Hydrodynamic limit in the Hyperbolic Space-Time Scale

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- Corresponding to different paramenters there are different *partial* equilibriums:
  - *mechanical equilibrium*: constant pressure or tension profiles,
  - *thermal equilibrium*: constant temperature profiles.
- These partial equilibriums may be reached at different time scales: *typically* mechanical equilibrium is reached faster than thermal equilibrium.

Mechanical and Thermal equilibrium

- **Mechanical Equilibrium** is reached in **hyperbolic** time scales (same rescaling of space and time), and is driven by Euler system of equations (for a compressible gas). It involves the ballistic evolution of the long waves (mechanical modes).

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- When thermal conductivity is finite, **Thermal Equilibrium** is reached later, in the **diffusive** time scales ($\text{time}^2 = \text{space}$), and temperature (or thermal energy) profiles evolve following *heat equation*.
- If thermal conductivity is infinite, **Thermal Equilibrium** is reached in a **super-diffusive** time scales ($\text{time}^\alpha = \text{space}, \alpha < 2$), and typically temperature (or thermal energy) profiles evolve following a *fractional heat equation*.

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Extermal forces or heat bath acting microscopically at the boundary on the system determine boundary conditions of the macroscopic equations.

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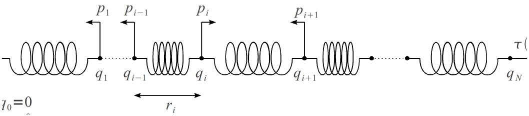
Most of non-equilibrium situation are obtained by

- changing boundary conditions in time
- applying boundary conditions corresponding to different equilibrium states, obtaining dynamics that have *non-equilibrium stationary states* (NESS).

Chain of oscillators



$$\dot r_x(t) = p_x(t) - p_{x-1}(t), \qquad x = 1, \dots, N$$

$$\dot p_x(t) = V'(r_{x+1}(t)) - V'(r_x(t)) \qquad x = 1, \dots, N-1$$

$$\dot p_N(t) = \tau(t/N) - V'(r_N(t))$$

$$p_0(t) = 0.$$

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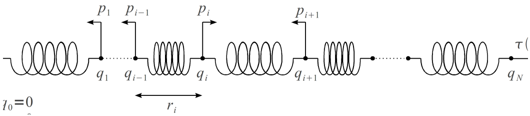

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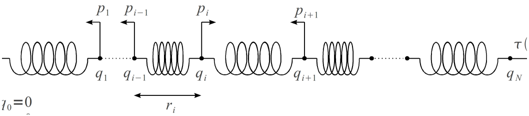

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We are interested in the *macroscopic* evolution of $(r_x(t), p_x(t), \mathcal{E}_x(t))$.

Gibbs measures and Thermodynamic Entropy

For $\tau(t) = \tau$ constant in time, a class of stationary measures is given by the Gibbs measures at temperature β^{-1}, tension τ

$$d\mu_{\beta,\tau,p} = \prod_{x=1}^{N} e^{-\beta(\mathcal{E}_x - \tau r_x) - \mathcal{G}(\beta,\tau)} dp_x dr_x$$

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Thermodynamic entropy is

$$S(u,r) = \inf_{\tau,\beta} \{-\beta\tau r + \beta u - \mathcal{G}(\beta,\tau)\}$$

$$\beta(u,r) = \partial_u S(u,r), \qquad \tau(u,r) = -\beta^{-1}\partial_r S(u,r).$$

Ergodicity (of the infinite system)

Consider the corresponding infinite dynamics:

$$\begin{aligned} \dot{r}_x(t) &= p_x(t) - p_{x-1}(t), \\ \dot{p}_x(t) &= V'(r_{x+1}(t)) - V'(r_x(t)) \qquad\qquad x \in \mathbb{Z} \end{aligned}$$

Any probability ν that is translation invariant, stationary and finite entropy density is a convex combination of Gibbs measures $d\mu_{\beta,\tau,\mathbf{p}}$.

