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DECOMPOSITION OF SOLENOIDAL VECTOR CHARGES INTO ELEMENTARY SOLENOIDS AND THE STRUCTURE OF NORMAL ONE-DIMENSIONAL CURRENTS

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ABSTRACT. We investigate the structure of vector charges whose divergency is a measure (i.e., the structure of normal one-dimensional flows). We prove that every vector charge with divergency 0 can be decomposed into elementary solenoids, i.e., the simplest charges of this kind (representable as an "averaged circulation" along a sufficiently good embedding of $\mathbb{R}$ into $\mathbb{R}^n$). The techniques used are those of the geometric measure theory, but the knowledge of this theory is not necessary to understand the statements and proofs.

§1. INTRODUCTION

1.1. Introductory notes. Let T be an $\mathbb{R}^n$-valued countably additive set function defined on the Borel σ-algebra $\mathscr{B}_n$ of $\mathbb{R}^n$:

$$T(E) = \left(T_1(E), \dots, T_n(E)\right), \quad E \in \mathscr{B}_n,$$

T_j being real measures on $\mathscr{B}_n$ (*scalar charges*). We call T *a vector charge.* We endow the set of all vector charges with the norm

$$\operatorname{var}(T) := \sup \sum_j |T(E_j)|,$$

where the supremum is taken over all Borel subdivisions of $\mathbb{R}^n$. Then we may identify this set with the space of *currents of finite mass* ([1], 4.1.7). The last term refers to linear functionals τ defined on the normed space $\mathscr{D}^1(\mathbb{R}^n)$ of all C^∞-vector fields $\varphi = (\varphi_1, \dots, \varphi_n)$ with compact support; $\mathscr{D}^1(\mathbb{R}^n)$ is endowed with the uniform norm

$$\|\varphi\|_\infty := \max_{\mathbb{R}^n} \sqrt{\sum_{j=1}^n |\varphi_j|^2}.$$

The functional τ corresponding to a charge T is defined by

$$\tau(\varphi) := \int_{\mathbb{R}^n} \sum_{j=1}^n \varphi_j \, dT_j, \quad \varphi \in \mathscr{D}^1(\mathbb{R}^n).$$

We often do not distinguish between τ and T and consider the space Ch of all charges as the conjugate space $\mathscr{D}_1(\mathbb{R}^n)$ of $\mathscr{D}^1(\mathbb{R}^n)$. The weak topology induced in Ch will be called the $\mathscr{D}$-topology.

Solenoidal charges (or simply *solenoids*) mentioned in the title of the article are divergence free charges T:

$$\operatorname{div} T = 0.$$

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This relation is understood in the sense of distributions; namely, it means that

$$\int_{\mathbb{R}^n} \sum_{j=1}^{n} \frac{\partial u}{\partial x_j} \, dT_j = 0$$

for every $u \in C_0^\infty(\mathbb{R}^n)$. The set of all solenoidal charges will be denoted by the symbol Sol.

Of particular interest is the space $\mathbb{N}_1(\mathbb{R}^n)$ of all charges enjoying the following properties:

$$\operatorname{spt} T \quad \text{is compact}, \qquad \operatorname{var}(\operatorname{div} T) < +\infty$$

($\operatorname{spt} T$ denotes the closed support of T). The latter property means that the distribution $\operatorname{div} T$ defined by the formula

$$\operatorname{div} T(u) := -T(\nabla u), \quad u \in C_0^\infty(\mathbb{R}^n),$$

is a scalar charge, and $\operatorname{var}(\operatorname{div} T)$ is the total variation of this charge. Using the terminology of [1], we call the charges of class $\mathbb{N}_1$ *normal*. Clearly, $\mathbb{N}_1$ contains the set Sol_c of all solenoids with compact support.

The problem of geometric structure of normal charges and solenoids arises in the geometric measure theory ([2], Problem 3.8) and in the homology theory with real coefficients [3]. The author has met this problem when studying approximation properties of various classes of vector fields and differential forms, as well as extension properties of fields and forms (see [4], where normal currents and solenoids arise in a natural way as dual objects).

Restricting ourselves to solenoidal charges first, we start with a heuristic discussion of their structure.

The simplest example of a solenoidal charge is *an oriented closed curve of finite length.* Roughly speaking, this is the circulation of a test field along an oriented rectifiable curve γ of finite length:

$$T_\gamma(\varphi) := \int_\gamma \langle \tau(x), \varphi(x) \rangle \, d\mathscr{H}^1(x), \quad \varphi \in \mathscr{D},$$

τ being the vector field of unit tangent vectors (the orientation of γ); we denote by $\mathscr{H}^m$ the m-dimensional Hausdorff measure. If $a, b \in \mathbb{R}^n$ are the endpoints of the curve, then

$$(\operatorname{div} T_\gamma)(u) = -T_\gamma(\nabla u) = -\int_\gamma \langle \tau(x), \varphi(x) \rangle \, d\mathscr{H}^1(x) = -\int_\gamma \dot{u} \, d\mathscr{H}^1(x) = u(a) - u(b),$$

where $\dot{u}$ is the derivative of the function $u \in C_0^\infty(\mathbb{R}^n)$ along γ. If γ is closed, then the divergency of T_γ is 0.

Is the stock of these simplest solenoids rich enough to create *all* solenoids? Is it possible to represent an arbitrary solenoid $T \in \mathrm{Sol}$ as a "continuous convex combination" of loops? We mean a representation of the form

$$T = \int T_\gamma \, d\mu(\gamma), \tag{1.1}$$

where μ is a nonnegative measure. "Convexity" means that

$$\operatorname{var}(T) = \int \operatorname{var}(T_\gamma) \, d\mu(\gamma) \tag{1.2}$$

($\operatorname{var}(T_\gamma)$ is the variation of the charge T_γ, i.e., the length of γ). The last relation implies, in particular, that μ-almost all curves γ stay in $\operatorname{spt} T$. Such a decomposition can be believed to exist because it does exist locally for smooth charges and, furthermore, vector measures with zero rotation do admit a decomposition into "hypersurfaces" (see the Fleming–Rishel formula below), which implies the existence of a decomposition of an arbitrary solenoidal charge into closed curves in the case of $\mathbb{R}^2$.

However, for $n > 2$ formulas (1.1) and (1.2) should be renounced. In the general case the set of "elementary solenoids" must include, besides closed curves, some of their generalizations (like an irrational winding of the torus or the Smale–Williams solenoid, see below). Then it turns out to be possible to justify the analogs of (1.1) and (1.2).

Before giving a rigorous formulation of the problem, we introduce the necessary notation and terminology.

The term *measure* will always mean a *nonnegative* countably additive set function defined on a σ-algebra of subsets of a space X.

We shall need not only charges, but also *local charges*, i.e., countably additive $\mathbb{R}^n$-valued set functions T defined on the ring of all bounded Borel sets in $\mathbb{R}^n$. The total variation of a local charge can be infinite; nevertheless, the following measure $\|T\|$ is defined on $\mathscr{B}_n$ (and is finite on each ball):

$$\|T\|(E) := \sup \sum_j |T(E_j)|,$$

the supremum being taken over all finite Borel subdivisions of E. The set of all local charges is denoted by $\mathrm{Ch}_{\mathrm{loc}}(\mathbb{R}^n)$. Any charge T is absolutely continuous with respect to $\|T\|$. Hence, by the Radon–Nikodym theorem $T = \mathbb{T}\|T\|$, where T is a Borel measurable field of unit vectors in $\mathbb{R}^n$ defined $\|T\|$-a.e. In other words,

$$T(\varphi) = \int_{\mathbb{R}^n} \langle \varphi(x), \mathbb{T}(x)\rangle \, d\|T\|(x), \quad \varphi \in \mathscr{D}^1(\mathbb{R}^n). \tag{1.3}$$

With any (local) charge T and with any Borel $E \subset \mathbb{R}^n$, we associate the restriction $T \llcorner E$ of T to E. This is the (local) charge $\chi_E T$, where χ_E denotes the characteristic function of E:

$$(T \llcorner E)(G) = T(E \cap G)$$

for every (bounded) $G \subset \mathscr{B}_n$. Clearly,

$$(T \llcorner E)(\varphi) = \int_E \langle \varphi, \mathbb{T}\rangle \, d\vec{T}, \quad \varphi \in \mathscr{D}^1(\mathbb{R}^n).$$

We shall also use the cartesian product $T \times S \in \mathrm{Ch}_{\mathrm{loc}}(\mathbb{R}^{n+l})$ of a charge $T \in \mathrm{Ch}_{\mathrm{loc}}(\mathbb{R}^n)$ and a scalar charge S on $\mathbb{R}^l$:

$$T \times S := (T_1 \times S, \dots, T_n \times S, \underbrace{0, \dots, 0}_{l}),$$

where $T_j \times S$ is the usual product of (real) measures and the T_j are the "coordinate" charges of T. It is easy to see that

$$\operatorname{div}(T \times S) = \operatorname{div} T \times S. \tag{1.4}$$

The same identity holds if T is a scalar charge and S is a charge (with an obvious definition of $T \times S$). A local charge T is called *locally normal* if $\operatorname{div} T$ (understood as a distribution) is a local (scalar) charge; in this case we write $T \in \mathrm{N}_{1,\mathrm{loc}}$. If $\operatorname{div} T = 0$, then we write $T \subset \mathrm{N}_{1,\mathrm{loc}}$. The set of local charges with zero divergency is denoted by $\mathrm{Sol}_{\mathrm{loc}}$.

