On the minimum time control of affine control systems

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Introduction and motivation

Optimal control problems

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$$\begin{cases} \dot{x} = f(x, u), \quad u(t) \in U \\ x(0) = x_0, \\ x(t_f) = x_f, \\ C(u) = \int_0^{t_f} \varphi(x(t), u(t)) dt \to \min . \end{cases}$$

Where $C : L^\infty([0, t_f], U) \to \mathbb{R}$ is the *cost function*.

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Where $C : L^\infty([0, t_f], U) \to \mathbb{R}$ is the *cost function.*
f continuous in u, x_u, the solution associated with a control $u \in L^\infty([0, t_f], U)$ uniquely well defined (Carathéodory).

Main questions

- Existence of any globally optimal trajectory ?
  → *Sufficient conditions*: Filippov's theorem.
- How can we find - characterize optimal trajectories ?
  → *Necessary conditions*: Pontrjagin's Maximum Principle, extremals.
- Local/global optimality of our extremal trajectories ?
  → *Second order conditions*: conjugate points, symplectic methods (Agrachev).
- Regularity of optimal trajectory ?
  → Techniques from *dynamical systems*: Normal hyperbolicity, invariant manifolds, normal forms.
- Are extremals computable - define an integrable system ?
  → *Galois differential theory* (but also, symbolic dynamics, Smale's horseshoe...)

Minimum time affine control systems

Set $\varphi = 1$, $f(x,u) = F_0(x) + u_1 F_1(x) + u_2 F_2(x)$ and $U = B$ Euclidean ball.

$$\begin{cases} \dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)),\ u_1^2 + u_2^2 \leq 1 \\ x(0) = x_0 \\ x(t_f) = x_f \\ t_f \to \min\,. \end{cases} \tag{1}$$

F_i smooth, $i = 0, 1, 2$, $x_0, x_f \in M$ a 4 dimensional manifold (can be generalized to $2n$ with n controls).

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F_i smooth, $i = 0, 1, 2$, $x_0, x_f \in M$ a 4 dimensional manifold (can be generalized to $2n$ with n controls).
Structure of Lie algebra **Lie**(F_0, F_1, F_2) is crucial.
Example: *Mechanical systems.*

$$\ddot{q} + \nabla V(q) = u,$$

V a smooth potential.

Motivating example: the controlled CR3BP

$$\ddot{q} + \nabla V_\mu(q) - 2i\dot{q} = u, \ \ \|u\| \leq 1 \tag{2}$$

in the rotating frame, u being the control (thrust of the engine) and

$$V_\mu(q) = \frac{1}{2}|q|^2 + \frac{1-\mu}{|q+\mu|} + \frac{\mu}{|q-1+\mu|}, \ \mu = \text{mass ratio.}$$

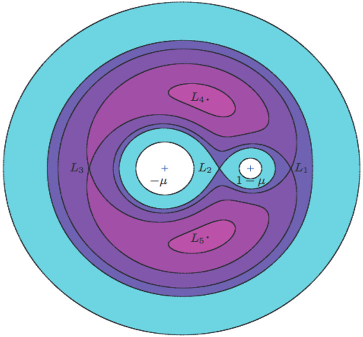

Figure: Hill's region and Lagrange points for the RC3BP

Singularities of minimum time affine control systems

Minimizing the cost generates *singularities.*
This thesis focuses on understanding of the consequences of these singularities:

- On the extremal flow, irregular behavior, non-uniqueness,
- On the optimality of the trajectories,
- On the computability of the possible optimal trajectories: destruction of first integrals.

Structure of the minimum time extremal flow

Hamiltonian formalism

Definition (Pseudo-Hamiltonian)

$\forall (x,p) \in T^*M,\ H(x,p,u) = \langle p, f(x,u) \rangle$

Control affine dynamics: $H(x,p,u) = H_0(x,p) + u_1 H_1(x,p) + u_2 H_2(x,p)$,
$H_i(x,p) = \langle p, F_i(x) \rangle,\ i = 0, 1, 2.$

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Theorem (Pontrjagin)

If (x, u) *is a minimum time trajectory then there exists an absolutely continuous Lipschitz curve* $p(t) \in T_{x(t)}M^* \setminus \{0\}$ *s.t.*

- (x, p) *is solution of :*

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p}(x, p, u) \\ \dot{p} = -\frac{\partial H}{\partial x}(x, p, u). \end{cases} \tag{3}$$

- $H(x(t), p(t), u(t)) = \max_{\tilde{u} \in U} H(x(t), p(t), \tilde{u})$.
- $H(x(t), p(t), u(t)) \geq 0$.

