Homoclinic Orbits in Families of Hypersurfaces with hyperbolic Periodic Orbits.

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Introduction

Variational methods provide interesting existence results on homoclinic orbits to hyperbolic fixed points of Hamiltonian systems under global conditions. The early result of Bolotin [3] about Lagrangian flows has been extended to Hamiltonian systems in $\mathbb{C}^n$ in [6], the hypothesis has been weakened in [10] and [15], and finally in [13] for autonomous systems. A natural generalization is the existence of homoclinic orbits to hyperbolic periodic motions of autonomous Hamiltonian systems. Very interesting results have been obtained by Bolotin in [4] and other papers, for Lagrangian systems on compact Riemannian manifolds, but there are no global results available for systems in $\mathbb{C}^n$ where the lack of topology makes Bolotin's methods inefficient. This paper is a first attempt in that direction.

A periodic motion of an autonomous Hamiltonian system always has at least two Floquet multipliers equal to 1. As a consequence it cannot be hyperbolic in the whole phase space, but only with respect to its energy shell, and it is not isolated, but included in a 1-parameter family of periodic motions, one motion on each energy shell. The union of the orbits of the family is an invariant two dimensional manifold, we call it the center manifold. It is normally hyperbolic in phase space and for that reason it is an easier problem to look for orbits homoclinic to that manifold than to look for orbits homoclinic to a prescribed periodic motion. An orbit homoclinic to the center manifold is homoclinic to one of the periodic motions, by energy conservation.

We study a model class of systems in $\mathbb{C}^n$ where the center manifold is a plane with harmonic oscillations on it. This situation is however quite general, as is explained in [2]. We prove that the periodic orbits having a homoclinic orbit are dense in the center manifold outside of a compat set. We obtain the homoclinics as accumulation points of sequences of periodic orbits. These periodic orbits are subharmonics of perturbed systems. Convergence of subharmonics has already been used to find homoclinics to hyperbolic fixed points, see [15].

One of the main features of homoclinic orbits is that they induce chaotic behavior. Indeed, it is well-known that a Bernoulli shift with positive entropy exists in periodically time-dependant systems containing a transverse homoclinic to a hyperbolic fixed point. This structure also exists in autonomous systems containing a hyperbolic orbit with a transversal homoclinic. It should be noted however that the orbit structure associated with a transversal homoclinic orbit to a hyperbolic fixed point of an autonomous system is not as well understood. It is chaotic in certain instances, see [7] or [5], but it can also be integrable. This is one of the reasons why we believe it is important to find some global existence results on homoclinic orbits to periodic orbits. We obtain classes of autonomous systems in $\mathbb{C}^n$ with this structure at many energy levels. We can study for example couplings between stable and unstable systems, and obtain large chaotic regions at high energy.

1 Results and examples

In the following, C will always be a positive constant, possibly different from one line to the other. The L^p norm of f will be noted $\|f\|_p$. We will often use technical results from [15] without proof. Let us define

$$J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; J_{2n} = \begin{bmatrix} J_2 & 0 & \cdots & 0 \\ 0 & J_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & J_2 \end{bmatrix},$$

and the associated symplectic form Ω_{2n} on $\mathbb{R}^{2n}$:

$$\Omega_{2n}(X, Y) = \langle J_{2n} X, Y \rangle.$$

We will omit the subscript $2n$. There is a splitting

$$(\mathbb{R}^{2n}, \Omega) = (\mathbb{R}^2, \Omega) \oplus (\mathbb{R}^{2n-2}, \Omega),$$

the subspaces $\mathbb{R}^2$ and $\mathbb{R}^{2n-2}$ are Ω-orthogonal and symplectic.

1.1 Main result

We consider the Hamiltonian system

$$\dot{X} = J \nabla H(X)$$

associated to the autonomous Hamiltonian

$$H(X) = H(x, z) = \frac{1}{2}\omega |x|^2 + \frac{1}{2}\langle Az, z\rangle + W(x, z), \tag{1}$$

$$X = (x, z) \in \mathbb{R}^2 \times \mathbb{R}^{2n-2},$$

where the pulsation ω is a positive number,

HA A is a $(2n-2) \times (2n-2)$ real symmetric matrix such that

$$\sigma(JA) \cap i\mathbb{R} = \emptyset,$$

and W is a C^2 function satisfying:

HW1 there is a $\alpha > 2$ and a continous function $C : \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ such that $W(x, z) \leqslant C(x)|z|^\alpha$ and $\nabla_z W(x, z) \leqslant C(x)|z|^{\alpha-1}$ in a neighborhood of $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^2 \times \mathbb{R}^{2n-2}$,

HW2 there is a $\mu \in (2, \alpha]$ such that

$$\mu W(X) \leqslant \langle \nabla W(X), X \rangle,$$

HW3 there is $B > 0$ such that

$$B|z|^\alpha \leqslant W(x,z).$$

We will introduce in the proof auxiliary systems satisfying

HW4 there exists a compact set outside of which

$$W(x,z) = a|z|^\alpha.$$

We obtain the useful inequalities

$$W(x,z) \leqslant C|z|^\alpha \ \ , |\nabla_z W(x,z)| \leqslant C|z|^{\alpha-1} \tag{2}$$

from [HW1] and [HW4]. The hypotheses [HA] and [HW1-3] are satisfied for example by the Hamiltonian

$$H(x_1,x_2,z_1,z_2) = x_1^2 + x_2^2 + z_1^2 - z_2^2 + \left(1+x_1^2\right)\left(z_1^2+z_2^2\right)^2.$$

More examples are given below.

A system satisfying [HA] and [HW1] has a two dimensional invariant space $\mathbb{R}^2 \times \{0\}$ which is foliated by periodic orbits O_r with equation

$$O_r(t) = (e^{J\omega t}(r,0),0)$$

all having the same period $T_0 = 2\pi/\omega$. The orbit O_r has energy $H = \omega r^2/2$ and is hyperbolic with respect to its energy shell. It has $n-1$ dimensional stable and unstable manifolds which in the $2n-1$ dimensional energy shell may intersect along a homoclinic orbit. In this paper we study this phenomenon, and prove:

Theorem 1 *Let us consider the Hamiltonian system (1) satisfying [HA], [HW1-3]. Let*

$$\mathcal{R} = \{r > 0 \text{ such that } O_r \text{ has a homoclinic orbit}\}.$$

There is a positive number M depending only on A, B and α such that

$$\left[\sqrt{\frac{M}{\pi}}, \infty\right) \subset \overline{\mathcal{R}},$$

where $\overline{\mathcal{R}}$ is the closure of $\mathcal{R}$.

Remarks :

1. There is an estimate for M, see (21) in the proof, which is enough to obtain that for fixed A and α
$$\lim_{B \longrightarrow \infty} M = 0.$$
2. It would be useful to obtain a more explicit estimate for M. We focus on a similar question in [1]. The setting is different and allows a better understanding of the constants. On the other hand, we obtain here infinitely many orbits while only one is obtained in [1].
3. We do not know whether $\overline{\mathcal{R}} = \mathbb{R}$. Since the origin does not have any homoclinic in general, it is not surprising that we can not find easily homoclinics close to the origin, but they may well exist.
4. The result cannot be improved to $[C,\infty) \subset \mathcal{R}$ without additional assumption, see example below.