The symbol $f_\sharp\mu$ will stand for the image of a measure μ under the mapping f (this image is defined on a suitable σ-algebra):

$$f_\sharp\mu(E) := \mu(f^{-1}E).$$

Some of the statements presented here are easier to formulate and prove if one makes use of the notion of the image $f_\sharp T \in \mathrm{Ch}_{\mathrm{loc}}(\mathbb{R}^n)$ of a charge T under a Lipschitz mapping $f: \mathbb{R}^k \to \mathbb{R}^n$ (for this image to exist it suffices, e.g., that T be of compact support or of finite variation; see [1, 4.1.7, 4.1.14] for more details).

Suppose f is a proper C^∞-mapping of $\mathbb{R}^k$ into $\mathbb{R}^n$ ("proper" means that the limit $\lim_{|x|\to\infty}|f(x)| = \infty$; an arbitrary C^∞-diffeomorphism of $\mathbb{R}^n$ onto itself is proper). Let $\mathbb{D}f$ be its Jacobi matrix. Then for $\varphi \in \mathscr{D}^1(\mathbb{R}^n)$

$$f_\sharp T(\varphi) := \int_{\mathbb{R}^n} \langle \mathbb{D}f \cdot \mathbb{T}, \varphi(f)\rangle \, d\|T\| = \int_{\mathbb{R}^n} \langle \mathbb{T}, (\mathbb{D}f)^* \cdot \varphi(f)\rangle \, d\|T\|, \tag{1.5}$$

or

$$f_\sharp T(\varphi) = T(f^\sharp \varphi),$$

where by $f^\sharp\varphi$ we denote the inverse image of a vector field φ under a C^∞-smooth mapping f:

$$(f^\sharp\varphi)(x) := (\mathbb{D}f)^*(x)\varphi(f(x)) \qquad (x \in \mathbb{R}^n).$$

In precisely one case (namely, $f: \mathbb{R} \to \mathbb{R}^n$) we shall need a nonsmooth but Lipschitz mapping f. Then $\|T\|(t)$ is a function (whose absolute value is 1 a.e.), and instead of (1.5) we use

$$f_\sharp T(\varphi) := \int_{\mathbb{R}} \mathbb{T}(t)\langle f'(t), \varphi(f(t))\rangle \, d\|T\|(t), \quad \varphi \in \mathscr{D}^n(\mathbb{R}^n). \tag{1.5'}$$

It can easily be verified that in these cases

$$(f \circ g)_\sharp T = f_\sharp g_\sharp T, \tag{1.6}$$

$$\mathrm{var}(f_\sharp T) \le \mathrm{Lip}(f)\,\mathrm{var}(T), \tag{1.7}$$

$$\mathrm{div}(f_\sharp T) = f_\sharp(\mathrm{div}\, T) \tag{1.8}$$

(Proofs under more general assumptions can be found in [1]).

We denote by δ_x the Dirac measure at the point x and by $\mathscr{L}^n$ the Lebesgue measure in $\mathbb{R}^n$. Let $\overrightarrow{[a\,;b]}$ denote a charge in $\mathrm{Ch}(\mathbb{R})$ defined as follows: if $a < b$, then

$$\overrightarrow{[a\,;b]}(\varphi) = \int_a^b \langle \varphi, e_1\rangle \, dt, \quad \varphi \in \mathscr{D}^1(\mathbb{R}),$$

and if $b > a$, then $\overrightarrow{[a\,;b]} := -\overrightarrow{[b\,;a]}$ ($e_1, \dots, e_n$ are the coordinate unit vectors in $\mathbb{R}^n$).

1.2. Statement of the problem. We return to the question of decomposition of a charge into simplest ones. This will be understood in the following precise meaning: a charge $T \in \mathrm{Ch}_{\mathrm{loc}}(\mathbb{R}^n)$ *decomposes* into charges lying in $J \subset \mathrm{Ch}_{\mathrm{loc}}(\mathbb{R}^n)$ if there is a measure μ on J such that

$$T = \int_J R \, d\mu(R), \tag{1.9}$$

$$\|T\| = \int_J \|R\| \, d\mu(R). \tag{1.10}$$

If, moreover, $T \in \mathbb{N}_{1,\mathrm{loc}}(\mathbb{R}^n)$, $J \subset \mathbb{N}_{1,\mathrm{loc}}(\mathbb{R}^n)$, and the following additional condition holds

$$\|\operatorname{div} T\| = \int_J \|\operatorname{div} R\| \, d\mu(R), \tag{1.11}$$

we say that T *completely decomposes* into charges lying in J.

Remark 1. All measures considered below will be Borel with respect to the $\mathscr{D}$-topology, therefore, the integrals in (1.9), (1.10), (1.11) will be understood in the weak sense. For example, $\int_J R\, d\mu(R)$ is a charge S defined by $S(\varphi) = \int_J R(\varphi)\, d\mu(R)$ for every field $\varphi \subset \mathscr{D}^1(\mathbb{R}^n)$.

Remark 2. If (1.9) holds, then (1.10) and (1.11) are equivalent to the following relations (respectively):

$$\operatorname{var}(T) = \int_J \operatorname{var}(R)\, d\mu(R), \tag{1.10'}$$

$$\operatorname{var}(\operatorname{div} T) = \int_J \operatorname{var}(\operatorname{div} R)\, d\mu(R). \tag{1.11'}$$

Remark 3. The general theory of convex sets guarantees the existence of a set J such that any solenoid can be decomposed into elements of J. Indeed, consider the unit ball B_{Sol} in the space Sol. This ball is metrizable (in the $\mathscr{D}$-topology), as a bounded set in the space Ch, conjugate to a separable space. So, the set $\operatorname{extr} B_{\mathrm{Sol}}$ of all its extreme points is Borel and nonempty (by the Krein–Milman theorem) and one can apply the Choquet theorem ([5]). It follows that for every $T \in$ Sol there exists a representing measure supported on $\operatorname{extr} B_{\mathrm{Sol}}$. For this measure statements (1.9) and (1.10′) hold. Hence there is at least a possibility to take $\operatorname{extr} B_{\mathrm{Sol}}$ for J, and the problem of finding J reduces to that of describing $\operatorname{extr} B_{\mathrm{Sol}}$.

By remark 3, the elements of $\operatorname{extr} B_{\mathrm{Sol}}$ are natural "elementary" solenoids. We (partly) describe them and obtain a "concrete" decomposition formula. From the same point of view, we also consider the structure of normal charges.

1.3. Examples and results. We start with several important examples of solenoids, and then state the results.

Example 1. A closed curve. The simplest example of a one-dimensional solenoidal charge is a *simple oriented closed curve* T *of finite length* $\operatorname{var}(T)$ mentioned above (all integer valued charges with zero boundary are decomposable into an at most countable sum of such curves, see [1], 4.2.18). This term is attributed to any charge $T \in \mathbb{N}_1(\mathbb{R}^n)$ for which there exists a function $f\colon [0; \operatorname{var}(T)] \to \mathbb{R}^n$ such that

$$\operatorname{Lip}(f) \le 1, \tag{1.12}$$

$$T = f_\sharp \overrightarrow{[0; \operatorname{var}(T)]}, \tag{1.13}$$

$$f \text{ is one-to-one on } [0; \operatorname{var}(T)), \tag{1.14}$$

$$f(0) = f(\operatorname{var}(T)). \tag{1.15}$$

It follows from (1.13) that

$$T(\varphi) = \int_0^{\operatorname{var}(T)} \langle f'(t), \varphi(f(t)) \rangle \, dt, \quad \varphi \in \mathscr{D}_1(\mathbb{R}^n).$$

If (1.14) (or (1.15)) does not hold, we shall omit the word 'simple' (resp., 'closed'). We recall that, if a curve is closed, then the boundary of the corresponding charge is zero.

Example 2. An irrational winding on the torus. The idea of this classical example is as follows (see, e.g., [6, pp. 92–94]). Consider the charge $e_1\mathscr{H}^2\llcorner \mathrm{Cyl}$ in $\mathbb{R}^3$, where $\mathrm{Cyl} := [0; 1] \times \{x_2^2 + x_3^2 = 1\}$ is the cylinder surface. Clearly, its divergency is supported by $\{0; 1\} \times \{x_2^2 + x_3^2 = 1\}$.

To "annihilate" the divergency, we "glue" our cylinder into a torus. We glue up the two boundary circles turning one of them by a π-irrational angle. To be a bit more precise, we put $Sq := [0, 1) \times [0, 1)$ and take a C^∞-transformation $f : \mathbb{R}^2 \to \mathbb{R}^3$ 1-periodic in x_1 and x_2 and such that $f|Sq$ is a bijection onto a torus. Consider the charge

$$T := f_\sharp(e_\theta \mathscr{L}^2\llcorner Sq),$$

where $e_\theta = (\cos\theta)e_1 + (\sin\theta)e_2$, $\sin\theta$ being irrational. It is not hard to see that $\operatorname{div} T = 0$. If we try to decompose T into curves by "gluing" local decompositions (which exist near every point of the torus), then, instead of loops, we obtain infinite curves, each one coiling round the torus and constituting its *"irrational" winding*. Finally, we remark that using this example it is not hard to construct a C^∞-smooth charge in $\mathbb{R}^3$ not decomposable into loops.