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By the *equivalence of ensembles (microcanonic to grand-canonic)*:

the only local translation invariant conserved quantities of the infinite systems are given by energy, momentum and density.

Completely integrable systems gives obvious conterexamples (Harmonic Oscillators, Toda Lattice,...).

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Theorem

(Fritz, Funaki, Lebowitz, PTRF 1994) Assume that a probability ν is translation invariant, stationary, finite entropy density, and the conditional measure $\nu(dp|r)$ is exchangeable.
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- $\nu(dp|r)$ maxwellian (Gallavotti-Verboven 1975)
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- Chaoticity of the dynamics, due to the non-linearity of V, should give such ergodic property
- Adding conservative noise (stochastic collisions) to the dynamics one obtain ergodicity.

Hyperbolic Scaling, Euler equations

3 conserved quantities: we expect the weak convergence to the hyperbolic system of PDE

$$\frac{1}{N}\sum_x G(x/N)\begin{pmatrix} r_x(Nt) \\ p_x(Nt) \\ \mathcal{E}_x(Nt) \end{pmatrix} \underset{N\to\infty}{\longrightarrow} \int_0^1 G(y)\begin{pmatrix} r(y,t) \\ p(y,t) \\ \mathfrak{e}(y,t) \end{pmatrix} dy$$

$$\begin{aligned} \partial_t r(t,y) &= \partial_y p(t,y) \\ \partial_t p(t,y) &= \partial_y \tau[u(t,y), r(t,y)] \\ \partial_t \mathfrak{e}(t,y) &= \partial_y (\tau[u(t,y), r(t,y)] p(t,y)) \end{aligned}$$

where $u = \mathfrak{e} - p^2/2$: internal energy.
and, *for smooth solutions*, the boundary conditions:

$$p(t,0) = 0, \qquad \tau[u(t,1), r(t,1)] = \tau(t)$$

Euristics

take $G : [0,1] \to \mathbb{R}$ with compact support in $(0,1)$,

$$\frac{d}{dt}\frac{1}{N}\sum_x G(x/N)\begin{pmatrix} r_x(Nt) \\ p_x(Nt) \\ \mathcal{E}_x(Nt) \end{pmatrix} = \sum_x G(x/N)\begin{pmatrix} \nabla p_{x-1}(Nt) \\ \nabla V'(r_x(Nt)) \\ \nabla\left[p_x(Nt)V'(r_x(Nt)\right] \end{pmatrix}$$

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assuming local equilibrium, we have

$$\sim -\int_0^1 G'(y)\begin{pmatrix} p(t,y) \\ \tau(u(t,y), r(t,y)) \\ p(t,y)\tau(u(t,y), r(t,y))\end{pmatrix} dy$$

Note that $y \in [0,1]$ is the *material (Lagrangian)* coordinate.

Results with conservative stochastic dynamics

- To prove some form of *local equilibrium* we need to add stochastic terms to the dynamics (the deterministic non-linear case is too difficult).

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- Random exchanges of velocities between nearest neighbor particles, conserve energy, momentum and volume, destroying all other (possible) conservation laws. It provides the *right ergodicity* property.
- With such noise in the dynamics, for smooth solutions the HL is proven in:
  - N. Even, S.O., ARMA (2014) (with boundary conditions),
  - S.O., SRS Varadhan, HT Yau, CMP (1993) (periodic bc).