1.2 Coupling stable and unstable systems

Let us consider the unstable system in $\mathbb{R}^2$ associated to the Hamiltonian

$$G(z) = \frac{1}{2}\langle Az, z\rangle + R(z),$$

where the matrix A satisfies [HA] and the nonlinearity R is superquadratic:

$$R(z) = o(|z|^2) \text{ near } 0,$$

$$R(z) \geqslant C|z|^\alpha \text{ with } \alpha > 2,$$
$$\langle \nabla R(z), z\rangle \geqslant \mu R(z), \text{ with } \mu > 2.$$

The origin is a hyperbolic fixed point and has a homoclinic orbit. It is well-known from Melnikov theory that a generic time-dependent perturbation creates transversal homoclinic orbits, which implies a chaotic behavior with positive topological entropy. A new way to introduce a chaotic behavior is to couple the system with a harmonic oscillator. Consider the system in $\mathbb{R}^4$ associated to the Hamiltonian

$$H(x, z) = |x|^2 + \frac{1}{2}\langle Az, z\rangle + (1 + F(x))G(z),$$

with a positive function F such that $\langle \nabla F(x), x\rangle \geqslant 0$. We can apply theorem 1 to that system, this provides homoclinics to many of the periodic motions $z = 0$ at high energy. By a small perturbation, these homoclinics can be made transversal, and then induce chaotic behavior in fast regions of phase space, that is in regions that contain no rest point.

1.3 Hypersurfaces of $\mathbb{R}^{2n}$

We now interpret our result in terms of hypersurfaces of $\mathbb{R}^{2n}$. Let Σ be a compact starshaped (with respect to the origin) hypersurface of $\mathbb{R}^{2n}$, let U_Σ be the bounded connected component of $\mathbb{R}^{2n} - \Sigma$, the notation $\Sigma' \preccurlyeq \Sigma$ means that $\Sigma' \subset \overline{U}_\Sigma$. It is well-known that a hypersurface carries a canonical direction field $D(x)$ satisfying

$$J\nabla H(x) \in D(x) \quad \forall x \in \Sigma$$

for any function H having Σ as a regular level hypersurface. Let us fix a matrix A satisfying [HA], for any $B > 0$ and $\alpha > 2$, we define the compact hypersurface

$$\Sigma(B, \alpha) = \left\{(x, z) \text{ such that } |x|^2 + \frac{1}{2}\langle Az, z\rangle + B|z|^\alpha = 1\right\}.$$

Let Σ be a starshaped hypersurface of $\mathbb{R}^{2n}$ such that there exist $0 < B \leqslant D$ and α with

$$\Sigma(D, \alpha) \preccurlyeq \Sigma \preccurlyeq \Sigma(B, \alpha),$$

it is not hard to see that $S = \Sigma \cap \{z = 0\}$ is an invariant circle of the canonical direction field. We can define the function

$$R(x, z) = 1 - |x|^2 - \frac{1}{2}\langle Az, z\rangle \geqslant B|z|^\alpha$$

on Σ. Since Σ is starshaped there is an α-homogeneous function $W : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ extending R. The surface Σ is then the regular level $H = 1$ of the Hamiltonian

$$H(x,z) = |x|^2 + \frac{1}{2}\langle Az, z\rangle + W(x,z),$$

and [HW2] holds. For any $(x,z) \in \mathbb{R}^{2n} - 0$, there is a $t > 0$ such that $(x/t, z/t) \in \Sigma$. We check [HW3] writing

$$W(x,z) = t^\alpha W(x/t, z/t) = t^\alpha R(x/t, z/t) \geqslant t^\alpha B\, |z/t|^\alpha \geqslant B|z|^\alpha,$$

[HW1] is also easily seen to hold. We can apply theorem 1 (with remark 1) to obtain:
theorem 1' *If $B \geqslant B_0$ there is a sequence $l_n \longrightarrow 1$ such that the hypersurface*

$$\Sigma_n = \big\{H = l_n\big\}$$

carries an orbit homoclinic to the periodic hyperbolic trajectory $\Sigma_n \cap \{z = 0\}$, where B_0 is a constant depending only on A and α.

Some comments may be useful. The main limitation of this result is that we do not obtain the existence of an orbit homoclinic to the prescribed closed invariant curve on the prescribed energy shell. It would be very interesting to find hypotheses implying such a conclusion. Our hypotheses are not sufficient, see example below. We can see our result in the following way: We give a constructive method to perturb smoothly the prescribed energy shell in order to create a homoclinic orbit (the hypersurfaces Σ_n are clearly converging in the C^∞ topology to the hypersurface Σ). For comparison, let us mention that, using local perturbation techniques of Hayashi, Xia has proved in a much more general setting that a homoclinic orbit can be created by a C^1 small perturbation of the hypersurface, ([9],[16]). As usual with these kinds of results, improving from the existence of a C^1-small perturbation to the existence of a C^∞-small perturbation is very hard and requires strong additional hypotheses, such as the ones we assume.

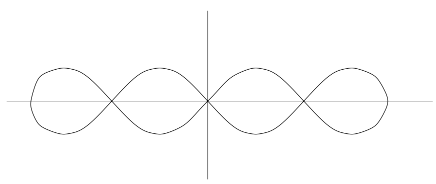
1.4 Example

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth function satisfying

$$z_2^2 - z_1^2 + C|z|^4 \leqslant F(z_1, z_2) \leqslant z_2^2 - z_1^2 + D|z|^4 \;\; \forall z,$$

$$F(z) = |z|^4 \;\text{ outside of a compact set},$$

the zero level of F having the shape shown in the figure below.



Consider the function

$$F_\lambda(z) = \frac{1}{\lambda^2} F(\lambda z),$$

the Hamiltonian $H_\lambda : \mathbb{R}^4 \longrightarrow \mathbb{R}$

$$H_\lambda(x,z) = |x|^2 + F_\lambda(z)$$

and the surface

$$\Sigma_\lambda = \Big\{ H_\lambda(x,z) = 1 \Big\}.$$

The vector field associated to H_λ has a product structure, its trajectories satisfy $\dot{z} = J\nabla F_\lambda(z)$. The origin is a hyperbolic rest point for this equation, but its stable and unstable manifolds are heteroclinic orbits connecting this fixed point to the two other ones and not homoclinic orbits. It follows that $\Sigma_\lambda \cap \{z = 0\}$ has no homoclinic orbit for the vector field, and thus no homoclinic either for the canonical direction field. Yet we now prove that for λ large enough it satisfies all hypotheses of theorem 1' with $\alpha = 4$. We first note that

$$F_\lambda = \frac{1}{2}\langle Az, z\rangle + O(|z|^4),$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

satisfies [HA]. To prove that Σ_λ is starshaped for λ large enough we observe that

$$\langle \nabla H_\lambda(x,z), (x,z)\rangle = 2|x|^2 + \langle \nabla F_\lambda(z), z\rangle = \langle \nabla F_\lambda(z), z\rangle - 2F_\lambda(z) + 2$$

on Σ_λ since $|x|^2 + F_\lambda(z) = 1$. This gives

$$\langle \nabla H_\lambda(x,z), (x,z)\rangle = \frac{1}{\lambda^2}\langle \nabla F(\lambda z), \lambda z\rangle - \frac{2}{\lambda^2}F(\lambda z) + 2 = \frac{1}{\lambda^2}\Big(\langle \nabla F(y), y\rangle - 2F(y) + 2\lambda^2\Big).$$

This is positive when λ is large enough, because $\langle \nabla F(y), y\rangle - 2F(y)$ has a lower bound, the surface is thus starshaped in this case.