Example 3. The Smale–Williams solenoid. This example is based on a construction of a well-known attractor. A detailed discussion of the method of constructing charges related to stable or nonstable manifolds of some diffeomorphisms can be found in [3] (see also [7] and a discussion of fractal structure of T_∞ in [8]). Therefore, we give only a brief intuitive description.

Consider a diffeomorphism f of a solid torus Tor into itself such that $f(\mathrm{Tor})$ "turns twice" inside Tor. The object $\mathrm{Tor}_\infty := \bigcap_{k=1}^\infty f^k(\mathrm{Tor})$ is known in the theory of dynamical systems as the Smale–Williams solenoid. Locally Tor_∞ is (topologically) the Cartesian product of a Kantor set by a segment (turning once around the axis of the solid torus, we mix these segments up and glue them together in a different order). We can orient these segments (i.e., prescribe a direction of rotation on Tor_∞) and define a dyadic measure μ on the Kantor set. This gives rise to a solenoid T, which coincides locally with $\mu \times \overrightarrow{[a; b]}$ (up to a Lipschitz homeomorphism). Thus, locally we can decompose T into curves. But mixing the segments up under "gluing" constitutes an obstacle to a global decomposition (a result of this mixing is, e.g., the fact that the Smale–Williams solenoid contains no loop).

Example 4. Almost periodic solenoids. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a Bohr almost periodic vector function. We assume for simplicity that the vector function f' is uniformly continuous and its absolute value does not exceed 1 everywhere. We define a charge T by the formula

$$T(\varphi) := \lim_{s\to\infty} \frac{1}{2s} \int_{-s}^{s} \langle f'(t), \varphi(f(t))\rangle\, dt, \quad \varphi \in \mathscr{D}^1(\mathbb{R}^n).$$

The limit exists, since the function $\langle f'(t), \varphi(f(t))\rangle$ is almost periodic and, consequently, admits averaging. For $\varphi \in \mathscr{D}^1(\mathbb{R}^n)$ we have

$$\begin{aligned}|T(\varphi)| &\le \limsup_{s\to\infty} \frac{1}{2s} \int_{-s}^{s} |\langle f'(t), \varphi(f(t))\rangle|\, dt \\ &\le \limsup_{s\to\infty} \frac{1}{2s} \int_{-s}^{s} \|\varphi(f(t))\|\, dt \le \|\varphi\|_\infty,\end{aligned}$$

whence $\operatorname{var}(T) \le 1$. Finally, for every $u \subset \mathscr{D}^0(\mathbb{R}^n)$,

$$\begin{aligned} -\operatorname{div} T(u) = T(du) &= \lim_{s\to\infty} \frac{1}{2s} \int_{-s}^{s} \langle \nabla f(t), du(f(t)) \rangle \, dt \\ &= \lim_{s\to\infty} \frac{1}{2s} \int_{-s}^{s} \frac{d}{dt} u(f(t)) \, dt \\ &= \lim_{s\to\infty} \frac{1}{2s} (u(f(s)) - u(f(-s))) = 0. \end{aligned}$$

So, the above charge T is normal and of divergence 0, and $\operatorname{var}(T) \le 1$. It is easy to see that all preceding examples fit into this pattern. The first one (for a curve of length 1) corresponds to a periodic function f (with period 1); in examples 2 and 3 a suitable function f can be constructed by gluing together the local decompositions.

This suggests taking the charges of Example 4 as "basic". It is natural to consider not only almost periodic functions f, but all functions for which the average exists. This leads to the following definition.

Definition. A charge T is called an *elementary solenoid* if there is a Lipschitz vector-function $f: \mathbb{R} \to \mathbb{R}^n$ enjoying the following properties:

$$\operatorname{Lip}(f) \le 1, \tag{1.16}$$

$$T = \mathscr{D} - \lim_{k\to\infty} \frac{1}{2k} F_\sharp \overrightarrow{[-k;k]}, \tag{1.17}$$

$$\operatorname{var}(T) = 1, \tag{1.18}$$

$$f(\mathbb{R}) \subset \operatorname{spt} T. \tag{1.19}$$

Remark 4. Conditions (1.16) and (1.17) are similar to the first two conditions in the definition of a curve; (1.17) means that for every field $\varphi \in \mathscr{D}^1(\mathbb{R}^n)$ the mean

$$T(\varphi) := \lim_{s\to\infty} \frac{1}{2s} \int_{-s}^{s} \langle f'(t), \varphi(f(t)) \rangle \, dt \tag{1.20}$$

exists. Condition (1.18) guarantees that after passage to the limit in (1.17) the solenoid will not "self-cancel", and (1.19) guarantees that when making our decompositions into solenoids we do not leave the support.

Now we state our main results. Denote by $\mathfrak{C}_l$ the set of all oriented curves of length l with $\mathscr{D}$-topology.

Theorem A. *Let $l > 0$. If $T \in \mathrm{Sol}$, then T can be decomposed into elements of $\mathfrak{C}_l$. To be more precise, there exists a finite Borel measure μ on $\mathfrak{C}_l$ with $\operatorname{var}\mu = \operatorname{var}(T)/l$ such that for $J = \mathfrak{C}_l$, relations* (1.9) *and* (1.10) *hold, and, moreover,*

$$\frac{2}{l}\|T\| \ge \int_{\mathfrak{C}_l} \|\operatorname{div} R\| \, d\mu(R), \tag{1.21}$$

$$\frac{\|T\|}{l} = \int_{\mathfrak{C}_l} \delta_{b(R)} \, d\mu(R) = \int_{\mathfrak{C}_l} \delta_{e(R)} \, d\mu(R) \tag{1.22}$$

($b(R)$ is the origin of the curve R, $e(R)$ is its end; the two parts of (1.21) *are measures).*

Theorem B. *Every solenoid can be decomposed into elementary solenoids.*

By virtue of Remark 3, Theorem B is equivalent to Theorem B′:

Theorem B′. *We have* $\operatorname{extr} B_{\mathrm{Sol}} \subset \mathrm{elem}$, *where* elem *denotes the set of all elementary solenoids.*

Theorem C. *Any charge* $T \in \mathbb{N}_1(\mathbb{R}^n)$ *can be completely decomposed into a sum of two charges* P *and* Q *such that* $\operatorname{div} P = 0$ (*and Theorem A is applicable to* P) *and* Q *can be completely decomposed into simple oriented curves of finite length* (we prove in fact Theorem C for charges $T \in \mathbb{N}_{1,\mathrm{loc}}(\mathbb{R}^{n+1})$ of finite variation).

Remark 5. Once more we emphasize the fact that *the supports of μ-almost all curves involved in the decomposition in Theorem A lie in* $\operatorname{spt} T$. This is very important for applications. The same is true for Theorems B and C.

Remark 6. Theorem A yields only a "noncomplete" decomposition (1.9)–(1.10), but it is often more convenient for applications, because only charges of simple structure (curves) are involved, and for big l there is a good estimate (1.22) for the variation of the divergence.

1.4. Currents of an arbitrary dimension. Our theme can be generalized to normal currents of an arbitrary dimension m (the question settled by theorems A–C corresponds to $m = 1$). In this more general setting the role of normal charges goes to the space $\mathbb{N}_m(\mathbb{R}^n)$ of m-dimensional normal currents (i.e., functionals defined on differential forms of degree m). The variation of a charge becomes the mass of a current; the divergence of a charge becomes the boundary (see the details in [1]):

$$\operatorname{div} T(\varphi) = \pm T(d\varphi),$$

where d is the ordinary exterior differential of the form φ. The role of "simple" charges played (for $m = 1$) by curves goes to the "integral currents" (rectifiable currents with rectifiable boundary; see [1; 4.1.24, 4.1.28] for a several other equivalent definitions are discussed. The space of all integral m-dimensional currents in $\mathbb{R}^n$ is denoted by $\mathbb{I}_m(\mathbb{R}^n)$. In accordance with [1, 4.2.25], in the case that we are studying here ($m = 1$) an arbitrary $T \in \mathbb{I}_1(\mathbb{R}^n)$ can be completely decomposed into a countable sum of simple oriented curves R_j, $j \in \mathbb{N}$, of finite length:

$$R = \sum_{j=1}^{\infty} R_j,$$

$$\operatorname{var}(R) = \sum_{j=1}^{\infty} \operatorname{var}(R_j),$$

$$\operatorname{var}(\operatorname{div} R) = \sum_{j=1}^{\infty} \operatorname{var}(\operatorname{div} R_j).$$

Hence the decomposability of a one-dimensional current into integral currents (see below) is equivalent to its decomposability into curves.

Let us briefly discuss decomposition problems for an arbitrary m.

We ask whether every $T \subset \mathbb{N}_m(\mathbb{R}^n)$ (or $T \subset \mathrm{Sol}_m$) can be decomposed into "simple" currents lying in a set $J \subset \mathbb{I}_m(\mathbb{R}^n)$ (in other words, whether we have $\operatorname{extr} B_N \subset \mathbb{I}_m(\mathbb{R}^n)$ or $\operatorname{extr} B_{\mathrm{Sol}} \subset \mathbb{I}_m(\mathbb{R}^n)$). If the answer is negative, it is desirable to find a description of admissible sets J.

The degenerate cases $m = 0$ and $m = n$, $\partial T = 0$, are not interesting (if $m = 0$, then we can decompose into point unit masses; if $m = n$, then $T = 0$).

