Thermal boundaries for energy superdiffusion

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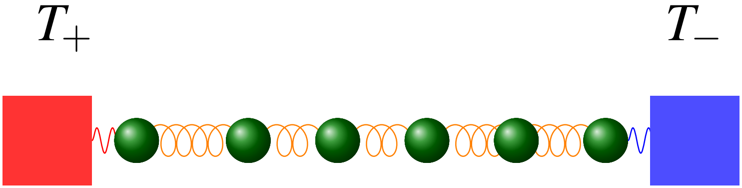
Thermal boundaries

Thermal boundaries appear in macroscopic equations for the evolution of energy or temperatures profiles.



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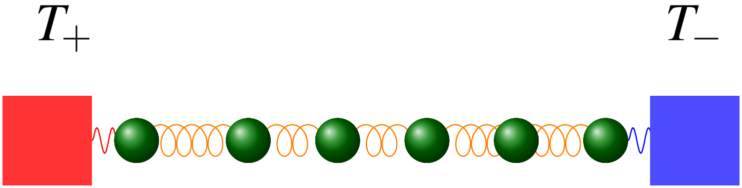

Basic example: *heat equation* for a material in contact with heat bath at the boundaries:

$$
\begin{aligned}
\partial_t u(t,y) &= \partial_y \left(D(u) \partial_y u(t,y) \right), \qquad y \in [0, L] \\
u(t,0) &= T_+, \qquad u(t,L) = T_-.
\end{aligned}
$$

But superdiffusion associated to fractional Laplacian can appear, and BC are more delicate.

Thermal boundary

Consider the usual heat equation with temperature fixed at $y = 0$:

$$\begin{aligned} \partial_t u(t,y) &= D\Delta_y u(t,y), \qquad y \in \mathbb{R} \smallsetminus \{0\} \\ u(t,0) &= T. \end{aligned}$$

This is very easy problem, and left side does not exchange energy with the right side.

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What about, for $\alpha < 1$,

$$
\begin{aligned}
\partial_t u(t,y) &= -D|\Delta_y|^\alpha u(t,y), \qquad y \in \mathbb{R} \setminus \{0\} \\
u(t,0) &= T \qquad ?
\end{aligned}
$$

There are many different definition of the BC for the fractional Laplacian. Which one emerges from the microscopic dynamics?

Microscopic modeling: Langevin stochastic thermostats

$\mathfrak{p} = \{\mathfrak{p}_x,\ x \in \mathbb{Z}\},\ \mathfrak{q} = \{\mathfrak{q}_x,\ x \in \mathbb{Z}\}$

$$\mathcal{H}(\mathfrak{p}, \mathfrak{q}) := \frac{1}{2}\sum_x \mathfrak{p}_x^2 + \frac{1}{2}\sum_{x,x'} \alpha_{x-x'}\mathfrak{q}_x\mathfrak{q}_{x'}$$

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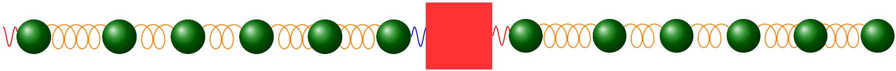

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$$d\mathfrak{q}_x(t) = \mathfrak{p}_x(t)dt, \quad x \in \mathbb{Z}$$

$$d\mathfrak{p}_x(t) = -(\alpha * \mathfrak{q}(t))_x dt + \Big(-\gamma\mathfrak{p}_0(t)dt + \sqrt{2\gamma T}dw(t)\Big)\delta_{0,x},$$

where $\{w(t),\ t \geq 0\}$ is a standard Wiener process.

Microscopic modeling: Poisson-Maxwell thermostats

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- $\{N(t),\ t \geq 0\}$ is a standard Poisson process,
- $\{\tilde{p}_j\}$ i.i.d. $\mathcal{N}(0, T)$,
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- for $\mu \to \infty$ it converges to the Langevin thermostat.
- for $\mu = 1/2$ flip sign of the velocity at Poisson times.