There remains to estimate

$$W_\lambda(x,z) = F_\lambda(z) - z_2^2 + z_1^2$$

on Σ_λ. From

$$C|z|^4 \leqslant W(x,z) \leqslant D|z|^4$$

we get

$$\lambda^2 C|z|^4 \leqslant W_\lambda(x,z) \leqslant \lambda^2 D|z|^4$$

and thus the condition

$$D|z|^4 \geqslant W_\lambda(x,z) \geqslant B_0|z|^4 \ \Rightarrow \Sigma(D,4) \preccurlyeq \Sigma_\lambda \preccurlyeq \Sigma(B_0,4)$$

is satisfied for λ large. $\square$

2 Convergence of periodic orbits

We prove theorem 1 in the sequel of this paper. We obtain the homoclinic orbits as limits of sequences of periodic orbits of H. It is useful to define the action of a T-periodic C^1 loop:

$$I_T(X) = \int_0^T \frac{1}{2}\langle JX(t), \dot{X}(t)\rangle - H(X(t))\, dt.$$

We have the following existence result, that will be proved in section 3.

Theorem 2 *There is a constant M depending only on A, B and α such that for any*

$$R_0 \geqslant \sqrt{M/\pi}\ , H_0 = \frac{1}{2}\omega R_0^2$$

and any $\epsilon > 0$ there is a $N(\epsilon) > 0$ and a sequence X_k of T_k-periodic orbits satisfying

$$T_k \longrightarrow \infty, \tag{3}$$

$$0 \leqslant I_{T_k}(X_k) \leqslant N(\epsilon), \tag{4}$$

$$|H(X_k) - H_0| \leqslant \epsilon, \tag{5}$$

$$z_k \not\equiv 0. \tag{6}$$

That N has to depend of ϵ in this lemma is what makes it impossible to obtain a homoclinic orbit on a given energy surface: we can not control in the same time the closeness and the action. We now prove that theorem 2 implies theorem 1, that is we study the convergence of the sequence $X_k = (x_k, z_k)$ obtained by theorem 2.

Lemma 1 *The sequences $\|z_k\|_\alpha$ and $\|X_k\|_{C^1}$ are bounded.*

Proof: Since the function H is proper, it follows from (5) that $\|X_k\|_\infty$ is bounded, as well as $\|X_k\|_{C^1}$ since X_k satisfies the equation

$$\dot{X}_k = J\nabla H(X_k).$$

To prove the first part of the lemma, let us write (4) and use [HW2,3]:

$$\begin{aligned} N \geqslant I(X_k) &= \int_0^{T_k} \frac{1}{2}\langle \nabla H(X_k), X_k\rangle - H(X_k)\, dt \\ &= \int_0^{T_k} \frac{1}{2}\langle \nabla W(X_k), X_k\rangle - W(X_k)\, dt \\ &\geqslant \int_0^{T_k} \left(\frac{\mu}{2} - 1\right) W(X_k)\, dt \\ &\geqslant B\left(\frac{\mu}{2} - 1\right) \|z_k\|_\alpha^\alpha. \end{aligned}$$

□

We are now in a position to use Ascoli's theorem to obtain a limit. Yet we first have to insure non triviality of the limit. It will result from

Lemma 2 *There is a $\delta > 0$ such that any periodic orbit of H staying in*

$$V_\delta = \{|z| \leqslant \delta\} \cap \{H \leqslant H_0 + 1\}$$

must satisfy $z \equiv 0$.

Proof: This lemma is a consequence of the fact that $z = 0$ is a normally hyperbolic manifold for H. To be more precise, let F_s and F_u be the stable and unstable spaces of JA, $\mathbb{R}^{2n-2} = F_s \oplus F_u$ by [HA]. We denote the projections by

$$P_s : \mathbb{R}^{2n-2} \longrightarrow F_s \text{ and } P_u : \mathbb{R}^{2n-2} \longrightarrow F_u.$$

There are Euclidean structures $|.|_s$ on F_s and $|.|_u$ on F_u and a $\lambda > 0$ such that $\langle JAz, z\rangle_s \leqslant -\lambda |z|_s^2$ when $z \in F_s$ and $\langle JAz, z\rangle_u \geqslant \lambda |z|_u^2$ when $z \in F_u$. From [HW1], we obtain a $\delta > 0$ such that

$$\langle P_s J\nabla H_l(x,z), P_s(z)\rangle_s \leqslant -\frac{\lambda}{2}|P_s(z)|^2$$

and

$$\langle P_u J\nabla H_l(x,z), P_u(z)\rangle_u \geqslant \frac{\lambda}{2}|P_u(z)|^2$$

when $|z| \leqslant \delta$ and $H \leqslant H_0 + 1$. It follows that if $X(t) = (x(t), z_s(t) + z_u(t))$ is a solution of the Hamiltonian equation lying in V_δ, $|z_u|_u$ is increasing or 0, and $|z_s|_s$ is decreasing or 0, thus the solution can not be periodic unless $z \equiv 0$. □

Since the equation is autonomous, we can change the time origin of X_k to obtain

$$z_k(0) \geqslant \delta/2.$$

For any fixed τ, the sequence $X_k|_{[-\tau,\tau]}$ has a uniform limit (up to taking a subsequence) and by diagonal extraction we can find a subsequence of X_k converging pointwise and uniformly on any compact set to a limit X_∞ satisfying

$$\dot{X}_\infty = J\nabla H(X_\infty).$$

We also see using Fatou's lemma that $\|z_\infty\|_\alpha$ is finite and since $\dot{z}_\infty$ is bounded,

$$\begin{aligned} &z_\infty(t) \longrightarrow 0 \text{ as } t \longrightarrow \pm\infty, \\ &z_\infty(0) \geqslant \delta/2. \end{aligned}$$

The energy $H_\infty = H(X_\infty)$ satisfies

$$H_\infty \in [H_0 - \epsilon, H_0 + \epsilon]$$

because of (5), and the equation

$$\frac{1}{2}\omega|x_\infty|^2 - H_\infty = -\frac{1}{2}\langle Az_\infty, z_\infty\rangle - W(x_\infty, z_\infty)$$

implies that $x_\infty(t)$ must go to

$$r = \sqrt{\frac{2H_\infty}{\omega}}$$

when $z_\infty(t)$ goes to 0. Thus the trajectory X_∞ is homoclinic to $\{z = 0\} \cap \{H = H_\infty\}$. This proves theorem 1, since $\epsilon > 0$ can be chosen as small as needed. □

3 Existence of periodic orbits

We prove theorem 2 in this section using variational methods. Let us fix a radius R_0 and the associated energy $H_0 = \omega R_0^2/2$. The functional I does not satisfy PS condition because the oscillations on the center manifold form a non-compact family of critical points of zero action. Moreover, we have to find a way to specify around which energy surface we are working. For these reasons, it will be useful to introduce a perturbation that will turn PS condition on, and that will confine critical points around the fixed energy surface.