If $m = n - 1$, $\operatorname{div} T = 0$, then $T = \operatorname{div} S$, where S is a current of dimension n with finite mass, which allows one to reduce this case to the case $m = n$. Here S

can be identified with a (scalar) locally summable function and T with its gradient; S turns out to belong to the class BV of functions with bounded variation (see [9]). The decomposition we are looking for is given by the Fleming-Rishel formula (see [9, 10] and [1], 4.5.9 (13)):

$$\nabla S = \int_{-\infty}^{+\infty} \nabla \chi_{E_t}\, dt,$$

$$\|\nabla S\| = \int_{-\infty}^{+\infty} \|\nabla \chi_{E_t}\|\, dt,$$

which yields a decomposition of S (and hence of T) into currents corresponding to the characteristic functions of the Lebesgue sets E_t of the function S.

If $m = n - 1$ and $\operatorname{div} T \neq 0$, then T can be decomposed into integral currents provided that its boundary is rectifiable: $\operatorname{div} T \in \mathbb{I}_{n-2}(\mathbb{R}^n)$; see [11]. But if $\operatorname{div} T \notin \mathbb{I}_{n-2}(\mathbb{R}^n)$, then this is impossible (in general), and, probably, no simple description of the extreme points of the corresponding unit ball exists. A counterexample (see the details in [12]) is based on the fact that for the current

$$T = dx \wedge dz \wedge \mathscr{L}^3 + dy \wedge dz \wedge \mathscr{L}^3 \llcorner \{z > 0\} \in \mathbb{N}_{2,\mathrm{loc}}(\mathbb{R}^3)$$

the decompositions into integral currents represented by "half-planes", naturally arising in the half-spaces $\{z < 0\}$ and $\{z > 0\}$, fail to "merge" on the border (the plane $z = 0$).

If $1 \le m \le n - 2$, then no decomposition into integral currents exists in general, even if $\operatorname{div} T = 0$. Already for $m = 1$, $n = 3$ counterexamples can easily be given for C^∞-currents (see Examples 2, 3). "Merging" in $\mathbb{R}^4$ the two counterexamples of the preceding paragraph disposed at a certain angle to each other, we can construct a current from $\mathbb{N}_{2,\mathrm{loc}}(\mathbb{R}^4)$ with zero boundary that cannot be decomposed even locally near the points of some 2-dimensional subspace in $\mathbb{R}^4$. Moreover, Zwarski have shown that for $2 \le m < n - 2$ local decompositions of C^∞-currents invoke some compatibility conditions (like in the Frobenius theorem emerging in these problems; see [8]); so, a C^∞-current can be nondecomposable into integral currents (even locally).

1.5. Sketch of the proof. There is a well-known correspondence between vector fields of zero divergency (smooth solenoidal charges) and noncompressible flows (they are also called 'currents', but we use the term 'flow' to avoid confusion). If a smooth solenoidal charge is given, then the functions determining the elementary solenoids it can be decomposed into, are trajectories of points moving in the corresponding flow with velocity one.

Therefore, it is desirable to learn to "trace the trajectories" for arbitrary $T \in \mathrm{Sol}$. If we simply smooth T out, then we will have problems with singular points of the corresponding vector field (it is difficult to follow the trajectories there). To avoid these difficulties, we extend T to a charge T' in $\mathbb{R}^{n+1}$, so as if we want to add the time coordinate to the corresponding flow (whose existence is not clear as yet) in order that the $(n+1)$-th coordinate of a moving point change uniformly. After smoothing in an appropriate way the charge obtained, we do not get singularities (the $(n+1)$-th component of the corresponding vector field is always positive), and a decomposition into "trajectories" presents no problem. Moreover, the angle between the vectors and the $(n+1)$-th coordinate axis will not exceed $45°$, which will immediately imply that a *global decomposition* of T' into trajectories "almost parallel" to the $(n+1)$-th axis exists. After that, approximating T' by smoothed charges, we arrive at a decomposition of T' itself, and it remains to project this decomposition to $\mathbb{R}^n$, obtaining a decomposition of T (in fact, the matter is somewhat more complicated: we prove Theorem A in this way, and then obtain elementary solenoids with the help of Birkhoff–Khinchin ergodic theorem).

In case $\operatorname{div} T \neq 0$ (Theorem C) our flow has a positive and a negative source (the boundary of T), and we prove Theorem C by checking (with the help of simple estimates) that almost every point coming from the "positive source" arrives eventually at the "negative" one (precisely the sum of such curves will give the second charge). This is done with the help of Theorem A, but at the beginning we add a charge of simple structure to T (embedded into $\mathbb{R}^{n+1}$), to ensure the triviality of the boundary.

We conclude with some words concerning a way to formalize intuitive arguments about flows. If T is a smooth solenoidal charge, then it does give rise to a flow (a group of transformations of $\mathbb{R}^n$) preserving the measure $\|T\|$. The generator of this group in the complex space $L_2(\|T\|)$ is a selfadjoint extension of the following symmetric operator A:

$$Af(x) = \frac{1}{i}\langle \nabla f, \mathbf{T}\rangle(x) = \frac{1}{i}\frac{\partial f}{\partial \mathbf{T}}, \quad f \in C_0^\infty(\mathbb{R}^n).$$

This definition of A makes sense for every solenoid T. It is natural to try to construct an appropriate unitary group and to obtain a "flow" in the general case, thus solving our main problem. Unfortunately, in the case of an arbitrary solenoidal charge T the operator A is only symmetric, but not selfadjoint (which would be necessary to construct a semigroup). However, it is possible to return to the problem of existence of a group or a semigroup after we have Theorem B: then with every solenoidal charge T, one can associate a certain Markov process with invariant measure $\|T\|$ (an analog of a flow) and the semigroup of operators related to this process.

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§2. The proof of Theorems A and B

2.1. Notation. We work with two spaces, $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$. The generic notation of points of $\mathbb{R}^{n+1}$ will be capital letters $(X, Y, Z, \dots)$, whereas points of $\mathbb{R}^n$ will be denoted by small letters $(x, y, z, \dots)$. We identify $\mathbb{R}^{n+1}$ with $\mathbb{R}^n \times \mathbb{R}$. So, the relation $X = (x, t)$, where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, means that

$$X = (X_1, \dots, X_{n+1}), \quad X_1 = x_1, \dots, X_n = x_n, X_{n+1} = t.$$

We identify a vector $v \in \mathbb{R}^n$ with $(v, 0) \in \mathbb{R}^{n+1}$. Accordingly, a set (a measure, a charge) $E \times F$ in $\mathbb{R}^{n+1}$ will mean the cartesian product of sets (measures, charges) E and F in $\mathbb{R}^n$ and $\mathbb{R}$, respectively. We denote by $\mathscr{P}$ and $\mathscr{Q}$ orthogonal projections of $\mathbb{R}^{n+1}$ onto $\mathbb{R}^n$ and $\mathbb{R}$ (the latter is often interpreted as the "time axis").

2.2. The class $\mathscr{A}$. We call a local charge R in $\mathbb{R}^{n+1}$ *almost parallel* to $\mathbb{R}$ if for $\|R\|$-almost every $X \in \mathbb{R}^{n+1}$

$$|\mathscr{P}\mathbf{R}(X)| \le \mathscr{Q}\mathbf{R}(X) \tag{2.1}$$

($|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^{n+1}$). If R is a curve (see 1.3, Example 1), then (2.1) means that the angle between the unit tangent vector $\mathbf{R}(X)$ and $\mathbb{R}$ does not exceed 45°.

We say that a charge $R \subset \mathbb{N}_{1,\mathrm{loc}}(\mathbb{R}^{n+1})$ *belongs to the class* $\mathscr{A}$ if it is almost parallel to $\mathbb{R}$ and if there is a Lipschitz vector function $F: \mathbb{R} \to \mathbb{R}^{n+1}$ such that

$$(2.2)\qquad \begin{aligned} &(\mathrm{a}) \quad \|F'(u)\| = 1, \\ &(\mathrm{b}) \quad R = F_\sharp\overrightarrow{(-\infty, +\infty)}, \\ &(\mathrm{c}) \quad \lim_{u\to+\infty} \mathscr{C}F(u) = +\infty, \quad \lim_{u\to-\infty} \mathscr{C}F(u) = -\infty. \end{aligned}$$

Relation (b) means that for every $\varphi \in \mathscr{D}^1(\mathbb{R}^{n+1})$

$$(2.3)\qquad R(\varphi) = \int_{-\infty}^{+\infty} \langle F'(u), \varphi(F(u))\rangle \, du.$$

Putting $N(= N(\varphi)) := \max\{\|\mathscr{C}X\| : X \in \operatorname{spt}\varphi\}$, $R_N := F_\sharp\overrightarrow{[-N; N]}$, we rewrite (2.3) in the following form:

$$R(\varphi) = R_N(\varphi).$$

Thus, R acts *locally* as a curve. Clearly, $\mathscr{A} \subset \mathrm{Sol}_{\mathrm{loc}}$, since for every function $G \subset C^\infty(\mathbb{R}^{n+1})$ with compact support

$$R(\nabla G) = \int_{-N}^{N} \frac{dG(F(t))}{dt}\, dt = G(F(N)) - G(F(-N)) = 0$$

if a positive number N is sufficiently large.

