

A Liouville theorem for vector valued semilinear heat equations with no gradient structure and applications to blow-up

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Abstract

We prove a Liouville Theorem for a vector valued semilinear heat equation with no gradient structure. Classical tools such as the maximum principle or energy techniques break down and have to be replaced by a new approach. We then derive from this theorem uniform estimates for blow-up solutions of that equation.

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1 Introduction

This paper is concerned with blow-up solutions of the semilinear heat equation

$$\partial_t u = \Delta u + F(u), \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \rightarrow \mathbb{R}^M$, Δ denotes the Laplacian and $F : \mathbb{R}^M \rightarrow \mathbb{R}^M$ is not necessarily a gradient. We say that $u(t)$ blows up in finite time T , if $u(t)$ exists for all $t \in [0, T)$ and

$$\lim_{t \rightarrow T} \|u(t)\|_{L^\infty} = +\infty.$$

We note that an extensive literature is devoted to the study of equation (1). Many results were found using monotonicity properties, maximal principle (valid for scalar equations) or energy techniques (valid when F is a gradient). See for example [Wei84], [Fuj66], [Bal77], [Lev73]. Unfortunately, there are important classes of PDEs where these techniques break down. For example, equations of the type (1), where F is not a gradient, or PDEs coming from geometric flows; see for example a review paper by Hamilton [Ham95].

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In this work, we would like to develop new tools for a class of equations where classical tools do not work, in particular, vector valued equations with no gradient structure. More precisely, we will consider the following reaction-diffusion equation.

$$u_t = \Delta u + (1 + i\delta)|u|^{p-1}u, \quad u(0, x) = u_0(x), \quad (2)$$

where $u(t) : \mathbb{R}^N \rightarrow \mathbb{C}$, $\delta \in \mathbb{R}$ and

$$p > 1 \text{ and } (N - 2)p < (N + 2). \quad (3)$$

Note that the nonlinearity in this equation is not a gradient. Note also that (2) is a particular case of the Complex Ginzburg-Landau equation

$$\partial_t u = (1 + i\beta)\Delta u + (\epsilon + i\delta)|u|^{p-1}u - \gamma u, \text{ where } (x, t) \in \mathbb{R}^N \times (0, T), \quad (4)$$

β , δ and γ real, $p > 1$ and $\epsilon = \pm 1$.

This equation is mostly famous when $\epsilon = -1$. It appears in the study of various physical problems (plasma physics, nonlinear optics). It is in particular used as an amplitude equation near the onset of instabilities in fluid mechanics (see for example Levermore and Oliver [LO96]). In this case, Plecháč and Šverák [PŠ01] used matching techniques and numerical simulations to give a strong evidence for the existence of blow-up solutions in the focusing case, namely $\beta\delta > 0$.

The case $\epsilon = 1$ is less famous. To our knowledge, there is only the work of Popp, Stiller, Kuznetsov and Kramer [PKK98], who use a formal approach to find blow-up solutions. More recently, Masmoudi and Zaag [MZ08a] gave a constructive method to show the existence of a stable blow-up solution under some conditions for the parameters.

Let us present in the following the known results for equation (2) and most importantly the research directions and open problems. In the study of the blow-up phenomenon for equation (2), we believe that there are two important issues:

Construction of examples of blow-up solutions: In this approach, one has to construct *examples* of solutions that blow up in finite time. In particular, one has to find conditions on initial data and/or parameters of the equation to guarantee that the solution blows up in finite time. For equation (2), we recall the result obtained by Zaag [Zaa98] (the range of δ has been widened in [MZ08a]):

- For each $\delta \in (-\sqrt{p}, \sqrt{p})$,
- i) equation (2) has a solution $u(x, t)$ on $\mathbb{R}^N \times [0, T)$ which blows up in finite time $T > 0$ at only one blow-up point $a \in \mathbb{R}^N$,
 - ii) moreover, we have

$$\lim_{t \rightarrow T} \|(T - t)^{\frac{1+i\delta}{p-1}} u(a + ((T - t)|\log(T - t)|)^{\frac{1}{2}} z, t) - f_\delta(z)\|_{L^\infty(\mathbb{R}^N)} = 0 \quad (5)$$

with

$$f_\delta(z) = (p - 1 + \frac{(p - 1)^2}{4(p - \delta^2)} |z|^2)^{-\frac{1+i\delta}{p-1}},$$

iii) there exists $u_* \in \mathcal{C}(\mathbb{R}^N \setminus \{a\}, \mathbb{C})$ such that $u(x, t) \rightarrow u_*(x)$ as $t \rightarrow T$ uniformly on compact subsets of $\mathbb{R}^N \setminus \{a\}$ and

$$u_*(x) \sim \left[\frac{8(p - \delta^2) |\log |x - a||}{(p - 1)^2 |x - a|^2} \right]^{\frac{1+i\delta}{p-1}} \quad \text{as } x \rightarrow a.$$

Remark: In [MZ08a], the same result was proved for equation (4), where the linearized operator around the expected profile is much more difficult to study.

Asymptotic behavior for any arbitrary blow-up solution: In this approach, one takes *any arbitrary* blow-up solution for equation (2) and tries to describe its blow-up behavior. More precisely, it consists in the determination of the asymptotic profile (that is a function from which, after a time dependent scaling, $u(t)$ approaches as $t \rightarrow T$) of the blow-up solution.

In earlier literature, the determination of the profile is done through the study of entire solutions (defined for all time and space) of the equation. See for example Grayson and Hamilton [GH96] for the case of the harmonic map heat flow and Giga and Kohn [GK85] for the heat equation (that is $\delta = 0$ in (2); there, the authors prove a Liouville Theorem which turns to be the trivial case of the Liouville Theorem proved by Merle and Zaag in [MZ98a] and [MZ00] and stated in Proposition 3.2 below). Let us remark that the use of *Liouville theorems* was successful for elliptic equations (see Gidas and Spruck [GS81a] and [GS81b]).

More recently, the characterization of entire solutions by means of *Liouville Theorems* allowed to obtain more than the blow-up profile, namely *uniform* estimates with respect to initial data and the singular point. See for the heat equation Merle and Zaag [MZ98b], [MZ98a], [MZ00], for the modified Korteweg de Vries equation Martel and Merle [MM00], for the nonlinear Schrödinger equation Merle and Raphael [MR04], [MR05] and for the wave equation Merle and Zaag [MZ08b] and [MZ08b].

The existence of a Lyapunov functional is traditionally a crucial tool in the proof of Liouville theorems, like for the heat equation [MZ00] or the wave equation [MZ08b]. One wonders whether it is possible to prove a Liouville theorem for a system with no Lyapunov functional. The first attempt was done by Zaag [Zaa01] for the following system

$$\partial_t u = \Delta u + v^p, \quad \partial_t v = \Delta v + u^q, \quad (6)$$

and its selfsimilar version

$$\begin{aligned} \partial_s \Phi &= \Delta \Phi - \frac{1}{2} y \cdot \nabla \Phi + \Psi^p - \left(\frac{p+1}{pq-1} \right) \Phi, \\ \partial_s \Psi &= \Delta \Psi - \frac{1}{2} y \cdot \nabla \Psi + \Phi^q - \left(\frac{q+1}{pq-1} \right) \Psi. \end{aligned} \quad (7)$$

This is the result of [Zaa01]:

Consider $p_0 > 1$ such that $(N-2)p_0 < N+2$ and $M > 0$. Then, there exists $\eta > 0$ such that if $|p - p_0| + |q - p_0| < \eta$, then for any nonnegative (Φ, Ψ) solution of (7) such that for all $(y, s) \in \mathbb{R}^N \times \mathbb{R}$, $\Phi(y, s) + \Psi(y, s) \leq M$, then either $(\Phi, \Psi) = (0, 0)$ or $(\Phi, \Psi) = (\Gamma, \gamma)$

or $(\Phi, \Psi) = \left(\Gamma (1 + e^{s-s_0})^{-\frac{p+1}{pq-1}}, \gamma (1 + e^{s-s_0})^{-\frac{q+1}{pq-1}} \right)$ for some $s_0 \in \mathbb{R}$, where (Γ, γ) is the only nontrivial constant solutions of (7) defined by

$$\gamma^p = \Gamma \left(\frac{p+1}{pq-1} \right) \text{ and } \Gamma^q = \gamma \left(\frac{q+1}{pq-1} \right). \quad (8)$$

Before [Zaa01], Andreucci, Herrero and Velázquez addressed the same question in [AHV97] but could not determine explicitly the third case. In some sense, they just gave the limits as $s \rightarrow \pm\infty$ (for a statement, see the remark after Proposition 2.2 below). The characterization of that third case is far more difficult than the rest. The lack of a Lyapunov functional was overcome thanks to an infinite dimensional blow-up criterion. Following system (6), it was interesting to address the case of equation (2) for $\delta \neq 0$. Like for system (6), there is no Lyapunov functional. On the contrary, no blow-up criterion is available and the set of non zero stationary solutions for the selfsimilar version is a continuum (see Proposition 2.1 below). For these two reasons, new tools have to be found, which makes our paper meaningful.

Another reason for our work is the full Ginzburg-Landau model (4) with $\beta \neq 0$. That case has one more difficulty since the linearized operator in selfsimilar variable becomes non selfadjoint, as one can see from [MZ08a]. Thus, this paper is a fundamental step towards the proof of a Liouville theorem for the full Ginzburg-Landau model (4), which we believe to be an open problem of great importance.

1.1 A Liouville theorem for system (2)

Our aim in this paper is to prove a Liouville theorem for equation (2). In order to do so, we introduce for each $a \in \mathbb{R}^N$, the following selfsimilar transformation:

$$w_a(y, s) = (T-t)^{\frac{(1+i\delta)}{p-1}} u(x, t), \quad y = \frac{x-a}{\sqrt{T-t}}, \quad s = -\log(T-t). \quad (9)$$

If u is a solution of (2), then the function $w = w_a$ satisfies for all $s \geq -\log T$ and $y \in \mathbb{R}^N$:

$$w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{(1+i\delta)}{(p-1)} w + (1+i\delta) |w|^{(p-1)} w. \quad (10)$$

We introduce also the Hilbert space

$$L_\rho^2 = \left\{ g \in L_{loc}^2(\mathbb{R}^N, \mathbb{C}), \int_{\mathbb{R}^N} |g|^2 e^{-\frac{|y|^2}{4}} dy < +\infty \right\} \text{ where } \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{N/2}}.$$

If g depends only on the variable $y \in \mathbb{R}^N$, we use the notation

$$\|g\|_{L_\rho^2}^2 = \int_{\mathbb{R}^N} |g(y)|^2 e^{-\frac{|y|^2}{4}} dy.$$

If g depends only on $(y, s) \in \mathbb{R}^N \times \mathbb{R}$, we use the notation

$$\|g(\cdot, s)\|_{L_\rho^2}^2 = \int_{\mathbb{R}^N} |g(y, s)|^2 e^{-\frac{|y|^2}{4}} dy.$$

The main result of the paper is the following Liouville theorem which classifies certain entire solutions (i.e. solutions defined for all $(y, s) \in \mathbb{R}^N \times \mathbb{R}$) of (10):

Theorem 1 (A Liouville theorem for equation (10)) Assuming (3), there exist $\delta_0 > 0$ and $M : [-\delta_0, \delta_0] \rightarrow (0, +\infty]$ with $M(0) = +\infty$, $M(\delta) \rightarrow +\infty$ as $\delta \rightarrow 0$ and the following property:

If $|\delta| \leq \delta_0$ and $w \in L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{C})$ is a solution of (10) with $\|w\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{C})} \leq M(\delta)$, then, either $w \equiv 0$ or $w \equiv \kappa e^{i\theta_0}$ or $w = \varphi_\delta(s - s_0)e^{i\theta_0}$ for some $\theta_0 \in \mathbb{R}$ and $s_0 \in \mathbb{R}$, where $\varphi_\delta(s) = \kappa(1 + e^s)^{-\frac{(1+i\delta)}{(p-1)}}$ and $\kappa = (p-1)^{-\frac{1}{p-1}}$.

Going back to the original variables $u(x, t)$, we rewrite this Liouville theorem in the following:

Theorem 2 (A Liouville theorem for equation (2)) Assuming (3), there exist $\delta_0 > 0$ and $M : [-\delta_0, \delta_0] \rightarrow (0, +\infty]$ with $M(0) = +\infty$, $M(\delta) \rightarrow +\infty$ as $\delta \rightarrow 0$ and the following property:

If $|\delta| \leq \delta_0$ and u is a solution of (2) satisfying $u(x, t)(T-t)^{\frac{1}{p-1}} \in L^\infty(\mathbb{R}^N \times (-\infty, T), \mathbb{C})$ and $\|u(x, t)(T-t)^{\frac{1}{p-1}}\|_{L^\infty(\mathbb{R}^N \times (-\infty, T), \mathbb{C})} \leq M(\delta)$, then, $u \equiv 0$ or there exists $T_0 \geq T$ and $\theta_0 \in \mathbb{R}$ such that for all $(x, t) \in \mathbb{R}^N \times (-\infty, T)$, $u(x, t) = \kappa(T_0 - t)^{-\frac{1+i\delta}{p-1}} e^{i\theta_0}$.

Remark: This result has already been proved by Merle and Zaag [MZ98a] and [MZ00] (see also Nouaili [Nou08]) when $\delta = 0$. In that case, $M(0) = +\infty$, which means that any L^∞ entire solution of (10), with no restriction on the size of its norm, is trivial (i.e. independent of space).

When $\delta \neq 0$, this conclusion holds only for "small" L^∞ norm (i.e. bounded by $M(\delta)$). We suspect that we cannot take $M(\delta) = +\infty$. In other words, we suspect that equation (10) under the condition (3) has nontrivial solutions in L^∞ with a high norm. We think that such solutions can be constructed in the form $w(y, s) = w_0(y)e^{i\omega s}$ with high ω and high $\|w_0\|_{L^\infty}$, as Popp *et al.* did through formal arguments in page 96 in [PKK98] when $\delta \sim \pm 3$ (which is outside our range).

Remark: Since this result was already known from [MZ98a] and [MZ00] when $\delta = 0$, the case $\delta \neq 0$ may appear as a not surprising interesting perturbation technique of the case $\delta = 0$. If this is clearly true for the statement, it is certainly not the case for the method and the techniques, mainly because the gradient structure breaks down and the linearized problem is no longer selfadjoint (see the beginning of Section 2 for more details). We have to invent new tools which are far from being a simple perturbation technique. This makes the main innovation of our work.

Remark: One may think that our result is completely standard in the context of dynamical systems. It happens that already in the case $\delta = 0$, standard methods such as the center manifold theory do not apply in our case as pointed by Filippas and Kohn in [FK92] page 834-835. In particular, Proposition 3.5 page 21 below, whose statement is standard, does not follow from center manifold theory because the nonlinear term is not quadratic in the function space L_ρ^2 .

1.2 Applications to type I blow-up solutions of (2)

As in previously cited blow-up recent literature ([MZ00], [MM00] and [MZ08b]), Liouville theorems have important applications to blow-up for the so called 'type I' blow-up

solutions of equation (2), that is, solutions satisfying

$$\forall t \in [0, T), \|u(t)\|_{L^\infty} \leq M(T-t)^{-\frac{1}{p-1}},$$

where T is the blow-up time. In other words, the blow-up rate is given by the associated ODE $u' = (1 + i\delta)|u|^{p-1}u$.

We know that the solution of (2) constructed in [Zaa98] is of type **I** (and the same holds for the solution of Ginzburg-Landau equation (4) constructed in [MZ08a]). However, we have been unable to prove whether *all* blow-up solutions of (2) are of type **I** or not. Note that when $\delta = 0$, Giga and Kohn [GK85] and Giga, Matsui and Sasayama [GMS04] prove that all blow-up solutions are of type **I**, provided that p is subcritical ($(N-2)p < N+2$). When $\delta \neq 0$, the methods of [GK85] and [GMS04] break down because we no longer have positivity or a Lyapunov functional.

However, following [MZ00] and [Zaa01], we can derive the following estimates for type **I** blow-up solutions of (2):

Proposition 3 (*Uniform blow-up estimates for type **I** solutions*) Assume (3), consider $|\delta| \leq \delta_0$ and a solution u of (2) that blows up at time T and satisfies

$$\forall t \in [0, T), \|u(t)\|_{L^\infty} \leq M(\delta)(T-t)^{-\frac{1}{p-1}},$$

where δ_0 and $M(\delta)$ are defined in Theorem 1. Then,

- (i) (L^∞ estimates for derivatives)

$$\|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}} \rightarrow \kappa \text{ and } \|\nabla^k u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1} + \frac{k}{2}} \rightarrow 0$$

as $t \rightarrow T$ for $k = 1, 2$ or 3.

- (ii) (*Uniform ODE Behavior*) For all $\varepsilon > 0$, there is $C(\varepsilon)$ such that $\forall x \in \mathbb{R}^N$, $\forall t \in [\frac{T}{2}, T)$,

$$\left| \frac{\partial u}{\partial t}(x, t) - (1 + i\delta)|u|^{p-1}u(x, t) \right| \leq \varepsilon|u(x, t)|^p + C.$$

Remark: When $\delta = 0$, this result was already derived from the *Liouville theorem* in [MZ98a] and [MZ00]. It happens that adapting that proof to the case $\delta \neq 0$ is not straightforward, because the gradient structure is missing. However, unlike for the Liouville theorem, the adaptation is mainly technical. For the reader's convenience, we show in Section 4 how to adapt the proof of [MZ98a] and [MZ00] in the case $\delta \neq 0$.

Our paper is organized as follows: Section 2 and Section 3 are devoted to the proof of the Liouville theorem (we only prove Theorem 2 since Theorem 1 follows immediately from the selfsimilar transformation (9)). Note that Section 2 contains the main arguments with no details and Section 3 includes the whole proof with all the technical steps. Finally, we prove in Section 4 the applications to blow-up stated in Proposition 3.

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2 The main steps and ideas of the proof of the Liouville Theorem

In this section, we adopt a pedagogical point of view and explain the main steps and ideas of the proof with no technical details. These details are presented in Section 3. The reader may think that our result is an interesting perturbation of the Liouville theorem proved in [MZ00] for $\delta = 0$. If this is true for the statement, it is certainly not the case for the proof for three structural reasons:

- the gradient structure breaks down when $\delta \neq 0$, which prevents us from using any energy method or blow-up criteria. To show blow-up, we need to find a very precise asymptotic behavior of the solution and show “by hand” that it cannot stay bounded.
- when $\delta \neq 0$, the linearized operator of (10) around the constant solution $w \equiv \kappa$ is no longer self adjoint and no general theory is applicable to derive eigenvalues directly. A careful decomposition of the solution is needed instead.
- since equation (2) is invariant under rotations in the complex plane ($u \rightarrow ue^{i\theta}$), this generates a null eigenvalue for the linear part of equation (10), and a precise modulation technique is needed, unlike the real valued case when $\delta = 0$.

The proof of the Liouville theorem is the same for $N = 1$ and $N \geq 2$ with subcritical p (see (3)). The only difference is in the multiplicity of the eigenvalues of the linearized operator of equation (10), which changes from 1 when $N = 1$ to a higher value when $N \geq 2$. In particular, one needs some extra notations and careful linear algebra in higher dimension. For the sake of clearness, we give here the proof when $N = 1$. The interested reader may find in section 4 (page 128) of [MZ00] how to get the higher dimensional case from the case $N = 1$. Clearly, the following statement is equivalent to Theorem 1:

For any $M > 0$, there exists $\delta'_0(M) > 0$ such that for all $|\delta| \leq \delta'_0(M)$, if $w(y, s)$ is an entire solution of (10), defined for all $(y, s) \in \mathbb{R} \times \mathbb{R}$ and

$$\|w(\cdot, s)\|_{L^\infty} \leq M, \tag{11}$$

then w depends only on the variable s .

In the following, we will prove this latter statement. Let us consider $M > 0$ and $w(y, s)$ satisfying (11), and prove that w is trivial provided that δ is small. As in [MZ00], the starting point is the investigation of the behavior of $w(y, s)$ as $s \rightarrow -\infty$.

Part 1: Behavior of $w(y, s)$ as $s \rightarrow -\infty$.

In the case $\delta = 0$, the method of Giga and Kohn [GK85] proves that $w(y, s)$ approaches the set of stationary solutions of (10) $\{0, \kappa e^{i\theta} | \theta \in \mathbb{R}\}$ as $s \rightarrow -\infty$ in L^2_ρ . We would like to do the same here, that is why we give the stationary solutions of (10) in the following.

Proposition 2.1 (*L^∞ stationary solutions of (10)*) *Consider $\delta \neq 0$ and $v \in L^\infty(\mathbb{R}^N)$ a solution of*

$$0 = \Delta v - \frac{1}{2}y \cdot \nabla v - \frac{1+i\delta}{p-1}v + (1+i\delta)|v|^{p-1}v. \tag{12}$$

Then, either $v \equiv 0$ or there exists $\theta_0 \in \mathbb{R}$ such that $v \equiv \kappa e^{i\theta_0}$.

Remark: When $\delta = 0$, the same result holds only for subcritical p verifying $(N - 2)p \leq N + 2$ and the proof due to Giga and Kohn is far from being trivial, see Theorem 1 (page 305) in [GK85].