Before we perturb the system, let us notice that since we are looking for phenomena taking place around a fixed energy surface, it is harmless to change the Hamiltonian at infinity. We use this remark following a well-known trick, see [12] for example. Let $K > 0$ be a large number, let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be a smooth increasing function such that $\chi(x) = 0$ for $x \leqslant K$, $\chi(x) = 1$ if $x \geqslant K + 1$ and $\chi' \leqslant 2$. We introduce the function

$$\tilde{W}(X) = (1 - \chi(|X|))W(X) + \chi(|X|)a|z|^\alpha$$

where

$$a = \max_{K \leqslant |X| \leqslant K+1} \frac{W(X)}{|z|^\alpha}.$$

It is not hard to check that this function satisfies [HW1-3], with the same constants. If K is large enough the hamiltonian have not been changed for $H(X) \leqslant H_0 + 1$ and it is the same to prove theorem 2 for $\tilde{W}$ or for W. In the following we will work with $\tilde{W}$ instead of W, but for simplicity we will still call it W, that is we will suppose that [HW4] holds. We are now in a position to introduce the perturbed hamiltonian we are going to study.

Let us take a function

$$\begin{array}{rcl} S : \mathbb{R}^{2n} & \longrightarrow & \mathbb{R} \\ (x, z) & \longmapsto & (H(x, z) - H_0)^4 \text{ when } H(x, z) \leqslant H_0 + 1, \\ (x, z) & \longmapsto & C(|x|^3 + |z|^\alpha) \text{ outside of a compact set.} \end{array}$$

We moreover assume that

$$H \geqslant H_0 + 1 \Rightarrow S \geqslant 1$$

and that there is a smooth and convex function f such that

$$S(x, 0) = f(|x|^2).$$

It is not hard to see that the above class of functions is not empty. Note that there exists a constant C such that for all $(x, z) \in \mathbb{R}^2 \times \mathbb{R}^{2n-2}$,

$$S(x, z) \leqslant C(H(x, z) - H_0))^4. \tag{7}$$

We consider the Hamiltonian

$$H_l(x, z) = H(x, z) + lS(x, z).$$

where l will always be chosen small enough so that the equation $H_l = E + l(E - H_0)^4$ has only one solution $E(H_l) \geqslant \min H$. The shell $H_l = h_l$ of H_l is the shell $H = E(h_l)$ of H when

$h_l \leqslant H_0 + 1$ thus the local structure of the flow has not been changed by the perturbation in this region, where there holds

$$\nabla H_l = (1 + 4l(H - H_0)^3)\nabla H. \tag{8}$$

Although H and H_l have the same periodic solutions in the region under interest, we will look for T-periodic trajectories of H_l, that are easier to be found as critical points of

$$I_l(x,z) \quad = \quad \int_0^T \langle -\frac{1}{2}J\dot{X}, X\rangle - H_l(X)\,dt$$

on a suitable function space. We will prove the following proposition, that leads to theorem 2.

Proposition 1 *There exists a constant M depending only on A, B and α, such that if*

$$\pi R_0^2 \geqslant M$$

there holds: For any $\Delta > 0$ and any

$$T \in \frac{2\pi}{\omega}\mathbb{N} \cap [1, \infty)$$

there exists $l(T)$ in the interval $(0, \Delta/T)$ and a T-periodic trajectory (x_T, z_T) of $H_{l(T)}$ such that

$$0 < I_{l(T)}(x_T, z_T) \leqslant M, \tag{9}$$

$$\int_0^T S(x_T, z_T)\,dt \leqslant \frac{TM}{\Delta} + 1, \tag{10}$$

$$z_T \not\equiv 0. \tag{11}$$

Before we prove this proposition, let us see that it implies theorem 2. Set

$$h_T = H_{l(T)}(X_T).$$

If Δ has been chosen large enough, (10) implies that S must take a value below one when T is large enough, thus X_T is contained in $H \leqslant H_0 + 1$ and has a fixed energy $E_T = H(X_T)$. We apply (10) once again and get

$$|E_T - H_0| \leqslant \left(2\frac{M}{\Delta}\right)^{\frac{1}{4}}$$

and

$$0 \leqslant h_T - E_T = l(E_T - H_0)^4 \leqslant 2\frac{M}{T}$$

when T is large enough. Let us now define the curve

$$\tilde{X}_T(t) = X_T\Big(\big(1 + 4l(E_T - H_0)^3\big)^{-1}t\Big),$$

it comes directly from (8) that $\tilde{X}_T$ is a trajectory of H, the period of which

$$\tilde{T} = \big(1 + 4l(E_T - H_0)^3\big)T$$

satisfies

$$\tilde{T} \geqslant T + 4l(E_T - H_0)^3 T \geqslant T - 8M^{\frac{3}{4}}\Delta^{\frac{1}{4}}.$$

We can estimate its action

$$I(\tilde{X}_T) = I_{l(T)}(X_T) + Th_T - \tilde{T}E_T = I_{l(T)}(X_T) + T(h_T - E_T) + (T - \tilde{T})E_T$$

and obtain

$$I(\tilde{X}_T) \leqslant 3M + 8M^{\frac{3}{4}}\Delta^{\frac{1}{4}}(H_0 + 1). \tag{12}$$

In addition, we have $I(\tilde{X}_T) \geqslant 0$ since $\tilde{X}_T$ is a trajectory of H. The sequence $\tilde{X}_T$ satisfies all the conclusions of theorem 2 which is finally proved. We remark that Δ appears in this estimate, so that we must fix it before passing to the limit, and that's why we can't reach the surface $H = H_0$ itself. □

We now have to prove proposition 1. Let us fix a period $T = \tau 2\pi/\omega$, $\tau \in \mathbb{N}$, and define the following functionals on smooth T-periodic arcs:

$$\begin{aligned}
e(x(t)) &= \int_0^T -\frac{1}{2}\langle J\dot{x}(t) + \omega x(t), x(t)\rangle dt, && (13)\\
h(z(t)) &= \int_0^T -\frac{1}{2}\langle J\dot{z}(t) + Az(t), z(t)\rangle dt, && (14)\\
b(x(t), z(t)) &= \int_0^T W(x(t), z(t))dt, && (15)\\
p(x(t)) &= \int_0^T S(x(t), z(t))dt. && (16)
\end{aligned}$$