Minimum time control-affine extremals

$$\begin{cases} \dot{x}(t) = F_0(x(t)) + u_1(t)F_1(x(t)) + u_2(t)F_2(x(t)), \ u_1^2 + u_2^2 \leq 1 \\ x(0) = x_0 \\ x(t_f) = x_f \\ t_f \to \min . \end{cases} \tag{4}$$

Proposition

For system (4) $\forall z = (x,p) \in T^*M, \quad H^{\max}(x,p) = H_0(x,p) + \sqrt{H_1^2(x,p) + H_2^2(x,p)^2}$

$u = \frac{1}{\sqrt{H_1^2+H_2^2}}(H_1, H_2)$: discontinuities of the control u are called **switchings**.

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Definition (Singular locus / Switching surface.)

$\Sigma = \{z = (x, p) \in T^*M, \; H_1(x, p) = H_2(x, p) = 0\} = F_1^\perp \cap F_2^\perp$ *smooth submanifold if* F_1 *and* F_2 *are linearly independent.*

Assumption

Set $\bar{z} \in \Sigma$, $\bar{x} = \pi(\bar{z})$, where $\pi : T^*M \to M$ the canonical projection.
Denote $F_{ij} = [F_i, F_j]$, $H_{ij} = \{H_i, H_j\}$.
Assume :

$$(\mathcal{A}) : \det(F_1(\bar{x}), F_2(\bar{x}), F_{01}(\bar{x}), F_{02}(\bar{x})) \neq 0$$

Checked for mechanical systems.
$\to$ $(\mathcal{A})$ + F_0 *recurrent* ($\mu = 0$ or certain Hill's regions of the RC3BP) $\Rightarrow$ Controllability.

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Proposition

Any system of the form $\ddot{q} + g(q, \dot{q}) = u$ *verifies* $(\mathcal{A})$.

In mechanical systems we also have $(\mathcal{B})$: $F_{12} = 0 \Rightarrow H_{12} = 0$.

Bifurcation system

Using $(\mathcal{A})$: $(x, p) \mapsto (x, H_1, H_2, H_{01}, H_{02})$ is a change of coordinates.
→ **Polar blow up:** $(H_1, H_2) = (\rho \cos\theta, \rho \sin\theta)$,

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$$(Y): \begin{cases} \dot{\rho} = -\sin\theta \\ \dot{\theta} = \frac{1}{\rho}(1 + \alpha - \cos\theta + f(\rho, \theta, \xi)) \\ \dot{\xi} = h(\rho, \theta, \xi), \end{cases} \tag{5}$$

$\xi = (x, H_{01}, H_{02})$.
(i) f, h smooth functions, h has values in $\mathbb{R}^k$;
(ii) $f(\bar{z}) = 0$, f has a nice local behavior around $\bar{z}$.

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(i) f, h smooth functions, h has values in $\mathbb{R}^k$;
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Definition (Partition of the singular locus)

$\Sigma = \Sigma_0 \cup \Sigma_- \cup \Sigma_+$ *with :*
$\Sigma_- = \{\alpha < 0\}$, $\Sigma_+ = \{\alpha > 0\}$, $\Sigma_0 = \{\alpha = 0\}$.

The case Σ_{-}: Mechanical systems

The case $\Sigma_- \Leftrightarrow \alpha < 0$

Theorem (Caillau, Fejoz, O.)

There exists unique solution for system (1) in a neighborhood $O_{\bar{z}}$ of $\bar{z}$, and there is at most one switch on $O_{\bar{z}}$.