If $R \in \mathscr{A}$ and F is the corresponding parametrization of R, then

$$F'_{n+1}(u)(= (\mathscr{C}F(u)')) \geq \sqrt{\sum_{j=1}^{n} (F'_j(u))^2}$$

a.e. on $\mathbb{R}$ (because R is almost parallel to $\mathbb{R}$). Hence $F'_{n+1} > 0$ a.e., and $F_{n+1}(v) = F_{n+1}(0) + \int_0^v F'_{n+1}(t)\, dt$ is a strictly increasing function of v mapping $\mathbb{R}$ onto itself (see (2.2 (c))). Taking $t = F_{n+1}(u)$ as a new parameter, we easily prove the existence of a Lipschitz vector function $f: \mathbb{R} \to \mathbb{R}^n$ such that

$$(2.4)\qquad \operatorname{Lip}(f) \leq 1, \quad R = \varphi_\sharp\overrightarrow{(-\infty; +\infty)}, \quad \text{where } \varphi := (f(t), t).$$

Conversely, every Lipschitz function $f: \mathbb{R} \to \mathbb{R}^n$ with $\operatorname{Lip}(f) \leq 1$ (i.e., every $f \in \mathrm{Lip}_1$) gives rise to a local charge.

Trying to project an element $R \in \mathscr{A}$ to $\mathbb{R}^n$, we put formally

$$(\mathscr{P}_\sharp R)(\varphi) = R(\mathscr{P}^\sharp\varphi), \quad \varphi \in \mathscr{D}^1(\mathbb{R}).$$

But this relation is senseless since $\mathscr{P}^\sharp\varphi \notin \mathscr{D}^1(\mathbb{R}^{n+1})$, whereas R is only a local charge in $\mathbb{R}^{n+1}$. To avoid this difficulty, we consider "pieces" R_Δ of R corresponding to compact segments $\Delta = [a, b] \subset \mathbb{R}^n$. Namely, we put

$$R_\Delta := \chi_{S_\Delta} R \quad (= R \llcorner S_\Delta),$$

where S_Δ denotes the strip $\mathbb{R}^n \times \Delta$. If R is defined by (2.4), then $R_\Delta = \varphi_\sharp\overrightarrow{[a; b]}$. Clearly, R_Δ is a curve; $\mathscr{P}_\sharp R_\Delta$ is also a curve:

$$\mathscr{P}_\sharp R_\Delta = f_\sharp\overrightarrow{[a; b]}.$$

We make some important observations:

$$(2.5)\qquad \begin{cases} (\mathrm{a}) \quad \operatorname{var} R_\Delta < \sqrt{2}(b - a); \\ (\mathrm{b}) \quad \operatorname{var} \mathscr{P}_\sharp R_\Delta \leq (b - a); \\ (\mathrm{c}) \quad \text{if } \operatorname{var} \mathscr{P}_\sharp R_\Delta = b - a, \text{ then } \mathscr{P}_\sharp R_\Delta \text{ is a curve of length } (b - a). \end{cases}$$

Indeed, $\operatorname{var} R_\Delta = \operatorname{var} g_t\overrightarrow{[a; b]}$, and we have (a) (see (1.7)); the same estimate gives (b) since $f \in \mathrm{Lip}_1$; (c) follows from our definition of a curve of given length (see Example 1 in 1.3). □

2.3. Slow motions in $\mathbb{R}^{n+1}$. Let Lip_C^n denote the class of all Lipschitz vector functions $f: \mathbb{R} \to \mathbb{R}^n$ with $\mathrm{Lip}(f) \le C$. We take some $f \in \mathrm{Lip}_1^n$ and consider the vector-function

$$\varphi: t \mapsto (f(t), t).$$

We call it a *slow motion in* $\mathbb{R}^{n+1}$ and write $\varphi \in \mathrm{Slow}(\mathbb{R}^{n+1})$. The term "slow" refers to the fact that the speed $|\dot\varphi(t)|$ does not exceed an absolute constant ($=\sqrt{2}$), while its direction does not change essentially since the vectors $\dot\varphi(t)$ remain almost parallel to $\mathbb{R}$. Every $\varphi \in \mathrm{Slow}(\mathbb{R}^{n+1})$ gives rise to an element R_φ of $\mathscr{A}$:

$$R_\varphi := \varphi_\sharp\overrightarrow{(-\infty; +\infty)}.$$

The mapping $\varphi \mapsto R_\varphi$ ($\varphi \in \mathrm{Slow}(\mathbb{R}^{n+1})$) is a parametrization of $\mathscr{A}$, because (2.4) implies that it is surjective. Now our aim is to endow Slow $(\mathbb{R}^{n+1})$ by the structure of a compact metric space.

2.3.1. Let $\widehat{\mathbb{R}}^m$ stand for the one-point compactification of $\mathbb{R}^m$:

$$\widehat{\mathbb{R}}^m := \mathbb{R}^m \cup \infty.$$

Topologically, the space $\widehat{\mathbb{R}}^m$ is $\mathbb{S}^m \subset \mathbb{R}^{m+1}$. Transferring the natural metric of $\mathbb{S}^m$ to $\widehat{\mathbb{R}}^m$, we obtain a metric d on $\widehat{\mathbb{R}}^m$; with this metric $\widehat{\mathbb{R}}^m$ is a compact metric space.

We also need the set $\widehat{C}_m$ of all continuous functions $f: \mathbb{R} \to \widehat{\mathbb{R}}^m$ such that either $f(\mathbb{R}) \subset \mathbb{R}^m$ or $f(\mathbb{R}) = \{\infty\}$. We set

$$\Delta_j(f, g) = \max\{d(f(t), g(t)) : t \in [-j, j]\},$$

$$\Delta(f, g) := \sum_{j=0}^{\infty} 2^{-j}\Delta_j(f, g)/(1 + \Delta_j(f, g)), \quad f, g \in \widehat{C}_m.$$

The convergence corresponding to the distance Δ is the uniform convergence on every compact subset of $\mathbb{R}$. Let $\mathrm{Lip}_{C,m}^{\wedge} := \mathrm{Lip}_C^n \cup \{f_\infty\}$ (here f_∞ denotes the constant function $f_\infty \equiv \infty$), and let $C_n := \widehat{C}_n \setminus \{f_\infty\}$.

2.3.2. $\mathrm{Lip}_{C,m}^{\wedge}$ *is a compact subset of* $\widehat{C}_m$. Indeed, if $f_k \in \mathrm{Lip}_{C,m}^{\wedge}$, $f \in \widehat{C}_m$, $f_k \xrightarrow{\widehat{C}_m} f$ and $f \ne f_\infty$, then f is a continuous $\mathbb{R}^m$-valued vector function and $f_k(t) \xrightarrow{\mathbb{R}^m} f(t)$ for every $t \in \mathbb{R}$, hence $f \in \mathrm{Lip}_C^m$; so, $\mathrm{Lip}_{C,m}^{\wedge}$ is closed in $\widehat{C}_m$. Now we take an arbitrary sequence (f_k) lying in Lip_C^m; we shall show that there is a subsequence (f_{k_l}) converging in $\widehat{C}_m$. We can assume that $f_k \ne f_\infty$ for all k. If (f_k) is uniformly bounded on every compact interval (i.e., $\sup\{|f_k(t)| : |t| \le j, k = 1, 2, \dots\} < +\infty$, $j = 1, 2, \dots$), then we apply the Arzela–Ascoli theorem to each segment $[-j, j]$, $j = 1, 2, \dots$, and the standard diagonal process yields the desired sequence (k_l). If (f_k) is unbounded on a segment $[-j^*, j^*]$, then there is a sequence of integers (k_l) and a sequence (t_l), $|t_l| \le j^*$, such that $f_{k_l}(t_l) > l$. This implies that for $j > j^*$ and $t \in [-j, j]$ we have

$$|f_{k_l}(t)| \ge |f_{k_l}(t_l)| - |f_{k_l}(t) - f_{k_l}(t_l)| > l - 2Cj.$$

Therefore, $\Delta_j(f_{k_l}, f_\infty) \xrightarrow[l\to\infty]{} 0$, whence $\Delta(f_{k_l}, f_\infty) \xrightarrow[l\to\infty]{} 0$.

2.3.3. We define "the improper slow motion" φ_∞ in $\widehat{\mathbb{R}}^n \times \mathbb{R}$ by

$$\varphi_\infty(t) \equiv (\infty, t).$$

Let $\widehat{S} := \mathrm{Slow}(\mathbb{R}^{n+1}) \cup \{\varphi_\infty\}$, $\widehat{\mathscr{P}}(\varphi) := \mathscr{P}(\varphi)$ if $\varphi \in \mathrm{Slow}(\mathbb{R}^{n+1})$, $\widehat{\mathscr{P}}(\varphi_\infty) := f_\infty$. If a metric $\widehat{\Delta}$ in $\widehat{S}$ is defined by

$$\widehat{\Delta}(\varphi, \psi) := \Delta(\widehat{\mathscr{P}}\varphi, \widehat{\mathscr{P}}\psi), \qquad \varphi, \psi \in \widehat{S},$$

then the mapping $\widehat{\mathscr{P}}$ becomes an isometry of $\widehat{S}$ onto $\mathrm{Lip}^{\wedge}_{1,n}$. Hence, by 2.3.2, $\widehat{S}$ *is a compact metric space.*

2.4. The continuity of the mapping $\varphi \mapsto R_\varphi$. Here we endow $\mathscr{A}$ with the $\mathscr{D}$-topology and show that the above mapping is continuous.