We are going to obtain T-periodic orbits of H_l as critical points of

$$I_l(x(t), z(t)) = e(x(t)) + h(z(t)) - b(x(t), z(t)) - lp(x(t), z(t)).$$

The proof of proposition 1 goes along the following line. We first take a good function space on which the above functional can be studied. We see that this functional has a "universal" linking structure, this allows us to define a critical level $c_T(l)$ which is a nonincreasing function of l. It will appear from the construction that $0 < c_T(l) \leqslant M$ for a constant M independent of l and T, this is (9). Since l is allowed to take values in the interval $(0, \Delta/T)$, there must be a l such that $c_T'(l)$ exists and $|c_T'(l)| \leqslant MT/\Delta$. Using Struwe's monotony method (see [14], II.9), it can be deduced that there is a critical point X at level $c_T(l)$ with

$$p(x(t)) = \left|\frac{\partial}{\partial l}H_l(X)\right| \leqslant 1 + MT/\Delta,$$

this is (10). To obtain (11) we just have to check that no T-periodic solution of H_l on $\mathbb{R}^2 \times \{0\}$ has its action in $(0, M]$ if $\pi R_0^2 \geqslant M$. Let us start with this program.

3.1 The analytical setting

We use Fourier series

$$x(t) = \sum_{k\in\mathbb{Z}} e^{Jk\omega t/\tau} x_k,\ x_k \in \mathbb{R}^2$$

to compute e:

$$e\left(\sum_{k\in\mathbb{Z}} e^{Jk\omega t/\tau} x_k\right) = \sum_{k\in\mathbb{Z}} \pi\,(k-\tau)\,|x_k|^2.$$

We define the inner product

$$\langle x, y\rangle_e = 2\pi\langle x_\tau, y_\tau\rangle + \sum_{k\in\mathbb{Z}} 2\pi\,|k-\tau|\,\langle x_k, y_k\rangle,$$

its associated norm $\|x\|_e^2 = \langle x, x\rangle_e$ and the space

$$E_e = \left\{x \in L^2(0,T;\mathbb{R}^2) \text{ such that } \|x\|_e < \infty\right\}.$$

It is classical that e can be extended to E_e as a continuous quadratic form, and there is an orthogonal splitting

$$E_e = E_e^+ \oplus E_e^0 \oplus E_e^-,$$

with

$$\begin{aligned} E_e^+ &= \{x \text{ such that } x_k = 0 \text{ for k } \leqslant \tau\}, \\ E_e^0 &= \{x \text{ such that } x_k = 0 \text{ for k } \neq \tau\}, \\ E_e^- &= \{x \text{ such that } x_k = 0 \text{ for k } \geqslant \tau\}, \end{aligned}$$

$P_e^\pm$, P_e^0 are the associated projections. We then obtain the nice expression

$$e(x) = \frac{1}{2}\|P_e^+(x)\|^2 - \frac{1}{2}\|P_e^-(x)\|^2,$$

and we can sum up some important properties:

Lemma 3 *The space E_e is the standard $H_T^{1/2}$ space, the norm $\|x\|_e$ is equivalent to the standard $\|x\|_{H^{1/2}}$ though non uniformly with respect to τ, thus for any $p > 1$ the embedding*

$$j_e^p : E_e \longrightarrow L_T^p(\mathbb{R}^2)$$

is compact. Moreover for any $x \in E_e$ there holds

$$\|x\|_e^2 \geqslant \frac{\omega}{\tau}\|x\|_2^2. \tag{17}$$

The proof is well-known, see [11] for a clear exposition. The last inequality follows directly from expressions in Fourier series. □

The quadratic form h can also be extended as

$$h(z) = \frac{1}{2}\|P_h^+(z)\|_h^2 - \frac{1}{2}\|P_h^-(z)\|_h^2$$

on a Hilbert space E_h, where $P_h^\pm$ are the projections on $E_h^\pm$ associated with the orthogonal splitting $E_h = E_h^+ \oplus E_h^-$.

Lemma 4 *The space E_h is the standard $H_T^{1/2}(\mathbb{R}^{2n-2})$ and the norm $\|z\|_h$ is uniformly equivalent to the standard $\|z\|_{H^{1/2}}$, that is there are constants C and C' independent of T such that*

$$C\|z\|_{H^{1/2}} \leqslant \|z\|_h \leqslant C'\|z\|_{H^{1/2}}.$$

As a consequence, the embeddings

$$j_h^p : E_h \longrightarrow L_T^p(\mathbb{R}^{2n-2})$$

are compact for any $p > 1$, moreover for $p \geqslant 2$ there are constants C_p and P_p independent of T such that

$$\|z\|_p \leqslant C_p\|z\|_h \tag{18}$$

and

$$\|P_h^{\pm} z\|_p \leqslant P_p\|z\|_p. \tag{19}$$

Proof: This is proposition 1.1 of [15]. $\square$

We can now define the total function space

$$E_T = E_e \times E_h\,, \quad \|(x,z)\|^2 = \|x\|_e^2 + \|z\|_h^2,$$

which is nothing but $H_T^{1/2}(\mathbb{R}^{2n})$ with an equivalent inner product (not uniformly in T). We have seen that e and h are continuous, and thus C^∞, quadratic forms. Let us now study the non quadratic parts. It is well-known that

$$\begin{array}{rcl} \tilde{p} : L^3(\mathbb{R}^2) \times L^\alpha(\mathbb{R}^{2n-2}) & \longrightarrow & \mathbb{R} \\ (x(t), z(t)) & \longmapsto & \displaystyle\int_0^T S(x(t), z(t))dt \end{array}$$

is C^1, and

$$p = \tilde{p} \circ j_T$$

also is, where

$$j_T(x,z) = \left(j_e^3(x), j_h^\alpha(z)\right) \in L^3 \times L^\alpha.$$

In the same line,

$$\begin{array}{rcl} \tilde{b} : L^3(\mathbb{R}^2) \times L^\alpha(\mathbb{R}^{2n-2}) & \longrightarrow & \mathbb{R} \\ (x(t), z(t)) & \longmapsto & \displaystyle\int_0^T W(x(t), z(t))dt \end{array}$$

is C^1 thanks to (2), and

$$b = \tilde{b} \circ j_T$$

also is.

Lemma 5 *The functional I_l is well defined and C^1 on E_T, it can be written*

$$I_l(x,z) = \frac{1}{2}\|P_e^+(x)\|^2 - \frac{1}{2}\|P_e^-(x)\|^2 + \frac{1}{2}\|P_h^+(z)\|^2 - \frac{1}{2}\|P_h^-(z)\|^2 - (\tilde{b} + l\tilde{p}) \circ j_T(x,z),$$

and its gradient is

$$\begin{aligned}\nabla I_l(x,z) &= P_e^+(x) - P_e^-(x) + P_h^+(z) - P_h^-(z) + j_T^*\left(\nabla(\tilde{b}+l\tilde{p})\circ j_T(x,z)\right),\\ &= L(x,z) + K(x,z),\end{aligned}$$

where K is continuous and maps bounded sets into relatively compact ones . The solutions of

$$\nabla I_l(X) = 0$$

are precisely the C^1 T-periodic trajectories of the system H_l.