- *If $\bar{z} \in \Sigma_-$: The local extremal flow $z : (t, z_0) \in [0, t_f] \times O_{\bar{z}} \mapsto z(t, z_0) \in M$ is piecewise smooth, and smooth on each strata :*

$$O_{\bar{z}} = S_0 \sqcup S^s \cup S^u \cup \Sigma$$

  - where S^s (resp. S^u) is the codimension one submanifold of initial conditions leading to the switching surface (resp. in negative times),
  - $S_0 = O_{\bar{z}} \setminus (S^s \cup S^u \cup \Sigma)$.

Constraint on the control → No singular flow inside Σ_-. Extend previous results from Agrachev and Biolo (2016) in a close context.

Idea of the proof

Regularize (Y) by rescaling the time $dt = \rho d\tau$.

$$(Z): \begin{cases} \rho' = -\rho \sin\theta \\ \theta' = 1 + \alpha - \cos\theta + O(\rho; \xi) \\ \xi' = \rho h(\rho, \theta, \xi) \end{cases} \tag{6}$$

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Then, in $\{\rho = 0\}$, for each ξ two parabolic equilibria $\theta_\pm$.
The manifolds $N_\pm = \{(0,\theta_\pm,\bar{\xi})\}$ are normally hyperbolic invariant submanifolds of *equilibria*, with stable and unstable manifolds of **dimension one**.
→ Finite initial time t_f: *existence* and *uniqueness*.

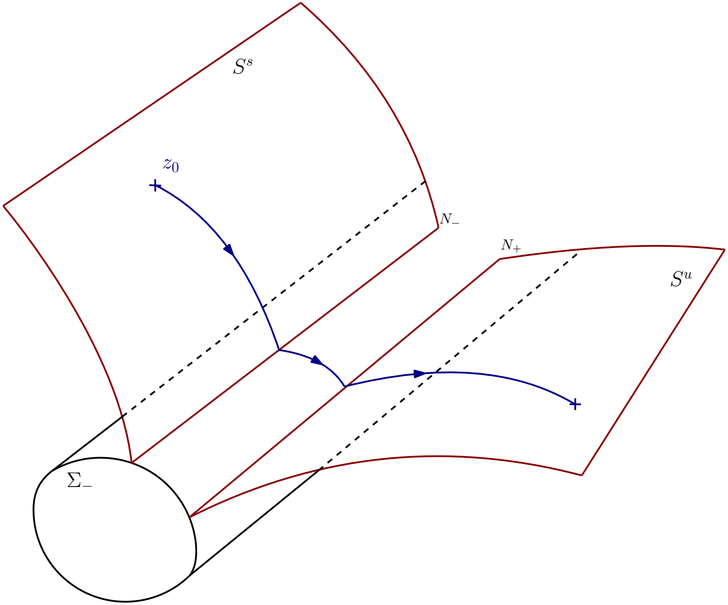

Figure: Blown-up phase portrait around Σ_-

Set

$$S^s = \cup_{z \in \mathrm{N}_-} W^s(z)$$

and

$$S^u = \cup_{z \in \mathrm{N}_+} W^u(z).$$

Eigenvalues on $\mathrm{N}_\pm$ are null, and in the direction of ρ, the Jacobian has a strictly negative eigenvalue: the spectral gap is infinite → S^s (S^u) is a C^∞-smooth foliation (Hirsch, Pugh, Shub).

The flow is smooth restricted to S^s and S_0: the regularized vector field is smooth, consequence of the dominated convergence theorem.