Proof of Proposition 2.1: Consider $v \in L^\infty(\mathbb{R}^N)$ a solution to (12). Multiplying (12) by $\bar{v}\rho$ and integrating over \mathbb{R}^N gives after integration by parts

$$0 = - \int |\nabla v|^2 \rho - \frac{(1 + i\delta)}{p - 1} \int |v|^2 \rho + (1 + i\delta) \int |v|^{p+1} \rho.$$

Since $\delta \neq 0$, identifying the imaginary and the real parts gives $\int |\nabla v|^2 \rho = 0$, hence $\nabla v \equiv 0$ and $\Delta v \equiv 0$. Plunging this in (12) yields the result. ■

To prove that the solution approaches the set of stationary solutions, the method of Giga and Kohn breaks down, since it heavily relies on the existence of the following Lyapunov functional for equation (10) in the case $\delta = 0$:

$$E(w) = \frac{1}{2} \int |\nabla w|^2 \rho dy + \frac{1}{2(p - 1)} \int |w|^2 \rho dy - \frac{1}{p + 1} \int |w|^{p+1} \rho dy. \quad (13)$$

When $\delta \neq 0$, we don't have such a Lyapunov functional. Fortunately, a perturbation method used by Andreucci, Herrero and Velázquez, works here and yields the following:

Proposition 2.2 *For any $M > 0$, there exists $\delta'_0(M)$ such that if $|\delta| \leq \delta'_0$ and w is an arbitrary solution of (10) satisfying for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, $|w(y, s)| \leq M$, then either (i) $\|w(\cdot, s)\|_{L^2_\rho} \rightarrow 0$ as $s \rightarrow -\infty$ or (ii) $\inf_{\theta \in \mathbb{R}} \|w(\cdot, s) - \kappa e^{i\theta}\|_{L^2_\rho} \rightarrow 0$ as $s \rightarrow -\infty$.*

The next parts of the strategy (parts 2 and 3) investigate case (i) and (ii) of Proposition 2.2, which are certainly not of the same degree of difficulty.

Part 2: Case where $w \rightarrow 0$ as $s \rightarrow -\infty$.

In this case, we have $w \equiv 0$. Rather than giving a proof, we simply explain here how the proof works. For the actual proof, we rely again on the method of Andreucci, Herrero and Velázquez (see Proposition 3.1, in section 3 of [AHV97]). Our argument is that the stationary solution of (10), which is identically zero, is stable in L^2_ρ , hence, no orbit can escape it, except the null orbit. To illustrate this, we write from equation (10) the following differential inequality for $h(s) \equiv \int_{\mathbb{R}} |w(y, s)|^2 \rho(y) dy$,

$$h'(s) \leq -\frac{2}{p - 1} h(s) + 2 \int_{\mathbb{R}} |w(y, s)|^{p+1} \rho(y) dy.$$

Using the regularizing effect of equation (10), we derive the following delay estimate :

$$\forall s \in \mathbb{R}, \int_{\mathbb{R}} |w(y, s)|^{p+1} \rho(dy) \leq C^* \left(\int_{\mathbb{R}} |w(y, s - 1)|^2 \rho(y) dy \right)^{\frac{p+1}{2}},$$

for some positive C^* . Therefore,

$$\forall s \in \mathbb{R}, h'(s) \leq -\frac{2}{p - 1} h(s) + C(M) h(s - 1)^{\frac{p+1}{2}}.$$

Using the fact that $h(s) \rightarrow 0$ as $s \rightarrow -\infty$ and delay ODE techniques, we show that $h(s)$ is driven by its linear part, hence for some $\varepsilon > 0$ small enough, we have

$$\forall \sigma \in \mathbb{R}, \forall s \geq \sigma + 1, h(s) \leq \varepsilon_0 e^{-\frac{2(s-\sigma)}{p-1}}.$$

Fixing $s \in \mathbb{R}$ and letting $\sigma \rightarrow -\infty$, we get that $h(s) \equiv 0$, hence $w \equiv 0$.

Now that case (i) of Proposition 2.2 has been handled, we consider case (ii) in the following.

Part 3: Case where $\inf_{\theta \in \mathbb{R}} \|w(\cdot, s) - \kappa e^{i\theta}\|_{L_\rho^2} \rightarrow 0$ as $s \rightarrow -\infty$.

The question to be asked here is the following: Does the solution converge to a particular $\kappa e^{i\theta_0}$ as $s \rightarrow -\infty$ or not?

The key idea is to classify the L_ρ^2 behavior of w as $s \rightarrow -\infty$. We proceed in 5 steps.

Step 1: Formulation of the problem.

Note that the degree of freedom in case (ii) of Proposition 2.2 comes from the invariance of equation (2) under the rotation ($u \rightarrow u e^{i\theta}$). This invariance generates a zero mode for equation (10), which is difficult to control. The idea to gain this control and show the convergence of $w(y, s)$ is to use a modulation technique by introducing the following parametrization of the problem:

$$w(y, s) = e^{i\theta(s)}(v(y, s) + \kappa) \text{ with } \kappa = (p-1)^{-\frac{1}{p-1}}. \quad (14)$$

A natural choice would be to take $\theta(s)$ such that $\|w(\cdot, s) - e^{i\theta(s)}\kappa\|_{L_\rho^2} = \inf_{\theta \in \mathbb{R}} \|w(y, s) - \kappa e^{i\theta}\|_{L_\rho^2}$. This is not our choice, we will instead choose $\theta(s)$ such that we kill the neutral mode mentioned above. More precisely, we claim the following:

Lemma 2.3 *There exists $s_1 \in \mathbb{R}$ and $\theta \in C^1((-\infty, s_1], \mathbb{R})$ such that*

(i) $\forall s \leq s_1, \int (\text{Im}(v) - \delta \text{Re}(v))\rho = 0$, where v is defined by (14).

(ii) We have $\|v(\cdot, s)\|_{L_\rho^2} \rightarrow 0$ as $s \rightarrow -\infty$.

(iii) For all $s \leq s_1$, we have

$$|\theta'(s)| \leq C \|v(\cdot, s)\|_{L_\rho^2}^2. \quad (15)$$

With the change of variables (14), we focus in the following steps on the description of the asymptotic behavior of $v(y, s)$ and $\theta(s)$ as $s \rightarrow -\infty$. Using (14), we write the equation satisfied by $v(= v_1 + i v_2)$ as

$$\partial_s v = \tilde{\mathcal{L}}v - i\theta_s(v + \kappa) + G, \quad (16)$$

$$\text{where } G = (1 + i\delta) \left\{ |v + \kappa|^{p-1}(v + \kappa) - \kappa^p - \frac{v}{p-1} - v_1 \right\}, \quad (17)$$

$$\text{satisfies } |G| \leq C|v|^2 \text{ and } \left| G - (1 + i\delta) \frac{1}{2\kappa} \{ (p-2)v_1^2 + v_2^2 + 2v_1 v_2 \} \right| \leq C|v|^3. \quad (18)$$

A good understanding of our operator $\tilde{\mathcal{L}}v = \Delta v - \frac{1}{2}y \cdot \nabla v + (1 + i\delta)v_1$ will be essential in our analysis. The following lemma provides us with the spectral properties of $\tilde{\mathcal{L}}$.

Lemma 2.4 (*Eigenvalues of $\tilde{\mathcal{L}}$*).

(i) $\tilde{\mathcal{L}}$ is a \mathbb{R} -linear operator defined on L_ρ^2 and its eigenvalues are given by

$$\left\{1 - \frac{m}{2} \mid m \in \mathbb{N}\right\}.$$

Its eigenfunctions are given by $\{(1 + i\delta)h_m, ih_m \mid m \in \mathbb{N}\}$ where

$$h_m(y) = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{n!(m-2n)!} (-1)^n y^{m-2n}. \quad (19)$$

We have: $\tilde{\mathcal{L}}((1 + i\delta)h_m) = (1 - \frac{m}{2})(1 + i\delta)h_m$ and $\tilde{\mathcal{L}}(ih_m) = -\frac{m}{2}ih_m$.

(ii) Each $r \in L_\rho^2$ can be uniquely written as

$$r(y) = (1 + i\delta)\tilde{r}_1(y) + i\tilde{r}_2(y) = (1 + i\delta) \left(\sum_{m=0}^{+\infty} \tilde{r}_{1m} h_m(y) \right) + i \left(\sum_{m=0}^{+\infty} \tilde{r}_{2m} h_m(y) \right),$$

where:

$$\begin{aligned} \tilde{r}_1(y) &= \operatorname{Re} \{r(y)\} \text{ and } \tilde{r}_2(y) = \operatorname{Im} \{r(y)\} - \delta \operatorname{Re} \{r(y)\} \\ \text{and for } i &= \{1, 2\}, \tilde{r}_{im} = \int \tilde{r}_i(y) \frac{h_m(y)}{\|h_m\|_{L_\rho^2}^2} \rho(y) dy. \end{aligned} \quad (20)$$

Remark: Note that the eigenvalues 1, 1/2 and 0 have a geometrical interpretation: they come from the invariance of equation (2) to translation in time ($\lambda = 1$) and space ($\lambda = 1/2$), dilations $u_\lambda(\xi, \tau) \rightarrow \lambda^{\frac{1}{p-1}} u(\xi\sqrt{\lambda}, \tau\lambda)$ and multiplications by $e^{i\theta}$ (the group S^1) for $\lambda = 0$.

Remark: Following (ii), we write each complex quantity (number or function) z as $z = z_1 + iz_2$ and $z = (1 + i\delta)\tilde{z}_1 + i\tilde{z}_2$ with $z_{j=1,2}, \tilde{z}_{j=1,2} \in \mathbb{R}$. In particular, we write

$$\begin{aligned} v(y, s) &= (1 + i\delta)\tilde{v}_1(y, s) + i\tilde{v}_2(y, s), \\ &= (1 + i\delta) \sum_{m=0}^{\infty} \tilde{v}_{1m}(s) h_m(y) + i \sum_{m=0}^{\infty} \tilde{v}_{2m}(s) h_m(y). \end{aligned} \quad (21)$$

Proof: Using the notation (21), we see that

$$\tilde{\mathcal{L}}v = \begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{L} - \mathcal{I} \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix}, \quad (22)$$

where

$$\mathcal{L}h = \Delta h - \frac{1}{2}y \cdot \nabla h + h, \quad (23)$$

is a well-known self adjoint operator of $L_\rho^2(\mathbb{R}, \mathbb{R})$ whose eigenfunctions are h_m (19), which are dilation of Hermite polynomials. Thus, the spectral properties of $\tilde{\mathcal{L}}$ directly derive from those of \mathcal{L} . The interested reader may find details in Lemma 2.2 page 590 from Zaag [Zaa98]. ■

Note from this Lemma that operator $\tilde{\mathcal{L}}$ has three nonnegative eigenvalues:

- $\lambda = 1$, with eigenfunction $(1 + i\delta)h_0(y) = (1 + i\delta)$.
- $\lambda = 1/2$, with eigenfunction $(1 + i\delta)h_1(y) = (1 + i\delta)y$.

• $\lambda = 0$, with two eigenfunctions $(1 + i\delta)h_2(y) = (1 + i\delta)(y^2 - 2)$ and $ih_0(y) = i$. From (21) and (20), the coordinate of $v(y, s)$ along the direction ih_0 is

$$\begin{aligned}\tilde{v}_{20}(s) &= \int (\operatorname{Im}(v(y, s)) - \delta \operatorname{Re}(v(y, s))) \frac{h_0(y)}{\|h_0\|_{L_\rho^2}^2} \rho(y) \\ &= \int (\operatorname{Im}(v(y, s)) - \delta \operatorname{Re}(v(y, s))) \rho(y).\end{aligned}$$

Using (i) of Lemma 2.3, we see that the choice of $\theta(s)$ guarantees that

$$\forall s \leq s_1, \tilde{v}_{20}(s) = 0. \quad (24)$$

In the next step, we will use the spectral information of $\tilde{\mathcal{L}}$ to derive the asymptotic behavior of v , then w as $s \rightarrow -\infty$.

Step 2: Asymptotic behavior as $s \rightarrow -\infty$.

As $s \rightarrow -\infty$, we expect that the coordinates of v on the eigenfunctions for $\lambda \geq 0$ will dominate. These eigenfunctions are $(1 + i\delta)$ when $\lambda = 1$, $(1 + i\delta)y$ when $\lambda = 1/2$, $(1 + i\delta)(y^2 - 2)$ or i when $\lambda = 0$. Note that for this latter case, the direction along i , already vanishes thanks to the choice of $\theta(s)$ (see (24)). So, if $\lambda = 0$ dominates, that is the coordinate of v on $(1 + i\delta)(y^2 - 2)$ dominates, since the linear part vanishes, the equation is driven by the quadratic approximation $\dot{x} \sim -x^2$, that is $x \sim \frac{1}{s}$. Using (iii) of Lemma 2.3, we see that $\theta(s)$ has a limit as $s \rightarrow -\infty$, hence w converges from (14). More precisely, we have:

Proposition 2.5 *There exists $\theta_0 \in \mathbb{R}$ such that $\theta(s) \rightarrow \theta_0$ and $\|w(\cdot, s) - \kappa e^{i\theta_0}\|_{L_\rho^2} \rightarrow 0$ as $s \rightarrow -\infty$. More precisely, one of the following situations occurs as $s \rightarrow -\infty$, for some $C_0 \in \mathbb{R}$ and $C_1 \in \mathbb{R}^*$,*

$$\begin{aligned}(i) \quad & \|w(\cdot, s) - \{\kappa + (1 + i\delta)C_0 e^s\} e^{i\theta_0}\|_{L_\rho^2} \leq C e^{\frac{3}{2}s}, \\ (ii) \quad & \|w(\cdot, s) - e^{i\theta_0} \left\{ \kappa + (1 + i\delta) \frac{\kappa}{4(p-\delta^2)s} (y^2 - 2) - i \frac{(1+\delta^2)\delta\kappa^2}{2(p-\delta^2)^2} \frac{1}{s} \right\}\|_{L_\rho^2} \leq C \frac{\log|s|}{s^2}, \\ (iii) \quad & \|w(\cdot, s) - \{\kappa + (1 + i\delta)C_1 e^{s/2} y\} e^{i\theta_0}\|_{L_\rho^2} \leq C e^{(1-\varepsilon)s}.\end{aligned} \quad (25)$$

In Step 3, we show that case (i) yields the explicit solution $\varphi_\delta(s - s_0)$ for some s_0 . In Steps 4 and 5, we rule out cases (ii) and (iii).

In comparison with the case $\delta = 0$, we can say that the difficulty in deriving Proposition 2.5 is only technical. One should bear in mind that the difficulty level is much lower than the obstacles we have in steps 4 and 5 to rule out cases (ii) and (iii) of Proposition 2.5.

Step 3: Case where (i) holds.

Like for step 2, there is no real novelty in this step, the difficulty is purely technical. First we recall (i) from Proposition 2.5:

$$\|w(\cdot, s) - \{\kappa + (1 + i\delta)C_0 e^s\} e^{i\theta_0}\|_{L_\rho^2} \leq C e^{\frac{3}{2}s}. \quad (26)$$

Let us remark that we already have a solution $\hat{\varphi}(s)e^{i\theta_0}$ of (10) defined in $\mathbb{R} \times (-\infty, \hat{s}]$ for some $\hat{s} \in \mathbb{R}$ and which satisfies the same expansion as w :

$$\begin{aligned} \text{(a) if } C_0 = 0, & \quad \text{just take } \hat{\varphi} \equiv \kappa, \\ \text{(b) if } C_0 < 0, & \quad \text{take } \hat{\varphi} \equiv \varphi_\delta(s - s_0), \text{ where } s_0 = -\log\left(-\frac{C_0(p-1)}{\kappa}\right), \\ \text{(c) if } C_0 > 0, & \quad \text{take } \hat{\varphi} \equiv \varphi_\delta^*(s - s_0), \text{ where } s_0 = \log\left(\frac{C_0(p-1)}{\kappa}\right), \\ & \quad \text{and } \varphi_\delta^*(s) = \kappa(1 - e^s)^{-\frac{(1+i\delta)}{(p-1)}} \end{aligned} \quad (27)$$

$\varphi_\delta^*(s)$ is a solution of (10) that blows up at $s = 0$, but is bounded for all $s \leq -1$. Note that, from (26) we have:

$$\|w(\cdot, s) - \varphi^*(s)e^{i\theta_0}\|_{L^2_p} \leq Ce^{\frac{3}{2}s}. \quad (28)$$

Since the difference between the two solutions of (10) is of order $e^{3s/2}$ and the largest eigenvalue of $\tilde{\mathcal{L}}$ is $1 < \frac{3}{2}$, this difference has to vanish leading to $w(y, s) = \hat{\varphi}(s)e^{i\theta_0}$ (remember that the largest eigenvalue matters, since $s \rightarrow -\infty$). Since case (c) violates the uniform bound (11), only cases (a) or (b) occur. More precisely, we have the following:

Proposition 2.6 *Assume that case (i) of Proposition 2.5 holds. Then, either $w \equiv \kappa e^{i\theta_0}$ or there exists $s_0 \in \mathbb{R}$ such that for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, $w(y, s) = \varphi_\delta(s - s_0)e^{i\theta_0}$ for some $\theta_0 \in \mathbb{R}$.*

Steps 4 and 5: Irrelevance of cases (ii) and (iii) of Proposition 2.5.

Step 4 and the following make the novelty of our work. Indeed, in the case $\delta = 0$ treated in [MZ00], cases (ii) and (iii) of Proposition 2.5 were ruled out thanks to a blow-up criterion based on energy methods. Indeed, when $\delta = 0$, Merle and Zaag used the Lyapunov functional for equation (10) introduced in (13). More precisely, they have the following blow-up criterion (see Proposition 2.1 page 111 in [MZ00]):

Lemma 2.7 *(A blow-up criterion for equation (10) when $\delta = 0$). Let W be a solution of (10), (with $\delta = 0$), which satisfies:*

$$E(W(y, s_0)) < \frac{p-1}{2(p+1)} \left(\int_{\mathbb{R}^N} |W(y, s_0)|^2 \rho(y) dy \right)^{\frac{p+1}{2}},$$

for some $s_0 \in \mathbb{R}$. Then, W blows-up at some time $S > s_0$.

Still for $\delta = 0$, it is shown in [MZ00], when case (ii) or (iii) hold in Proposition 2.5, that

for some a_0 and s_0 , we have

$$E(w_{a_0}(\cdot, s_0)) < \frac{p-1}{2(p+1)} \left(\int w_{a_0}(y, s_0)^2 \rho(y) dy \right)^{\frac{p+1}{2}}, \quad (29)$$

where

$$w_{a_0}(y, s) = w(y + a_0 e^{\frac{s}{2}}, s), \quad (30)$$

is also a solution of (10).

A contradiction follows then since in the same time w_{a_0} is defined for all $s \in \mathbb{R}$ from (30) and has to blow-up by condition (29) and Lemma 2.7.

When $\delta \neq 0$, all this collapses. No perturbation method can allow us to use in any sense the Lyapunov functional or the blow-up criterion. We have to invent a new method to rule out cases (ii) and (iii) of Proposition 2.5. Let us explain our strategy only for case (ii), since it is quite similar for case (iii). From rotation invariance of equation (10), we assume that $\theta_0 = 0$.

Our source of inspiration is the study of (10), when $\delta = 0$ and $w \rightarrow \kappa$ as $s \rightarrow +\infty$ (and not $-\infty$) by Herrero and Velázquez [HV93] and Velázquez [Vel92], to obtain the (supposed to be generic) profile, starting with the following profile

$$w(y, s) = \kappa + \frac{\kappa}{2ps} \left(1 - \frac{1}{2}|y|^2\right) + o\left(\frac{1}{s}\right) \text{ as } s \rightarrow \infty.$$

The convergence here takes place in L^2_ρ and $L^\infty(|y| < R)$ for any $R > 0$.

Herrero and Velázquez extended this convergence to a larger set of the form $|y| \leq K\sqrt{s}$, for any $K > 0$. They obtained:

$$\sup_{|y| < K\sqrt{s}} \left| w(y, s) - f_0\left(\frac{y}{\sqrt{s}}\right) \right| \rightarrow 0 \text{ as } s \rightarrow +\infty,$$

where $f_0 = \left((p-1) + \frac{(p-1)^2}{4p} \frac{|y|^2}{s} \right)^{-\frac{1}{p-1}}$ is a solution of

$$0 = \frac{1}{2}\xi \cdot \nabla f_0(\xi) - \frac{1}{p-1}f_0(\xi) + |f_0|^{p-1}f_0(\xi), \text{ where } \xi = \frac{y}{\sqrt{s}},$$

In some sense, we can say that f_0 is an approximate solution of (10) when $s \rightarrow \infty$, because

$$\|\partial_s f_0 - \left\{ \Delta f_0 + \frac{1}{2}\xi \cdot \nabla f_0 - \frac{1}{p-1}f_0 + |f_0|^{p-1}f_0 \right\}\|_{L^\infty} = \|\partial_s f_0 - \Delta f_0\|_{L^\infty} \leq \frac{C}{s}.$$

We note that Velázquez's method is a kind of characteristic's method applied to the parabolic equation (10), where the Laplacian term is dropped down because the profile is flat. Here, we will use ideas from Velázquez to find the profiles of the solution in the variables $\frac{y}{\sqrt{-s}}$ ($ye^{s/2}$ in Step 5). We hope to find singular profiles, which violate the upper bound (11) on $w(y, s)$.