The proof is classical, see [11]. □

There remains to study the behavior of Palais Smale sequences. The unperturbed functional I_0 does not satisfy PS condition, but

Lemma 6 *the functional I_l satisfies the PS condition for any $l > 0$.*

Proof: The proof follows the line of [12], chapter 6. We go in it since many details are different. Let X_m be a bounded PS sequence,

$$\nabla I_l(X_m) = L(X_m) + K(X_m) \longrightarrow 0$$

implies that $L(X_m) = (P_e^+(x_m) - P_e^-(x_m), P_h^+(z_m) - P_h^-(z_m))$ has a convergent subsequence, but then $P_e^0(X_m) = X_m - P_e^+(X_m) - P_e^-(X_m) - P_h^+(X_m) - P_h^-(X_m)$ is bounded and thus has a convergent subsequence since E_e^0 is finite dimensional. Thus any bounded PS sequence has a convergent subsequence. There remains to prove that all PS sequences are bounded. It will be useful to estimate

$$I_l(X) - \frac{1}{2}\langle\nabla I_l(X), X\rangle = \int_0^T \frac{1}{2}\langle\nabla W_l(X), X\rangle - W_l(X)\,dt$$

where $W_l = W + lS = A|z|^\alpha + D|x|^3$ at infinity, thus

$$\begin{aligned}I_l(X) - \frac{1}{2}\langle\nabla I_l(X), X\rangle &\geqslant \int_0^T A\left(\frac{\alpha}{2}-1\right)|z|^\alpha + \frac{D}{2}|x|^3 - C\,dt\\ &\geqslant C(\|z\|_\alpha^\alpha + \|x\|_3^3 - 1)\end{aligned}$$

Applying the above to a PS sequence X_m gives

$$\|z_m\|_\alpha^\alpha + \|x_m\|_3^3 \leqslant C(1 + \epsilon_m\|X_m\|), \tag{20}$$

with $\epsilon_m \longrightarrow 0$. Next

$$|\langle\nabla I_l(X_m), z_m^+\rangle| = \left|2\|z_m^+\|_e^2 - \int_0^T \langle\nabla W_l(X_m), z_m^+\rangle\,dt\right| \leqslant \epsilon_m\|z_m^+\|_e$$

gives

$$\begin{aligned}2\|z_m^+\|_e^2 &\leqslant \left|\int_0^T \langle\nabla_z W_l(X_m), z_m^+\rangle\,dt\right| + \epsilon_m\|z_m^+\|_e\\ &\leqslant C\int_0^T (1 + |z_m|^{\alpha-1})|z_m^+|\,dt + \epsilon_m\|z_m^+\|_e\\ &\leqslant C\left\|1 + |z_m|^{\alpha-1}\right\|_{\frac{\alpha}{\alpha-1}}\|z_m^+\|_\alpha + \epsilon_m\|z_m^+\|_e\\ &\leqslant C(1 + \|z_m\|_\alpha^{\alpha-1})\|z_m^+\|_\alpha + \epsilon_m\|z_m^+\|_e\\ &\leqslant C(1 + \|z_m\|_\alpha^\alpha)\,\|z_m^+\|_e\,,\end{aligned}$$

combining this with (20) yields

$$\|z_m^+\|_e \leqslant C(1+\epsilon_m\|X_m\|).$$

The same can be written for z_m^- and $x_m^\pm$, there just remains to deal with x_m^0, which is done noticing that (20) gives

$$\|x_m^0\|_e = \frac{\omega}{\tau}\|x_m^0\|_2 \leqslant \frac{\omega}{\tau}\|x_m\|_2 \leqslant C\|x_m\|_3 \leqslant C(1+\epsilon_m\|X_m\|).$$

All this together

$$\|X_m\|^2 = \|x_m^0\|^2 + \|x_m^+\|^2 + \|x_m^-\|^2 + \|z_m^+\|^2 + \|z_m^-\|^2 \leqslant C(1+\epsilon_m\|X_m\|)^2$$

implies that $\|X_m\|$ is bounded. □

We are now ready to apply classical variational methods to I_l.

3.2 The topology.

The topological argument is inspired from the one in [15]. Yet the center of our linking is not the origin as usual, but the distinguished orbit $O_{R_0}(t)$. It is not hard to check that $O_{R_0}(t)$ is a critical point of our variational problem. As usual (see [8]) , we introduce a group Γ of homeomorphisms of E_T:

Definition 1 *A homeomorphism $\gamma : E_T \to E_T$ belongs to Γ iff it can be written in the form*

$$\gamma(x,z) = e^{a_e^+(x)}P_e^+(x) + e^{a_e^-(x)}P_e^-(x) + P_e^0(x) + e^{a_h^+(z)}P_h^+(x) + e^{a_h^-(z)}P_h^-(z) + k(x,z)$$

where $a_{e,h}^\pm : E_T \to \mathbb{R}$ are continuous and map bounded sets into bounded sets, and $k : E_T \to E_T$ is continuous and maps bounded sets into relatively compact ones. In addition there exists a $\rho > 0$ such that the support of $a_{e,h}^\pm$ and k is contained in

$$\{(x,z) \in E_T \text{ such that } e(x)+h(x) > 0 \text{ and } \|(x,z)\| \leqslant \rho\}.$$

The functionals e and h are defined in (13) and (14) above.

It is not hard to see that Γ is a group, see [11], 5.3 for related material. Let us now introduce the sphere

$$S^+ = \{(x,z) \in E_e^+ + E_h^+ \text{ such that } \|(x,z)\| = 1\}.$$

We shall link $\tilde{S}^+ = O_{R_0} + S^+$ with an affine subspace of E_T of the form $O_{R_0} + E_e^- + E_e^0 + E_h^- + \mathbb{R}z_T$, with $z_T \in E_h^+$. We follow Tanaka [15] for the choice of z_T, and take $z_T = P_h^+(\phi)$, where $\phi \in C_0^\infty((0,1), \mathbb{R}^{2n-2})$ is extended by 0 to $[0,T]$ and satisfies

$$\int_0^1 \langle J\dot\phi + A\phi, \phi\rangle dt < 0.$$

Lemma 7 *There are positive constants C_p and C_p' independent of $T \geqslant 1$ such that for all $p > 1$*

$$\begin{aligned} C_0 \leqslant \quad & \|z_T\| & \leqslant C_0' \\ C_p \leqslant \quad & \|z_T\|_p & \leqslant C_p'. \end{aligned}$$

This is lemma 1.4 of [15]. □

Let

$$V = E_e^- + E_e^0 + E_h^- + \mathbb{R} z_T,$$

The spaces V and S^+ link with respect to Γ:

Lemma 8 (Intersection property) *For $\gamma \in \Gamma$, we have*

$$\gamma(S^+) \cap V \neq \emptyset.$$

Proof: This is classical, see for example [8], proposition 1. □

It is therefore natural to define:

Definition 2

$$c_T(l) = \sup_{\gamma \in \Gamma} \left(\inf_{S^+} I_l \circ \gamma \right).$$

Before we prove that $c_T(l)$ is a critical value, it is of interest for us to estimate it.