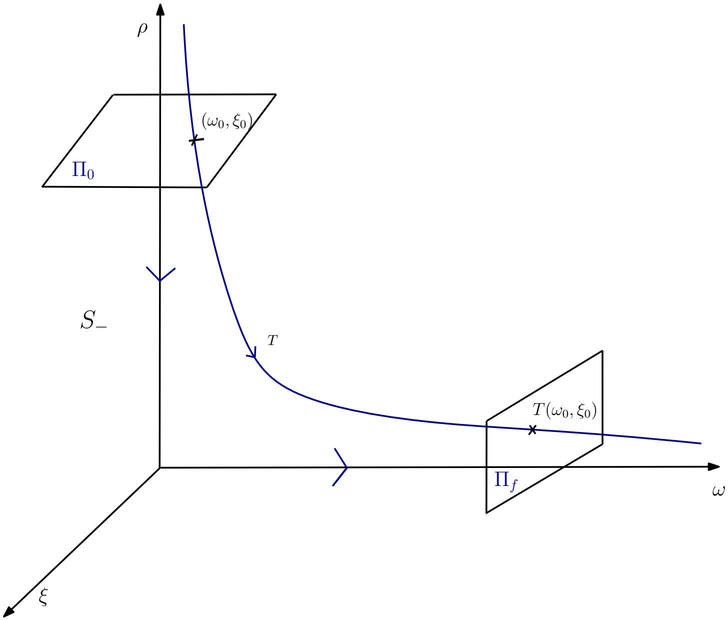

Figure: The regular-singular transition

Transition map

$\Pi_0 \subset \{\rho = \rho_0\}$, $\Pi_f \subset \{\omega = \omega_f\}$.

Theorem (Caillau, Fejoz, O., Roussarie)

Let $T : \Pi_0 \to \Pi_f$ *be the Poincaré mapping between the two sections.*
Then, T *is smooth in* $(\omega_0 \ln \omega_0, \omega_0, \xi_0)$*, that*

$$T(\omega_0, \xi_0) = (R(\omega_0 \ln \omega_0, \omega_0, \xi_0), X(\omega_0 \ln \omega_0, \omega_0, \xi_0)).$$

R, X *smooth.*

Idea : Straightening via a normal form + Blow up

Bounding the number of switchings for orbit transfers

Previous theorems apply to the CR3BP but not to the non-autonomous Restricted Elliptic Three Body Problem:

$$\ddot{q} + \nabla V_\mu(t, q) = u,$$

with $V_\mu(t, q) = \frac{1-\mu}{\|q-q^1(t)\|} + \frac{\mu}{\|q-q^2(t)\|}$. q^1, q^2 position vectors of the two primaries.

CR3BP and ER3BP

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Definition

Define $\delta = \inf_{[0,t_f]} |q(t)|$, $\delta_1 = \inf_{[0,t_f]} |q(t) - q^1(t)|$, $\delta_2 = \inf_{[0,t_f]} |q(t) - q^2(t)|$. *This quantities represents the distance to the collisions in the two body, and restricted three-body problems respectively. Finally note* $\delta_{12}(\mu) = \frac{\delta_1 \delta_2}{((1-\mu)\delta_2^3 + \mu \delta_1^3)^{1/3}}$.

Upper bound on the number of switchings → bounds the number of heteroclinic connections between S^s and S^u.

Sturm theorem

Theorem (Morse)

Let $a, b \in \mathbb{R}$, *with* $b > a$. *Consider the two linear second order equations*

$$z'' + P(t)z = 0, \tag{7}$$

$$z'' + Q(t)z = 0, \tag{8}$$

with $P(t), Q(t) \in S_n(\mathbb{R})$ *s. t.* $Q(t) - P(t) \geq 0$, *there exists* $\bar{t}$ *with* $Q(\bar{t}) - P(\bar{t}) > 0$. *If (7) has a non trivial solution* y, $y(a) = y(b) = 0$, *then (8) has a non trivial solution which vanishes in* a *and* $c < b$.

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Proposition (Caillau, Fejoz, O.)

- *Keplerian case, the maximum amount of switchings is* $N_0 = [\frac{t_f}{\pi\delta^{3/2}}]$ *on* $[0, t_f]$.
- *Controlled Elliptic Three-Body Problem with a mass ratio* μ, *On a time interval* $[0, t_f]$ *the maximum amount of such singularities is* $N_\mu = [\frac{t_f}{\pi\delta_{12}(\mu)^{3/2}}]$.