Our candidate of the profile is $G\left(\frac{y}{\sqrt{-s}}\right)$, with $G(\xi) = \kappa \left(1 - \frac{(p-1)}{4(p-\delta^2)}\xi^2\right)^{-\frac{(1+i\delta)}{(p-1)}}$. In fact G is a solution of

$$0 = -\frac{1}{2} \cdot \xi \nabla G(\xi) - \frac{1+i\delta}{p-1}G + (1+i\delta)|G|^{p-1}G.$$

We can note (as in the case of f_0 defined below) that G is an approximate solution of (10) (for $|y| < K_0\sqrt{-s}$, where $K_0 = \sqrt{\frac{4(p-\delta^2)}{(p-1)}}$). We see also that G is singular for $|y| = K_0\sqrt{-s}$. Using Velázquez's technique to extend the convergence in (ii) of Proposition 2.5 from $|y| < R$ to larger regions $|y| < a_0\sqrt{-s}$, with $a_0 < K_0$, we can prove the following:

Proposition 2.8 *Assume that case (iii) from Proposition 2.5 holds, then there exists $a_0 > 0$, such that:*

$$\lim_{s \rightarrow -\infty} \sup_{|y| \leq a_0 \sqrt{-s}} \left| w(y, s) - G\left(\frac{y}{\sqrt{-s}}\right) \right| = 0, \quad (31)$$

where $G(\xi) = \kappa \left(1 - \frac{(p-1)}{4(p-\delta^2)} \xi^2 \right)^{-\frac{(1+i\delta)}{(p-1)}}$.

Imagine for a second that (31) holds for any arbitrary $a_0 < K_0$. Since $|G(\xi)| \rightarrow \infty$ as $\xi \rightarrow K_0$, we can fix a_0 large enough so that $|G(a_0)| \geq 3M$. Taking $|s_0|$ large enough in (31), we then see that

$$|w(a_0 \sqrt{-s_0}, s_0)| \geq 2M,$$

which contradicts the upper bound (11). It happens that unlike the case $s \rightarrow \infty$, where $\xi = 0$ realizes the maximum of the profile f_0 , here $\xi = 0$ realizes the minimum, which obliges us to take a_0 small enough in order to use Velázquez' method of convergence extension. Since a_0 is small in our approach, we remark from (31) that $w(y, s)$ is flat (i.e. close to a constant) in a large region, in the sense that

$$\sup_{|y - \frac{a_0}{2} \sqrt{-s_0}| \leq 4|s_0|} \left| w(y, s_0) - G\left(\frac{a_0}{2}\right) \right| \rightarrow 0 \text{ as } s_0 \rightarrow -\infty.$$

Using a kind of continuity with respect to initial data for equation (10), we can show that for any $\varepsilon > 0$

$$\sup_{s_0 \leq s \leq s_0^* - \varepsilon} \left| w\left(\frac{a_0}{2} \sqrt{-s_0}, s\right) - W_{a_0}(s) \right| \rightarrow 0 \text{ as } s_0 \rightarrow \infty, \quad (32)$$

where $s_0^* < +\infty$ is the lifespan of $W_{a_0}(s)$ the space independent solution of (10), with $W_{a_0}(s_0) \equiv G\left(\frac{a_0}{2}\right)$. It happens that W_{a_0} can be computed explicitly:

$$W_{a_0}(s) = \kappa \left(1 - e^{s-s_0} \frac{(p-1)a_0^2}{16(p-\delta^2)} \right)^{-\frac{(1+i\delta)}{p-1}}$$

and that it blows-up at time $s = s_0^* - \log\left(\frac{(p-1)a_0^2}{16(p-\delta^2)}\right) > s_0$, because a_0 is small.

Taking $s_0^* = s_0 - \varepsilon_0$, where $\varepsilon_0 > 0$ is small enough such that $|W_{a_0}(s_0^* - \varepsilon_0)| \geq 3M$, we see from (32) that $|w\left(\frac{a_0}{2} \sqrt{-s_0}, s_0^* - \varepsilon_0\right)| \geq 2M$, which violates the upper bound (11).

Conclusion of Part 3 and the sketch of proof of the Liouville theorem:

From Step 4 and 5 we see that cases (ii) and (iii) of Proposition 2.5 are ruled out. By Step 3, we obtain that $w \equiv \kappa e^{i\theta_0}$ or $w \equiv \varphi_\delta(s - s_0) e^{i\theta_0}$ for some real s_0 and θ_0 , where φ_δ is defined in Theorem 1, which is the desired conclusion of Theorem 1. In Section 3, we give the details of the proof.

3 Details of the proof of the Liouville theorem

In this section, we give the whole proof of the Liouville theorem. We only prove Theorem 1 since Theorem 2 immediately follows though the selfsimilar transformation (9). Note

that in Section 2, we already gave a sketch of the proof stressing only the main arguments. Thus this section is intended only to readers interested in technical details.

We adopt here the same sectioning as in Section 2: three parts and Part 3 is divided in five steps. Hence, we recommend that the reader reads first a given step in section 2 before reading the corresponding step in Section 3. As in Section 2, we prove Theorem 1 in its form given in the statement around (11). We consider $M > 0$ and a global solution $w(y, s)$ of (10), defined for all $(y, s) \in \mathbb{R} \times \mathbb{R}$ such that

$$\|w(y, s)\|_{L^\infty} \leq M.$$

Our goal is to find $\delta'_0(M) > 0$, such that if $|\delta| \leq \delta'_0(M)$, w depends only on the variable s . We proceed in three parts:

In Part 1, we show that when $s \rightarrow -\infty$, either $w \rightarrow 0$ or w approaches the set $\{\kappa e^{i\theta} | \theta \in \mathbb{R}\}$.

In Part 2, we handle the first case and show that $w \equiv 0$.

In Part 3, we linearize the equation around $\kappa e^{i\theta(s)}$, for some well chosen $\theta(s)$, and show that either $w \equiv \kappa e^{i\theta_0}$ or $w \equiv \varphi_\delta(s - s_0)e^{i\theta_0}$ for some real s_0 and θ_0 , where φ_δ is defined in Theorem 1, which concludes the proof.

It happens that we rely on the analysis performed by Andreucci, Herrero and Velázquez [AHV97] for the system (7). That is the reason why we give Part 1 and Part 2 at once.

Parts 1 and 2: Behavior of $w(y, s)$ as $s \rightarrow \infty$ and conclusion in the case where $w \rightarrow 0$ as $s \rightarrow -\infty$

In these parts, we investigate the behavior of w as $s \rightarrow -\infty$ and reach a conclusion in the easiest case. Following what we wrote in Part 1 of Section 2, we know from Proposition 2.1 that the set of stationary solutions of (10) consists in 0 and $\kappa e^{i\theta}$, where $\theta \in \mathbb{R}$. In order to prove that w approaches this set as $s \rightarrow -\infty$, we rely completely on the analysis performed in [AHV97] for the system (7). Indeed, no extra arguments is necessary for the present equation (10). That is why we only give the main arguments which make the proof of [AHV97] hold for equation (10) and refer the interested reader to [AHV97] for the details. Now, using the perturbation method of [AHV97], we have the following:

Proposition 3.1 *(A primary classification) For any $M > 0$, there exists $\delta'_0(M)$ such if $|\delta| \leq \delta'_0$ and w is an arbitrary solution to (10) satisfying for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, $|w(y, s)| \leq M$, then, either (i) ($\|w(s)\| \equiv 0$) or (ii) ($\inf_{\theta \in \mathbb{R}} \|w(s) - \kappa e^{i\theta}\| \rightarrow 0$) as $s \rightarrow -\infty$.*

Remark: This result replaces Proposition 2.2 and Part 2 in section 2.

Remark: In [AHV97], the conclusion of the authors in Theorem 2 for system (7) is more accurate: either (Φ, Ψ) is $(0, 0)$ or (Γ, γ) defined in (8), or

$$(\Phi, \Psi) \rightarrow (\Gamma, \gamma) \text{ at } -\infty \text{ and } (\Phi, \Psi) \rightarrow (0, 0) \text{ at } +\infty.$$

Using the same technique for our equation (10), we get Proposition 3.1. Indeed, due to the fact that the set of non trivial stationary solutions is a continuum (see Proposition 2.1), we need a modulation technique to derive the case $w \equiv \kappa e^{i\theta}$, this case will be treated in Part 3.

Proof of Proposition 3.1: This Proposition follows from the arguments developed for the twin system (7) in [AHV97], no more. To keep our paper in a reasonable length

limit, we don't give the proof. However, we should mention the 3 fundamental features of (10) that one needs to check to be convinced that the proof of Andreucci, Herrero and Velázquez works here.

- Both systems are of parabolic type involving the same linear operator

$$\mathcal{L}_0 v = \frac{1}{\rho} \operatorname{div}(\rho \nabla v) = \Delta v - \frac{1}{2} y \cdot \nabla v,$$

if $v = (1 + i\delta)\tilde{v}_1 + i\tilde{v}_2$ with $\tilde{v}_1, \tilde{v}_2 \in \mathbb{R}$, then

$$\begin{aligned} \partial_s \tilde{v}_1 &= \mathcal{L}_0 \tilde{v}_1 - a\tilde{v}_1 + f_1(\tilde{v}_1, \tilde{v}_2) \\ \partial_s \tilde{v}_2 &= \mathcal{L}_0 \tilde{v}_2 - b\tilde{v}_2 + f_2(\tilde{v}_1, \tilde{v}_2) \end{aligned}$$

with $a < 0$, $b < 0$ and $|f_i(\tilde{v}_1, \tilde{v}_2)| \leq C|v|^\alpha$ for some $\alpha > 1$.

- When $p = q = p_0$ in (7), the authors give in (3.12) and Lemma 3.2 of [AHV97] a classification of entire solutions. In our case, when $\delta = 0$ in (10), we have the following Liouville theorem (see Theorem 1 in [MZ00])

Proposition 3.2 (Merle-Zaag [MZ00]; *A Liouville theorem for equation (10) with $\delta = 0$ and subcritical p*) Assume (3) and let $w \in L^\infty(\mathbb{R}^N \times \mathbb{R}, \mathbb{C})$ be a solution of

$$\frac{\partial w}{\partial s} = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w.$$

Then necessarily, one of the following cases occur:

- a) $w \equiv 0$,
- b) $\exists \theta \in \mathbb{R}$ such that $w(y, s) = \kappa e^{i\theta}$,
- c) there exists $s_0 \in \mathbb{R}$, such that for all $(y, s) \in \mathbb{R}^N \times \mathbb{R}$, $w(y, s) = \varphi(s - s_0) e^{i\theta_0}$ where $\theta_0 \in \mathbb{R}$ and

$$\varphi(s) = \kappa(1 + e^s)^{-\frac{1}{p-1}}.$$

Remark: Note that φ is the unique global solution (up to a translation) of

$$\varphi_s = -\frac{\varphi}{p-1} + \varphi^p,$$

satisfying $\varphi \rightarrow \kappa$ as $s \rightarrow -\infty$ and $\varphi \rightarrow 0$ as $s \rightarrow \infty$. The method of Andreucci, Herrero and Velázquez in [AHV97] is in fact a perturbation method around this result.

- The property of equation (2) saying "small L_ρ^2 norm implies no blow-up locally" (note that this property replaces the Giga-Kohn property "small local energy implies no blow-up locally", which breaks down because we no longer have a gradient structure). This the property:

Proposition 3.3 For all $M > 0$, there exist positive η_0 , C_0 and M_0 such that if $|\delta| \leq 1$ and v is a solution of (2) satisfying

$$\forall t \in [0, 1), \quad \|v(t)\|_{L^\infty} \leq M(1-t)^{-\frac{1}{p-1}} \text{ and if } \forall |x_0| \leq 1, \quad \|w_{x_0}(\cdot, 0)\|_{L^2_\rho} \leq \eta, \quad (33)$$

for some $0 < \eta \leq \eta_0$, where

$$y = \frac{\xi - x_0}{\sqrt{1-\tau}}, \quad s = -\log(1-\tau), \quad w_{x_0}(y, s) = (1-\tau)^{\frac{1+i\delta}{p-1}} v(\xi, \tau),$$

then:

(i) For all $|x_0| \leq 1$ and $s \in [0, +\infty)$,

$$\|w_{x_0}(\cdot, s)\|_{L^2_\rho} \leq C_0 \eta e^{-\frac{s}{p-1}}. \quad (34)$$

(ii) For all $|x| \leq 1$ and $t \in [0, 1)$, we have $|v(x, t)| \leq M_0$.

Now, we write the following Lemma which will be useful in the proof of the proposition above.

Lemma 3.4 (Regularizing effect of the operator \mathcal{L}) Assume that $\psi(y, s)$ satisfies

$$\forall s \in [a, b], \quad \forall y \in \mathbb{R}, \quad \psi_s \leq (\mathcal{L} + \sigma)\psi \text{ and } 0 \leq \psi(y, s),$$

for some $a \leq b$ and $\sigma \in \mathbb{R}$, where

$$\mathcal{L}\psi = \Delta\psi - \frac{1}{2}y \cdot \nabla\psi + \psi = \frac{1}{\rho} \operatorname{div}(\rho \nabla\psi) + \psi. \quad (35)$$

Then for any $r > 1$, there exists $C^* = C^*(r, \sigma) > 0$ and $s^* = s^*(r) > 0$ such that

$$\forall s \in [a + s^*, b], \quad \left(\int_{\mathbb{R}} |\psi(y, s)|^r \rho(y) dy \right)^{1/r} \leq C^* \|\psi(\cdot, s - s^*)\|_{L^2_\rho}. \quad (36)$$

Proof: See Lemma 2.3 in [Vel93]. ■

Proof of Proposition 3.3:

Consider $M > 0$, $|\delta| \leq 1$ and a solution v of (2) such that (33) holds for some $\eta > 0$, $|x_0| \leq 1$.

(i) For simplicity, we write w instead of w_{x_0} . Since w is a solution of (10), we multiply (10) by $\partial_s w \rho$ and integrate to get

$$I'(s) \leq -\frac{2}{p-1} I(s) + \int |w(y, s)|^{p+1} \rho(y) dy, \text{ where } I(s) = \int |w(y, s)|^2 \rho(y) dy. \quad (37)$$

If we note $w = w_1 + iw_2$, then using Kato's inequality ($\Delta w_i \cdot \operatorname{sgn}(w_i) \leq \Delta |w_i|$ with $i = 1, 2$) and the fact that w is bounded, we obtain by equation (10)

$$\partial_s(|w_1| + |w_2|) \leq \Delta(|w_1| + |w_2|) - \frac{y}{2} \cdot \nabla(|w_1| + |w_2|) + C(|w_1| + |w_2|),$$

for some $C = C(M) > 0$.

Using Lemma 3.4, we see that there exist $C^*(M) > 0$ and $s^* = s^*(p+1) > 0$ such that for all $s \geq s^*$

$$\int |w(y, s)|^{p+1} \rho(y) dy \leq C^* \|w(\cdot, s - s^*)\|_{L_\rho^2}^{p+1} \quad (38)$$

Now, we divide the proof in two steps:

Step 1: $0 \leq s \leq s^*$. Using (37) and the fact that w is bounded by $M > 0$ (see (33)), we get

$$I'(s) \leq \lambda I(s) \text{ for some } \lambda = \lambda(M) > 0,$$

hence $I(s) \leq e^{\lambda s} I(0) \leq e^{\lambda s} \eta^2 \leq \frac{C_0^2}{2} \eta^2 e^{-\frac{2s}{p-1}}$, where we define $C_0^2 = 2e^{(\lambda + \frac{2}{p-1})s^*}$. This gives (34) for $0 \leq s \leq s^*$.

Step 2: $s \geq s^*$. In this step, we argue by contradiction to prove (34) for all $s \geq s^*$. We suppose that there exists $s_1 > s^*$, such that

$$I(s) < (C_0 \eta)^2 e^{-\frac{2s}{p-1}}, \text{ for all } s^* \leq s < s_1 \quad (39)$$

$$I(s_1) = (C_0 \eta)^2 e^{-\frac{2s_1}{p-1}}. \quad (40)$$

Let $F(s) = I(s)(C_0 \eta)^{-2} e^{\frac{2s}{p-1}}$. From (37), (38), (39) and Step 1, we have for all $s^* \leq s \leq s_1$,

$$\begin{aligned} F'(s) &\leq C^*(C_0 \eta)^{-2} e^{\frac{2s}{p-1}} I(s - s^*)^{\frac{p+1}{2}} \\ &\leq C^*(C_0 \eta)^{p-1} e^{\frac{2s}{p-1}} e^{-(s-s^*)\frac{p+1}{p-1}} \leq C^*(C_0 \eta)^{p-1} e^{\frac{p+1}{p-1}s^*} e^{-s}. \end{aligned}$$

Since $F(s^*) \leq \frac{1}{2}$ from the step above, we integrate the last inequality to obtain

$$\begin{aligned} F(s_1) &\leq C^*(C_0 \eta)^{p-1} e^{\frac{p+1}{p-1}s^*} \left(e^{-s^*} - e^{-s_1} \right) + F(s^*), \\ &\leq C^*(C_0 \eta)^{p-1} e^{\frac{2s^*}{p-1}} + \frac{1}{2} \leq \frac{3}{4}, \end{aligned}$$

for $\eta \leq \eta_0(M)$ small enough. This contradicts (40). Therefore, (34) holds

(ii) Applying parabolic regularity to equation (10) and using estimate (33), we get for all $|x_0| \leq 1$, $R > 0$ and $|y| < R$, $|w_{x_0}(y, s)| \leq M_0 e^{-\frac{s}{p-1}}$, hence for all $t \in [0, 1)$, $|v(x_0, t)| \leq M_0$, for some $M_0 = M_0(M)$. This ends the proof of Proposition 3.3. ■

Part 3: Case where $\inf_{\theta \in \mathbb{R}} \|w(\cdot, s) - \kappa e^{i\theta}\|_{L_\rho^2} \rightarrow 0$ **as** $s \rightarrow -\infty$.

We study case (ii) of Proposition 3.1. As we wrote in Part 3 of section 2, the natural question is to know whether w converges to a particular $\kappa e^{i\theta_0}$ as $s \rightarrow -\infty$ or not. A modulation technique will be essential to classify the L_ρ^2 behavior for w and prove the convergence. We proceed in five steps.

- Step 1 is intended to the modulation technique.
- In Step 2, we show that the linearized problem of (10) around $\kappa e^{i\theta}$ has 3 nonnegative directions as $s \rightarrow -\infty$ ($\lambda = 1, 1/2$ or 0), and that the component along one direction dominates the others. This gives a kind of profile for w with a uniform convergence on every compact sets.

- In Step 3, we show that the case where $\lambda = 1$ dominates corresponds either to $w = \kappa e^{i\theta_0}$ or $w = \varphi_\delta(s - s_0)e^{i\theta_0}$ for some $\theta_0 \in \mathbb{R}$ and $s_0 \in \mathbb{R}$, where φ_δ is defined in Theorem 1.
- Steps 4 and 5: To rule out cases where the directions $\lambda = 1/2$ or $\lambda = 0$ dominates, we use a geometrical method where the key idea is Velázquez's work [Vel92] to extend the convergence from compact sets to larger zones, where the profile appears to be singular, which violates the uniform bound (11) in w . These steps make the innovation of our work.

Step 1: Formulation of the problem

Let us recall Lemma 2.3 from Section 2.

Lemma 2.3 *There exists $s_1 \in \mathbb{R}$ and $\theta \in C^1((-\infty, s_1], \mathbb{R})$ such that*

(i) $\forall s \leq s_1$, $\int (\operatorname{Im}(v) - \delta \operatorname{Re}(v))\rho = 0$, where v is defined by

$$w(y, s) = e^{i\theta(s)}(v(y, s) + \kappa) \text{ with } \kappa = (p-1)^{-\frac{1}{(p-1)}}. \quad (41)$$

(ii) We have $\|v(\cdot, s)\|_{L^2_\rho} \rightarrow 0$ as $s \rightarrow -\infty$.

(iii) For all $s \leq s_1$, we have

$$|\theta'(s)| \leq C\|v(\cdot, s)\|_{L^2_\rho}^2. \quad (42)$$

Proof of Lemma 2.3:

(i) Since $\inf_{\theta \in [0, 2\pi]} \|w(\cdot, s) - \kappa e^{i\theta}\|_{L^2_\rho} \rightarrow 0$ as $s \rightarrow -\infty$ and $\|w(\cdot, s) - \kappa e^{i\theta}\|_{L^2_\rho}$ is continuous as a function of θ and w , there exists $\tilde{\theta}(s)$ such that

$$\|g\|_{L^2_\rho}^2 = \int_{\mathbb{R}^N} |g|^2 e^{-\frac{|y|^2}{4}} dy.$$

$$\|w(\cdot, s) - \kappa e^{i\tilde{\theta}(s)}\|_{L^2_\rho} = \inf_{\theta \in [0, 2\pi]} \|w(\cdot, s) - \kappa e^{i\theta}\|_{L^2_\rho} \rightarrow 0 \text{ as } s \rightarrow -\infty. \quad (43)$$

we will slightly modify $\tilde{\theta}(s)$, so that if we define $v(y, s)$ by (41) for some $\theta(s)$ close to $\tilde{\theta}(s)$, then we have (i) of Lemma 2.3. We apply the implicit function theorem to $F : L^2_\rho \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $F(w, \theta) = \int (\operatorname{Im}(we^{-i\theta} - \kappa) - \delta \operatorname{Re}(we^{-i\theta} - \kappa))\rho$. Since we have $F(\kappa e^{i\tilde{\theta}}, \tilde{\theta}) = 0$ and $\frac{\partial F}{\partial \theta} = -\int (\operatorname{Re}(we^{-i\theta}) + \delta \operatorname{Im}(we^{-i\theta}))\rho$, hence $\frac{\partial F}{\partial \theta}(\kappa e^{i\tilde{\theta}}, \tilde{\theta}) = -\kappa \neq 0$, using the implicit function theorem and (43), we obtain the existence and uniqueness of C^1 $\theta(w)$ such that $F(w, \theta(w)) = 0$ and $|e^{i\theta(w)} - e^{i\tilde{\theta}}| \leq C_0\|w(\cdot, s) - \kappa e^{i\tilde{\theta}}\|_{L^2_\rho}$.