Proposition 2 *There is a constant M that depends only on A, B and α such that for all $l > 0$*

$$0 < c_T(l) \leqslant M.$$

Proof: For all $\eta > 0$ there exists $\gamma \in \Gamma$ such that $\gamma(S^+) = O_{R_0} + \eta S^+$. On the other hand, the intersection property above implies that $c_T(l) \leqslant \sup_V I_l$. For these reasons, proposition 2 follows from lemma 9 and 10 below. □

Lemma 9 *Let us fix all parameters. There are $\eta > 0$ and $\delta > 0$ such that*

$$I_l(O_{R_0} + x^+, z^+) > \delta$$

whenever $(x^+, z^+) \in E_e^+ \times E_h^+$ satisfy $\|(x^+, z^+)\| = \eta$.

Proof: Since H is clearly lipschitz continous on compact sets, there exists a constant C such that, for all t, and all sufficiently small $(x, z) \in \mathbb{R}^2 \times \mathbb{R}^{2n-2}$,

$$|H(O_{R_0}(t) + x, z) - H_0| \leqslant C(|x| + |z|).$$

On the other hand, recalling that [HW4] is now assumed,

$$|H(O_{R_0}(t) + x, z) - H_0| \leqslant C(|x|^2 + |z|^\alpha)$$

holds at infinity, so that, for all x and z,

$$|H(O_{R_0}(t) + x, z) - H_0| \leqslant C(|x| + |z| + |x|^2 + |z|^\alpha).$$

Combining this with estimate (7) gives

$$\begin{aligned} S(O_{R_0}(t) + x, z) &\leqslant C\Big(H(O_{R_0}(t) + x, z) - H_0\Big)^4 \\ &\leqslant C\Big(|x| + |x|^2 + |z| + |z|^\alpha\Big)^4 \\ &\leqslant C(|x|^4 + |x|^8 + |z|^4 + |z|^{4\alpha}). \end{aligned}$$

Noticing that $O_{R_0} \in E_e^0$ this yields, for small η,

$$\begin{aligned}
I_l(O_{R_0}+x^+,z^+) &= \frac{1}{2}\|x^+\|^2+\frac{1}{2}\|z^+\|^2-b(O_{R_0}+x^+,z^+)-lp(X)\\
&\geqslant \frac{1}{2}\|x^+\|^2+\frac{1}{2}\|z^+\|^2\\
&-C\|z^+\|_\alpha^\alpha-C\|z^+\|_4^4-C\|z^+\|_{4\alpha}^{4\alpha}-C\|x^+\|_4^4-C\|x^+\|_8^8\\
&\geqslant \frac{1}{2}\|x^+\|^2+\frac{1}{2}\|z^+\|^2\\
&-C\|z^+\|_\alpha^\alpha-C\|z^+\|_4^4-C\|z^+\|_{4\alpha}^{4\alpha}-C\|x^+\|_2^4-C\|x^+\|_2^8\\
&\geqslant \frac{1}{2}\eta^2-C(\eta^\alpha+\eta^{4\alpha}+\eta^4+\eta^8).
\end{aligned}$$

We have used (17) and (18) for the last inequality. □

Lemma 10 *There is a constant M that depends only on A, B and α such that for all $l>0$*

$$I_l|_V \leqslant M.$$

Proof: Let $X=(x^-+x^0,z^-+rz_T)\in V$, from [HW3] we get

$$\begin{aligned}
I_l(X) &= -\frac{1}{2}\|x^-\|^2-\frac{1}{2}\|z^-\|^2+\frac{1}{2}\|rz_T\|^2-b(X)-lp(X)\\
&\leqslant \frac{1}{2}\|z_T\|^2r^2-B\|z^-+rz_T\|_\alpha^\alpha.
\end{aligned}$$

Using (19) gives:

$$\|rz_T\|_\alpha^\alpha=\|P_h^+(z^-+rz_T)\|_\alpha^\alpha \leqslant P_\alpha^\alpha\|z^-+rz_T\|_\alpha^\alpha,$$

combining these equations yields

$$I_l(X)\leqslant \frac{1}{2}\|z_T\|^2r^2-BP_\alpha^{-\alpha}\|z_T\|_\alpha^\alpha r^\alpha,$$

and we obtain the lemma setting

$$M=\sup_{T\in[1,\infty)}\sup_{r\in\mathbb{R}^+}\left(\frac{1}{2}\|z_T\|^2r^2-BP_\alpha^{-\alpha}\|z_T\|_\alpha^\alpha r^\alpha\right) \tag{21}$$

which is finite according to lemma 7. In addition, we see that

$$\lim_{B\to\infty}M=0.$$

□

3.3 The critical point

We will now prove that there exists $l(T)\in]0,\Delta/T[$ and a critical point X_T of $I_{l(T)}$ at level $c_T(l(T))$ such that $p(x_T)\leqslant 1+TM/\Delta$. Let us first chose $l(T)$.

Lemma 11 *There exists $l(T)\in(0,\Delta/T)$ such that $l\longmapsto c_T(l)$ is differentiable in $l(T)$ and*

$$|c_T'(l(T))|\leqslant TM/\Delta.$$

Proof: From its definition, $c_T(l)$ is a nonincreasing function of l, it is thus differentiable almost everywhere in $]0, \Delta/T[$ and there holds

$$\int_0^{\Delta/T} c_T'(l)dl \geqslant -M.$$

$\square$

We are now going to
suppose that there is no critical point at level $c_T(l(T))$ satisfying $p(x_T) \leqslant 1 - c_T'(l(T))$,
and prove that this leads to a contradiction. Let $l_n \longrightarrow l(T)$ be a decreasing sequence, $I_n = I_{l_n}$, $c_n = c_T(l_n)$, $c' = |c_T'(l(T))|$ and $c = c_T(l(T))$. Using the supposition above and the fact that PS is satisfied for $I_{l(T)}$ we can prove the following lemma by a deformation argument:

Lemma 12 *There is an ϵ in the interval $(0, c/2)$ such that for any K there is an homeomorphism $\gamma_K \in \Gamma$ satisfying*

$$I_{l(T)}(\gamma_K(X)) \geqslant I_{l(T)}(X)$$

for all $X \in E_T$, and such that

$$I_{l(T)}(\gamma_K(X)) \geqslant c + \epsilon$$

for all X satisfying the following three inequalities

$$\begin{aligned} p(X) &\leqslant c' + 1/2, \\ I_{l(T)}(X) &\geqslant c - \epsilon, \\ \|X\| &\leqslant K. \end{aligned}$$

$\square$

From the definition of c_n, we can choose $\gamma_n \in \Gamma$ such that

$$\inf_{S^+} I_n \circ \gamma_n \geqslant c_n - (l_n - l)/10.$$

For n large enough there holds

$$\begin{aligned} I_{l(T)} \circ \gamma_n\big|_{S^+} \geqslant I_n \circ \gamma_n|_{S^+} &\geqslant c_n - (l_n - l)/10 \\ &\geqslant c - (c' + 1/10)(l_n - l) - (l_n - l)/10 \\ &\geqslant c - (c' + 1/5)(l_n - l). \end{aligned}$$