Coupled Harmonic Oscillators

$$\hat{f}(k) = \sum_x f_x e^{-i2\pi kx} \qquad k \in \Pi \sim [0,1]$$

- $\hat{\alpha}(k) \in \mathbb{C}^\infty(\Pi)$.
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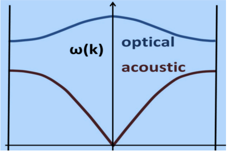
$$\alpha_x = 2\delta_{x,0} - \delta_{x,\pm 1}, \qquad \omega(k) = 2|\sin(\pi k)|.$$



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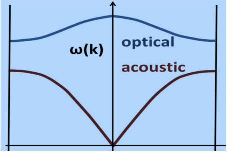

$$\hat{\psi}(t,k) := \omega(k)\hat{\mathfrak{q}}(t,k) + i\hat{\mathfrak{p}}(t,k), \qquad \int |\hat{\psi}(t,k)|^2 = \mathcal{H}(\mathfrak{p},\mathfrak{q}).$$

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$$\hat{\psi}(t,k) = e^{-i\omega(k)t}\hat{\psi}(0,k) - i\gamma\int_0^t e^{-i\omega(k)(t-s)}\mathfrak{p}_0(s)ds + i\sqrt{2\gamma T}\int_0^t e^{-i\omega(k)(t-s)}dw(t)$$

Wigner distribution

$\eta \in \mathbb{R}$, $\varepsilon > 0$ -hyperbolic rescaling of space–time

$$\widehat{W}_\varepsilon(t,\eta,k) \;:=\; \frac{\varepsilon}{2}\mathbb{E}\left[\hat{\psi}^*\left(\varepsilon^{-1}t, k - \frac{\varepsilon\eta}{2}\right)\hat{\psi}\left(\varepsilon^{-1}t, k + \frac{\varepsilon\eta}{2}\right)\right]$$

$$\widehat{W}_\varepsilon(t,\eta,k) \;:=\; \widehat{W}_\varepsilon(t,-\eta,k)^*$$

and the inverse Fourier transform in η

$$W_\varepsilon(t,y,k) = \int \widehat{W}_\varepsilon(t,\eta,k)e^{i2\pi\eta y}\,d\eta \;\in\mathbb{R}, \qquad y\in\mathbb{R},$$

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$$W_\varepsilon(t,y,k) \underset{\varepsilon\to 0}{\rightharpoonup} W(t,y,k) \ \geq \ 0, \quad \text{as distribution}$$

When $\gamma = 0$ it is easy to prove that

$$\partial_t W(t,y,k) + \frac{\omega'(k)}{2\pi}\partial_y W(t,y,k) = 0$$

Easy case: $\gamma = 0$, no thermostat

From waves to particle:

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$$W(t,y,k) = W\left(0, y - \frac{\omega'(k)}{2\pi}t, k\right)$$

Phonon of wave number k moves freely with velocity $\frac{\omega'(k)}{2\pi}$.

$\gamma > 0$ Explicit solution (microscopic)

$$J(t) = \int_{\mathbb{T}} \cos\left(\omega(k)t\right)\, dk,$$

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The Laplace transform of $J(t)$ is given by

$$\tilde{J}(\lambda) = \int_0^\infty e^{-\lambda t} J(t) dt = \int_{\mathbb{T}} \frac{\lambda}{\lambda^2 + \omega^2(k)} dk.$$

$$\tilde{g}(\lambda) = \left(1 + \gamma \tilde{J}(\lambda)\right)^{-1} = \int_0^\infty e^{-\lambda t} g(dt). \qquad |\tilde{g}(\lambda)| < 1$$

$$\phi(t,k) = \int_0^t e^{i\omega(k)\tau} g(d\tau) \underset{t\to\infty}{\longrightarrow} \tilde{g}(-i\omega(k)) := \nu(k)$$

$$\hat{\psi}(t,k) = e^{-i\omega(k)t}\Big[\hat{\psi}(0,k) - i\gamma \int_0^t \phi(t-s,k) e^{i\omega(k)s} \mathfrak{p}_0^0(s)\ ds + i\sqrt{2\gamma T} \int_0^t \phi(t-s,k) e^{i\omega(k)s}\ dw(s)\Big]$$

$\mathfrak{p}_0^0(i)$: moment of particle 0 under the free evolution for $\gamma = 0$.