Harmonic Oscillators Chain

This is an example of a non-ergodic dynamics:

$$V(r) = r^2/2$$

in fact it is a *completely integrable dynamics*:

$$\dot{q}_x = p_x, \qquad \dot{p}_x = \Delta q_x = q_{x+1} + q_{x-1} - q_x,$$

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Take here $x = 1, \dots, N$,

$$\hat{f}(k) = \sum_x f_x e^{i2\pi kx} \qquad k \in \{0, 1/N, \dots, (N-1)/N\}$$

$\omega(k) = 2|\sin(\pi k)|$ dispersion relation:

$$\mathcal{H} = \sum_x \mathcal{E}_x = \frac{1}{2N} \sum_k \left[\omega(k)^2 |\hat{q}(k)|^2 + |\hat{p}(k)|^2 \right]$$

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$$\frac{d}{dt}\hat{\psi}(t,k) = -i\omega(k)\hat{\psi}(t,k) \qquad \hat{\psi}(t,k) = e^{-i\omega(k)t}\hat{\psi}(0,k)$$

Harmonic Oscillators Chain: Quantum Dynamics

$$p_x = -i\partial_{q_x} = -i\left(\partial_{r_{x+1}} - \partial_{r_x}\right)$$

$$\mathcal{E}_x = \frac{1}{2}\left(p_x^2 + r_x^2\right)$$

$$a_k = \frac{1}{\omega(k)}\hat{\psi}(k), \qquad a_k^* = \frac{1}{\omega(k)}\hat{\psi}(k)^*$$

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Heisenberg evolution $\frac{d}{dt}A(t) = i[\mathcal{H}, A(t)]$

$$a_k(t) = e^{-i\omega(k)t}a_k, \qquad a_k^*(t) = e^{i\omega(k)t}a_k^*.$$

Harmonic Chain: Thermal Equilibrium (Classic case)

Consider the chain in *thermal* equilibrium: initial distribution with covariances

$$\langle\ r_x(0); r_{x'}(0)\ \rangle = \langle\ p_x(0); p_{x'}(0)\ \rangle = \beta^{-1}\delta_{x,x'}, \qquad \langle\ q_x; p_{x'}\ \rangle = 0,$$

for some inverse temperature β, while in *mechanical local equilibrium*:

$$\langle\ r_{[Ny]}(0)\ \rangle \longrightarrow r(0,y), \qquad \langle\ p_{[Ny]}(0)\ \rangle \longrightarrow p(0,y).$$

Harmonic Chain: Thermal Equilibrium (classic case)

thermal equilibrium is conserved by the dynamics: for any $t \geq 0$

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Proof.

Thermal equilibrium is Fourier space is:

$$\langle\, \hat\psi(k,0)^*; \hat\psi(k',0)\,\rangle = 2\beta^{-1}\delta(k-k'), \qquad \langle\, \hat\psi(k,0); \hat\psi(k',0)\,\rangle = 0.$$

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Consequently

$$\langle\, \hat\psi(k,t)^*; \hat\psi(k',t)\,\rangle = e^{i(\omega(k)-\omega(k'))t}\langle\, \hat\psi(k,0)^*; \hat\psi(k',0)\,\rangle = 2\beta^{-1}\delta(k-k'$$

$$\langle\, \hat\psi(k,t); \hat\psi(k',t)\,\rangle = e^{-i(\omega(k)+\omega(k'))t}\langle\, \hat\psi(k,0); \hat\psi(k',0)\,\rangle = 0.$$

□

Harmonic Chain: Thermal Equilibrium implies Euler Equation limit

$r_{[Ny]}(Nt)$ and $p_{[Ny]}(Nt)$ converge weakly to the solution of the linear wave equation

$$\partial_t \mathbf{r}(y,t) = \partial_y \mathbf{p}(y,t), \qquad \partial_t \mathbf{p}(y,t) = \partial_y \mathbf{r}(y,t).$$

This is the Euler equation for this system since here $\tau(u,r) = r$.