2.4.1. Putting $\widehat{\mathscr{A}} := \mathscr{A} \cup \{0\}$, we make $\widehat{\mathscr{A}}$ a topological space taking the $\mathscr{D}$-topology from $\mathrm{Ch}_{\mathrm{loc}}$. The space $\widehat{\mathscr{A}}$ is metrizable. Indeed, let a countable set $\Gamma \subset \mathscr{D}^1(\mathbb{R}^{n+1})$ be uniformly dense in $\mathscr{D}^1(\mathbb{R}^{n+1})$. We can assume, moreover, that for any triple $(N, \varphi, \varepsilon)$, where $N > 0$ is an integer, $\varphi \in \mathscr{D}^1(\mathbb{R}^{n+1})$, $\mathrm{spt}\,\varphi \subset S_{[-N,N]} = \mathbb{R}^n \times [-N, N]$, and $\varepsilon > 0$, there is $\gamma \in \Gamma$ with support in $S_{[-N,N]}$ such that $\max_{\mathbb{R}^{n+1}} |\varphi - \gamma| < \varepsilon$. We fix some $T_0 \in \widehat{\mathscr{A}}$. Every neighborhood of T_0 contains its neighborhood of the form

$$\nu(T_0, A, \varepsilon) := \{\, T \in \widehat{\mathscr{A}} : \max_{\varphi \in A} |(T - T_0)(\varphi)| < \varepsilon \,\},$$

where $A \subset \mathscr{D}^1(\mathbb{R}^{n+1})$ is finite, $\varepsilon > 0$. We choose a large integer $N = N_A$ such that $\cup_{\varphi \in A} \mathrm{spt}\,\varphi \subset S_{[-N,N]}$. For every $\varphi \in A$ we can pick some $\gamma_\varphi \in \Gamma$ satisfying the following conditions: $\mathrm{spt}\,\gamma_\varphi \subset S_{[-N,N]}$ and $\max|\varphi - \gamma_\varphi| < \varepsilon/8N\sqrt{2}$. Let Γ' ($= \Gamma'(A, \varepsilon)$) be the set of all functions γ_φ, $\varphi \in A$. Then

$$\nu_0 := \nu(T_0, \Gamma', \varepsilon/2) \subset \nu(T_0, A, \varepsilon).$$

To prove this inclusion, we take $T \in \nu_0$ and put $R := T - T_0$; clearly, for every $\varphi \in A$

$$\begin{aligned} |R(\varphi)| &\le |R(\gamma_\varphi)| + |R(\varphi) - R(\gamma_\varphi)| \\ &< \varepsilon/2 + (\mathrm{var}\, T_{[-N,N]} + \mathrm{var}(T_0)_{[-N,N]}) \max|\varphi - \gamma_\varphi| \\ &< \varepsilon/2 + 4\sqrt{2}N \cdot \max|\varphi - \gamma_\varphi| < \varepsilon \end{aligned}$$

(we use (2.5 (a))). Hence, the $\mathscr{D}$-topology on $\widehat{\mathscr{A}}$ coincides with the Γ-topology (the latter can be defined by the metric

$$d(T, T_0) := \sum_{j=0}^{\infty} 2^{-j} \frac{|R(\gamma_j)|}{1 + |R(\gamma_j)|},$$

where $\Gamma = \{\gamma_1, \gamma_2, \dots\}$).

2.4.2. Now we consider a mapping $B : \widehat{S} \to \widehat{\mathscr{A}}$ defined by

$$B(\varphi) := R_\varphi(\varphi \in \mathrm{Slow}(\mathbb{R}^{n+1})), \qquad B(\varphi_\infty) := 0$$

(see 2.3.3), and prove its continuity. The spaces $\widehat{S}$ and $\widehat{\mathscr{A}}$ being metric, it suffices to prove the sequential continuity. We take $\Phi \in \mathscr{D}^1(\mathbb{R}^{n+1})$, $\varphi \in \mathrm{Slow}(\mathbb{R}^{n+1})$, and a sequence (φ_i) of slow motions tending to φ (in $\mathrm{Slow}(\mathbb{R}^{n+1})$, i.e., in C_{n+1}). To prove that $R_{\varphi_j}(\Phi) \to R_\varphi(\varphi)$, for an arbitrary sequence $j_1 < j_2 < \dots$ we shall find a sequence $k_1 < k_2 < \dots$ such that

$$R_{\varphi_{J(l)}}(\varphi) \to R_\varphi(\varphi),$$

where $J(l) = j_{k_l}$. Let $f_j = \mathscr{P}(\varphi_j)$, then $\varphi_j(t) = (f_j(t), t)$. Clearly, $\varphi(t) = (f(t), t)$, where $f_n \xrightarrow{C_n} f$. We denote $N_\varphi = \max\{|\mathscr{O}X| : X \in \operatorname{spt}(\varphi)\}$ and choose (k_l) in such a way that $f_{J(l)}$ converges weakly in $L^2([-N_\varphi, N_\varphi])$ to a vector-function λ. This is possible, since $|f'| \le 1$ a.e. Passing to the limit in the identity

$$f_{J(l)}(v) = f_{J(l)}(0) + \int_0^v f'_{J(l)}(t)\,dt, \qquad v \in [-N_\varphi, N_\varphi],$$

we see that $f(v) = f(0) + \int_0^v \lambda(t)\,dt$, $v \in [-N_\varphi, N_\varphi]$, whence $\lambda = f'$ a.e. Thus,

$$R_{\Phi_{J(l)}}(\varphi) = \int_{-N_\varphi}^{N_\varphi} \langle \mathscr{P} f'_{J(l)}(t), \varphi(f_{J(l)}(t), t)\rangle dt + \int_{-N_\varphi}^{N_\varphi} \varphi_{n+1}(f_{J(l)}(t), t) dt.$$

Now let $\varphi_j \to \varphi_\infty$ in $\widehat{S}$ and $\varphi \in \mathscr{D}^1(\mathbb{R}^{n+1})$. Then

$$d(\mathscr{P}(\varphi_j(t)), \infty) \xrightarrow[j\to\infty]{} 0$$

uniformly on $[-N_\varphi, N_\varphi]$. Hence $\operatorname{spt}\varphi \cap \operatorname{spt} R_{\varphi_j} = \varnothing$ for large values of j, and $B(\varphi_j)(\varphi) = 0$. But this means that $B(\varphi_j) \to B(\varphi_\infty) = 0$.

2.4.3. Being the image of a compact space $\widehat{S}$ under a continuous mapping B, the space $\widehat{\mathscr{A}}$ is compact (in the $\mathscr{D}$-topology). In what follows, we shall also need restrictions of slow motions to compact time intervals $\Delta = [a, b]$. For $R \in \mathscr{A}$ we put $r_\Delta(R) = R \llcorner S_\Delta$, where $S_\Delta := \mathbb{R} \times \Delta$; we also put $r_\Delta(\varphi_\infty) := 0$. It is easy to see that r_Δ is continuous on $\widehat{\mathscr{A}}$. Consequently, $\widehat{\mathscr{A}_\Delta} := r_\Delta(\widehat{\mathscr{A}})$ is $\mathscr{D}$-compact. The set $\mathscr{A}_\Delta := r_\Delta(\mathscr{A})$ consists of curves. If $R \in \mathscr{A}_\Delta$, then $\operatorname{spt}(R) \subset S_\Delta$, $\mathscr{O}b(R) = a$, $\mathscr{O}e(R) = b$.

Now we turn to the proof of Theorems A and B.

2.5. The first step: extending T to a local charge on $\mathbb{R}^{n+1}$. Let $T \in \operatorname{Sol}(\mathbb{R}^n)$ be the given nondegenerate solenoid in $\mathbb{R}^{n+1}$.

We define a *local* charge T' on $\mathbb{R}^{n+1}$ ("the extension of T") by

$$\begin{aligned} T' &:= T \times \mathscr{L}^1 + \|T\| \times e_{n+1}\mathscr{L}^1 \\ &= (\mathbf{T} + e_{n+1})(\|T\| \times \mathscr{L}^1) = (T_1 \times \mathscr{L}^1, \dots, T_n \times \mathscr{L}^1, \mathbb{T} \times \mathscr{L}^1). \end{aligned}$$

In other words,

$$(*) \qquad T'(\varphi) = \int_{-\infty}^{+\infty} T(\varphi^t)\,dt + \int_{-\infty}^{+\infty} dt\Big(\int_{\mathbb{R}^n} \varphi(x, t)\,d\mathbf{T}(x)\Big), \qquad \varphi \in D^1(\mathbb{R}^{n+1}),$$

where $\varphi^t(x) := \varphi(x, t)$. Since the unit vectors $\mathbf{T}$ and e_{n+1} are orthogonal $(\|T\| \times \mathscr{L}^1)$-a.e., we obtain

$$\|T'\| = \sqrt{2}\|T\| \times \mathscr{L}^1, \qquad T' = \frac{1}{\sqrt{2}}(T + e_{n+1}).$$

Also, we need the following properties of T':

1) $T' \in \operatorname{Sol}_{1,\mathrm{loc}}(\mathbb{R}^{n+1})$;
2) T' *is almost parallel to* $\mathbb{R}$;

3) *if* $\Delta := [a, b]$, $S_\Delta := \mathbb{R}^n \times \Delta$, $T'_\Delta := T' \llcorner S_\Delta$, *then*

$$\operatorname{div} T'_\Delta = \|T\| \times (\delta_a - \delta_b), \qquad \operatorname{var} T'_\Delta = l\sqrt{2}\operatorname{var} T;$$

4) $\mathscr{P}_1 T'_\Delta = (b-a)T$.