Let us set $K_n = \sup_{S^+} \|\gamma_n\|$, and let $\varphi_n = \gamma_{K_n}$ be the homeomorphism given by the lemma. Take $X \in S^+$:
Either $I_{l(T)}(\gamma_n(X)) \leqslant c + (l_n - l)/5$,
and since

$$\begin{aligned} (l_n - l)p(\gamma_n(X)) &= I_{l(T)}(\gamma_n(X)) - I_n(\gamma_n(X)) \leqslant c + (l_n - l)/5 - \big(c + (c' + 1/5)(l_n - l)\big) \\ &\leqslant (c' + 1/2)(l_n - l), \end{aligned}$$

we can apply the lemma for n large enough and get

$$I_{l(T)}(\varphi_n \circ \gamma_n(X)) \geqslant c + \epsilon;$$

or

$$I_{l(T)}(\varphi_n \circ \gamma_n(X)) \geqslant I_{l(T)}(\gamma_n(X)) \geqslant c + (l_n - l)/5.$$

In both cases we have, for n large enough,

$$I_{l(T)}(\varphi_n \circ \gamma_n(X)) \geqslant c + (l_n - l)/5,$$

which means that there exists $\gamma = \varphi_n \circ \gamma_n \in \Gamma$ such that

$$\inf_{S^+} I_{l(T)} \circ \gamma > c,$$

this is in contradiction with the definition of c. We have proved the existence of a critical point satisfying (9) and (10). There remains to prove (11).

3.4 Non triviality

In this subsection, we prove conclusion (11). We point out that this is the only part in the proof of proposition 1 where the condition $\pi R_0^2 \geqslant M$ is used. In fact, the critical point constructed always exists, but it may be contained in the plane $z = 0$. That it is not the case under our hypotheses is a key ingredient for the non triviality of the homoclinic obtained after convergence. We first observe that

$$\nabla_z H_l(x, 0) = 0 \;\Rightarrow\; \nabla_z I_l(x, 0) = 0,$$

which means that the plane $z = 0$ is left invariant by the flow, and that the subspace $E_e \times \{0\}$ is transversally critical. As a consequence, the critical points of I_l that are on the form $(x(t), 0)$ are precisely the critical points of $I_l|_{E_e \times \{0\}}$, and they are the T-periodic orbits of the flow contained in $z = 0$.

Lemma 13 *Let $(x, 0)$ be a critical point, then the set $\{x(t), t \in \mathbb{R}\}$ is a circle $S(r)$, where r satisfies*

$$lf'(r^2) \in \frac{\pi}{T}\mathbb{Z},$$

and we have

$$I_{l(T)}(x, 0) = T\left(r^2 lf'(r^2) - lf(r^2)\right) \notin (0, \pi R_0^2].$$

Proof: The plane $z = 0$ is invariant, and the equation on it is

$$\dot{x} = J(\omega + 2lf'(|x|^2))x,$$

the solutions of which

$$X_r(t) = re^{J(\omega + 2lf'(r^2))t}$$

have period

$$T(r) = \left|\frac{2\pi}{\omega + 2lf'(r^2)}\right|.$$

These solutions are critical points only if $T \in \mathbb{N}T(r)$. This implies

$$lf'(r^2) + \frac{\omega}{2} \in \frac{\pi}{T}\mathbb{Z},$$

hence the lemma since $\omega = 2\pi/T$. The computation of the action is straightforward, that it can not take values in the forbidden interval when r is critical is a consequence of the convexity of f: the function

$$x \longmapsto g(x) = xf'(x) - f(x)$$

is increasing and thus $g(x) \leqslant 0$ when $x \leqslant R_0^2$ since $f(R_0^2) = f'(R_0^2) = 0$. On the other hand, the function

$$x \longmapsto (x - R_0^2)f'(x) - f(x)$$

is increasing for $x \geqslant R_0^2$, which implies that

$$g(x) > R_0^2 f'(x)$$

when $x > R_0^2$. Either $r > R_0$ and we must have

$$I = Tlg(r^2) > TlR_0^2 f'(r^2) \geqslant \pi R_0^2$$

or $r \leqslant R_0$ and $I \leqslant 0$. □

The proposition 1 follows from the fact that $c_T(l(T))$ is in the hole if $\pi R_0^2 \geqslant M$, and thus can not be one of the "bad" critical points. □

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References

[1] Bernard P. : Homoclinic orbit to a center manifold, to appear in Var. Meth. & P. D. E..

[2] Bernard P. : Recherche variationnelle d'orbites homoclines dans les systèmes dynamiques hamiltoniens, thèse, 2000.

[3] Bolotin S.V. : Libration Motions of Natural Dynamical Systems, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **6**, (1978), 72-77.

[4] Bolotin S.V. : The Existence of Homoclinic Motions, Viestnik Mosk. Univ. Mat., **38,6** (1983), 98-103.

[5] Buffoni B. - Séré E. : A global condition for quasi-random behavior in a class of conservative systems, Comm. Pure and App. Math. , **49** (1996), 285-305.

[6] Coti Zelati V., Ekeland I., and Séré E. : A Variational Approach to Homoclinic Orbits in Hamiltonian Systems, Math. Ann. **288** (1990), 133-160.

[7] Devaney R. L. : Homoclinic orbits in hamiltonian systems, J. Diff. Eq. **21** (1976), 431-438.

[8] Ekeland I. and Hofer H. : Symplectic Topology and Hamiltonian Dynamics, Math Z, **200** (1989), 355-378.

[9] Hayashi S. : Connecting Invariant Manifolds and the Solution of C^1 Stability and Ω Stability Conjecture for Flows, Annals of Maths, **145** (1997), 81-137.

[10] Hofer H. and Wysocki K. : First Order Elliptic Systems and the Existence of Homoclinic Orbits in Hamiltonian Systems, Math. Ann. **288** (1990), 483-503.

[11] Hofer H. and Zehnder E. : *Symplectic Invariants and Hamiltonian Dynamics*, Birkhauser Advanced Texts (1994).

[12] Rabinowitz P. H. : *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS, **65**, AMS, Providence (1986).

[13] Séré E. : Homoclinic Orbits on Contact Hypersurfaces in $\mathbb{R}^{2n}$ of Restricted Contact Type, Comm. Math. Phys. **172** (1995), 293-316.

[14] Struwe M. : *Variational Methods*, Springer, Second Edition, (1996).

[15] Tanaka K. : Homoclinic Orbits in a First Order Superquadratic Hamiltonian System: Convergence of Subharmonic Orbits, J. Diff. Equ. **94** (1991), 315-339.

[16] Xia Z. : Homoclinic Points in Symplectic and Volume Preserving Diffeomorphisms, Comm. Math. Phys. **177** (1996) 435-449.

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