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For the energy, because of the thermal equilibrium, for any $t \geq 0$:

$$\langle \mathcal{E}_x(t) \rangle = \beta^{-1} + \frac{1}{2}\left(\langle p_x(t) \rangle^2 + \langle r_x(t) \rangle^2\right)$$

$$\langle \mathcal{E}_{[Ny]}(Nt) \rangle \longrightarrow \mathbf{e}(y,t) = \beta^{-1} + \frac{1}{2}\left(\mathbf{p}^2(y,t) + \mathbf{r}^2(y,t)\right),$$

$$\partial_t \mathbf{e}(y,t) = \partial_y \left(\mathbf{p}(y,t)\,\mathbf{r}(y,t)\right).$$

Quantum Harmonic Chain: Thermal Equilibrium

Initial density matrix ρ_β, define

$$\langle A \rangle = \operatorname{tr}\big(A\rho_\beta\big)), \qquad \langle A; B \rangle = \langle AB \rangle - \langle A \rangle \langle B \rangle$$

such that

$$\langle r_x(0); r_{x'}(0) \rangle = \langle p_x(0); p_{x'}(0) \rangle = C_\beta(x-x'), \qquad \langle q_x; p_{x'} \rangle = \frac{i}{2}\delta(x-x')$$

$$C_\beta(x) = \frac{1}{N}\left[\beta^{-1} + \sum_{k\neq 0} e^{2\pi ikx}\Big(\frac{\omega_k}{e^{\beta\omega_k}-1} + \frac{\omega_k}{2}\Big)\right] \qquad (1)$$

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$$\langle r_{[Ny]}(0) \rangle \longrightarrow r(0,y), \qquad \langle p_{[Ny]}(0) \rangle \longrightarrow p(0,y).$$

$$\langle \mathcal{E}_{[Ny]} \rangle \longrightarrow \mathbf{e}(y) = \bar{C}(\beta) + \frac{1}{2}\left(\mathbf{p}^2(y) + \mathbf{r}^2(y)\right),$$

$$\bar{C}(\beta) = \int_0^1 \omega(k)\left(\frac{1}{e^{\beta\omega(k)}-1} + \frac{1}{2}\right) dk \underset{\beta\to 0}{\sim} \beta^{-1}$$

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$$\partial_t \mathbf{e}(y,t) = \partial_y \left(\mathbf{p}(y,t)\,\mathbf{r}(y,t)\right).$$

Harmonic Chain: Local Thermal Equilibrium is not conserved

The argument fails dramatically if the system is not in thermal equilibrium, even local thermal Gibbs

$$\langle\, r_x(0); r_{x'}(0)\,\rangle = \langle\, p_x(0); p_{x'}(0)\,\rangle = \beta^{-1}\left(\frac{x}{N}\right)\delta_{x,x'}, \quad \langle\, q_x(0); p_{x'}(0)\,\rangle = 0 \tag{2}$$

is not conserved, and correlations between $p_x(t)$ and $r_x(t)$ build up in time.

No autonomous macroscopic equation for the energy!

There are infinite many conservation laws.

Wigner distribution

$\xi \in \mathbb{R},\ k \in [0,1],$

$$\widehat{W}_N(\xi, k, t) \ := \ \frac{2}{N} \left\langle\!\!\left\langle \hat{\psi}^* \left(Nt, k - \frac{\xi}{2N}\right) \hat{\psi} \left(Nt, k + \frac{\xi}{2N}\right) \right\rangle\!\!\right\rangle$$

$$W_N(y, k, t) = \int \widehat{W}_N(t, \eta, k) e^{-i2\pi\xi y} \, d\eta, \qquad y \in \mathbb{R},$$

Wigner distribution

$\xi \in \mathbb{R}, \; k \in [0,1]$,

$$\widehat{W}_N(\xi, k, t) \; := \; \frac{2}{N} \left\langle\!\!\left\langle \hat{\psi}^* \left(Nt, k - \frac{\xi}{2N} \right) \hat{\psi} \left(Nt, k + \frac{\xi}{2N} \right) \right\rangle\!\!\right\rangle$$

$$W_N(y, k, t) = \int \widehat{W}_N(t, \eta, k) e^{-i2\pi\xi y} \, d\eta, \qquad y \in \mathbb{R},$$