Results in presence of the Langevin thermostat in 0 ($\gamma > 0$)

$$\nu(k) = \left(1 + \gamma \tilde{J}(-i\omega(k))\right)^{-1}, \qquad \tilde{J}(\lambda) = \int_{\mathbb{T}} \frac{\lambda}{\lambda^2 + \omega^2(k)} dk.$$

$$\mathrm{Re}\, \nu(k) = \left(1 + \frac{\gamma\pi}{|\omega'(k)|}\right) |\nu(k)|^2$$

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$$\mathfrak{g}(k) = \frac{2\pi\gamma|\nu(k)|^2}{|\omega'(k)|} \qquad \text{absorbtion probability}$$

$$p_+(k) = \left|1 - \frac{\gamma\pi\nu(k)}{|\omega'(k)|}\right|^2 \qquad \text{transmission probability}$$

$$p_-(k) = \left|\frac{\gamma\pi\nu(k)}{|\omega'(k)|}\right|^2 \qquad \text{reflection probability}$$

$$\mathfrak{g}(k) + p_+(k) + p_-(k) = 1$$

Results for Langevin Thermostat

T.Komorowski, L.Ryzhik, S.O., H.Spohn, ARMA (2020)

$$\partial_t W(t,y,k) + \frac{\omega'(k)}{2\pi}\partial_y W(t,y,k) = 0, \qquad y \in \mathbb{R} \smallsetminus \{0\}$$

with boundary conditions:

$$W(t,0^+,k) = p_-(k)W(t,0^+,-k) + p_+(k)W(t,0^-,k) + \mathfrak{g}(k)T, \quad 0 < k < 1/2$$
$$W(t,0^-,k) = p_-(k)W(t,0^-,-k) + p_+(k)W(t,0^+,k) + \mathfrak{g}(k)T, \quad -1/2 < k < 0$$

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Since $\mathfrak{g}(k) + p_+(k) + p_-(k) = 1$, we have that

$$W(t,y,k) = T \qquad \text{is a stationary solution.}$$

No dispersion in the macroscopic scattering!

Results for Poisson Thermostat

T.Komorowski, S.O., arXiv:2101.04360 (2021)

$$\partial_t W(t,y,k) + \frac{\omega'(k)}{2\pi}\partial_y W(t,y,k) = 0, \qquad y \in \mathbb{R} \setminus \{0\}$$

with boundary conditions $0 < k < 1/2$:

$$W(t,0^+,k) = p_-(k)W(t,0^+,-k) + p_+(k)W(t,0^-,k) + p_{abs}\mathfrak{g}(k)T$$
$$+\mathfrak{g}(k)\int_0^{1/2} W(t,0^-,\ell)p_{sc}(\ell)d\ell + \mathfrak{g}(k)\int_0^{1/2} W(t,0^+,-\ell)p_{sc}(\ell)d\ell$$

$$p_+(k) + p_-(k) + p_{abs}\mathfrak{g}(k) + \mathfrak{g}(k)\int_0^1 p_{sc}(\ell)d\ell = 1$$

Dispersion in the macroscopic scattering!

$p_{abs} \to 1$ and $p_{sc}(\ell) \to 0$ as $\mu \to \infty$ (converges to the Langevin scattering).

