}\int \beta(|v|)B : BMdv\,. \tag{3}$$

**Proof.** Start from

$$\epsilon\,\partial_t g_\epsilon + v\cdot\nabla g_\epsilon + \frac{1}{\epsilon^q}\mathcal{L}(g_\epsilon) = \epsilon^{1-q}\frac{1}{2}M^{-1}\mathcal{B}(Mg_\epsilon, Mg_\epsilon)$$

Multiply by $\epsilon^q$,

$$\epsilon \to 0 \Rightarrow \mathcal{L}(g) = 0 \Rightarrow g = \rho + v\cdot u + (\frac{1}{2}|v|^2 - \frac{3}{2})\theta$$

$$\epsilon\partial_t\langle g_\epsilon\rangle + \nabla\langle v g_\epsilon\rangle = 0 \Rightarrow \nabla\cdot u = 0$$

$$\epsilon\,\partial_t\langle v g_\epsilon\rangle + \nabla\langle v\otimes v g_\epsilon\rangle = 0 \Rightarrow \nabla(\rho+\theta) = 0\,.$$

For the moments:

$$\partial_t\langle v g_\epsilon\rangle + \frac{1}{\epsilon}\nabla_x\langle v\otimes v g_\epsilon\rangle = 0$$
$$\partial_t\langle v g_\epsilon\rangle + \frac{1}{\epsilon}\nabla_x\langle A(v)\otimes v g_\epsilon\rangle + \nabla\frac{1}{\epsilon}\langle\frac{1}{3}|v|^2 g_\epsilon\rangle = 0$$
$$\partial_t\langle\frac{1}{2}(|v|^2-5)g_\epsilon\rangle + \frac{1}{\epsilon}\nabla_x\langle B(v)\otimes v g_\epsilon\rangle = 0$$

Use $\mathcal{L}^{-1}A$ and $\mathcal{L}^{-1}B$

$$\frac{1}{\epsilon}\langle A(v)g_\epsilon\rangle = \frac{1}{\epsilon}\langle\mathcal{L}^{-1}A(v)\mathcal{L}g_\epsilon\rangle$$
$$= -\epsilon^q\langle\mathcal{L}^{-1}A(v)\partial_t g_\epsilon\rangle - \epsilon^{q-1}\nabla_x\langle\mathcal{L}^{-1}A(v)vg_\epsilon\rangle + \langle\mathcal{L}^{-1}A(v)\mathcal{B}(Mg_\epsilon, Mg_\epsilon)\rangle$$
$$\frac{1}{\epsilon}\langle B(v)g_\epsilon\rangle = \frac{1}{\epsilon}\langle\mathcal{L}^{-1}B(v)\mathcal{L}g_\epsilon\rangle$$
$$= -\epsilon^q\langle\mathcal{L}^{-1}B(v)\partial_t g_\epsilon\rangle - \epsilon^{q-1}\nabla_x\langle\mathcal{L}^{-1}B(v)vg_\epsilon\rangle + \langle\mathcal{L}^{-1}B(v)\mathcal{B}(Mg_\epsilon, Mg_\epsilon)\rangle$$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle A(v)g_\epsilon\rangle = -\lim_{\epsilon\to 0}\epsilon^{q-1}\nabla_x\langle\mathcal{L}^{-1}A(v)vg\rangle + \langle\mathcal{L}^{-1}A(v)\mathcal{B}(Mg, Mg)\rangle$$
$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\langle B(v)g_\epsilon\rangle = -\lim_{\epsilon\to 0}\epsilon^{q-1}\nabla_x\langle\mathcal{L}^{-1}B(v)vg\rangle + \langle\mathcal{L}^{-1}B(v)\mathcal{B}(Mg, Mg)\rangle$$

$$\int \alpha(|v|)A_{ij}(v)A_{kl}(v)Mdv = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl})$$

$$\int \beta(|v|)B_i(v)B_j(v)Mdv = \kappa\delta_{ij}$$

For $q \in \ker \mathcal{L}_M$

$$\mathcal{B}(Mg, Mg) = \frac{1}{2}\mathcal{L}_M(g^2)$$

$$\Rightarrow \langle \mathcal{L}^{-1}A(v)\mathcal{B}(Mg, Mg)\rangle = \frac{1}{2}\langle A(v)g^2\rangle = u \otimes u - \frac{1}{3}|u|^2\,,$$

$$\Rightarrow \langle \mathcal{L}^{-1}B(v)\mathcal{B}(Mg, Mg)\rangle = \frac{1}{2}\langle B(v)g^2\rangle = \frac{5}{2}u\theta$$

## Convergence

•Let $F_\epsilon^{in} \geq 0$ be any sequence of measurable functions satisfying the entropy bound $H(F_\epsilon^{in}|M) \leq C^{in}\epsilon^2$, and let $F_\epsilon$ be a renormalized solution of the scaled Boltzmann equation

$$\epsilon\partial_t F_\epsilon + v \cdot \nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \quad F_\epsilon\Big|_{t=0} = F_\epsilon^{in}$$

•Let $g_\epsilon \equiv g_\epsilon(x, v)$ be such that $G_\epsilon := 1 + \epsilon g_\epsilon \geq 0$ a.e.. We say that $g_\epsilon \to g$ entropically at rate $\epsilon$ as $\epsilon \to 0$ iff

$$g_\epsilon \to g \text{ in } w - L^1_{loc}(M dv dx)\,, \text{ and } \frac{1}{\epsilon^2}H(MG_\epsilon|M) \to \tfrac{1}{2}\iint g^2 M dv dx$$

**Theorem.** *Assume that*

$$\frac{F_\epsilon^{in}(x, v) - M(v)}{\epsilon M(v)} \to u^{in}(x) \cdot v$$

*entropically at rate $\epsilon$. Then the family of bulk velocity fluctuations*

$$\frac{1}{\epsilon}\int_{\mathbf{R}^3} vF_\epsilon dv$$

*is relatively compact in $w-L^1_{loc}(dtdx)$ and each of its limit points as $\epsilon \to 0$ is a Leray solution of*

$$\partial_t u + \mathrm{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u\,, \quad \mathrm{div}_x\, u = 0\,, \quad u\Big|_{t=0} = u^{in}$$

$$H(F|M) = \iint F \log(\frac{F}{M}) - F + M) dxdv$$
$$= \iint M((\frac{F}{M}) \log(\frac{F}{M}) - (\frac{F}{M}) + 1) dxdv$$