In the limit it decompose in a thermal and a mechanical part:

$$\lim_{N \to \infty} \widehat{W}_N(\xi, k, t) = \widehat{W}_{th}(\xi, k, t) + \widehat{W}_m(\xi, t) \, \delta_0(dk) \qquad (3)$$

The mechanical part $\widehat{W}_m(\xi, t)$ is the Fourier transform of the mechanical energy

$$\widehat{W}_m(\xi, t) = \int \frac{1}{2} \left(\mathbf{p}^2(y, t) + \mathbf{r}^2(y, t) \right) e^{i2\pi\xi y} \, dy,$$

Wigner distribution

For the thermal Wigner distribution it holds the transport equation:

$$\partial_t W_{th}(y,k,t) + \frac{\omega'(k)}{2\pi} \partial_y W_{th}(y,k,t) = 0.$$

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in fact for $k \neq 0$

$$\begin{aligned} \widehat{W}_N(\xi,k,t) &:= e^{i\left[\omega\left(k-\frac{\xi}{2N}\right)-\omega\left(k+\frac{\xi}{2N}\right)\right]Nt}\,\widehat{W}_N(\xi,k,0) \\ &\underset{N\to\infty}{\sim} e^{-i\omega'(k)\xi t}\,\widehat{W}_{th}(\xi,k,0) \end{aligned}$$

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$$W(t,y,k) = W(0, y - \frac{\omega'(k)}{2\pi} t, k)$$

Phonon of wave number k moves freely with velocity $\frac{\omega'(k)}{2\pi}$.

Wigner distribution

Consequently the thermal energy $\tilde{\mathbf{e}}(y,t)$ (i.e. temperature) evolves non autonomously following the equation

$$\partial_t \tilde{\mathbf{e}}(y,t) + \partial_y J(y,t) = 0, \qquad J(y,t) = \int \omega'(k) W_{th}(y,k,t)\, dk.$$

We say that the system is in *local equilibrium* if $W_{th}(y,k) = \beta^{-1}(y)$ constant in k.
Starting in thermal equilibrium means $W_{th}(y,k,0) = \beta^{-1}$ and trivially $W_{th}(y,k,t) = \beta^{-1}$ for any $t > 0$.
But starting with local equilibrium, i.e. $W(y,k,0) = \beta^{-1}(y)$ constant in k, we have a non autonomous evolution of $\tilde{e}(y,t)$.

Harmonic Chain with Random Masses

The problem with the harmonic chain is that thermal waves of wavenumber k *move* with speed $\omega'(k)$, if they are not uniformed distributed (i.e. the system is not in thermal equilibrium), the temperature profile will not remain constant, as it should be following the Euler equations.

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If the masses are random, the thermal modes remains localized (frozen), by Anderson localization. This allows to close the energy equation, **without local equilibrium**.

Harmonic Chain with Random Masses

(F. Huveneers, C. Bernardin, S.Olla, Comm.Math.Phys. 2019)

$\{m_x\}$ i.i.d. with absolutely continuous distribution μ,

$$0 < m_- \le m_x \le m_+, \qquad \overline{m} = \mathbb{E}_\mu(m_x).$$

$$m_x \dot{q}_x(t) = p_x(t), \qquad \dot{p}_x(t) = \Delta q_x(t), \qquad x = 1, \dots, N$$

with $q_0 = q_1$ and $q_{N+1} = q_N$ as boundary conditions.