(ii) Since, we have from (41)

$$\begin{aligned} \|v(\cdot, s)\|_{L^2_\rho} &= \|w(\cdot, s) - \kappa e^{i\theta(s)}\|_{L^2_\rho} \leq \|w(\cdot, s) - \kappa e^{i\tilde{\theta}}\|_{L^2_\rho} + \kappa|e^{i\theta} - e^{i\tilde{\theta}}| \\ &\leq (1 + C_0\kappa)\|w(\cdot, s) - \kappa e^{i\tilde{\theta}(s)}\|_{L^2_\rho}, \end{aligned}$$

using (43), we conclude that $\|v(\cdot, s)\|_{L^2_\rho} \rightarrow 0$ as $s \rightarrow -\infty$.

(iii) writing $v = (1 + i\delta)\tilde{v}_1 + i\tilde{v}_2$, we rewrite (16) as follows

$$\tilde{v}_{1s} = \mathcal{L}\tilde{v}_1 + \theta'(s)(\delta\tilde{v}_1 + \tilde{v}_2) + \tilde{G}_1 \quad (44)$$

$$\tilde{v}_{2s} = (\mathcal{L} - 1)\tilde{v}_2 - \theta'(s)((1 + \delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa) + \tilde{G}_2 \quad (45)$$

where \mathcal{L} is given in (35),

$$\tilde{G}_1 = \frac{p - \delta^2}{2\kappa} \tilde{v}_1^2 + \frac{1}{2\kappa} \tilde{v}_2^2 + O(|v|^3) \quad (46)$$

$$\tilde{G}_2 = (1 + \delta^2) \frac{\tilde{v}_1(\delta\tilde{v}_1 + \tilde{v}_2)}{\kappa} + O(|v|^3). \quad (47)$$

Note that $(1 + i\delta)\tilde{G}_1 + i\tilde{G}_2 = G$ is defined in (17).

Now, we multiply (45) by ρ and integrate over \mathbb{R} to get

$$\int \tilde{v}_{2s}\rho = \int \operatorname{div}(\rho\nabla\tilde{v}_2) - \int \theta'(s)((1 + \delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa)\rho + \int \tilde{G}_2\rho.$$

From (20), we have $\tilde{v}_2 = \operatorname{Im}(v) - \delta \operatorname{Re}(v)$, we get from (i) of Lemma 2.3 $\int \tilde{v}_{2s}\rho = 0$.

Since $\int \operatorname{div}(\rho\nabla\tilde{v}_2) = 0$ we obtain

$$\theta'(s) \int ((1 + \delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa)\rho = \int \tilde{G}_2\rho. \quad (48)$$

Using (47), we have

$$\left| \int \tilde{G}_2\rho \right| \leq C \int |v|^2\rho. \quad (49)$$

Recalling from (ii) that $\lim_{s \rightarrow -\infty} \|v\| = 0$, we have

$$\int ((1 + \delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa)\rho \rightarrow \int \kappa\rho \text{ as } s \rightarrow -\infty.$$

Thus, the conclusion follows from (48) and (49). ■

Step 2: Asymptotic behavior of v as $s \rightarrow -\infty$.

First, we recall the decomposition (21):

$$\begin{aligned} v(y, s) &= (1 + i\delta)\tilde{v}_1(y, s) + i\tilde{v}_2(y, s), \\ &= (1 + i\delta) \sum_{m=0}^{\infty} \tilde{v}_{1m}(s)h_m(y) + i \sum_{m=0}^{\infty} \tilde{v}_{2m}(s)h_m(y), \end{aligned}$$

and introduce

$$v_-(y, s) = (1 + i\delta) \sum_{m=3}^{\infty} \tilde{v}_{1m}(s)h_m(y) + i \sum_{m=1}^{\infty} \tilde{v}_{2m}(s)h_m(y).$$

As we saw in Step 2 of Section 2, the modulation techniques gives $\tilde{v}_{20}(s) = 0$. Therefore, we have

$$v(y, s) = (1 + i\delta)(\tilde{v}_{10}(s)h_0(y) + \tilde{v}_{11}(s)h_1(y) + \tilde{v}_{12}(s)h_2(y)) + v_-(y, s)$$

Using ODE techniques, we are able to prove the following:

Proposition 3.5 (*Classification of the behavior of $v(y, s)$ as $s \rightarrow -\infty$)* As $s \rightarrow -\infty$, one of the following situations occurs:

$$\begin{aligned}
& (i) |\tilde{v}_{11}(s)| + |\tilde{v}_{12}(s)| + \|v_-(\cdot, s)\|_{L_\rho^2} = o(\tilde{v}_{10}(s)), \\
& \|v(\cdot, s) - (1 + i\delta)C_0 e^s\|_{L_\rho^2} = O(e^{\frac{3}{2}s}) \text{ and } |\theta'(s)| = O(e^{2s}) \text{ for some } C_0 \in \mathbb{R}. \\
& (ii) |\tilde{v}_{10}(s)| + |\tilde{v}_{12}(s)| + \|v_-(\cdot, s)\|_{L_\rho^2} = o(\tilde{v}_{11}(s)), \\
& \|v(\cdot, s) - (1 + i\delta)C_1 e^{s/2} y\|_{L_\rho^2} = O(e^{(1-\varepsilon)s}) \text{ and } |\theta'(s)| = O(e^s) \text{ for some } C_1 \in \mathbb{R} \setminus 0 \text{ and } \varepsilon > 0. \\
& (iii) |\tilde{v}_{10}(s)| + |\tilde{v}_{11}(s)| + \|v_-(\cdot, s)\|_{L_\rho^2} = o(\tilde{v}_{12}(s)), \\
& \|v(\cdot, s) + (1 + i\delta) \frac{\kappa}{4(p - \delta^2)s} (y^2 - 2)\|_{L_\rho^2} = O\left(\frac{\log |s|}{s^2}\right) \text{ and } \theta'(s) = \frac{(1 + \delta^2)\delta\kappa}{2(p - \delta^2)} \frac{1}{s^2} + O\left(\frac{\log |s|}{s^2}\right).
\end{aligned}$$

Proof: As already pointed out by Filippas and Kohn in page 834-835 in [FM95] in the case $\delta = 0$, we can't use center manifold theory to get the result. In some sense, we are not able to say that the nonlinear terms in (44) and (45) are quadratic in the function space L_ρ^2 . However, using ODE techniques similar to those of [MZ98a] and [FM95], we manage to conclude. Since we add no real novelty, we leave the proof to Appendix A. ■

Now, we recall Proposition 2.5 as it is a direct consequence of the Proposition above. **Proposition 2.5** *There exists $\theta_0 \in \mathbb{R}$ such that $\theta(s) \rightarrow \theta_0$ and $\|w(\cdot, s) - \kappa e^{i\theta_0}\|_{L_\rho^2} \rightarrow 0$ as $s \rightarrow -\infty$. More precisely, one of the following situations occurs as $s \rightarrow -\infty$, for some $C_0 \in \mathbb{R}$ and $C_1 \in \mathbb{R}^*$,*

$$\begin{aligned}
(i) \quad & \|w(\cdot, s) - \{\kappa + (1 + i\delta)C_0 e^s\} e^{i\theta_0}\|_{L_\rho^2} \leq C e^{\frac{3}{2}s}, \\
(ii) \quad & \|w(\cdot, s) - e^{i\theta_0} \left\{ \kappa + (1 + i\delta) \frac{\kappa}{4(p - \delta^2)s} (y^2 - 2) - i \frac{(1 + \delta^2)\delta\kappa^2}{2(p - \delta^2)^2} \frac{1}{s} \right\}\|_{L_\rho^2} \leq C \frac{\log |s|}{s^2}, \\
(iii) \quad & \|w(\cdot, s) - \{\kappa + (1 + i\delta)C_1 e^{s/2} y\} e^{i\theta_0}\|_{L_\rho^2} \leq C e^{(1-\varepsilon)s}.
\end{aligned} \tag{50}$$

Proof of Proposition 2.5: From Proposition 3.5, we have $\|v(\cdot, s)\|_{L_\rho^2} = o(1/|s|)$ in all cases. Then using (42), we obtain $|\theta'(s)| \leq C/s^2$. Consequently, there exists a θ_0 such that $\theta(s) \rightarrow \theta_0$ as $s \rightarrow -\infty$. Using the definition (41), we get the convergence for w . We will just prove (iii), since the proof for (i) and (ii) is the same and even easier. Integrating the estimate for θ' (see (iii) of Proposition 3.5), we get

$$\theta(s) = \theta_0 - \frac{(1 + \delta^2)\delta\kappa}{2(p - \delta^2)^2} \frac{1}{s} + O\left(\frac{\log |s|}{s^2}\right) \tag{51}$$

and

$$e^{i\theta(s)} = e^{i\theta_0} \left\{ 1 - i \frac{(1 + \delta^2)\delta\kappa}{2(p - \delta^2)^2} \frac{1}{s} + O\left(\frac{\log |s|}{s^2}\right) \right\} \tag{52}$$

Using the fact that $w(y, s) = e^{i\theta(s)}(\kappa + v(y, s))$ (see(41)), the desired estimate follows from (52) and the L_ρ^2 expansion of v from (ii) of Proposition 3.5. This concludes the proof of Proposition 2.5. ■

Step 3: Case where (i) of Proposition 2.5 holds

We prove Proposition 2.6, more precisely, we will prove that either $w \equiv \kappa e^{i\theta_0}$ or there exists $s_0 \in \mathbb{R}$, such that $w = \varphi_\delta(s - s_0) e^{i\theta_0}$.

As we wrote in Step 3 of section 2, if $\hat{\varphi}$ defined by (27), then we have

$$\forall s \leq \hat{s}, \quad \|w(\cdot, s) - \hat{\varphi}(s) e^{i\theta_0}\|_{L_\rho^2} \leq C e^{\frac{3}{2}s}. \tag{53}$$

Our goal is to prove that $w \equiv \hat{\varphi}$ on $\mathbb{R} \times (-\infty, s_*]$. If we introduce $V = w - \hat{\varphi}e^{i\theta_0}$, then we see from (10) V satisfies:

$$\partial_s V = \left(\tilde{\mathcal{L}} + l(s) \right) V + B, \quad (54)$$

where

$$\tilde{\mathcal{L}}V = \Delta V - \frac{1}{2}y \cdot \nabla V + (1 + i\delta)V, \quad |l(s)| \leq Ce^s \text{ and } |B| \leq C|V|^2 \text{ for all } s \leq s_1. \quad (55)$$

As we saw in Lemma 19 and (22), $\tilde{\mathcal{L}}$ is diagonal with respect to $(\tilde{V}_1, \tilde{V}_2)$ such that $V = (1 + i\delta)\tilde{V}_1 + i\tilde{V}_2$ and 1 is its largest eigenvalue.

Therefore, if we define $\|V\|_{\mathcal{L}} = \sqrt{\int (\tilde{V}_1^2 + \tilde{V}_2^2)\rho}$, an equivalent norm to $\|V(\cdot, s)\|_{L^2_\rho}$, then we get from (54) and (55)

$$\partial_s \|V\|_{\mathcal{L}} \leq (1 + Ce^s)\|V\|_{\mathcal{L}} + C\|V^2\|_{\mathcal{L}}.$$

To estimate $\|V^2\|_{\mathcal{L}}$, it is easy to see from (54) and the fact that V is bounded that

$$\partial_s (|\tilde{V}_1| + |\tilde{V}_2|) \leq \Delta(|\tilde{V}_1| + |\tilde{V}_2|) - \frac{y}{2} \cdot \nabla(|\tilde{V}_1| + |\tilde{V}_2|) + C(|\tilde{V}_1| + |\tilde{V}_2|).$$

Therefore, we can apply the regularizing effect of Lemma 3.4 to $|\tilde{V}_1| + |\tilde{V}_2|$ and obtain the existence of $C^* > 0$ and s^* , such that $\|V(\cdot, s)\|_{\mathcal{L}}^2 \leq C^*\|V(\cdot, s - s^*)\|_{\mathcal{L}}^2$. Then, we obtain

$$\forall s \leq s_2, \quad I'(s) \leq \frac{5}{4}I(s) + CI(s - s^*)^2, \quad (56)$$

where $I(s) = \|V(\cdot, s)\|_{\mathcal{L}}$. Since $I(s) \leq Ce^{3/2s}$ from (53), the following lemma from [MZ98a] allows us to conclude.

Lemma 3.6 *Consider $I(s)$ a positive C^1 function such that (56) is satisfied and $0 \leq I(s) \leq Ce^{3/2s}$ for all $s \leq s_2$, for some s_2 . Then, for some $s_3 \leq s_2$, we have $I(s) = 0$ for all $s \leq s_3$.*

Proof: By a trivial induction, we prove that

$$\forall n \in \mathbb{N}^* \text{ and } s \leq s_2 \quad I(s) \leq \frac{(Ce^{3/4s})^{2^n}}{C}.$$

Taking $s \leq s_3$, where $Ce^{3/4s_3} = 1/2$ and making $n \rightarrow +\infty$, we see that for all $s \leq s_3$, $I(s) = 0$. ■

Using Lemma 3.6, we see that $V \equiv 0$ on $\mathbb{R} \times (-\infty, s_3]$. Consequently, we have

$$\forall (y, s) \in \mathbb{R} \times (-\infty, s_3], \quad w(y, s) = \hat{\varphi}(s)e^{i\theta_0}. \quad (57)$$

From the uniqueness of the Cauchy problem for equation (10) and since w is defined for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, $\hat{\varphi}$ is also defined for $(y, s) \in \mathbb{R} \times \mathbb{R}$ and (57) holds for all $(y, s) \in \mathbb{R} \times \mathbb{R}$. Therefore, case (c) in (27) cannot hold and for all $(y, s) \in \mathbb{R}^2$, $w(y, s) = \kappa e^{i\theta_0}$ or $w(y, s) = \varphi_\delta(s - s_0)e^{i\theta_0}$. This concludes the proof of Proposition 2.6 and finishes Step 3.

Step 4: Irrelevance of the case (iii) of Proposition 2.5

As we said in step 4 of Section 2, it is enough to prove Proposition 2.8 (which we recall here) to conclude this case:

Proposition 2.8 *Assume that case (iii) from Proposition 2.5 holds, then there exists $a_0 > 0$, such that:*

$$\lim_{s \rightarrow -\infty} \sup_{|y| \leq a_0 \sqrt{-s}} \left| w(y, s) - G\left(\frac{y}{\sqrt{-s}}\right) \right| = 0, \quad (58)$$

where $G(\xi) = \kappa \left(1 - \frac{(p-1)}{4(p-\delta^2)} \xi^2 \right)^{-\frac{(1+i\delta)}{(p-1)}}$.

Indeed, let us first use Proposition 2.8 to find a contradiction ruling out case (iii) of Proposition 2.5, and then prove Proposition 2.8.

We define u_{s_0} by

$$u_{s_0}(\xi, \tau) = (1 - \tau)^{-\frac{1+i\delta}{p-1}} w(y, s) \text{ where } y = \frac{\xi + \frac{\varepsilon_0}{2} \sqrt{-s_0}}{\sqrt{1 - \tau}} \text{ and } s = s_0 - \log(1 - \tau). \quad (59)$$

We note that u_{s_0} is defined for all $\tau \in [0, 1)$ and $\xi \in \mathbb{R}$. u_{s_0} satisfies equation (2). The initial condition at time $\tau = 0$ is $u_{s_0}(\xi, 0) = w(\xi + \frac{\varepsilon_0}{2} \sqrt{-s_0}, s_0)$. From (11), we have

$$\forall \tau \in [0, 1), \|u_{s_0}(\cdot, \tau)\|_{L^\infty} \leq M(1 - \tau)^{-\frac{1}{p-1}}. \quad (60)$$

Using Proposition 2.8, we get:

$$\sup_{|\xi| < 4|s_0|^{1/4}} \left| u_{s_0}(\xi, 0) - G\left(\frac{\xi}{2}\right) \right| \equiv g(s_0) \rightarrow 0 \text{ as } s_0 \rightarrow -\infty.$$

If we define v , the solution of:

$$\begin{cases} v' &= (1 + i\delta)|v|^{p-1}v, \\ v(0) &= G\left(\frac{\varepsilon_0}{2}\right), \end{cases}$$

then $v(\tau) = \kappa \left(1 - \frac{(p-1)\varepsilon_0^2}{16(p-\delta^2)} - \tau \right)^{-\frac{(1+i\delta)}{p-1}}$, which blows up at time $1 - \frac{(p-1)\varepsilon_0^2}{16(p-\delta^2)} < 1$. Therefore, there exists $\tau_0 < 1$, such that $|v(\tau_0)| = 2M(1 - \tau_0)^{-\frac{1}{p-1}}$. Now, we consider the function $z = u_{s_0} - v$, then we have for all $\tau \in [0, \tau_0]$:

$$\partial_\tau |z| \leq \Delta |z| + C(\varepsilon_0) |z|. \quad (61)$$

We recall Lemma 2.11 (page 1063) from [MZ98b]:

Lemma 3.7 *Assume that $z(\xi, \tau)$ satisfies for all $|\xi| \leq 4B_1$ and $\tau \in [0, \tau_*]$:*

$$\begin{cases} \partial_\tau z &\leq \Delta z + \lambda z + \mu, \\ z(\xi, 0) &\leq z_0, z(\xi, \tau) \leq B_2, \end{cases}$$

where $\tau_* \leq 1$. Then, for all $|\xi| \leq B_1$ and $\tau \in [0, \tau_*]$,

$$z(\xi, \tau) \leq e^{\lambda\tau} (z_0 + \mu + CB_2 e^{-B_1^2/4}).$$

Using the fact that z is bounded for all $\tau \in [0, \tau_0]$, by $B_2 = B_2(\varepsilon_0)$ (use (60)), we apply this Lemma with $B_1 = |s_0|^{1/4}$, $\tau_* = \tau_0$, $z_0 = g(s_0)$ and $\lambda = 0$. Then, we get for all $\tau \in [0, \tau_0]$,

$$\sup_{|\xi| \leq |s_0|^{1/4}} |z(\xi, \tau)| \leq g(s_0) + C(\varepsilon_0)e^{-|s_0|^{1/2}/4} \rightarrow 0 \text{ as } s_0 \rightarrow -\infty.$$

For $|s_0|$ large enough and $\xi = 0$, we get: $|z(0, \tau_0)| \leq \frac{M}{2}(1 - \tau_0)^{-1/(p-1)}$ and

$$|u_{s_0}(0, \tau_0)| \geq |v(\tau_0)| - |z(0, \tau_0)| \geq \frac{3M}{2}(1 - \tau_0)^{-\frac{1}{p-1}},$$

which is in contradiction with (60). Thus case (iii) of Proposition 2.5 cannot occur. Remains to prove Proposition 2.8.

Proof of Proposition 2.8: We note $f(y, s) = G\left(\frac{y}{\sqrt{-s}}\right)$, then f satisfies

$$-\frac{y}{2} \cdot \nabla f - \frac{(1+i\delta)}{(p-1)}f + (1+i\delta)|f|^{p-1}f = 0.$$

Consider some arbitrary $\varepsilon_0 \in (0, R^*)$, where $R^* = \sqrt{\frac{4(p-\delta^2)}{(p-1)}}$. The parameter ε_0 will be fixed later in the proof small enough. If we note

$$F(y, s) = f(y, s) + (1+i\delta)\frac{\kappa}{2(p-\delta^2)}\frac{1}{s} - i\frac{(1+\delta^2)\delta\kappa^2}{(p-\delta^2)^2}\frac{1}{s}, \quad (62)$$

then, we see from (iii) of Proposition 2.5 that

$$\|(F(\cdot, s) - w(\cdot, s))(1 - \chi_{\varepsilon_0})\|_{L_p^2} = O\left(\frac{\log|s|}{s^2}\right) \text{ as } s \rightarrow -\infty, \quad (63)$$

where

$$\chi_{\varepsilon_0}(y, s) = 1 \text{ if } \frac{|y|}{\sqrt{|s|}} \geq 3\varepsilon_0 \text{ and zero otherwise.} \quad (64)$$

The formal idea of this proof is that F solve in an approximate way the same equation as w for $s \rightarrow -\infty$. By (63), w and F are very close in the region $|y| \sim 1$. Our task is to prove that they remain close in the larger region $|y| \leq \varepsilon_0\sqrt{-s}$, for some ε_0 chosen later. Let us consider a cut-off function

$$\gamma(y, s) = \gamma_0\left(\frac{y}{\sqrt{-s}}\right), \quad (65)$$

where $\gamma_0 \in \mathcal{C}^\infty(\mathbb{R})$ is such that $\gamma_0(\xi) = 1$ if $|\xi| \leq 3\varepsilon_0$ and $\gamma_0(\xi) = 0$ if $|\xi| \geq 4\varepsilon_0$. We introduce

$$\nu = (w - F). \quad (66)$$

We note $\nu = (1+i\delta)\tilde{\nu}_1 + i\tilde{\nu}_2$ and $Z = \gamma(|\tilde{\nu}_1| + |\tilde{\nu}_2|)$. Our proof is the same as Velázquez [Vel92], except for the fact that we need to perform a cut-off, since our profile $F(y, s)$ defined by (62) is singular on the parabola $y = R^*\sqrt{-s}$. The cut-off function will generate an extra term, difficult to handle. Let us present the major steps of the proof in the following. The proof of the presented Lemmas, will be given at the end of this step.