Idea

The controlled ER3BP dynamics:

$$H(q, v, p_q, p_v) = p_q.v - p_v.\nabla V_\mu(t, q) + p_v.u,$$

and

$$H^{max}(q, v, p_q, p_v) = p_q.v - p_v.\nabla V_\mu(t, q) + \|p_v\|.$$

Linear equation in p_v:

$$\ddot{p}_v + \nabla_q^2 V_\mu(t, q) p_v = 0. \tag{9}$$

Compare $V_\mu(t, q)$ with a well-chosen matrix.

$$A_t(q) = \begin{pmatrix} 1 + \frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} & 0 \\ 0 & \frac{1-\mu}{|q-q^1(t)|^3} + \frac{\mu}{|q-q^2(t)|^3} \end{pmatrix} \in \mathcal{S}_2(\mathbb{R}).$$

The general affine case

Nilpotent equilibrium, $\alpha = 0$ case

$$(\alpha = 0) \Rightarrow \begin{cases} \dot\rho = -\rho \sin\theta \\ \dot\theta = 1 - \cos\theta + O(\rho;\xi) \\ \dot\xi = \rho \tilde h(\rho, \theta, \xi) \end{cases} \qquad (10)$$

a_0 smooth on $\mathbb{R}^k$. Set $\tilde h_1(0,0,0) = c$.

Theorem (O., Roussarie)

Let $\bar z$ be in Σ_0. If $c > 0$, there exist extremals passing through $\bar z$, these extremals are connected to the singular flow in Σ_0.

Idea: Dimensional reduction + quasi-homogeneous blow up + dynamical study in the plane.

Sketch of the proof

Under generic hypothesis, ∃ coordinates $\tilde{\xi} = (\zeta, \tilde{\xi}_2 \dots, \tilde{\xi}_k)$ s.t.

$$\begin{cases} \rho' = -\rho\theta + O(\rho\theta^3) \\ \theta' = \zeta + \theta^2/2 + O(\rho + |\theta|^4) \\ \zeta' = c\rho + \rho O(\rho + |\theta| + |\tilde{\xi}|). \end{cases} \tag{11}$$

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Blow up. Nilpotent equilibrium $(\rho, s, \zeta) = (0, 0, 0)$, we will use a specific blow-up:

$$\begin{cases} \rho = R^3\bar{\rho} \\ \theta = R\bar{\theta} \\ \zeta = R^2\bar{\zeta} \end{cases}$$

with $(\bar{\rho}, \bar{s}, \bar{\zeta}) \in S^2_+$ the *hemisphere* $\rho \geq 0$, $R \in \mathbb{R}_+$.

Sketch of the proof

The chart (i). Interior of S^2_+: One hyperbolic equilibrium m_0, unstable node restricted to S^2_+.

The chart (ii). *4 semi-hyperbolic equilibria* $\in \partial S^2_+ = \{R = 0, \bar{\rho} = 0\} \cong \mathbb{S}^1$.

- $-\pi/2$: unstable node in restriction to S^2_+
- $\pi/2$: stable node in restriction to S^2_+
- $\omega_0 \in]\pi/2, \pi[$ (unstable) and $-\omega_0$ (stable) nodes restricted to $\mathbb{S}^1$.

$\pm\omega_0$: hyperbolic restricted to S^2_+, with lines of zeros in the plane $\bar{\rho} = 0$.

Local-global portrait on S^2_+

The stable and unstable manifolds of these equilibria can be connected using

Theorem (Poincaré-Bendixson)

Let X be a vector field in the plane, any maximal solution of $\dot{x} = X(x)$ contained in a compact set, is either converging to an equilibrium point or a limit cycle.

No periodic orbit: choosing a transverse domain containing m_0 and using Poincaré-Hopf formula.
→ stable and unstable manifolds can be connected in a unique way: the stable manifold from ω_0 is connected to m_0.
→ Finite initial time.