Proof. Property 1) follows from (∗), where we put $\varphi = \nabla u$, u being a C^∞-function on $\mathbb{R}^{n+1}$ with compact support (one can also use (1.3), (1.4)).

Property 2) is obvious (since $\mathscr{P}\mathbf{T}' = \mathbf{T}/\sqrt{2}$, $\mathscr{Q}\mathbf{T}' = e_{n+1}/\sqrt{2}$). To prove 3), we note that

$$\begin{aligned}\operatorname{div} T'_\Delta &= \operatorname{div} T \times \mathscr{L}^1 + \|T\| \times \operatorname{div}[a; b] = \|T\| \times (\delta_b - \delta_a);\\ \operatorname{var}(T'_\Delta) &= \|T'\|(\mathscr{S}_\Delta) = (\sqrt{2}\|T\| \times \mathscr{L}^1)(\mathscr{S}_\Delta)\\ &= \sqrt{2}\|T\|(\mathbb{R}^n)\mathscr{L}^1[a, b] = \sqrt{2}\operatorname{var}(T)(b-a) = l\sqrt{2}\operatorname{var}(T).\end{aligned}$$

To prove 4), we recall that

$$(\mathscr{P}^t\varphi)(X_1, \dots, X_n, X_{n+1}) = \varphi(X_1, \dots, X_n), \quad \varphi \in \mathscr{D}^1(\mathbb{R}^n),$$

hence for every charge R in $\mathbb{R}^{n+1}$

$$(\mathscr{P}_1 R)(\varphi) = R(\mathscr{P}^t\varphi) = \int_{\mathbb{R}^{n+1}} \langle \mathbf{R}(X_1, \dots, X_n, X_{n+1}), \varphi(X_1, \dots, X_n)\rangle \, d\|R\|(X).$$

For $R = T'_\Delta$ we get

$$\begin{aligned}(\mathscr{P}_1 T'_\Delta)(\varphi) &= \int_{\mathbb{R}^n \times [a;b]} \langle \mathbf{T}', \varphi(\mathscr{P}X)\rangle \, d\|T'\| \times \mathscr{L}^1(X)\\ &= \int_a^b \int_{\mathbb{R}^n} \langle \mathbf{T}, \varphi(\mathscr{P}X)\rangle \, d\|T\|(X)\, dt = lT(\varphi). \quad \square\end{aligned}$$

2.6. The second step: a regularization of T'. The charge T and the scalar measure $\|T\|$ will be regularized separately. We choose an everywhere positive function $\Phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \Phi(x)\, d\mathscr{L}^n(x) = 1$. If we put

$$\Phi_\varepsilon(x) := \varepsilon^{-n}\Phi\left(\frac{x}{\varepsilon}\right),$$

then a standard argument shows that for every vector charge T of finite variation we have

$$\begin{gathered}\operatorname{var}(T * \Phi_\varepsilon) \le \operatorname{var}(T);\\ T * \Phi = \sigma\mathscr{L}^n, \quad \text{where } \sigma \text{ is a } C^\infty\text{-smooth vector field},\\ T * \Phi_\varepsilon \xrightarrow{\mathscr{D}} T \quad \text{as } \varepsilon \to 0.\end{gathered}$$

We denote

$$\begin{aligned}\|T\|_\varepsilon &:= \|T\| * \Phi_\varepsilon = t_\varepsilon\mathscr{L}^n,\\ T_\varepsilon &:= T * \Phi_\varepsilon = \tau'_\varepsilon\mathscr{L}^n,\end{aligned}$$

where t_ε is an everywhere positive C^∞-function (because $\|T\| \ne 0$ and Φ_ε is everywhere positive), and τ'_ε is a C^∞-smooth vector field on $\mathbb{R}^n$. Finally, we introduce a charge T'_ε on $\mathbb{R}^{n+1}$ by setting

$$\begin{aligned}T'_\varepsilon &:= T_\varepsilon \times \mathscr{L}^1 + \|T\|_\varepsilon \times e_{n+1}\mathscr{L}^1\\ &= \left((T_1)_\varepsilon \times \mathscr{L}^1, \dots, (T_n)_\varepsilon \times \mathscr{L}^1, \|T\|_\varepsilon \times \mathscr{L}^1\right) = (\tau'_\varepsilon, t_\varepsilon)\mathscr{L}^{n+1}.\end{aligned}$$

Precisely this charge will be regarded as a regularization of T'.

We shall use the following properties of T'_ε:

1) $T'_\varepsilon \in \mathrm{Sol}_{\mathrm{loc}}$;

2) $T'_\varepsilon = \tau_\varepsilon$, *where* τ_ε *is a* C^∞*-smooth vector field on* $\mathbb{R}^{n+1}$;

3) *for every* $X \subset \mathbb{R}^{n+1}$

$$\tau_\varepsilon(X) \neq 0, \qquad |\mathscr{P}\tau_\varepsilon(X)| \leq \mathscr{Q}\tau_\varepsilon(X).$$

Property 3) means that τ_ε is almost parallel to the $(n+1)$-th axis. Next, putting $T'_{\varepsilon,\Delta} := T'_\varepsilon \llcorner \mathscr{S}_\Delta$ we have

4) $T'_{\varepsilon,\Delta} \subset \mathbb{N}_{1,\mathrm{loc}}(\mathbb{R}^{n+1})$, $\mathrm{var}(T'_{\varepsilon,\Delta}) < l\sqrt{2}\,\mathrm{var}(T)$, $\mathrm{var}(\mathrm{div}\, T'_{\varepsilon,\Delta}) < 2\,\mathrm{var}(T)$;

5) $T'_{\varepsilon,\Delta} \xrightarrow{\mathscr{D}} T'_\Delta$ *as* $\varepsilon \to 0$.

Proof. Clearly, $\mathrm{div}\, T_\varepsilon = \mathrm{div}(T * \Phi_\varepsilon) = (\mathrm{div}\, T) * \Phi_\varepsilon = 0$; now the proof of 1) can be completed as the proof of property 1) of T'. To prove 2), one can simply put $\tau_\varepsilon := (\tau'_\varepsilon, t_\varepsilon)$. Then $\tau_\varepsilon(X) \neq 0$ since $t_\varepsilon(X) > 0$. Moreover, $\|\mathscr{P}\tau_\varepsilon\| = \|\tau'_\varepsilon\| < t_\varepsilon = \mathscr{Q}\tau_\varepsilon$, and we have 3). The first part of 4) is obvious. To prove the second, we notice that, by 3),

$$\|\tau_\varepsilon\| = \sqrt{\|\tau'_\varepsilon\|^2 + t_\varepsilon{}^2} \leq \sqrt{2}t_\varepsilon,$$

whence

$$\begin{aligned}\mathrm{var}\, T'_{\varepsilon,\Delta} = \|T'_\varepsilon\|(S_\Delta) &= \int_{S_\Delta} |\tau_\varepsilon|\, d\mathscr{L}^{n+1} \leq \sqrt{2}\int_{S_\Delta} t_\varepsilon(x)\, d\mathscr{L}^{n+1}(X) \\ &= l\sqrt{2}\int_{\mathbb{R}^n} t_\varepsilon(x)\, d\mathscr{L}^n(x) = l\sqrt{2}\,\mathrm{var}\,\|T\|_\varepsilon \leq l\sqrt{2}\,\mathrm{var}\, T.\end{aligned}$$

To finish the proof of 4) it remains to note that

$$\mathrm{div}\, T'_{\varepsilon,\Delta} = \|T\|_\varepsilon \times (\delta_a - \delta_b).$$

To prove 5), we observe that

$$T'_{\varepsilon,\Delta} = T_\varepsilon \times (\mathscr{L}^1 \llcorner \Delta) + \|T\|_\varepsilon \times \overrightarrow{[a\,;\,b]},$$
$$T'_\Delta = T \times (\mathscr{L}^1 \llcorner \Delta) + \|T\| \times \overrightarrow{[a\,;\,b]}.$$

But $T_\varepsilon \xrightarrow{\mathscr{A}} T$ and $\|T_\varepsilon\| \xrightarrow{\mathscr{A}} \|T\|$, whence $T'_{\varepsilon,\Delta} \xrightarrow{\mathscr{A}} T'_\Delta$.

2.7. The third step: a decomposition of the smooth charge T'_ε into elements of $\mathscr{A}$. The field τ_ε is solenoidal and almost parallel to the $(n+1)$-th axis. We start analyzing it with the following obvious remark. If a C^∞-smooth vector field σ on $\mathbb{R}^{n+1}$ is solenoidal and parallel to $\mathbb{R}$ (i.e., if $\mathscr{P}\sigma(X) = 0$), then σ is constant on every line $z \times \mathbb{R}$. Indeed, $\mathrm{div}\sigma = \frac{\partial \sigma_{n+1}}{\partial x_{n+1}}$, hence $\sigma(X) = (0, \ldots, 0, \sigma_{n+1}(X))$ depends on X_{n+1} only. Therefore, the solenoidal charge $\sigma\mathscr{L}^{n+1}$ corresponding to the field σ can be decomposed into oriented lines parallel to the $(n+1)$-th axis. Marking every such line by the point at which it intersects $\mathbb{R}^n$, we can say that the corresponding "decomposition measure" is $\sigma_{n+1}\mathscr{L}^n$.