Lemma 3.8 (Estimates in modified L_ρ^2 spaces.) *There exists $\varepsilon_0 > 0$ such that the function Z satisfies for all $s \leq s_*$ and $y \in \mathbb{R}$:*

$$\partial_s Z - \Delta Z + \frac{1}{2}y \cdot \nabla Z - (1 + \sigma)Z \leq C \left(Z^2 + \frac{(y^2 + 1)}{s^2} + \chi_{\varepsilon_0} \right) - 2 \operatorname{div} ((|\tilde{\nu}_1| + |\tilde{\nu}_2|) \nabla \gamma), \quad (67)$$

where $s_* \in \mathbb{R}$, $\sigma = 1/100$ and χ_{ε_0} is defined in (64). Moreover,

$$N_{2\varepsilon_0\sqrt{|s|}}^2(Z(s)) = o(1) \text{ as } s \rightarrow -\infty, \quad (68)$$

where the norm $N_r^q(\psi)$ is defined, for all $r > 0$ and $1 \leq q < \infty$, by

$$N_r^q(\psi) = \sup_{|\xi| \leq r} \left(\int |\psi(y)|^q \exp\left(-\frac{(y - \xi)^2}{4}\right) dy \right)^{1/q}. \quad (69)$$

Using the regularizing effect of the operator \mathcal{L} , we derive the following pointwise estimate, which allows us to conclude the proof of Proposition 2.8:

Lemma 3.9 (An upper bound for $Z(y, s)$ in $\{|y| \leq \varepsilon_0\sqrt{-s}\}$.) *We have:*

$$\sup_{|y| \leq \varepsilon_0\sqrt{-s}} Z(y, s) = o(1) \text{ as } s \rightarrow -\infty.$$

Indeed, we have by definition of Z , for all $|y| < \varepsilon_0\sqrt{-s}$, $|w - F| = \bar{\nu} \leq CZ(y, s)$. Thus, Proposition 2.8 follows from Lemma 3.9. Remains to prove Lemma 3.8 and Lemma 3.9.

Proof of Lemma 3.8: The proof of (67) is straightforward and a bit technical. We leave it to Appendix B. Let us then prove (68). We take $s_0 < s_*$ and $s_0 \leq s < s_*$ such that $e^{\frac{s-s_0}{2}} \leq \sqrt{-s}$. We use the variation of constant formula in (67) to write

$$\begin{aligned} Z(y, s) &\leq S_\sigma(s - s_0)Z(\cdot, s_0) \\ &\quad + \int_{s_0}^s S_\sigma(s - \tau) \left(C \left\{ Z^2 + \frac{(y^2 + 1)}{\tau^2} + \chi_{\varepsilon_0} \right\} - 2 \operatorname{div} ((|\tilde{\nu}_1| + |\tilde{\nu}_2|) \nabla \gamma) \right) d\tau, \end{aligned}$$

where S_σ is the semigroup associated to the operator $\mathcal{L}_\sigma \phi = \Delta \phi - \frac{1}{2}y \cdot \nabla \phi + (1 + \sigma)\phi$, defined on $L_\rho^2(\mathbb{R})$. The kernel of the semigroup $S_\sigma(\tau)$ is

$$S_\sigma(\tau, y, z) = \frac{e^{(1+\sigma)\tau}}{(4\pi(1 - e^{-\tau}))^{1/2}} \exp \left[-\frac{|ye^{-\tau/2} - z|^2}{4(1 - e^{-\tau})} \right]. \quad (70)$$

Setting

$$r \equiv r(s, s_0) = 2\varepsilon_0 e^{\frac{s-s_0}{2}} = R_1 e^{\frac{s-s_0}{2}} \quad (71)$$

and taking the N_r^2 -norm we obtain

$$\begin{aligned}
N_r^2(Z(\cdot, s)) &\leq N_r^2(S_\sigma(s - s_0)Z(\cdot, s_0)) + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)Z(\cdot, \tau)^2) d\tau \\
&\quad + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau) \left(\frac{(y^2 + 1)}{\tau^2} \right)) d\tau \\
+ C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)\chi_{\varepsilon_0}(y, \tau)) d\tau &+ C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)(\operatorname{div}(|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma)) d\tau \\
&\equiv J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

In comparison with [Vel92], we have a new term J_5 coming from the cut-off terms. Therefore, we just recall in the following claim the estimates on $J_1 \dots J_4$ from [Vel92], and treat J_5 , which is a new ingredient in our proof:

Claim 3.10 *We obtain as $s \rightarrow -\infty$*

$$\begin{aligned}
|J_1| &\leq C e^{(1+\sigma)(s-s_0)} \frac{\log |s_0|}{|s_0^2|}, \\
|J_2| &\leq C \int_{s_0}^{s_0 + ((s-R_0) - s_0)_+} \frac{e^{(1+\sigma)(s-\tau-R_0)}}{(1 - e^{s-\tau-R_0})^{1/20}} (L_r^2(Z(\cdot, \tau)^2)) d\tau + C \frac{e^{(s-s_0)(1+\sigma)}}{s_0^2}, \\
&\quad \text{with } R_0 = 4\varepsilon_0, \\
|J_3| &\leq C \frac{e^{(s-s_0)(1+\sigma)}}{s_0^2} (1 + (s - s_0)), \\
|J_4| &\leq C e^{(s-s_0)(1+\sigma)} e^{\alpha s}, \text{ where } \alpha > 0, \\
|J_5| &\leq C e^{(s-s_0)(1+\sigma)} e^{\beta s}, \text{ where } \beta > 0.
\end{aligned}$$

Proof: See page 1578 in [Vel92] for $J_1 \dots J_4$.

Now, we treat J_5 . We have from (70):

$$\begin{aligned}
&S_\sigma(s - \tau) (-\operatorname{div}(|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma), \\
&= -\frac{C e^{(s-\tau)(1+\sigma)}}{(1 - e^{s-\tau})^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{4(1 - e^{-(s-\tau)})}\right) \operatorname{div}(|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma d\lambda, \\
&= \frac{C e^{(s-\tau)(1+\sigma)}}{(1 - e^{s-\tau})^{1/2}} \int_{\mathbb{R}} -\frac{(ye^{-(s-\tau)/2} - \lambda)}{2(1 - e^{-(s-\tau)})} \exp\left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{4(1 - e^{-(s-\tau)})}\right) (|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma d\lambda.
\end{aligned} \tag{72}$$

Since w and F are bounded for $\frac{|y|}{\sqrt{-\tau}} \leq R^*/2$ and $\operatorname{supp}(\nabla\gamma) \subset (-4\varepsilon_0\sqrt{-\tau}, -3\varepsilon_0\sqrt{-\tau}) \cup (3\varepsilon_0\sqrt{-\tau}, 4\varepsilon_0\sqrt{-\tau})$, we have

$$\begin{aligned}
|(|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma| &\leq C(|\tilde{\nu}_1| + |\tilde{\nu}_2|)\mathbb{I}_{\{3\varepsilon_0 \leq \frac{|y|}{\sqrt{-\tau}} \leq 4\varepsilon_0\}}, \\
&\leq C(\mathbb{I}_{3\varepsilon_0 \leq \frac{|y|}{\sqrt{-\tau}} \leq 4\varepsilon_0}) \leq C\chi_{\varepsilon_0}.
\end{aligned}$$

Using Cauchy-Schwartz inequality, we obtain:

$$|S_\sigma(s-\tau)(-\operatorname{div}(|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma)| \leq \frac{Ce^{(s-\tau)(1+\sigma)}}{(1-e^{s-\tau})^{3/2}}\mathcal{I}_1\mathcal{I}_2,$$

where,

$$\begin{aligned}\mathcal{I}_1 &= \left(\int_{\mathbb{R}} (ye^{-(s-\tau)/2} - \lambda)^2 \exp\left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{4(1-e^{-(s-\tau)})}\right) d\lambda \right)^{1/2}, \\ \mathcal{I}_2 &= \left(\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{4(1-e^{-(s-\tau)})}\right) \chi_{\varepsilon_0} d\lambda \right)^{1/2}.\end{aligned}$$

Doing a change of variables, we obtain $\mathcal{I}_1 = C(1-e^{-(s-\tau)})^{3/4}$. Furthermore, we have:

$$\mathcal{I}_2^2 \leq \mathcal{I}_3 \left(\int_{\mathbb{R}} \chi_{\varepsilon_0} e^{-\frac{\lambda^2}{4}} d\lambda \right)^{1/2},$$

where,

$$\mathcal{I}_3 = \left(\int \exp\left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{2(1-e^{-(s-\tau)})} + \frac{\lambda^2}{4}\right) d\lambda \right)^{1/2}.$$

We introduce $\theta = ye^{-(s-\tau)/2}$, by completing squares, we readily check that:

$$\frac{\lambda^2}{4} - \frac{(\theta - \lambda)^2}{2(1-e^{-(s-\tau)})} = -\frac{(1+e^{-(s-\tau)})}{4(1-e^{-(s-\tau)})} \left(\lambda - \frac{2\theta}{1+e^{-(s-\tau)}} \right)^2 + \frac{\theta^2}{2(1+e^{-(s-\tau)})},$$

then we obtain:

$$\mathcal{I}_3^2 = C \left(\frac{(1-e^{-(s-\tau)})}{(1+e^{-(s-\tau)})} \right)^{1/2} \exp\left(\frac{\theta^2}{2(1+e^{-(s-\tau)})}\right).$$

Therefore,

$$|N_r^2(S_\sigma(s-\tau)\operatorname{div}(|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma)| \leq C \frac{e^{(s-\tau)(1+\sigma)}}{(1+e^{-(s-\tau)})^{1/8}(1-e^{-(s-\tau)})^{5/8}} \|\chi_{\varepsilon_0}\|^{1/2} \mathcal{I}_4,$$

$$\text{where } \mathcal{I}_4 = N_r^2 \left(\exp\left(\frac{y^2 e^{-(s-\tau)}}{8((1+e^{-(s-\tau)})^2)}\right) \right).$$

Let us compute \mathcal{I}_4 . Using the fact that

$$\begin{aligned}-\frac{(y-\mu)^2}{4} + \frac{y^2 e^{-(s-\tau)}}{4(1-e^{-(s-\tau)})} &= \\ \frac{1}{4} \left(-\left(y(1+e^{-(s-\tau)})^{-1/2} - \mu(1+e^{-(s-\tau)})^{1/2}\right)^2 + \mu^2 e^{-(s-\tau)} \right),\end{aligned}$$

and doing a change of variables, we obtain:

$$\begin{aligned} & \int_{\mathbb{R}} \exp\left(-\frac{(y-\mu)^2}{4} + \frac{y^2 e^{-(s-\tau)}}{4(1-e^{-(s-\tau)})}\right) dy \\ & \leq C \exp\left(\frac{\mu^2 e^{-(s-\tau)}}{4}\right) \int_{\mathbb{R}} \exp\left(-\frac{1}{4}\left(y(1+e^{-(s-\tau)})^{-1/2} - \mu(1+e^{-(s-\tau)})^{1/2}\right)^2\right) dy. \end{aligned}$$

Hence $\mathcal{I}_4 \leq C(1+e^{-(s-\tau)})^{1/8}$ and

$$N_r^2(S_\sigma(s-\tau)) \operatorname{div}((\tilde{\nu}_1 + \tilde{\nu}_2) \nabla \gamma) \leq C \frac{e^{(s-\tau)(1+\sigma)}}{(1-e^{-(s-\tau)})^{5/8}} \left(\int_{|\lambda| \geq R_1 \sqrt{-\tau}} e^{-\frac{\lambda^2}{4}} d\lambda \right).$$

This gives

$$|J_5| = \int_{s_0}^s N_r^2(S_\sigma(s-\tau)) (\operatorname{div}(\tilde{\nu}_1 + \tilde{\nu}_2)) d\tau \leq C(\eta) e^{(s-s_0)(1+\sigma)} e^{\alpha s_0},$$

where $\alpha > 0$. This concludes the proof of the claim 3.10. ■

Summing up $J_{i=1..5}$, from claim 3.10 we obtain

$$\begin{aligned} N_r^2(Z(\cdot, s)) & \leq \\ & e^{(s-s_0)(1+\sigma)} C \frac{\log|s_0|}{s_0^2} + C \int_{s_0}^{s_0 + ((s-R_0)-s_0)_+} \frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1-e^{s-\tau-R_0})^{1/20}} (N_r^2(Z(\cdot, \tau)))^2 d\tau. \end{aligned}$$

Now, we recall the following from [Vel92]:

Lemma 3.11 *Let $\varepsilon, C, R, \sigma$ and α be positives constants, $0 < \alpha < 1$ and assume that $H(s)$ is a family of continuous functions satisfying:*

$$H(s) \leq \varepsilon e^{s(1+\sigma)} + C \int_0^{(s-R)_+} \frac{e^{(s-\tau)(1+\sigma)} H(\tau)^2}{(1-e^{(s-\tau-R)})^\alpha} d\tau \text{ for } s > 0.$$

Then there exists $\xi = \xi(R, C, \alpha)$ such that for any $\varepsilon \in (0, \varepsilon_1)$ and any s for which $\varepsilon e^{s(1+\sigma)} \leq \xi$, we have

$$H(s) \leq 2\varepsilon e^{s(1+\sigma)}.$$

Proof: See the proof of Lemma 2.2 from [Vel92]. Note that the proof of [Vel92] is done in the case $\sigma = 0$, but it can be adapted to some $\sigma > 0$ with no difficulty. ■

We conclude that $N_{r(\tau, s_0)}^2(Z(\cdot, s)) \leq C e^{(s-s_0)(1+\sigma)} \frac{\log|s_0|}{s_0^2}$ as $s \rightarrow -\infty$. If we fix $s = -e^{(s-s_0)}$, then we obtain $s \sim s_0$, $\log|s| \sim \log|s_0|$ and $N_{R_1 \sqrt{-s}}^2(Z(\cdot, s)) \leq C s^{1+\sigma} \frac{\log|s_0|}{s_0^2} \leq C \frac{\log|s|}{s^{1-\sigma}} \rightarrow 0$ as $s \rightarrow -\infty$. Since $\sigma = \frac{1}{100}$, we get $N_{R_1 \sqrt{-s}}^2(Z(\cdot, s)) = o(1)$, as $s \rightarrow -\infty$. This concludes the proof of Lemma 3.8. ■

Proof of Lemma 3.9.

We aim at bounding $Z(y, s)$ for $|y| \leq R_2\sqrt{-s}$ in terms of $N_{R_1\sqrt{-s'}}(Z(s'))$, where $R_2 = \varepsilon_0$ and $R_1 = 2\varepsilon_0$, for some $s' < s$. Starting from equation (67), we do as in [Vel92]:

$$\begin{aligned} Z(\cdot, s) &\leq \{e^{CR_0} S(R_0) Z(\cdot, s - R_0)\} \\ &\quad + \left\{ C \int_{s-R_0}^s e^{C(s-\tau)} S(s-\tau) \left(\frac{(y^2+1)}{\tau^2} + \chi_{\varepsilon_0} \right) d\tau \right\} \\ &\quad - \left\{ 2 \int_{s-R_0}^s e^{C(s-\tau)} S(s-\tau) (\operatorname{div}((\tilde{\nu}_1 + \tilde{\nu}_2)\nabla\gamma)) d\tau \right\} \\ &= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3, \text{ where } R_0 = 4\varepsilon_0, \end{aligned}$$

where S is the semigroup associated to the operator \mathcal{L} defined in (35). The terms \mathcal{M}_1 and \mathcal{M}_2 are estimated in the following:

Claim 3.12 (Velázquez) *There exists s_0 , such that for all $s \leq s_0$*

$$\begin{aligned} \sup_{|y| \leq R_2\sqrt{-s}} |\mathcal{M}_1| &= \sup_{|y| \leq R_2\sqrt{-s}} \int_{s-R_0}^s \left(\frac{|y|^2+1}{s^2} + \chi_{\varepsilon_0} \right) \leq \frac{C}{|s|}, \\ \sup_{|y| \leq R_2\sqrt{-s}} |\mathcal{M}_2| &= \sup_{|y| \leq R_2\sqrt{-s}} |e^{CR_0} S(R_0) Z(\cdot, s - R_0)| = o(1) \text{ as } s \rightarrow -\infty. \end{aligned} \quad (73)$$

Proof: See page 1581 from [Vel92] and Lemma 6.5 in [HV93] in a similar case. ■

It remains to estimate \mathcal{M}_3 . Using page (72), we write

$$\begin{aligned} &|S(s-\tau) (-\operatorname{div}((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma))| \\ &= \left| \frac{Ce^{s-\tau}}{(1-e^{s-\tau})^{1/2}} \int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{4(1-e^{-(s-\tau)})}\right) \operatorname{div}((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma) d\lambda \right|, \\ &= \left| \frac{Ce^{s-\tau}}{(1-e^{s-\tau})^{1/2}} \int_{\mathbb{R}} -\frac{(ye^{-(s-\tau)/2} - \lambda)}{2(1-e^{-(s-\tau)})} \exp\left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{4(1-e^{-(s-\tau)})}\right) (|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma d\lambda \right|, \\ &\leq \frac{Ce^{s-\tau}}{(1-e^{s-\tau})^{3/2}} \int_{\mathbb{R}} |ye^{-(s-\tau)/2} - \lambda| \exp\left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{4(1-e^{-(s-\tau)})}\right) \chi_{\varepsilon_0} d\lambda, \\ &\leq \frac{Ce^{s-\tau}\sqrt{-\tau}}{(1-e^{s-\tau})^{3/2}} \int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{4(1-e^{-(s-\tau)})}\right) \chi_{\varepsilon_0} d\lambda. \end{aligned}$$

We make the change of variables $z = (1 - e^{-(s-\tau)})^{-1/2}(\lambda - e^{-(\tau-s)/2}y)$ and we obtain

$$\int_{\mathbb{R}} \exp\left(-\frac{(ye^{-(s-\tau)/2} - \lambda)^2}{4(1-e^{-(s-\tau)})}\right) \chi_{\varepsilon_0} d\lambda \leq (1 - e^{s-\tau})^{1/2} \int_{\Sigma} e^{-z^2/4} dz,$$

where,

$$\Sigma = \left\{ z \in \mathbb{R} : \left| z + e^{-(\tau-s)/2}(1 - e^{s-\tau})^{-1/2}y \right| \geq 3\varepsilon_0(1 - e^{s-\tau})^{-1/2}\sqrt{-\tau} \right\}.$$

Since $|ye^{-(\tau-s)/2}| \leq \varepsilon_0\sqrt{-s}$, we readily see that $\Sigma \subset \{z \in \mathbb{R} : |z| \geq \varepsilon_0\sqrt{-s}\}$. Then we conclude that

$$|S(s-\tau) (-\operatorname{div}((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma))| \leq \frac{Ce^{s-\tau}}{(1-e^{s-\tau})} e^{\beta s}, \text{ where } \beta > 0,$$

and we obtain

$$\sup_{|y| \leq R_2 \sqrt{-s}} |\mathcal{M}_3| = o\left(\frac{1}{|s|}\right) \text{ as } s \rightarrow -\infty.$$

Putting together $\mathcal{M}_{i=1,3}$, the proof of lemma 3.9 is complete. This concludes also the proof of Proposition 2.8 and rules out case (iii) of Proposition 2.5. ■

Step 5: Irrelevance of the case (ii) of Proposition 2.5

To conclude the proof of Theorem 1, we consider case (ii) of Proposition 2.5. We assume as in the previous case that $\theta_0 = 0$. We claim that the following proposition allows us to reach a contradiction in this case.

Proposition 3.13 *There exists $\varepsilon_0 > 0$ such that*

$$\lim_{s \rightarrow -\infty} \sup_{|y| \leq \varepsilon_0 e^{-s/2}} \left| w(y, s) - G(ye^{s/2}) \right| = 0, \text{ where } G(\xi) = \kappa(1 - C_1 \kappa^{-p} \xi)^{-\frac{(1+i\delta)}{(p-1)}}. \quad (74)$$

Indeed, as in the previous Step, first, we will find a contradiction ruling out case (ii) of Proposition 2.5 and then prove Proposition 3.13.

We define u_{s_0} by

$$u_{s_0}(\xi, \tau) = (1 - \tau)^{-\frac{1+i\delta}{p-1}} w(y, s) \text{ where } y = \frac{\xi + \frac{\varepsilon_0}{2} e^{-s_0/2}}{\sqrt{1 - \tau}} \text{ and } s = s_0 - \log(1 - \tau). \quad (75)$$

u_{s_0} is defined for all $\tau \in [0, 1)$ and $\xi \in \mathbb{R}$. u_{s_0} satisfies equation (2). The initial condition at time $\tau = 0$ is $u_{s_0}(\xi, 0) = w(\xi + \frac{\varepsilon_0}{2} e^{-s_0/2}, s_0)$. From (11), we have

$$\forall \tau \in [0, 1), \|u_{s_0}(\cdot, \tau)\|_{L^\infty} \leq M(1 - \tau)^{-\frac{1}{p-1}}. \quad (76)$$

Using Proposition 3.13, we get:

$$\sup_{|\xi| < 4e^{-s_0/4}} |u_{s_0}(\xi, 0) - G(\varepsilon_0/2)| \equiv g(s_0) \rightarrow 0 \text{ as } s_0 \rightarrow -\infty.$$

If we define v , the solution of:

$$\begin{cases} v' &= (1 + i\delta)|v|^{p-1}v, \\ v(0) &= G\left(\frac{\varepsilon_0}{2}\right), \end{cases}$$

then $v(\tau) = \kappa \left(1 - C_1 \kappa^{-p} \frac{\varepsilon_0}{2} - \tau\right)^{-\frac{(1+i\delta)}{p-1}}$, which blows up at time $1 - C_1 \kappa^{-p} \frac{\varepsilon_0}{2} < 1$. Therefore, there exists $\tau_0 < 1$, such that $|v(\tau_0)| = 2M(1 - \tau_0)^{-\frac{1}{p-1}}$. Now, we consider the function $z = u_{s_0} - v$, then we have for all $\tau \in [0, \tau_0]$:

$$\partial_\tau |z| \leq \Delta |z| + C(\varepsilon_0) |z|. \quad (77)$$

Using the fact that z is bounded for all $\tau \in [0, \tau_0]$ by $B_2 = B_2(\varepsilon_0)$ (use (76)), we use Lemma 3.7 with $B_1 = e^{-s_0/4}$, $\tau_* = \tau_0$, $z_0 = g(s_0)$ and $\lambda = 0$. We obtain for all $\tau \in [0, \tau_0]$,

$$\sup_{|\xi| \leq e^{-\varepsilon_0/4}} |z(\xi, \tau)| \leq g(s_0) + C(\varepsilon_0) e^{-e^{-|s_0|/2}/16} \rightarrow 0 \text{ as } s_0 \rightarrow -\infty.$$

For $|s_0|$ large enough and $\xi = 0$, we get $|z(0, \tau_0)| \leq M/2(1 - \tau_0)^{-\frac{1}{p-1}}$ and

$$|u_{s_0}(0, \tau_0)| \geq \frac{3}{2}M(1 - \tau_0)^{-\frac{1}{p-1}},$$

which by (75) is in contradiction with (76) and case (ii) of Proposition 2.5 is ruled out. Now, we prove Proposition 3.13.