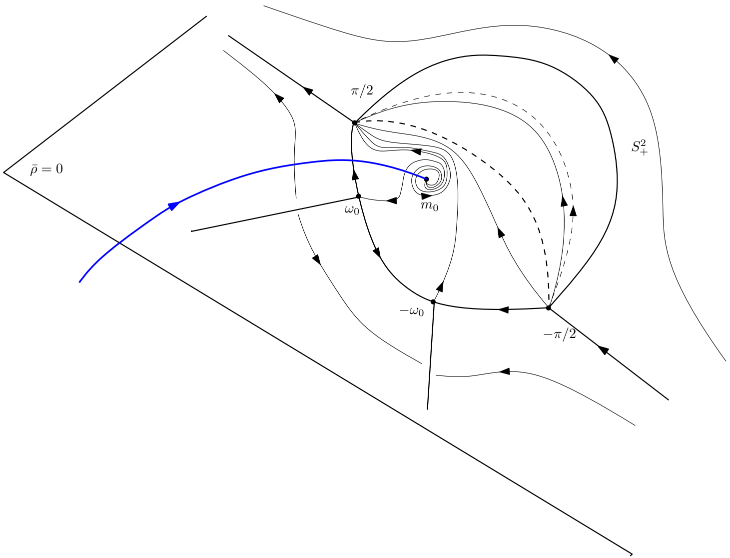

Figure: Phase portrait around the nilpotent equilibrium

Stratification of the flow

Theorem (O., Roussarie)

In a neighborhood $O_{\bar{z}}$ of a point $\bar{z} \in \Sigma_0$, the flow is well defined, continuous, and piecewise smooth. More precisely, there exists a stratification:

$$O_{\bar{z}} = S_0 \cup S^s \cup S^u \cup S_0^s$$

where

- S_0^s *is the submanifold of codimension 2 of initial conditions leading to* Σ_0*,*
- S^s *is the submanifold of codimension 1 of initial conditions leading to* Σ_-*,*
- $S_0 = O_{\bar{z}} \setminus (S_1^0 \cup S_1)$*.*

The extremal flow is smooth on each stratum.

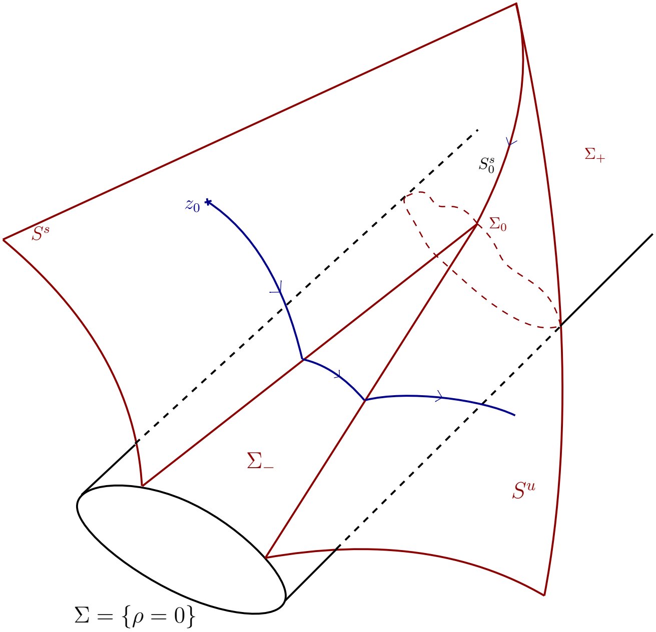

Figure: Stratification of the extremal flow around Σ_0

Σ_+ and the singular flow

The case Σ_+.

Proposition

No switching at Σ_+. In a neighborhood of a point $\bar{z}$ in Σ_+, there is no switch, and the extremal flow is smooth, i.e., Σ_+ is never crossed. In other words, ρ does not vanish in (5).

The singular flow (flow inside Σ):

- There exists a smooth singular flow inside Σ_+, which cannot be optimal (Goh condition).
- There cannot exist any admissible singular extremal in Σ_-.
- There exists a singular flow in Σ_0.

Jump on the control

$r := \sqrt{H_{01}^2 + H_{02}^2}$

Remark

In the case $\bar{z} \in \Sigma_-$,the jump on the control at a switching time $\bar{t}$ is:

$$u(\bar{t}_\pm) = (\cos\theta_\pm, \sin\theta_\pm) = \frac{1}{r^2}(-H_{02}H_{12} \pm H_{01}\sqrt{r^2 - H_{12}^2}, H_{01}H_{12} \pm H_{02}\sqrt{r^2 - H_{12}^2}).$$

$(\mathcal{B}) \Rightarrow \pi$-singularities for mechanical systems.

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($\mathcal{B}$) $\Rightarrow$ π-singularities for mechanical systems.