Gibbs States, Local Gibbs States

The Gibbs states are characterized by three parameters: $\beta > 0$ and $p, r \in \mathbb{R}$. Its probability density writes

$$\prod_{x=1}^{N} \frac{e^{-\frac{\beta m_x}{2}\left(\frac{p_x}{m_x}-\frac{p}{\overline{m}}\right)^2-\frac{\beta}{2}(r_x-r)^2}}{Z(\beta,p,r,m_x)}.$$

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We start with a local Gibbs state:

$$\prod_{x=1}^{N} \frac{e^{-\frac{\beta(x/N)m_x}{2}\left(\frac{p_x}{m_x}-\frac{p(x/N)}{\overline{m}}\right)^2-\frac{\beta(x/N)}{2}(r_x-r(x/N))^2}}{Z(\beta(x/N), p(x/N), r(x/N), m_x)}.$$

Harmonic Chain with Random Masses: hydrodynamic limit

(F. Huveneers, C. Bernardin, S.Olla, Comm.Math.Phys. 2019)
Almost surely with respect to $\{m_x\}$:

$$< r_{[Ny]}(Nt) >, < p_{[Ny]}(Nt) >, < \mathcal{E}_{[Ny]}(Nt) > \rightharpoonup \ (\mathbf{r}(y,t), \mathbf{p}(y,t), \mathfrak{e}(y,t))$$

$$\begin{aligned} \partial_t \mathbf{r}(t,y) &= \frac{1}{\overline{m}} \partial_y \mathbf{p}(t,y) \\ \partial_t \mathbf{p}(t,y) &= \partial_y \mathbf{r}(t,y) \\ \partial_t \mathfrak{e}(t,y) &= \frac{1}{\overline{m}} \partial_y \left(\mathbf{r}(t,y) \mathbf{p}(t,y) \right) \end{aligned}$$

with initial conditions:

$$\mathbf{r}(y,0) = r(y), \qquad \mathbf{p}(y,0) = p(y), \qquad \mathfrak{e}(y,0) = \frac{1}{\beta(y)} + \frac{p^2(y)}{2\overline{m}} + \frac{r^2(y)}{2}.$$

Quantum Harmonic Chain with Random Masses: hydrodynamic limit

(Amirali Hannani, 2020, arXiv:2011.07552) same Euler equations:

$$\partial_t \mathbf{r}(t,y) = \frac{1}{\overline{m}} \partial_y \mathbf{p}(t,y)$$
$$\partial_t \mathbf{p}(t,y) = \partial_y \mathbf{r}(t,y)$$
$$\partial_t \mathfrak{e}(t,y) = \frac{1}{\overline{m}} \partial_y \left(\mathbf{r}(t,y) \mathbf{p}(t,y) \right)$$

but

$$\mathfrak{e}(t,y) = \mathfrak{f}^{\mu}(\beta(y)) + \frac{p^2(t,y)}{2\overline{m}} + \frac{r^2(t,y)}{2}.$$

Here $\mathfrak{f}^{\mu}(\beta(y))$ is the quantum thermal energy, that depends on the distribution μ of the random masses (not an explicit function, except for deterministic equal masses).

Random Masses: Localization of Thermal Modes

Equation of motion can be written as

$$\ddot{r}_x = -(\nabla^* M^{-1} \nabla r)_x \quad (1 \le x \le N-1), \qquad \ddot{p}_x = \left(\Delta M^{-1} p\right)_x \quad (1 \le x \le N),$$

where $M_{x,x'} = \delta_{x,x'} m_x$.

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$$M^{-1/2}(-\Delta) M^{1/2} \varphi^k = \omega_k^2 \, \varphi^k, \qquad k = 0, \dots, N-1.$$

$$\psi^k = M^{-1/2} \varphi^k, \qquad M^{-1} \Delta \psi^k = \omega_k^2 \psi_k$$

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$$\psi^k = M^{-1/2} \varphi^k, \qquad M^{-1} \Delta \psi^k = \omega_k^2 \psi_k$$

$$r(t) = \sum_{k=1}^{N-1} \left(\frac{\langle \nabla \psi^k, r(0) \rangle}{\omega_k} \cos \omega_k t + \langle \psi^k, p(0) \rangle \sin \omega_k t \right) \frac{\nabla \psi^k}{\omega_k},$$

$$p(t) = \sum_{k=0}^{N-1} \left(\langle \psi^k, p(0) \rangle \cos \omega_k t - \frac{\langle \nabla \psi^k, r(0) \rangle}{\omega_k} \sin \omega_k t \right) M \psi^k.$$

Localization of Thermal Modes

Localization length ξ_k diverges with N:

$$\xi_k^{-1} \sim \omega_k^2 \sim \left(\frac{k}{N}\right)^2,$$

only the modes $k > \sqrt{N}$ are localized.