When random bulk scattering is present

If we add a (slow) random scattering in the bulk of the system: random flip of sign of velocities, random exchange of velocity of n.n. particles, with rates $\sim \gamma'\varepsilon$.
T. Komorowski, S.O. (J. Funct. Analys. 2020)

$$\partial_t W(t,y,k) + \frac{\omega'(k)}{2\pi}\partial_y W(t,y,k) = \gamma' LW(t,y,k)$$

$$LW(k) = \int \mathfrak{R}(k,k')\left(W(k') - W(k)\right)$$

$$R(k,k') = R(K)R(k'), \qquad R(k) \sim |k|^2,\ k \sim 0$$

same boundary conditions:

$$W(t,0^+,k) = p_-(k)W(t,0^+,-k) + p_+(k)W(t,0^-,k) + \mathfrak{g}(k)T, \quad 0 < k < 1/2$$
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Without the thermostat: Basile, O., Spohn, ARMA 2009.

Diffusive and superdiffusive behavior for $\gamma' > 0$, $\gamma = 0$

For random exchange of nearest neighbor velocities

$$R(k,k') = R(K)R(k'), \; R(k) \sim |k|^2, \qquad k \sim 0$$

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. Without thermal boundary:

- if $\omega'(k) \sim k$ (optical chain) : $D < +\infty$, diffusive behaviour

$$W(\lambda^2 t, \lambda y, k) \underset{\lambda \to 0}{\longrightarrow} e(t,y)$$

$$\partial_t e = D \partial_{yy} e, \qquad D = \frac{1}{4\pi^2 \gamma'} \int \frac{\omega'(k)^2}{R(k)} dk$$

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- if $\omega'(k) \sim 1$ (acustic chain) : $D = +\infty$, superdiffusive behaviour (Jara-Komorowski-Olla AAP 2009, Basile-Bovier MPRF 2010):

$$W(\lambda^{3/2} t, \lambda y, k) \underset{\lambda \to 0}{\longrightarrow} e(t,y)$$

$$\partial_t e = -\hat{c}|\Delta|^{3/4} e,$$

Diffusive behaviour with thermal BC: $\gamma' > 0$, $\gamma > 0$

Giada Basile, Tomasz Komorowki, S.O.,
Kinetic and Related Models, AIMS, (2019)
In the cases of bulk diffusive behavior: $\omega'(k) \sim k$ (optical chain)

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Reflection and transmission of phonons are irrelevant in this time scale: phonons gets absorbed and created such that energy density is T at $y = 0$.

Superdiffusive behaviour with thermal boundary

Tomasz Komorowski, S.O., Lenya Rhyzik, Ann. of Prob. (2020)

Situation is different in the super-diffusive case.

$$\partial_t W(t,y,k) + \frac{\omega'(k)}{2\pi}\partial_y W(t,y,k) = \int \mathfrak{R}(k,k')\left(W(k') - W(k)\right)$$

$$R(k,k') = R(K)R(k'), \qquad R(k) \sim |k|^2, \qquad |\omega'(k)| \sim 2, \qquad k \sim 0,$$

$$W(t,0^+,k) = p_-(k)W(t,0^+,-k) + p_+(k)W(t,0^-,k) + \mathfrak{g}(k)T, \quad 0 < k < 1/2$$
$$W(t,0^-,k) = p_-(k)W(t,0^-,-k) + p_+(k)W(t,0^+,k) + \mathfrak{g}(k)T, \quad -1/2 < k < 0$$

Superdiffusive behaviour with thermal boundary

Tomasz Komorowski, S.O., Lenya Rhyzik, Ann. of Prob. (2020)
Situation is different in the super-diffusive case.

$$\partial_t W(t,y,k) + \frac{\omega'(k)}{2\pi}\partial_y W(t,y,k) = \int \mathfrak{R}(k,k')\big(W(k') - W(k)\big)$$

$$R(k,k') = R(K)R(k'), \qquad R(k) \sim |k|^2, \qquad |\omega'(k)| \sim 2, \qquad k \sim 0,$$

$$W(t,0^+,k) = p_-(k)W(t,0^+,-k) + p_+(k)W(t,0^-,k) + \mathfrak{g}(k)T, \quad 0 < k < 1/2$$
$$W(t,0^-,k) = p_-(k)W(t,0^-,-k) + p_+(k)W(t,0^+,k) + \mathfrak{g}(k)T, \quad -1/2 < k < 0$$

$$R(k,k') \sim 0, \qquad k \sim 0,$$

the phonon is crossing the thermostat when $k \sim 0$.