Proposition

In Σ_0. Consider the extremal $z(t)$ entering the singular locus in $z(\bar{t}) = \bar{z} \in \Sigma_0$,

- *If $H_{12}(\bar{z}) = r(\bar{z})$, the extremal control is continuous on $[0, t_f]$,*
- *If $H_{12} = -r(\bar{z})$, the extremal control has a π-singularity at the switching time $\bar{t}$.*

Optimality of minimum time extremals

Exponential mapping

Definition (exponential map)

We call exponential mapping from x_0, *the map*

$$\exp_{x_0} : (t, p_0) \in [0, t_f] \times T^*_{\bar{x}_0}M \cap S^s \to \pi(z(t, x_0, p_0)) = x(t, x_0, p_0) \in M$$

Assumption: $T^*_{x_0}M \pitchfork S^s$, then $T^*_{x_0}M \cap S^s$ is a smooth submanifold of dimension 3.

Theorem (O.)

Denote $M(t) := d\exp_{\bar{x}_0}(t, \bar{p}_0)$. *If*

(i) *The reference extremal is normal,*

(ii) $\det M(t) \neq 0$ *for all* $t \in]0, \bar{t}[\cup]\bar{t}, \bar{t}_f]$ *and* $\det M(\bar{t}_-) \det M(\bar{t}_+) \neq 0$,

then the reference trajectory is a $\mathcal{C}^0$*-local minimizer among all trajectories with same endpoints.*

Propagate a Lagrangian perturbation

$\exists$ Lagrangian submanifold $\mathcal{L}$ transverse to $T^*_{x_0}M$, s. t. $\mathcal{S}_0 = \mathcal{L} \cap S^s$ is a smooth submanifold of dimension 3 $\rightarrow$ *Regularity* on $\mathcal{S}_0$.

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The canonical projection π is a **homeomorphism** on

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onto its image. The same holds for

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and $\mathcal{S}_1 \cup \mathcal{S}_2$ (extremals cut Σ transversally).
→ Cost comparison using the Liouville form $\lambda = p dx$ exact on $\mathcal{S}_i$: Extremals are locally optimal.

Integrability of minimum time Hamiltonian in the Kepler problem

Kepler problem

Definition (Liouville integrability)

Let H be a smooth function on a $2n$-dimensional symplectic manifold. The associated Hamiltonian system is integrable iff there exists n independent first integrals (constant of motion) in involution.

Classical reduction of the two body problem $\mu = 0$,

$$\ddot{q} + \frac{q}{||q||^3} = \varepsilon u.$$

Uncontrolled two body problem ($\varepsilon = 0$) is well known to be integrable.

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Three body problem is **not integrable** (Poincaré).

Minimum time Kepler problem

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Maximized Hamiltonian:

$$H(q, v, p_q, p_v) = p_q.v - \frac{p_v.q}{\|q\|^3} + \|p_v\|. \tag{15}$$

→ **Liouville integrability** of H ?

Theorem (Morales-Ramis)

Let us consider a Hamiltonian H *analytic on a complex analytic symplectic manifold and a particular solution* Γ *not reduced to a point. If* H *is integrable in the Liouville sense with meromorphic first integrals, then the first order variational equation near* Γ *has a virtually Abelian Galois group over the base field of meromorphic functions on* Γ*.*

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Theorem (Caillau, Combot, Fejoz, O.)

The minimum time Kepler problem is not meromorphically Liouville integrable on $\mathcal{M}$*.*

Strategy

Invariant manifold of collisions

$$\color{red}{S} = \{q_2 = v_2 = p_{q_2} = p_{v_2} = 0\} \cap \mathcal{M}$$

→ H is integrable on S: collision trajectory $\Gamma(t)$.
→ Compute the Normal Variational Equation along $\Gamma(t)$.
→ Its Galois group contains the group of a hypergeometric equation: Contains $SL_2(\mathbb{C})$, not even solvable.
→ The variational equation is Fuchsian: Non-integrability in the class of meromorphic functions (Schlesinger's density theorem).

Perspectives

Perspectives

- Find a generic condition to treat the general case.
- Numerical experimentation for the extremal.
- Real non integrability for the minimum time Kepler problem.
- KAM theory: non geodesic convexity in the Kepler configuration?

Thank you for
your attention !