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Localization length ξ_k diverges with N:

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only the modes $k > \sqrt{N}$ are localized.
More precisely: for $0 < \alpha < \frac{1}{2}$

$$\mathbb{E}\left(\sum_{k=N^{1-\alpha}}^{N-1} |\psi_x^k \psi_{x'}^k|\right) \leq Ce^{-cN^{-2\alpha}|x-x'|}.$$

This estimate is enough to prove that thermal modes remains localized and do not *move* macroscopically.

Random masses: Larger time scales

Assume for simplicity that we are in a *mechanical equilibrium*:

$$\langle \; r_x(0) \; \rangle = 0, \qquad \langle \; p_x(0) \; \rangle = 0,$$

(only thermal energy present)
but not in thermal equilibrium, then, for any $\alpha \geq 1$

$$< \mathcal{E}_{[Ny]}(N^{\alpha} t) > \underset{N \ to \infty}{\longrightarrow} \mathbf{e}(0, y) = \bar{C}(\beta(y))$$

NO evolution for the temperature profile at any scale!

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NO evolution for the temperature profile at any scale!
In particular, for $\alpha = 2$ (diffusive scaling), thermal diffusivity is null.

Open questions for the quantum case

- In order to deal with the *anharmonic* interaction, in the classical case, conservative noise is added to obtain ergodicity of the infinite dynamics (unique characterization of the translational invariant stationary states)
  (cf B. Nachtergaele, and H-T Yau, CMP 2003).
  How to add *conservative noise* in the quantum dynamics in order to obtain similar result?

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  (cf B. Nachtergaele, and H-T Yau, CMP 2003).
  How to add *conservative noise* in the quantum dynamics in order to obtain similar result?
- Boundary tension? More generally boundary conditions, thermostat etc.

entropy evolution

$$\partial_t r = \partial_x p \qquad \partial_t p = \partial_x \tau \qquad \partial_t \mathfrak{e} = \partial_x(\tau p)$$
$$p(t,0) = 0, \qquad \tau(r(1,t), u(1,t)) = \tau(t)$$

$U = \mathfrak{e} - p^2/2$, $\beta = \frac{\partial S}{\partial U}$, $\tau = -\frac{1}{\beta}\frac{\partial S}{\partial r}$

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For smooth solutions:

$$\begin{aligned} \frac{d}{dt} S(u(y,t), r(y,t)) &= \beta\left(\partial_t e - p\partial_t p\right) - \beta\tau\partial_t r \\ &= \beta\left(\partial_x(\tau p) - p\partial_x \tau - \tau\partial_x p\right) = 0 \end{aligned}$$

The evolution is *isoentropic* in the smooth regime.

Shocks, contact discontinuities, weak solutions, entropy solutions

Even starting with initial smooth profiles, hyperbolic non-linear systems develops discontinuities:

- shocks: discontinuities in the tension profile,

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- entropy solutions
- viscosity solutions

Isothermal Dynamics

- J. Fritz, *Microscopic theory of isothermal elasticity*, ARMA 2011, infinite volume

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The system is in contact with a heat bath that keeps it at a constant temperature β^{-1}. Energy is not conserved anymore. Macroscopically we have a non-linear wave equation:

$$\partial_t r(t,y) = \partial_y p(t,y)$$
$$\partial_t p(t,y) = \partial_y \tau[\beta, r(t,y)]$$