Superdiffusive behaviour with thermal boundary

Tomasz Komorowski, S.O., Lenya Rhyzik, Ann. of Prob. (2020)

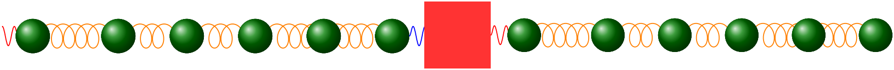
For $k \to 0$, $\mathfrak{g}(k) \to \mathfrak{g}_0 > 0$, $\quad p_-(k) \to p_-(0) > 0$.

$$W(\lambda^{3/2} t, \lambda y, k) \underset{\lambda \to 0}{\longrightarrow} e(t, y)$$

$$\begin{aligned}
\partial_t e(t,y) = & - \hat{c}|\Delta|^{3/4} e(t,y) \\
& + \hat{c}\mathfrak{g}_0 \int_{yy'<0} q(y - y')(T - e(t,y'))dy' \\
& + \hat{c} p_-(0) \int_{yy'<0} q(y - y')(e(t,-y') - e(t,y))dy'.
\end{aligned}$$

$$q(y) = \frac{c}{|y|^{5/4}} \qquad \text{kernel of } |\Delta|^{3/4}.$$

Direct Hydrodynamic Limit



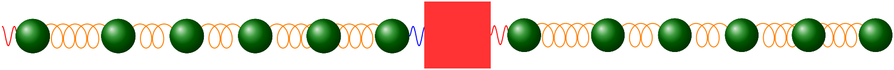
$$\mathfrak{p} = \{\mathfrak{p}_x,\ x \in \mathbb{Z}\},\ \mathfrak{q} = \{\mathfrak{q}_x,\ x \in \mathbb{Z}\}$$

$$\mathcal{H}(\mathfrak{p},\mathfrak{q}) := \frac{1}{2}\sum_x \mathfrak{p}_x^2 + \frac{1}{2}\sum_{x,x'} \alpha_{x-x'}\mathfrak{q}_x\mathfrak{q}_{x'}$$

$$\begin{aligned} d\mathfrak{q}_x(t) &= \mathfrak{p}_x(t)dt, \qquad x \in \mathbb{Z} \\ d\mathfrak{p}_x(t) &= -(\alpha * \mathfrak{q}(t))_x + \left(-\gamma \mathfrak{p}_0(t)dt + \sqrt{2\gamma T}dw(t)\right)\delta_{0,x}, \end{aligned}$$

$$\mathcal{E}_x = \frac{\mathfrak{p}_x^2}{2} + \frac{\mathfrak{q}_x\,(\alpha \star \mathfrak{q})_x}{2}$$

Direct Hydrodynamic Limit



$\mathfrak{p} = \{\mathfrak{p}_x,\ x \in \mathbb{Z}\}$, $\mathfrak{q} = \{\mathfrak{q}_x,\ x \in \mathbb{Z}\}$

$$\mathcal{H}(\mathfrak{p},\mathfrak{q}) := \frac{1}{2}\sum_x \mathfrak{p}_x^2 + \frac{1}{2}\sum_{x,x'} \alpha_{x-x'}\mathfrak{q}_x\mathfrak{q}_{x'}$$

$$d\mathfrak{q}_x(t) = \mathfrak{p}_x(t)dt, \qquad x \in \mathbb{Z}$$
$$d\mathfrak{p}_x(t) = -(\alpha * \mathfrak{q}(t))_x + \left(-\gamma\mathfrak{p}_0(t)dt + \sqrt{2\gamma T}dw(t)\right)\delta_{0,x},$$

$$\mathcal{E}_x = \frac{\mathfrak{p}_x^2}{2} + \frac{\mathfrak{q}_x\,(\alpha * \mathfrak{q})_x}{2}$$

$$\mathcal{E}_{[\lambda y]}(\lambda^{-3/2}t) \underset{\lambda\to 0}{\rightharpoonup} e(t,y) \qquad \text{weakly}$$