Proof of Proposition 3.13: The proof is very similar to that of Proposition 2.8. We note $f(y, s) = G(ye^{s/2})$, then f satisfies

$$\partial_s f - \frac{1}{2}y \cdot \nabla f - (1 + i\delta)\frac{f}{(p-1)} + (1 + i\delta)|f|^{p-1}f = 0. \quad (78)$$

Consider an arbitrary $\varepsilon_0 \in (0, \frac{R^*}{10})$, where $R^* = \frac{\kappa^p}{C_1}$. ε_0 will be fixed small enough later. Let us consider a cut-off function $\gamma(y, s) = \gamma_0(ye^{s/2})$, where $\gamma_0 \in C^\infty(\mathbb{R})$ such that $\gamma_0(\xi) = 1$ if $|\xi| \leq 3\varepsilon_0$ and $\gamma_0(\xi) = 0$ if $|\xi| \geq 4\varepsilon_0$. We note $\nu = (w - f)$ and $Z = \gamma(|\tilde{\nu}_1| + |\tilde{\nu}_2|)$. From (ii) of Proposition 2.5, we have

$$\|Z\| \leq Ce^{s(1-\varepsilon)} \text{ as } s \rightarrow -\infty, \text{ for some } \varepsilon > 0. \quad (79)$$

As in the previous case, we divide our proof in two parts given in the following lemmas.

Lemma 3.14 (Estimates in the modified L_ρ^2 spaces.) *There exists $\varepsilon_0 > 0$ such that the function Z satisfies for all $s \leq s_*$ and $y \in \mathbb{R}$,*

$$\partial_s Z - \Delta Z + \frac{1}{2}y \cdot \nabla Z - (1 + \sigma)Z \leq C(Z^2 + e^s + \chi_{\varepsilon_0}) - 2\operatorname{div}((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma), \quad (80)$$

where $s_* \in \mathbb{R}$, $\sigma = \frac{1}{100}$ and

$$\chi_{\varepsilon_0}(y, s) = 1 \text{ if } |y|e^{s/2} \geq 3\varepsilon_0 \text{ and zero otherwise.} \quad (81)$$

Moreover, we have

$$N_{2\varepsilon_0 e^{-s/2}}^2(Z(s)) = o(1) \text{ as } s \rightarrow -\infty. \quad (82)$$

As in Step 4, the following lemma allows us to conclude the proof of Proposition 3.13:

Lemma 3.15 (An upper bound for $Z(y, s)$ in $|y| \leq \varepsilon_0 e^{-s/2}$.) *We have:*

$$\sup_{|y| \leq \varepsilon_0 e^{-s/2}} Z(y, s) = o(1) \text{ as } s \rightarrow -\infty. \quad (83)$$

Remains to prove Lemmas 3.14 and 3.15 to conclude the proof of Proposition 3.13. Here, we only sketch the proof of Lemma 3.14, since it is completely similar to Step 4. We don't give the proof of Lemma 3.15. We refer the reader to Step 4 and Proposition 2.4 from Velázquez [Vel92] for similar situations.

Proof of Lemma 3.14: As in the previous step, we leave the proof of (80) to Appendix B. Let us now apply variation of constants formula and take the norm $N_{r(s,s_0)}^2$, where $r(s, s_0)$ is as in (71). Assume that $s_0 < 2s_*$, then for all $s_0 \leq s \leq \frac{s_0}{2}$, we have

$$\begin{aligned} N_r^2(Z(\cdot, s)) &\leq N_r^2(S_\sigma(s - s_0)Z(\cdot, s_0)) + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)(Z(\cdot, \tau)^2))d\tau \\ &\quad + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)(e^\tau))d\tau + C \int_{s_0}^s N_r^2(S_\sigma(s - \tau)(\chi_{\varepsilon_0}(\cdot, \tau)))d\tau \\ &\quad - 2 \int_{s_0}^s N_r^2(S_\sigma(s - \tau)(\operatorname{div}(|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla\gamma))d\tau \\ &= J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

Arguing as in Step 4 and using (79), we prove:

Claim 3.16

$$\begin{aligned} |J_1| &\leq C e^{(s-s_0)(1+\sigma)} e^{s_0(1-\varepsilon)}, \\ |J_2| &\leq C \int_{s_0}^{s_0+((s-R_0)-s_0)_+} \frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1 - e^{s-\tau-R_0})^{1/20}} (L_r^2(Z(\cdot, s)^2))d\tau + C e^{(s-s_0)(1+\sigma)} e^s \\ \text{with } R_0 &= 4\varepsilon_0, \\ |J_3| &\leq C e^{(s-s_0)(1+\sigma)} e^s, \\ |J_4| &\leq C e^{(s-s_0)(1+\sigma)} e^{-\alpha e^{-s}} \text{ where } \alpha > 0, \\ |J_5| &\leq C e^{(s-s_0)(1+\sigma)} e^{-\beta e^{-s}} \text{ where } \beta > 0. \end{aligned}$$

Proof: To estimate $J_{i=1..4}$, see page 1584 in [Vel92]. To treat J_5 , we proceed as in the proof of Lemma 3.8 of the previous Step. ■

Summing up $J_{i=1..5}$, we obtain:

$$\begin{aligned} N_r^2(Z(\cdot, s)) &\leq \\ &C e^{(s-s_0)(1+\sigma)} e^{(1-\varepsilon)s} + C \int_{s_0}^{s_0+((s-R_0)-s_0)_+} \frac{e^{(s-\tau-R_0)(1+\sigma)}}{(1 - e^{s-\tau-R_0})^{1/20}} (L_r^2(Z(\cdot, s)^2))d\tau, \end{aligned}$$

then using Proposition 3.11, we get $N_{r(s,s_0)}^2(Z(\cdot, s)) \leq C e^{(s-s_0)(1+\sigma)} e^{(1-\varepsilon)s}$ as $s \rightarrow -\infty$ for $s_0 \leq s \leq \frac{s_0}{2}$. If we fix $s = s_0/2$, then we obtain $N_{r(s,s_0)}^2(Z(\cdot, s)) \leq C e^{s(2(1-\varepsilon)-(1+\sigma))} \leq C e^{s(1-(2\varepsilon+\sigma))} \rightarrow 0$ as $s \rightarrow -\infty$, since ε is small enough and $\sigma = \frac{1}{100}$. This concludes the proof of Lemma 3.14. ■

As announced earlier, we don't give the proof of Lemma 3.15 and refer the reader to Step 4 and Section 2 from [Vel92]. This concludes the proof of Proposition 3.13 and rules out case (ii) of Proposition 2.5.

Conclusion of Part 3 and the sketch of proof of the Liouville theorem:

As we wrote in Section 2, we conclude from Step 4 and 5 that cases (ii) and (iii) of Proposition 2.5 are ruled out. By Step 3, we obtain that $w \equiv \kappa e^{i\theta_0}$ or $w \equiv \varphi_\delta(s - s_0) e^{i\theta_0}$ for some real s_0 and θ_0 , where φ_δ is defined in Theorem 1, which is the desired conclusion of Theorem 1.

4 Applications of the Liouville Theorem for a type I blow-up solution of (2)

In this section we say how to adapt to the case $\delta \neq 0$, the proof of Proposition 3 given in [MZ98a] and [MZ00] in the case $\delta = 0$.

Proof of (i) of Proposition 3: The proof is exactly the same as in the case $\delta = 0$ (see page 148 in [MZ98a]). However, one needs the following lower bound which is a bit tricky to get and which we give for the reader's convenience.

Lemma 4.1 (*Sharp lower bound on the blow-up rate*) For all $t \in [0, T)$,

$$\|u(t)\|_{L^\infty} \geq \kappa(T-t)^{-\frac{1}{p-1}}.$$

Remark: This bound is sharp, since there is equality for the solutions of the ODE $v' = (1+i\delta)|v|^{p-1}v$, which are particular solutions of (2).

Proof: we introduce $\tilde{\rho} = \sqrt{1+|u|^2}$, we claim that $\tilde{\rho}$ satisfies

$$\partial_t \tilde{\rho} \leq \Delta \tilde{\rho} + \tilde{\rho}^p. \quad (84)$$

Indeed, we can easily prove that $\partial_t |u|^2 = \bar{u}\Delta u + u\Delta \bar{u} + 2|u|^{p+1}$. Then we have:

$$\begin{aligned} \partial_t \tilde{\rho} &= \frac{\partial_t |u|^2}{2(1+|u|^2)^{1/2}}, \\ \Delta \tilde{\rho} &= \frac{\Delta |u|^2}{2(1+|u|^2)^{1/2}} - \frac{|\nabla |u|^2|^2}{4(1+|u|^2)^{3/2}}, \\ &= \frac{\bar{u}\Delta u + u\Delta \bar{u} + 2|\nabla u|^2}{2(1+|u|^2)^{1/2}} - \frac{|u \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u|^2}{4(1+|u|^2)^{3/2}}. \end{aligned}$$

Using the fact that $|u \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u|^2 \leq 4|u|^2|\nabla u|^2 \leq 4(1+|u|^2)|\nabla u|^2$, we have $\Delta \tilde{\rho} \geq \frac{\bar{u}\Delta u + u\Delta \bar{u}}{2(1+|u|^2)^{1/2}}$, hence

$$\partial_t \tilde{\rho} \leq \Delta \tilde{\rho} + \frac{|u|^{p+1}}{(1+|u|^2)^{1/2}} \leq \Delta \tilde{\rho} + \tilde{\rho}^p,$$

which gives (84).

Now we prove that $\tilde{\rho} \geq \kappa(T-t)^{-\frac{1}{p-1}}$, for all $t \in [0, T)$. For this, we argue by contradiction. Assume that $\|\tilde{\rho}\|_{L^\infty} < \kappa(T-t_0)^{-\frac{1}{p-1}}$, for some $t_0 < T$. Then, there exists $T_0 > T$ such that $\|\tilde{\rho}(t_0)\|_{L^\infty} \leq \kappa(T_0-t_0)^{-\frac{1}{p-1}}$. Using the maximum principle, the inequality remains valid after T_0 and we have $\limsup_{t \rightarrow T} \|\tilde{\rho}(t)\|_{L^\infty} \leq \kappa(T_0-T)^{-\frac{1}{p-1}} < \infty$, which is a contradiction. This concludes the proof of Lemma 4.1. ■

Proof of (ii) of Proposition 3: Consider $|\delta| \leq \delta_0$ and a solution $u(t)$ of (2), that blows up in finite time $T > 0$ such that

$$\forall t \in [0, T), \|u(t)\|_{L^\infty} \leq M(\delta)(T-t)^{-\frac{1}{p-1}}, \quad (85)$$

where δ_0 and $M(\delta)$ are defined in Theorem 1. Let us prove now the uniform pointwise control of the diffusion term by the nonlinear term, which asserts that the solution $u(t)$ behaves everywhere like the ODE $u' = (1 + i\delta)|u|^{p-1}u$ (up to a constant).

The plan of the proof is the same as in [MZ98a] and [MZ00]. However, the Giga-Kohn property "small local energy implies no blow-up locally" breaks down because we no longer have a gradient structure. The property has to be replaced by a new idea of ours "small L^2_ρ norm implies no blow-up locally" which is stated in Proposition 3.3.

We argue by contradiction and assume that for some $\varepsilon_0 > 0$, there exists $(x_n, t_n)_{n \in \mathbb{N}}$, a sequence of elements of $\mathbb{R} \times [\frac{T}{2}, T)$, such that

$$\forall n \in \mathbb{N}, |\Delta u(x_n, t_n)| \geq \varepsilon_0 |u(x_n, t_n)|^p + n. \quad (86)$$

From the uniform estimates and the parabolic regularity, since $\|\Delta u\|_{L^\infty}$ is bounded on compact sets of $[\frac{T}{2}, T)$, we have

$$T - t_n \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Part (i) of Proposition 3 implies that $|u(x_n, t_n)|(T - t_n)^{\frac{1}{p-1}}$ is uniformly bounded, therefore, we can assume that it converges as $n \rightarrow +\infty$. Let us consider two cases:

i) *Estimates in the very singular region:* $|u(x_n, t_n)|(T - t_n)^{\frac{1}{p-1}} \rightarrow \kappa_0 > 0$. From (86), it follows that

$$\|\Delta u(t_n)\|_{L^\infty} \geq |\Delta u(x_n, t_n)| \geq \varepsilon_0 \left(\frac{\kappa_0}{2}\right)^p (T - t_n)^{-\frac{p}{p-1}},$$

with $t_n \rightarrow T$, which contradicts (i) of Proposition 3.

ii) *Estimates in the singular region:* $u(x_n, t_n)(T - t_n)^{\frac{1}{p-1}} \rightarrow 0$.

We consider n large enough, such that

$$|u(x_n, t_n)|(T - t_n)^{\frac{1}{p-1}} \leq \frac{\eta_0}{3}, \text{ where } \eta_0 \text{ is defined in Proposition 3.3.}$$

We take $t_n^0 \rightarrow T$ such that

$$(T - t_n^0)^{-\frac{p}{p-1}} = \sqrt{n}. \quad (87)$$

Using (86) and uniform estimates, we obtain:

$$n \leq |\Delta u(x_n, t_n)| \leq C_0 (T - t_n)^{-\frac{p}{p-1}},$$

hence $t_n^0 < t_n$. Now we distinguish two cases:

Case 1. We assume that (up to extracting a subsequence) there exists $t'_n \in (t_n^0, t_n)$, such that $|u(x_n, t'_n)|(T - t'_n)^{\frac{1}{p-1}} = \frac{2}{3}\eta_0$. If we consider

$$v_n(\xi, \tau) = (T - t'_n)^{\frac{1}{p-1}} u(x_n + \xi\sqrt{T - t'_n}, t'_n + \tau(T - t'_n)), \quad (88)$$

then, we have from (i) of Proposition 3 and (85)

$$|v_n(0, 0)| = \frac{2}{3}\eta_0, \|\nabla v_n(0)\|_{L^\infty} + \|\Delta v_n(0)\|_{L^\infty} \rightarrow 0, \quad (89)$$

$$\forall \tau < 1, \|v_n(\tau)\|_{L^\infty} \leq M(\delta)(1-\tau)^{-\frac{1}{p-1}} \text{ and } \partial_\tau v_n = \Delta v_n + (1+i\delta)|v_n|^{p-1}v_n.$$

Using parabolic regularity, we can extract a subsequence (still denoted by t_n) such that, $v_n(\xi, \tau) \rightarrow \hat{v}(\xi, \tau)$ in $\mathcal{C}^{2,1}$ of every compact set of $\mathbb{R} \times (-\infty, 1)$, with

$$\partial_\tau \hat{v} = \Delta \hat{v} + (1+i\delta)|\hat{v}|^{p-1}\hat{v}, |\hat{v}(0,0)| = 2/3\eta_0 \text{ and } \|\hat{v}\|_{L^\infty} \leq M(\delta)(1-\tau)^{-\frac{1}{p-1}}.$$

Using the Liouville Theorem (see Theorem 2), we get

$$\hat{v}(\xi, \tau) = \kappa \left(\left(\frac{3\kappa}{2\eta_0} \right)^{p-1} - \tau \right)^{-\frac{1+i\delta}{p-1}} e^{i\theta_0}, \text{ for some } \theta_0 \in \mathbb{R}.$$

We claim that it is enough to extend the convergence of $v_n \rightarrow \hat{v}$ to all $\tau \in [0, 1)$ (and $\xi = 0$), to conclude. Indeed, if we have this extended convergence, then we write from (86) and the definition (88) of v_n ,

$$|\Delta v_n(0, \tau_n)| = (T - t'_n)^{\frac{p}{p-1}} |\Delta u(0, t_n)| \geq \frac{\varepsilon_0}{2} |u(0, t_n)|^p (T - t'_n)^{\frac{p}{p-1}} \geq \frac{\varepsilon_0}{2} |v_n(0, \tau_n)|^p,$$

with $\tau_n = \frac{t_n - t'_n}{T - t'_n}$. Letting $n \rightarrow \infty$, we obtain

$$0 \geq \frac{\varepsilon_0}{2} \min_{\tau \in [0,1]} |\hat{v}(\tau)|^p \geq \frac{\varepsilon_0}{2} \left(\frac{2}{3}\eta_0 \right)^p, \quad (90)$$

which is a contradiction.

Let us then extend the convergence. If we consider the following similarity variables,

$$y = \frac{\xi - \xi_0}{\sqrt{1-\tau}}, s = -\log(1-\tau), w_{n,\xi_0}(y, s) = (1-\tau)^{\frac{1}{p-1}} v_n(\xi, \tau), \quad (91)$$

then, we see from (89) that for all $|\xi_0| \leq 1$, $\|w_{n,\xi_0}(\cdot, 0)\|_{L^2_\rho} \leq \eta_0$, for n large enough. Using Proposition 3.3, we get for all $|\xi| \leq 1$ and $\tau \in [0, 1)$, $|v_n(\xi, \tau)| \leq M_0$. Using the parabolic regularity, we can extend the convergence, and then reach the contradiction (90). This concludes Case 1.

Case 2. We assume that for some $n_0 \in \mathbb{N}$, for all $n \geq n_0$ and $t \in [t_n^0, t_n]$, we have:

$$(T - t)^{\frac{1}{p-1}} |u(x_n, t)| < \frac{2}{3}\eta_0.$$

Then, we take $t'_n = t_n^0$ and introduce v_n by (88). As in Case 1, we obtain by Proposition 3.3 and the parabolic regularity:

$$\forall |\xi| \leq 1 \text{ and } \tau \in [0, 1), |v_n(\xi, \tau)| \leq M_0, \quad |\Delta v_n(0, \tau_n)| \leq C_0 \eta_0 \text{ where } \tau_n = \frac{t_n - t_n^0}{T - t_n^0}.$$

Therefore, we get from (86), (88) and (87):

$$n \leq |\Delta u_n(x_n, t_n)| = (T - t_n^0)^{-\frac{p}{p-1}} |\Delta v_n(0, \tau_n)| \leq C_0 \eta_0 (T - t_n^0)^{-\frac{p}{p-1}} = C_0 \eta_0 \sqrt{n},$$

which is a contradiction, as $n \rightarrow \infty$. This ends Case 2 and concludes the proof of Proposition 3. ■

A Proof of Proposition 3.5

We prove Proposition 3.5 here, we recall from (44,...,47):

$$\tilde{v}_{1s} = \mathcal{L}\tilde{v}_1 + \theta'(s)(\delta\tilde{v}_1 + \tilde{v}_2) + \tilde{G}_1, \quad (92)$$

$$\tilde{v}_{2s} = (\mathcal{L} - 1)\tilde{v}_2 - \theta'(s)((1 + \delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa) + \tilde{G}_2, \quad (93)$$

where \mathcal{L} is given in (35) and

$$\tilde{G}_1 = \frac{p - \delta^2}{2\kappa}\tilde{v}_1^2 + \frac{1}{2\kappa}\tilde{v}_2^2 + O(|v|^3), \quad (94)$$

$$\tilde{G}_2 = (1 + \delta^2)\frac{\tilde{v}_1(\delta\tilde{v}_1 + \tilde{v}_2)}{\kappa} + O(|v|^3). \quad (95)$$

A primary idea to deal with system (92,...,95) is to confirm that it is driven by its linear part $\partial_s(\tilde{v}_1, \tilde{v}_2) = (\mathcal{L}\tilde{v}_1, (\mathcal{L} - 1)\tilde{v}_2)$ (except for the neutral modes \tilde{v}_{12} where the second order terms matter, and $\tilde{v}_{20} = 0$ by the choice of the modulation parameter; see (24)).