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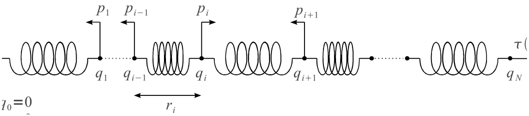
Microscopic isothermal dynamics



$$\begin{cases} dr_1 = Np_1 dt + N\sigma_N \left(V'(r_2) - V'(r_1)\right) dt - \sqrt{2\beta^{-1}N\sigma_N} d\widetilde{w}_1 \\ dr_i = N(p_i - p_{i-1})dt + N\sigma_N \left(V'(r_{i+1}) + V'(r_{i-1}) - 2V'(r_i)\right) dt + \sqrt{2\beta^{-1}N\sigma_N}(d\widetilde{w}_{i-1} - d\widetilde{w}_i) \\ dr_N = N(p_N - p_{N-1})dt + N\sigma_N \left(V'(r_{N-1}) - V'(r_N)\right) dt + \sqrt{2\beta^{-1}N\sigma} d\widetilde{w}_{N-1} \\ dp_1 = N(V'(r_2) - V'(r_1))dt + N\sigma_N \left(p_2 - p_1\right) dt - \sqrt{2\beta^{-1}N\sigma_N} dw_1 \\ dp_i = N(V'(r_{i+1}) - V'(r_i))dt + N\sigma_N \left(p_{i+1} + p_{i-1} - 2p_i\right) dt + \sqrt{2\beta^{-1}N\sigma_N}(dw_{i-1} - dw_i) \\ dp_N = N(\bar{\tau}(t) - V'(r_N))dt + N\sigma_N \left(p_{N-1} - p_N\right) dt + \sqrt{2\beta^{-1}N\sigma_N} dw_{N-1}, \end{cases} ,$$

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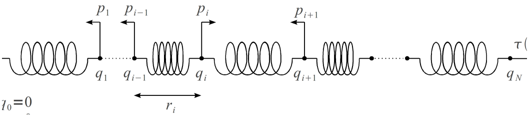

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\end{cases}
,
$$

$$
\lim_{N\to+\infty} \frac{\sigma_N}{N} = \lim_{N\to\infty} \frac{N}{{\sigma_N}^2} = 0.
$$

Hydrodynamic Limit

$$\frac{1}{N}\sum_x G(x/N)\begin{pmatrix} r_x(t) \\ p_x(t) \end{pmatrix} \underset{N\to\infty}{\longrightarrow} \int_0^1 G(y)\begin{pmatrix} r(y,t) \\ p(y,t) \end{pmatrix} dy$$

L^2-valued weak solution of

$$\begin{aligned} \partial_t r(t,y) &= \partial_y p(t,y) \\ \partial_t p(t,y) &= \partial_y \tau_\beta[r(t,y)] \\ & p(t,0) = 0, \quad \tau(r(t,1)) = \bar{\tau}(t), \end{aligned}$$

with boundary conditions that satisfy the *Clausius inequality* between the *work* done by the boundary force and the change in the *free energy*.

S. Marchesani, S. Olla, *On the existence of L2-valued thermodynamic entropy solutions for a hyperbolic system with boundary conditions*, Comm. Partial Diff. Eq. (2020).

Entropy production and Clausius inequality

Free energy at time t:

$$\mathcal{F}(t) = \int_0^1 \left[\frac{p(t,y)^2}{2} + F_\beta(r(t,y)) \right] dy,$$

$$\partial_r F_\beta(r) = \tau_\beta(r),$$

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$$\partial_r F_\beta(r) = \tau_\beta(r),$$

$$\mathcal{F}(t) - \mathcal{F}(0) \geq W(t) = \int_0^t \tau(s) p(s,1)\, ds$$

where $W(t)$ is the work done by the boundary force $\tau(t)$.