Direct Hydrodynamic Limit

$$\mathcal{E}_x = \frac{\mathfrak{p}_x^2}{2} + \frac{\mathfrak{q}_x\,(\alpha \star \mathfrak{q})_x}{2}, \qquad x \in \mathbb{Z}$$

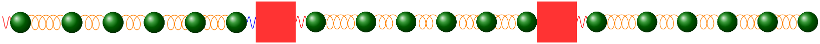
$$\mathcal{E}_{[\lambda y]}(\lambda^{-3/2} t) \underset{\lambda\to 0}{\rightharpoonup} e(t,y) \qquad \text{weakly} \quad y \in \mathbb{R}.$$

$$\partial_t \tilde{e}(t,y) = \mathfrak{L}\tilde{e}(t,y), \qquad \tilde{e}(t,0) = 0,$$

$$\mathfrak{L} \sim -(-\Delta_y)^{3/4}$$

- Without the thermostat: M.Jara,T.Komoroswki, S.Olla, CMP (2015).
- With the thermostat, still open.

Two Thermostats

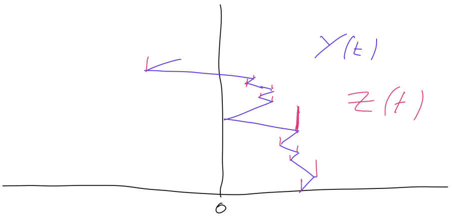


at distance $N = [\lambda^{-1}]$.

$$
\begin{aligned}
d\mathfrak{q}_x(t) &= \mathfrak{p}_x(t)dt, \qquad && x \in \mathbb{Z} \\
d\mathfrak{p}_x(t) &= -(\alpha * \mathfrak{q}(t))_x && + \left(-\gamma \mathfrak{p}_0(t)dt + \sqrt{2\gamma T_-}dw_1(t)\right)\delta_{0,x} \\
& && + \left(-\gamma \mathfrak{p}_N(t)dt + \sqrt{2\gamma T_+}dw_2(t)\right)\delta_{N,x}
\end{aligned}
$$

If $T_+ \neq T_-$: non-equilibrium stationary state.

Idea of the proof



- Process $Y(t)$ is generated by the linear Boltzmann equation (continuous trajectories),
- Process $Z(t)$ is a pure jump Levy process obtained from $Y(t)$, characterized by a Levy measure $r(y)$ given by the law of

$$\frac{\bar{\omega}'(K)}{R(K)}$$

where K is a random variable on S^1 with law $R(k)dk$.

We only need to control the convergence of $Z(t)$!

Articles

A review article:

- T. Komorowski, S. Olla, *Thermal Boundaries in Kinetic and Hydrodynamic Limits*, http://arxiv.org/abs/2010.04721, to appear in "Recent advances in kinetic equations and applications", F. Salvarani ed., **Springer INdAM Series**, 2021.

Articles

A review article:

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Results with thermostat:

- T. Komorowski, S. Olla, L. Ryzhyk, H. Spohn *High frequency limit for a chain of harmonic oscillators with a point Langevin thermostat.*, **Arch. Rational Mech. An.**, 237, 497543 (2020).
- T. Komorowski, S. Olla, *Asymptotic Scattering by Poissonian Thermostatst*, arXiv:2101.04360 (2021),
- T. Komorowski, S. Olla, *Kinetic limit for a chain of harmonic oscillators with a point Langevin thermostat*, **J. Funct. Anal.**, 279, n.12, 108764, (2020)
- T. Komorowski, S. Olla, L. Ryzhyk, *Fractional Diffusion limit for a kinetic equation with an interface*, **Annals of Probability**, Vol. 48, No. 5, 22902322, (2020),
- G. Basile, T. Komorowski, S. Olla, *Diffusive limits for a kinetic equation with a thermostatted interface*, **Kinetic and Related Models**, AIMS, (2019), 12(5): 1185-1196,