To this end, let us decompose \tilde{v}_1 and \tilde{v}_2 , respectively with respect to the spectrum of \mathcal{L} (with a positive ($\lambda = 1$ or $\lambda = 1/2$), zero and nonnegative part ($\lambda \leq -1/2$)) and $\mathcal{L} - 1$ (with zero eigenvalue and a nonnegative part ($\lambda \leq -1/2$)). Let us introduce some notations

$$\begin{aligned} \tilde{v}_{1+}(y, s) &= \tilde{v}_{10}(s)h_0(y) + \tilde{v}_{11}(s)h_{11}(y), \quad z(s) = \|\tilde{v}_{1+}(\cdot, s)\|_{L_\rho^2}, \\ \tilde{v}_{1null}(y, s) &= \tilde{v}_{12}(s)h_2(y), \quad x(s) = \|\tilde{v}_{1null}(\cdot, s)\|_{L_\rho^2}, \\ \tilde{v}_{1-}(y, s) &= \sum_3^{+\infty} \tilde{v}_{1m}(s)h_m(y), \quad y_1(s) = \|\tilde{v}_{1-}(\cdot, s)\|_{L_\rho^2}, \end{aligned}$$

and we note by

$$\tilde{v}_{2\perp}(y, s) = \sum_1^{+\infty} \tilde{v}_{2m}(s)h_m(y), \quad y_2(s) = \|\tilde{v}_{2\perp}(\cdot, s)\|_{L_\rho^2}.$$

Since we have $\tilde{v}_{20}(s) = 0$ from (24), it follows that

$$\tilde{v}_{2\perp}(y, s) = \tilde{v}_2(y, s) \text{ and } y_2(s) = \|\tilde{v}_2(\cdot, s)\|_{L_\rho^2}.$$

Finally, we note

$$\begin{aligned} N_1(s) &= \|\theta_s(\delta\tilde{v}_1 + \tilde{v}_2) + \tilde{G}_1\|_{L_\rho^2}, \\ N_2(s) &= \|\theta_s((1 + \delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa) + \tilde{G}_2\|_{L_\rho^2}, \end{aligned}$$

We proceed in 3 steps:

- In step 1, we use ODE techniques to show that either z or x dominates as $s \rightarrow -\infty$.
- In step 2, we consider the case where z dominates and show that it leads to case (i) or (ii) of Proposition 3.5.
- In step 3, we show that (iii) of Proposition 3.5 holds in the case where x dominates.

A.1 Step 1: Either $\|\tilde{v}_{1+}(\cdot, s)\|_{L^2_\rho}$ or $\|\tilde{v}_{1null}(\cdot, s)\|_{L^2_\rho}$ dominates as $s \rightarrow -\infty$

Projecting (92) onto the unstable subspace of \mathcal{L} forming the L^2_ρ -inner product with \tilde{v}_{1+} , and using standard inequalities, we get

$$\dot{z} \geq \frac{1}{2}z - N_1.$$

Working similarly with $\tilde{v}_{10}(s)$, $\tilde{v}_{1-}(y, s)$ and $\tilde{v}_2(y, s)$ we arrive at the system

$$\begin{aligned} \dot{z} &\geq \frac{1}{2}z - N_1, \\ |\dot{x}| &\leq N_1, \\ \dot{y}_1 &\leq -\frac{1}{2}y_1 + N_1, \\ \dot{y}_2 &\leq -\frac{1}{2}y_2 + N_2. \end{aligned} \tag{96}$$

Using the fact that v is bounded (see (11) and (14)), and (15), we obtain easily

$$N_1^2 + N_2^2 \leq C \int |v|^4 \rho, \tag{97}$$

for some positive constant C . Thus, it follows from (96) that

$$\begin{aligned} \dot{z} &\geq \frac{1}{2}z - CN \\ |\dot{x}| &\leq CN \\ \dot{y} &\leq -\frac{1}{2}y + CN, \end{aligned} \tag{98}$$

where

$$y \equiv y_1 + y_2 \text{ and } N^2 \equiv \int |v|^4 \rho. \tag{99}$$

If we knew that for $|s|$ large enough

$$N \leq \varepsilon(x + y + z), \tag{100}$$

which is equivalent to $\int |v|^4 \rho \leq \varepsilon^2 \int |v|^2 \rho$, we could use ODE techniques to conclude the step. The meaning of estimate (100) is essentially that the L^2_ρ -norm of quadratic term $|v|^2$ is small compared to the norm of the linear term $|v|$. However, we do not have this information at this stage. We thus estimate N as follows. Given any $\varepsilon > 0$, and any $\alpha > 0$ (both will be chosen small in the sequel), there is a time s_* such that:

$$\int |v|^4 \rho = \int_{|y| > \alpha^{-1}} |v|^4 \rho + \int_{|y| < \alpha^{-1}} |v|^4 \rho \leq \alpha^k \int |v|^4 |y|^k \rho + \varepsilon^2 \int |v|^2 \rho \text{ for all } s \leq s_*. \tag{101}$$

Here we use the fact that $v(y, s)$ goes to zero uniformly on the compact set $|y| < \alpha^{-1}$, which follows from (ii) of Lemma 2.3 and parabolic regularity. The exponent k which appears in (101) is an arbitrary positive integer (later we will choose it to be large). We set

$$J^2 \equiv \int |v|^4 |y|^k \rho,$$

so that (101) can be rewritten as

$$\int |v|^4 \rho \leq \alpha^k J^2 + \varepsilon^2 \int |v|^2 \rho \text{ for all } s \leq s_*.$$

From the inequalities above, we get that

$$N \leq \alpha^{k/2}J + \varepsilon(x + y + z) \text{ for all } s \leq s_*. \quad (102)$$

We next estimate J . Multiplying (92) by $\tilde{v}_1|v|^2|y|^k\rho$, and (45) by $\tilde{v}_2|v|^2|y|^k\rho$, integrating over all \mathbb{R} , we get after some calculations:

$$\dot{J} \leq -\theta J + \varepsilon'(x + y + z) + c(x + y + z)^2,$$

where

$$\theta = \frac{k}{4} - c - \frac{k\alpha^2}{2}(k-1) \text{ and } \varepsilon' = \frac{1}{2}\varepsilon\alpha^{2-k/2}k(k+n-2). \quad (103)$$

Using the fact that $x, y, z \rightarrow 0$ as $s \rightarrow -\infty$, we end up with

$$\dot{J} \leq -\theta J + 4\varepsilon'(x + y + z), \quad (104)$$

where θ is still given by (103) with a different value of the constant c , to end the proof we choose k large enough (certainly $k > 4$), so that for some $\alpha^*(k) > 0$, we have for $0 < \alpha < \alpha^*$, $\theta \geq \frac{1}{2}$. We obtain from (98), (102) and (104):

$$\begin{aligned} \dot{z} &\geq \left(\frac{1}{2} - \hat{\varepsilon}\right) - \hat{\varepsilon}(x + \tilde{y}), \\ |\dot{x}| &\leq \hat{\varepsilon}(x + \tilde{y} + z), \\ \dot{\tilde{y}} &\leq -\left(\frac{1}{2} - \hat{\varepsilon}\right)\tilde{y} + \hat{\varepsilon}(x + z) \end{aligned}$$

where

$$\tilde{y} \equiv y + J, \quad \hat{\varepsilon} \equiv C \max(\varepsilon + \varepsilon\alpha^{2-k/2}, \alpha^{k/2}).$$

Note that $\hat{\varepsilon}$ can be made arbitrarily small by choosing first α and then ε sufficiently small. Now, we conclude using the following Lemma

Lemma A.1 *Let $x(s)$, $y(s)$ and $z(s)$ be absolutely continuous, real valued functions that are non negative and satisfy:*

i) $(x, y, z)(s) \rightarrow 0$ as $s \rightarrow -\infty$,

ii) For all $\varepsilon > 0$, there exists $s_0 \in \mathbb{R}$ such that for all $s \leq s_0$

$$\begin{aligned} \dot{z} &\geq c_0z - \varepsilon(x + y) \\ |\dot{x}| &\leq \varepsilon(x + y + z) \\ \dot{y} &\leq -c_0y + \varepsilon(x + z). \end{aligned} \quad (105)$$

Then either $x + y = o(z)$ or $y + z = o(x)$, as $s \rightarrow -\infty$.

Proof: Here, we adapt the proof of Lemma A.1 (page 172) from [MZ98a]. By rescaling in time, we may assume $c_0 = 1$.

Part 1. Let $\varepsilon > 0$. We show in this part that either

$$\exists s_2(\varepsilon) \text{ such that } \forall s \leq s_2, z(s) + y(s) \leq C\varepsilon x(s), \quad (106)$$

or

$$\exists s_2(\varepsilon) \text{ such that } \forall s \leq s_2, x(s) + y(s) \leq C\varepsilon z(s). \quad (107)$$

We show that for all $s \leq s_0(\varepsilon)$, $\beta(s) \leq 0$ where $\beta = y - 2\varepsilon(x + z)$. We argue by contradiction and suppose that there exists $s_* \leq s_0(\varepsilon)$ such that $\beta(s_*) > 0$. Then, if $s \leq s_*$ and $\beta(s) > 0$,

we have from (105) $\dot{\beta} = \dot{y} - 2\varepsilon(\dot{x} + \dot{z}) \leq 0$. Therefore, for all $s \leq s_*$, $\beta(s) \geq \beta(s_*) > 0$, which contradicts $\beta(s) \rightarrow 0$ as $s \rightarrow -\infty$. Thus, for all $s \leq s_0(\varepsilon)$

$$y \leq 2\varepsilon(x + z). \quad (108)$$

Therefore, (105) yields

$$\dot{z} \geq \frac{1}{2}z - 2\varepsilon x, \quad (109)$$

$$|\dot{x}| \leq 2\varepsilon(x + z). \quad (110)$$

Let $\gamma(s) = 8\varepsilon x(s) - z(s)$. Two cases then arise:

- Case 1. There exists $s_2 \leq s_0(\varepsilon)$ such that $\gamma(s_2) > 0$. Then we compute $\dot{\gamma} = 8\varepsilon\dot{x} - \dot{z} \leq 16\varepsilon^2(x + z) - \frac{1}{2}z + 2\varepsilon x = \gamma(s) \left(\frac{1}{4} + 2\varepsilon\right) - z(s) \left(\frac{1}{4} - 2\varepsilon - 16\varepsilon^2\right)$. Therefore, for all $s \leq s_2$, $\gamma(s) \geq \gamma(s_2)e^{(\frac{1}{4}-2\varepsilon)(s-s_2)} > 0$, that is, $8\varepsilon x(s) > z(s)$. Together with (108), this yields (106).
- Case 2. For all $s \leq s_0(\varepsilon)$, $\gamma(s) \leq 0$, that is, $8\varepsilon x(s) \leq z(s)$. In this case (110) yields

$$\forall s \leq s_0(\varepsilon), \dot{z} \geq \frac{1}{4}z \text{ and } \dot{x} \leq \left(2\varepsilon + \frac{1}{4}\right)z, \text{ hence } \dot{x} \leq (1 + 8\varepsilon)\dot{z}. \quad (111)$$

By integration, we get $x(s) \leq (8\varepsilon + 1)z(s)$. We inject this in (110) and get from (111) $\dot{x} \leq 2\varepsilon(x + z) \leq 2\varepsilon z(2 + 8\varepsilon) \leq 8\varepsilon(2 + 8\varepsilon)\dot{z}(s)$ which gives $x(s) \leq 8\varepsilon(2 + 8\varepsilon)z(s)$ by integration.

Part 2. It is easy to see that if for some $\varepsilon > 0$, (106) holds, then it holds for all $\varepsilon' < \varepsilon$ and the same with (107). This concludes the proof of Lemma A.1. ■

Applying Lemma A.1, we get either

$$\|\tilde{v}_{12}\|_{L^2_\rho} + \|\tilde{v}_{1-}(\cdot, s)\|_{L^2_\rho} + \|\tilde{v}_2(\cdot, s)\|_{L^2_\rho} = o(\|\tilde{v}_{1+}(\cdot, s)\|_{L^2_\rho})$$

or

$$\|\tilde{v}_{1+}(\cdot, s)\|_{L^2_\rho} + \|\tilde{v}_{1-}(\cdot, s)\|_{L^2_\rho} + \|\tilde{v}_2(\cdot, s)\|_{L^2_\rho} = o(\|\tilde{v}_{12}\|_{L^2_\rho}).$$

A.2 Step 2: Case where $\|\tilde{v}_{1+}(\cdot, s)\|_{L^2_\rho}$ dominates

Now, we focus on the case $\|\tilde{v}_{1null}(\cdot, s)\|_{L^2_\rho} + \|\tilde{v}_{1-}(\cdot, s)\|_{L^2_\rho} + \|\tilde{v}_2(\cdot, s)\|_{L^2_\rho} = o(\|\tilde{v}_{1+}(\cdot, s)\|_{L^2_\rho})$. We will show that it leads to either case (i) or case (ii) of Proposition 3.5. We want to derive from (92) the equations satisfied by \tilde{v}_{10} and \tilde{v}_{11} . For this, we estimate in the following lemma, $\int \tilde{G}_1 k_m(y) \rho(y) dy$ for $m = 0, 1$ where

$$k_m(y) = h_m(y) / \|h_m\|_{L^2_\rho}^2$$

and \tilde{G}_1 is given by (94).

Lemma A.2 *There exists $\beta_0 > 0$, and an integer $k' > 4$ such that for all $\beta \in (0, \beta_0)$, $\exists s_0 \in \mathbb{R}$ such that $\forall s \leq s_0$, $\int v^2 |y|^{k'} \rho \leq c_0(k') \beta^{4-k'} z(s)^2$.*

Proof: This lemma is analogous to Lemma A.3 p 175 from [MZ98a], which handles the real case with $\delta = 0$. One can adapt with no difficulty the proof of the present context. ■ Proceeding as in Appendix.A from [MZ98a] and doing the projection of equation (44), respectively on $k_0(y)$ and $k_1(y)$, we obtain

$$\tilde{v}'_{10}(s) = \tilde{v}_{10}(s) + \frac{p - \delta^2}{2\kappa}(1 + \alpha(s))z^2(s), \quad (112)$$

and

$$\tilde{v}'_{11}(s) = \frac{1}{2}\tilde{v}_{11}(s) + \eta(s)z(s)^2, \quad (113)$$

where $z(s) = \|\tilde{v}_{1+}(\cdot, s)\|_{L^2_\rho}$, $\alpha(s) \rightarrow 0$ as $s \rightarrow +\infty$ and η is bounded. Then, from standard ODE techniques, we get

$$\forall \varepsilon > 0, \tilde{v}_{10}(s) = O(e^{(1-\varepsilon)s}) \text{ and } \tilde{v}_{11} = C_1 e^{\frac{s}{2}} + O(e^{(1-\varepsilon)s}). \quad (114)$$

Since $z(s)^2 = \|\tilde{v}_{1+}(\cdot, s)\|_{L^2_\rho}^2 = \tilde{v}_{10}^2 + 2\tilde{v}_{11}^2$, we write (112) as

$$\tilde{v}'_{10}(s) = \tilde{v}_{10}(s) + \frac{p - \delta^2}{\kappa}|C_1|^2 s e^s (1 + \alpha(s)) + \gamma(s),$$

where $\gamma(s) = O(e^{2(1-\varepsilon)s})$ and $\alpha(s) \rightarrow 0$ as $s \rightarrow -\infty$, which gives by integration

$$\tilde{v}_{10}(s) = \frac{p - \delta^2}{\kappa}|C_1|^2 s e^s (1 + o(s)) + C_0 e^s + O(e^{2(1-\varepsilon)s}), \text{ as } s \rightarrow -\infty. \quad (115)$$

Two cases then arise:

- If $C_1 \neq 0$, then $\tilde{v}_{11} \equiv C_1 e^{\frac{s}{2}} \gg \tilde{v}_{10} = O(s e^s)$, from (115). Note first that applying Lemma 3.4 to $|v_1| + |v_2|$ (this is possible from equations (92) and (93) and the boundedness of v), we have for all $|s|$ large enough (and $s < 0$),

$$N^2 = \int |v(y, s)|^4 \rho(y) dy \leq C * \|v(\cdot, s - s^*)\|_{L^2_\rho}^2, \quad (116)$$

for some positive s^* and C^* .

Recalling system (98) and using (116), we obtain, $\dot{y} \leq -\frac{1}{2}y + c\|v(\cdot, s - s^*)\|_{L^2_\rho}^2 \leq -\frac{1}{2}y + c e^s$. Then, we obtain $y = O(e^s)$, similarly, we obtain $x = \|\tilde{v}_{1null}(\cdot, s)\|_{L^2_\rho} = O(e^s)$.

We conclude that $\|v(\cdot, s) - (1 + i\delta)C_1 e^{s/2} y\|_{L^2_\rho} = O(e^{s(1-\varepsilon)})$ as $s \rightarrow -\infty$, for some $\varepsilon > 0$. Using (15), we get $|\theta_s| \leq C e^s$. This is case (ii) of Proposition 3.5.

- If $C_1 = 0$, we obtain case (i) of Proposition 3.5. Indeed, let us first improve the estimate of v . In fact, from (115) we have $\tilde{v}_{10} = C_0 e^s + O(e^{3/2s})$ and from (113) $\tilde{v}_{11} = O(e^{3/2s})$.

We note $y = \|\tilde{v}_{1-}(\cdot, s)\|_{L^2_\rho} + \|\tilde{v}_2(\cdot, s)\|_{L^2_\rho}$ and $x = \|\tilde{v}_{1null}(\cdot, s)\|_{L^2_\rho}$. Recalling system (98) and using (116), we obtain

$$\dot{y} \leq -\frac{1}{2}y + c\|v(\cdot, s - s^*)\|_{L^2_\rho}^2 \leq -\frac{1}{2}y + c e^{2s}.$$

Then, we have that $y = \|\tilde{v}_{1-}(\cdot, s)\|_{L_\rho^2} + \|\tilde{v}_2(\cdot, s)\|_{L_\rho^2} = O(e^{3/2s})$. Similarly, we obtain that

$$x = \|\tilde{v}_{1null}(\cdot, s)\|_{L_\rho^2} = O(e^{3/2s})$$

and we conclude

$$\begin{aligned} & \|v(\cdot, s) - (1 + i\delta)\tilde{v}_{10}(s)\|_{L_\rho^2} \\ &= \|(1 + i\delta)(\tilde{v}_{11}(s) + \tilde{v}_{1null}(\cdot, s) + \tilde{v}_{1-}(\cdot, s)) + i\tilde{v}_2(\cdot, s)\|_{L_\rho^2} \\ &= O(e^{3/2s}). \end{aligned}$$

Using (15), we get $|\theta'(s)|_{L_\rho^2} \leq Ce^{2s}$. This is case (i) of Proposition 3.5.

A.3 Step 3: Case where $\|\tilde{v}_{1null}(\cdot, s)\|_{L_\rho^2}$ dominates

In the following we prove that (iii) of Proposition 3.5 holds. First, we prove the following Lemma:

Lemma A.3 *Assume that*

$$\|\tilde{v}_{1+}(\cdot, s)\|_{L_\rho^2} + \|\tilde{v}_{1-}(\cdot, s)\|_{L_\rho^2} + \|\tilde{v}_2(\cdot, s)\|_{L_\rho^2} = o(\|\tilde{v}_{1null}(\cdot, s)\|_{L_\rho^2}) \quad (117)$$

holds. Then

$$v(y, s) = -(1 + i\delta)\frac{\kappa}{4(p - \delta^2)s}(y^2 - 2) + o\left(\frac{1}{s}\right),$$

in L_ρ^2 as $s \rightarrow -\infty$.

Proof: Since $\tilde{v}_{1null} = \tilde{v}_{12}(s)h_2(y)$, we note that $\tilde{v}_{12} = \int \tilde{v}_1 k_2 \rho$. Projecting equation (92) onto $h_2(y)$ we get

$$\begin{aligned} \frac{d}{ds}(\tilde{v}_{12}) &= \frac{p - \delta^2}{2\kappa} \int \tilde{v}_1^2 k_2(y) \rho(y) \\ &\quad + \theta'(s) \int (\delta\tilde{v}_1 + \tilde{v}_2) k_2(y) \rho(y) + \int \frac{1}{2\kappa} \tilde{v}_2^2 k_2(y) \rho(y) + O\left(\int |v|^3 k_2(y) \rho(y)\right), \\ &= \frac{p - \delta^2}{2\kappa} \int \tilde{v}_{1null}^2 k_2(y) \rho(y) - \frac{p - \delta^2}{2\kappa} \int (\tilde{v}_{1null}^2 - \tilde{v}_1^2) k_2(y) \rho(y) \\ &\quad + \theta'(s) \int (\delta\tilde{v}_1 + \tilde{v}_2) k_2(y) \rho(y) + \int \frac{1}{2\kappa} \tilde{v}_2^2 k_2(y) \rho(y) + O\left(\int |v|^3 k_2(y) \rho(y)\right), \\ &\equiv \frac{(p - \delta^2)}{2\kappa} 8\tilde{v}_{12}^2 + \frac{p - \delta^2}{2\kappa} \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4, \end{aligned}$$

where we use the fact that $\int \tilde{v}_{1null} k_2 \rho = \tilde{v}_{12}^2 \int h_2^2 k_2 \rho = 8\tilde{v}_{12}^2$. We next estimate \mathcal{E}_1 , \mathcal{E}_2 , \mathcal{E}_3 and \mathcal{E}_4 . For this, we need the following lemma:

Lemma A.4 *There exist $\alpha_0 > 0$ and an integer $k' > 4$ such that for all $\alpha \in (0, \alpha_0)$, there exists $s_0 \in \mathbb{R}$ such that for all $s \leq s_0$,*

$$\int |v|^2 |y|^{k'} \rho dy \leq c_0(k') \alpha^{4-k'} \int \tilde{v}_{1null}^2 \rho dy.$$

Proof: See proof of Lemma C.1 in [MZ98a] (page 187). ■

Recalling that $\tilde{v}_1 = \tilde{v}_{1-} + \tilde{v}_{1+} + \tilde{v}_{1null}$, we write on the one hand:

$$\begin{aligned} |\mathcal{E}_1| &\leq \int |\tilde{v}_{1+} + \tilde{v}_{1-}| \times |\tilde{v}_1 + \tilde{v}_{1null}| |k_2(y)| \rho, \\ &\leq c \left(\int |\tilde{v}_{1+} + \tilde{v}_{1-}|^2 \rho \right)^{1/2} \left\{ \left(\int \tilde{v}_1^2 k_2^2(y) \rho \right)^{1/2} + \left(\int \tilde{v}_{1null}^2 k_2^2(y) \rho \right)^{1/2} \right\}. \end{aligned}$$

We have from (117) $(\int |\tilde{v}_{1+} + \tilde{v}_{1-}|^2 \rho)^{1/2} = o(\tilde{v}_{12})$ and

$$\left(\int \tilde{v}_1^2 k_2^2(y) \rho \right)^{1/2} + \left(\int \tilde{v}_{1null}^2 k_2^2(y) \rho \right)^{1/2} \leq \left(\int |v|^2 k_2^2 \rho \right)^{1/2} + c|\tilde{v}_{12}| \equiv I_1 + I_2.$$

On the other hand, we have:

$$\begin{aligned} \mathcal{E}_3 &= \int \frac{1}{2\kappa} \tilde{v}_2^2 k_2(y) \rho(y) \leq c \left(\int \tilde{v}_2^2 \rho \right)^{1/2} \left(\int \tilde{v}_2^2 k_2^2 \rho \right)^{1/2}, \\ &\leq o(\tilde{v}_{12}) \underbrace{\left(\int |v|^2 k_2^2 \rho \right)^{1/2}}_{I_1}. \end{aligned}$$

To treat I_1 , we have from A.4:

$$\int \tilde{v}_1^2 k_2^2 \rho \leq c \int |v|^2 \rho + c \int |v|^2 |y|^{k'} \rho \leq c \left(\int |v|^2 \rho \right) \leq c \tilde{v}_{12}^2.$$

We conclude that $\mathcal{E}_1 = o(\tilde{v}_{12}^2)$ and $\mathcal{E}_3 = o(\tilde{v}_{12}^2)$. We can see easily that $\mathcal{E}_2 = o(\tilde{v}_{12}^2)$, because of Lemma 2.3.

It remains to estimate \mathcal{E}_4 , we consider $\alpha \in (0, \alpha_0)$ and we proceed as in Appendix C from [MZ98a], (page 189). We write for $m = 0$ or $m = 2$:

$$\begin{aligned} \int |v|^3 |y|^m \rho dy &\leq \int_{|y| \leq \alpha^{-1}} |v|^3 |y|^m \rho dy + \int_{|y| \geq \alpha^{-1}} |v|^3 |y|^m \rho dy, \\ &\leq \varepsilon \alpha^{-m} \int_{|y| \leq \alpha^{-1}} |v|^2 \rho dy + CM \alpha^{k'-m} \int_{|y| \geq \alpha^{-1}} |v|^2 |y|^{k'} \rho dy, \\ &\leq C(\varepsilon \alpha^{-m} + M c_0(k') \alpha^{4-m}) \int \tilde{v}_{null}^2 \rho dy, \end{aligned}$$

where, we used the fact that $|v| \rightarrow 0$ as $s \rightarrow -\infty$ in $L^\infty(B(0, \alpha^{-1}))$, $|v(y, s)| \leq M$, Lemma A.4 and $\int |v|^2 \rho dy \leq \int \tilde{v}_{null}^2 \rho dy$. We can then choose ε and α such that for $s \leq s_0$, $\int |v|^3 |y|^m \rho \leq \varepsilon \int \tilde{v}_{null}^2 \rho$ and we obtain $\mathcal{E}_4 = o(\tilde{v}_{12}^2)$.

So finally, we have

$$\frac{d}{ds}(\tilde{v}_{12}) = \frac{(p - \delta^2)}{\kappa} 4\tilde{v}_{12}^2 + o(\tilde{v}_{12}^2).$$

Solving the above, we obtain

$$\tilde{v}_{1null} = -\frac{\kappa}{4(p - \delta^2)s} (1 + o(1))(y^2 - 2).$$

This concludes the proof of Lemma A.3. ■

In order to finish the proof of (iii) of Proposition 3.5. We need to refine the estimates of Lemma A.3 to catch the $O(\frac{|s|}{s^2})$.

Recalling system (98) and using (116), we obtain,

$$y' \leq -\frac{1}{2}y + c\|v(\cdot, s - s^*)\|_{L^2_\rho}^2 \leq -\frac{1}{2}y + c\frac{1}{s^2}.$$

Then, integrating $(ye^{s/2})' \leq C\frac{e^{s/2}}{s^2}$ between $-\infty$ and s , we get $y \leq \frac{C}{s^2}$. Doing the same

for $z = \|\tilde{v}_{1+}(\cdot, s)\|_{L^2_\rho}$, we obtain $(ze^{-s/2})' \geq C\frac{e^{-s/2}}{s^2}$, integrating between s and $s_0 \geq s$, we have $z \leq \frac{C}{s^2}$.

Proceeding as in the proof of Proposition A.3, we write:

$$\begin{aligned} \frac{d}{ds}(\tilde{v}_{12}) &= \frac{p - \delta^2}{2\kappa} \int \tilde{v}_{1null}^2 k_2(y) \rho(y) - \frac{p - \delta^2}{2\kappa} \int (\tilde{v}_{1null}^2 - \tilde{v}_1^2) k_2(y) \rho(y) \\ &+ \theta'(s) \int (\delta \tilde{v}_1 + \tilde{v}_2) k_2(y) \rho(y) + \int \frac{1}{2\kappa} \tilde{v}_2^2 k_2(y) \rho(y) + O\left(\int |v|^3 k_2(y) \rho(y)\right), \quad (118) \\ &\equiv \frac{4(p - \delta^2)}{\kappa} \tilde{v}_{12}^2 + \frac{p - \delta^2}{2\kappa} \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4. \end{aligned}$$

Then, we have:

$$\begin{aligned} |\mathcal{E}_1| &\leq \int |\tilde{v}_{1+} + \tilde{v}_{1-} + \tilde{v}_2| \times |v + \tilde{v}_{1null}| |k_2(y)| \rho, \\ &\leq \left(\int |\tilde{v}_{1+} + \tilde{v}_{1-} + \tilde{v}_2|^2 \rho \right)^{1/2} \left\{ \left(\int v^2 k_2^2(y) \rho \right)^{1/2} + \left(\int \tilde{v}_{1null}^2 k_2^2(y) \rho \right)^{1/2} \right\}, \\ &\leq \varepsilon \left(\int \tilde{v}_{1null}^2 \rho \right)^{1/2} \left\{ c \left(\int v^4 \rho \right)^{1/4} + c \left(\int \tilde{v}_{1null}^2 \rho \right)^{1/2} \right\}. \end{aligned}$$

Using the fact that $\|\tilde{v}_{1null}(\cdot, s)\|_{L^2_\rho} \sim \frac{C}{s}$ and (116), we have

$$\int v^4 \rho \leq c \left(\int v^2(\cdot, s - s^*) \rho \right)^2 \leq \frac{c}{(s - s^*)^2} \leq \frac{c}{s^2}.$$

Thus, $\mathcal{E}_1 \leq \frac{C}{s^3}$. Similarly, we obtain $\mathcal{E}_2 \leq \frac{C}{|s|^3}$, $\mathcal{E}_3 \leq y^2 \leq \frac{C}{s^4}$ and $\mathcal{E}_4 \leq \frac{C}{|s|^3}$. Then, we have from (118):

$$\frac{d}{ds}(\tilde{v}_{12}) = \frac{4(p - \delta^2)}{\kappa} \tilde{v}_{12}^2 + O\left(\frac{1}{s^3}\right) = \frac{4(p - \delta^2)}{\kappa} \tilde{v}_{12}^2 \left(1 + O\left(\frac{1}{s}\right)\right).$$

By integrating, we conclude that:

$$\tilde{v}_{12} = -\frac{\kappa}{4(p - \delta^2)s} + O\left(\frac{\log |s|}{s^2}\right).$$

Finally, we get $\|v(\cdot, s) - (1 + i\delta)\frac{\kappa}{4(p - \delta^2)s}(y^2 - 2)\|_{L^2_\rho} = O\left(\frac{\log |s|}{s^2}\right)$ as $s \rightarrow -\infty$. Remains to prove the estimate for $\theta'(s)$ to conclude.

Integrating equation (93) with respect to ρdy , we obtain:

$$\theta'(s) \int ((1 + \delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa)\rho = \int \tilde{G}_2\rho.$$

On the one hand, we have $((1 + \delta^2)\tilde{v}_1 + \delta\tilde{v}_2 + \kappa) = \kappa + O(\frac{1}{s})$. On the other hand, using (95), we get

$$\int \tilde{G}_2\rho = \frac{(1 + \delta^2)\delta}{\kappa} \int \tilde{v}_1^2\rho + \frac{(1 + \delta^2)}{\kappa} \int \tilde{v}_1\tilde{v}_2\rho,$$

where we have from (iii) of Proposition 3.5, $\int \tilde{v}_1\tilde{v}_2\rho = O(\frac{\log|s|}{s^3})$,

$$\begin{aligned} \int \tilde{v}_1^2\rho &= \int \tilde{v}_{12}^2 h_2^2\rho + \int (\tilde{v}_1^2 - \tilde{v}_{12}^2 h_2^2)\rho \\ &= 8\tilde{v}_{12}^2 + \int ((\tilde{v}_1 - \tilde{v}_{12}h_2))(\tilde{v}_1 + \tilde{v}_{12})\rho, \end{aligned}$$

$$\tilde{v}_{12} = \frac{\kappa}{4(p-\delta^2)s} + O(\frac{\log|s|}{s^2}), \int ((\tilde{v}_1 - \tilde{v}_{12}h_2))(\tilde{v}_1 + \tilde{v}_{12})\rho \leq C\frac{\log|s|}{s^2} \times \frac{1}{s} = C\frac{\log|s|}{s^3}.$$

$$\theta'(s) = \frac{(1 + \delta^2)\delta}{\kappa} \left(\frac{\kappa}{2(p-\delta^2)} \right)^2 \frac{1}{s^2} + O(\frac{\log|s|}{s^3}).$$

Consequently, we obtain the desired estimate for θ_s . This concludes the proof of Proposition 3.5. ■

B Equations of Z in Step 4 and 5

Equation of Z in Step 4: In this part we establish the equation satisfied by Z in Step 4 of the proof of Theorem 1. We note $\phi : \mathbb{C} \rightarrow \mathbb{C}$ the function defined by $\phi(x) = |x|^{p-1}x$. If we introduce $\nu = (w - F)$, where F is defined by (62), then we see from (10) that ν satisfies the following equation for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, such that for $|y| < 4\varepsilon_0\sqrt{-s}$

$$\partial_s\nu = (\mathcal{L} - 1)\nu + l(\nu) + B(\nu) + R(y, s), \quad (119)$$

where \mathcal{L} is defined in (35), $\nu = \nu_1 + i\nu_2$,

$$\begin{aligned} l(\nu) &= (1 + i\delta) \left[-\frac{\nu}{p-1} + (p-1)|F|^{p-3}F(F_1\nu_1 + F_2\nu_2) + |F|^{p-1}\nu \right], \\ B(\nu) &= (1 + i\delta) \left[\phi(F + \nu) - \phi(F) - (p-1)|F|^{p-3}F(F_1\nu_1 + F_2\nu_2) - |F|^{p-1}\nu \right], \\ R(y, s) &= -\partial_s F + \Delta F - \frac{1}{2}y \cdot \nabla F - (1 + i\delta)\frac{F}{p-1} + (1 + i\delta)|F|^{p-1}F. \end{aligned}$$

Using Taylor's formula and the fact that w and F are bounded for $|y| \leq 4\varepsilon_0\sqrt{-s}$, we readily obtain for all $s \leq s_0$ and $|y| < 4\varepsilon_0\sqrt{-s}$

$$\begin{aligned} |B(\nu)| &\leq C|\nu|^2, \\ |R(y, s)| &\leq C \left(\frac{|y|^2+1}{s^2} + \chi_{\varepsilon_0} \right), \end{aligned}$$

with χ_{ε_0} defined in (64). If we write $\nu = (1 + i\delta)\tilde{\nu}_1 + i\tilde{\nu}_2$, $B = (1 + i\delta)\tilde{B}_1 + i\tilde{B}_2$ and $R = (1 + i\delta)\tilde{R}_1 + i\tilde{R}_2$, then we have:

$$\partial_s \tilde{\nu}_1 = \mathcal{L}\tilde{\nu}_1 + l_{1,1}\tilde{\nu}_1 + l_{1,2}\tilde{\nu}_2 + \tilde{B}_1 + \tilde{R}_1 \quad (120)$$

$$\partial_s \tilde{\nu}_2 = (\mathcal{L} - 1)\tilde{\nu}_2 + l_{2,2}\tilde{\nu}_2 + l_{2,1}\tilde{\nu}_1 + \tilde{B}_2 + \tilde{R}_2, \quad (121)$$

where

$$\begin{cases} l_{1,1}(y, s) &= (1 - \delta^2)(|F|^{p-1} - \frac{1}{p-1}) + (p-1)|F|^{p-3}(F_1^2 - \delta^2 F_2^2) - 1, \\ l_{1,2}(y, s) &= -\delta(|F|^{p-1} - \frac{1}{p-1}) + (p-1)|F|^{p-3}(F_1 - \delta F_2)F_2, \\ l_{2,1}(y, s) &= (1 + \delta^2)(|F|^{p-1} - \frac{1}{p-1}) + (p-1)|F|^{p-3}(F_1 + \delta F_2)F_2, \\ l_{2,2}(y, s) &= (1 + \delta^2)(|F|^{p-1} - \frac{1}{p-1}) + (p-1)|F|^{p-3}F_2^2. \end{cases}$$

Proceeding as in the proof of Lemma B.1 from [Zaa98] (page 615), we obtain for all $|y| \leq 4\varepsilon_0\sqrt{-s}$

$$|l_{i,j}(y, s)| \leq C \min \left[\frac{(1 + |y^2|)}{|s|}, 1 \right], \text{ for any } i, j \in \{1, 2\}.$$

Therefore, we write for $|s|$ large enough and $|y| \leq 4\varepsilon_0\sqrt{-s}$:

$$|l_{i,j}(y, s)| \leq C \left\{ \frac{(1 + \varepsilon_0^2|s|)}{|s|} + \chi_{\varepsilon_0} \right\} \leq C \{2\varepsilon_0^2 + \chi_{\varepsilon_0}\}.$$

Now, we multiply (120) and (121) respectively by $\text{sgn}(\tilde{\nu}_1)$ and $\text{sgn}(\tilde{\nu}_2)$. Using Kato's inequality, we obtain for $z = |\tilde{\nu}_1| + |\tilde{\nu}_2|$, $|s|$ large enough and $|y| \leq 4\varepsilon_0\sqrt{-s}$:

$$\partial_s z - \Delta z + \frac{1}{2}y \cdot \nabla z - (1 + \sigma)z \leq C \left(z^2 + \frac{(y^2 + 1)}{s^2} + \chi_{\varepsilon_0} \right),$$

where we fix ε_0 small enough so that $\sigma = C\varepsilon_0^2 = \frac{1}{100}$.

Now, we consider the cut-off function γ (65), we define $Z = z\gamma$ and we obtain for $|s|$ large enough:

$$\begin{aligned} \partial_s Z - \Delta Z + \frac{1}{2}y \cdot \nabla Z - (1 + \sigma)Z &\leq C \left(Z^2 + \frac{(y^2 + 1)}{s^2} + \chi_{\varepsilon_0} \right) \\ &\quad + z \left(\partial_s \gamma - \Delta \gamma + \frac{y}{2} \cdot \nabla \gamma \right) - 2\nabla \gamma \cdot \nabla z, \end{aligned}$$

(here, we used the fact that $\gamma z^2 = Z^2 + (\gamma - \gamma^2)z^2 \leq Z^2 + C\chi_{\varepsilon_0}$). The last terms in this equation are the cut-off terms. Using the fact that $z(\partial_s \gamma - \Delta \gamma + \frac{y}{2} \cdot \nabla \gamma) - 2\nabla \gamma \cdot \nabla z \leq C\chi_{\varepsilon_0} - 2\text{div}(z\nabla \gamma)$, we obtain for $|s|$ large enough:

$$\partial_s Z - \Delta Z + \frac{1}{2}y \cdot \nabla Z - (1 + \sigma)Z \leq C \left(Z^2 + \frac{(y^2 + 1)}{s^2} + \chi_{\varepsilon_0} \right) - 2\text{div}((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla \gamma),$$

which is the desired equation in Lemma 3.8.

Equation of Z in Step 4: In the following, we determine the equation satisfied by Z in Step 4. We note by $\nu = w - f$. We can see from (78), that ν satisfies the following equation for all $(y, s) \in \mathbb{R} \times \mathbb{R}$, such that for $|y| < 4\varepsilon_0 e^{-s/2}$

$$\partial_s \nu = \Delta \nu - \frac{1}{2}y \cdot \nabla \nu + l(\nu) + B(\nu) + R(y, s),$$

where

$$\begin{aligned} l(\nu) &= -(1+i\delta)\frac{\nu}{p-1} + (1+i\delta)\{(p-1)|f|^{p-3}f(f_1\nu_1 + f_2\nu_2) + |f|^{p-1}\nu\}, \\ B(\nu) &= (1+i\delta)\{|f + \nu|^{p-1}(f + \nu) - |f|^{p-1}f - (p-1)|f|^{p-3}f(f_1\nu_1 + f_2\nu_2) - |f|^{p-1}\nu\}, \\ R(y, s) &= e^s \Delta G(ye^{s/2}). \end{aligned}$$

Using a Taylor formula, we prove that for $|s|$ large and $|y| \leq \frac{R^*}{2}e^{-s/2}$

$$|B(\nu)| \leq C|\nu|^2, |R(y, s)| \leq Ce^s + \chi_{\varepsilon_0}(y, s),$$

with χ_{ε_0} is defined by (81). If we write $\nu = (1+i\delta)\tilde{\nu}_1 + i\tilde{\nu}_2$, $B = (1+i\delta)\tilde{B}_1 + i\tilde{B}_2$ and $R = (1+i\delta)\tilde{R}_1 + i\tilde{R}_2$, then we have:

$$\partial_s \tilde{\nu}_1 = \mathcal{L}\tilde{\nu}_1 + l_{1,1}\tilde{\nu}_1 + l_{1,2}\tilde{\nu}_2 + \tilde{B}_1 + \tilde{R}_1 \quad (122)$$

$$\partial_s \tilde{\nu}_2 = (\mathcal{L} - 1)\tilde{\nu}_2 + l_{2,2}\tilde{\nu}_2 + l_{2,1}\tilde{\nu}_1 + \tilde{B}_2 + \tilde{R}_2, \quad (123)$$

where

$$\begin{cases} l_{1,1}(y, s) &= (1 - \delta^2)(|f|^{p-1} - \frac{1}{p-1}) + (p-1)|f|^{p-3}(f_1^2 - \delta^2 f_2^2) - 1, \\ l_{1,2}(y, s) &= -\delta(|f|^{p-1} - \frac{1}{p-1}) + (p-1)|f|^{p-3}(f_1 - \delta f_2)f_2, \\ l_{2,1}(y, s) &= (1 + \delta^2)(|f|^{p-1} - \frac{1}{p-1}) + (p-1)|f|^{p-3}(f_1 + \delta f_2)f_2, \\ l_{2,2}(y, s) &= (1 + \delta^2)(|f|^{p-1} - \frac{1}{p-1}) + (p-1)|f|^{p-3}f_2^2. \end{cases}$$

Proceeding as in the proof of Lemma B.1 from [Zaa98] (page 615), we obtain for $|y|e^{s/2} \leq 4\varepsilon_0$ and s large

$$|l_{i,j}(y, s)| \leq C \min\left[|y|e^{s/2}, 1\right], \text{ for any } i, j \in \{1, 2\}.$$

If we consider χ_{ε_0} defined in (81), then, we write for $|s|$ large and $|y| \leq 4\varepsilon_0 e^{-s/2}$:

$$|l_{i,j}| \leq C \left\{ |y|e^{s/2} + \chi_{\varepsilon_0} \right\} \leq C \{ \varepsilon_0 + \chi_{\varepsilon_0} \}.$$

Now, we multiply (122) and (123) respectively by $\text{sgn}(\tilde{\nu}_1)$ and $\text{sgn}(\tilde{\nu}_2)$. Using Kato's inequality, we obtain for $z = |\tilde{\nu}_1| + |\tilde{\nu}_2|$, $|s|$ large enough and $|y|e^s \leq 4\varepsilon_0$,

$$\partial_s z - \Delta z + \frac{1}{2}y \cdot \nabla z - (1 + \sigma)z \leq C(z^2 + e^s + \chi_{\varepsilon_0}),$$

where $\sigma = C\varepsilon_0 = \frac{1}{100}$. Now, we consider the cut-off function γ , we define $Z = z\gamma$ and we obtain for $|s|$ large:

$$\begin{aligned} & \partial_s Z - \Delta Z + \frac{1}{2}y \cdot \nabla Z - (1 + \sigma)Z \\ & \leq C(Z^2 + e^s + \chi_{\varepsilon_0}) - z \left(\partial_s \gamma - \Delta \gamma + \frac{y}{2} \cdot \nabla \gamma \right) + 2\nabla \gamma \nabla z. \end{aligned}$$

The last terms in this equation are the cut-off terms. Using $z(\partial_s \gamma - \Delta \gamma + \frac{y}{2} \cdot \nabla \gamma) - 2\nabla \gamma \nabla z \leq C\chi_{\varepsilon_0} + 2\text{div}(z\nabla \gamma)$, we obtain for $|s|$ large:

$$\partial_s Z - \Delta Z + \frac{1}{2}y \cdot \nabla Z - (1 + \sigma)Z \leq C(Z^2 + e^s + \chi_{\varepsilon_0}) - 2\text{div}((|\tilde{\nu}_1| + |\tilde{\nu}_2|)\nabla \gamma),$$

which is the desired equation in (80).

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