2.7.1. Now we return to the solenoidal vector field τ_ε, almost parallel to the $(n+1)$-th axis. We are going to show that a suitable rectifying diffeomorphism G makes it parallel to $\mathbb{R}$ (but still solenoidal). Applying the inverse transformation $H = G^{-1}$, we decompose T'_ε into (infinite) curves, namely, into the images of the lines parallel to the $(n+1)$-th axis.

Since τ_ε is almost parallel to the $(n+1)$-th axis (property 3) of T'_ε), such a global rectification does exist, i.e., *there is a C^∞-diffeomorphism $G\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ preserving the $(n+1)$-th coordinate and such that the field $\sigma := \mathbf{D}G \cdot \tau_\varepsilon$ is parallel to the $(n+1)$-th axis ($\mathbf{D}G$ is the Jacobi matrix of G).*

Proof. We consider the following Cauchy problem:

$$\begin{cases} Y(t_\circ) = Y^\circ, & t_\circ \in \mathbb{R},\ Y^\circ \in \mathbb{R}^{n+1}, \\ \dot{Y} = \tilde{\tau}_\varepsilon(Y), & \tilde{\tau}_\varepsilon := \dfrac{\tau_\varepsilon}{(\tau_\varepsilon)_{n+1}}. \end{cases} \tag{2.6}$$

It is equivalent to

$$\begin{cases} Y_j(t_\circ) = Y_j^\circ, \\ \dfrac{dY_j}{dt}(t) = (\tilde{\tau}_\varepsilon)_j(Y_1, \dots, Y_n, t), \quad j = 1, \dots, n, \end{cases} \tag{2.7}$$

since (2.6) implies $\dot{Y}_{n+1} = 1$. From the obvious estimate

$$\sum_{j=1}^{n} (\tilde{\tau}_\varepsilon)_j^2 = \|\tilde{\tau}_\varepsilon\|^2 \le 1$$

and the Picard theorem it follows that the Cauchy problem (2.7) has a unique C^∞-solution defined everywhere on $\mathbb{R}$. Hence the same is true for the problem (2.6); its solution will be denoted by $Y_{t_\circ, Y^\circ}$.

The transformation

$$G\colon Y \longmapsto \left(\mathscr{P} Y_{Y_{n+1}, Y}(0),\ Y_{n+1}\right) \in \mathbb{R}^{n+1}$$

is a C^∞-diffeomorphism of $\mathbb{R}^{n+1}$ onto itself preserving the $(n+1)$-th coordinate and transforming the trajectory of every solution of (2.6) into a line parallel to $\mathbb{R}$, since for $Y^\circ \in \mathbb{R}^{n+1}$ we have

$$\mathscr{P} G\left(Y_{t_\circ, Y^\circ}(t)\right) = \mathscr{P} Y_{t_\circ, Y^\circ}(0), \quad t \in \mathbb{R}.$$

Differentiating this identity in t and putting $t = t_\circ$ yields

$$\mathscr{P}\left(\mathbf{D}G(Y^\circ) \cdot \tilde{\tau}_\varepsilon(Y^\circ)\right) = 0$$

for every $Y^\circ \in \mathbb{R}^{n+1}$. Hence the field $\tilde{\sigma} := \mathbf{D}G \cdot \tilde{\tau}_\varepsilon$ is orthogonal to the hyperplane $\mathbb{R}^n$. The same is true for $\sigma := \mathbf{D}G \cdot \tau_\varepsilon$.

2.7.2. Substituting G and $T'_\varepsilon = \tau_\varepsilon \mathscr{L}^{n+1}$ to (1.5) and changing the variable under the integral sign, we can write

$$\begin{aligned} (G_\sharp T'_\varepsilon)(\varphi) &= \int_{\mathbb{R}^n} \langle \mathbf{D}G \cdot \tau_\varepsilon, \varphi(G) \rangle \, d\mathscr{L}^{n+1} \\ &= \int_{\mathbb{R}^n} \langle \mathbf{D}G(H) \cdot \tau_\varepsilon(H), \varphi \rangle |\mathbf{J}H| \, d\mathscr{L}^{n+1}, \end{aligned}$$

where $H := G^{-1}$ and $\mathbf{J}H$ is the Jacobian of H. In other words,

$$G_\sharp T'_\varepsilon = \mathbf{D}G(H) \cdot \tau_\varepsilon(H) \cdot |\mathbf{J}H| \mathscr{L}^{n+1}.$$

The field on the right-hand side is parallel to the $(n+1)$-th axis (we have proved this for a similar field without the factor $\mathbf{J}H$). Besides, this field is solenoidal, since, in accordance with (1.8),

$$\operatorname{div} G_\sharp T'_\varepsilon = G_\sharp \operatorname{div} T'_\varepsilon - G_\sharp 0 = 0.$$

Hence, by the remark at the beginning of Section 2.7, this field does not depend on X_{n+1}:

$$\mathbf{D}G(H)\cdot\tau_\varepsilon(H)\cdot\mathbf{J}H = \rho(z)dX_{n+1}, \quad z\in\mathbb{R}^n,$$

and we get a decomposition of $G_\sharp T'_\varepsilon$ into lines parallel to $\mathbb{R}$:

$$G_\sharp T'_\varepsilon = \int_{\mathbb{R}^n} \delta_z\times\overrightarrow{(-\infty;\infty)}\cdot\rho(z)\,d\mathscr{L}^n(z). \tag{2.9}$$

2.7.3. Recalling that, by (1.6),

$$H_\sharp G_\sharp T'_\varepsilon = (HG)_\sharp T'_\varepsilon = (Id)_\sharp T'_\varepsilon = T'_\varepsilon,$$

we apply H to the two sides of (2.9):

$$T'_\varepsilon = \int_{\mathbb{R}^n} h_z\rho(z)\,d\mathscr{L}^n(z), \quad \rho=\rho_\varepsilon; \tag{2.9$'$}$$

here $h_z := H_\sharp(\delta_z\overrightarrow{(-\infty;+\infty)})\in\mathscr{A}$ is a local charge corresponding to the slow motion φ_z,

$$\varphi_z(t) := H(z,t) = (f(z,t),t). \tag{2.10}$$

The point $f(z,t)=\mathscr{P}H(z,t)$ is determined by the conditions

$$\dot f(z,t)\equiv\tau^*(f(z,t)) \quad ((z,t)\in\mathbb{R}^{n+1},\ f(z,0)=z), \tag{2.11}$$

where $\tau^* := \mathscr{P}\tau = \left(\frac{\tau_1}{\tau_{n+1}}, \frac{\tau_2}{\tau_{n+1}},\dots,\frac{\tau_n}{\tau_{n+1}}\right)$ (see (2.6) and (2.7)).

2.7.4. Now we "embed" the decomposition (2.9′) into the space $\widehat{S}$ of slow motions (see 2.3.3). The mapping $\Phi\colon\mathbb{R}^n\to\mathrm{Slow}(\mathbb{R}^{n+1})(\subset\widehat{S})$ defined by

$$z\mapsto\varphi_z \quad (z\in\mathbb{R}^n)$$

is continuous, because $f(z,t)\to f(z_0,t)$ uniformly in $t\in[a,b]$ as $z\to z_0$ (for every segment $[a,b]$). Putting $\Phi(\infty):=\Phi_\infty$, we see that Φ becomes a continuous mapping of $\widehat{\mathbb{R}}^n$ into $\widehat{S}$. This mapping is one-to-one, hence it is a homeomorphism.

The image $\nu(=\nu_\varepsilon)$ of the measure $\rho\mathscr{L}^{n+1}$ under Φ is

$$\nu_\varepsilon(\mathscr{E}) := \int_{\Phi^{-1}(\mathscr{E})}\rho\,d\mathscr{L}^{n+1}, \quad \mathscr{E}\in\mathscr{B}(\widehat{S}). \tag{2.10$'$}$$

Clearly, ν is a Borel measure on $\widehat{S}$ supported by a finite-dimensional compact subset $\Phi(\mathbb{R}^n)$ of $\widehat{S}$ (parameterized by $\widehat{\mathbb{R}}^n$ via Φ).

Now we can rewrite (2.9′) as

$$T'_\varepsilon = \int_{\widehat{S}} R_\gamma\,d\nu_\varepsilon(\gamma), \tag{2.11$'$}$$

where $R_\gamma =: B(\gamma)\in\mathscr{A}$ denotes the local charge on $\mathbb{R}^{n+1}$ corresponding to $\gamma\in\widehat{S}$ (see (2.4.2)).

2.7.5. The measure ν depends on ε ($\nu=\nu_\varepsilon$); we shall show that *the variation of ν_ε is uniformly bounded:*

$$\operatorname{var}\nu_\varepsilon\le\operatorname{var}T \quad \text{for all } \varepsilon>0. \tag{2.12}$$

For a local charge σ on $\mathbb{R}^{n+1}$, we put $\sigma_1 := \sigma \llcorner (\mathbb{R}^n \times [0, +\infty))$. Identifying the slow motion φ_z with the local charge $B(\varphi_z)$ (see 2.4.2), we have

$$\begin{aligned}\operatorname{var} \nu_\varepsilon &= \operatorname{var} \Phi_\sharp(\rho_\varepsilon, \mathscr{L}^n) - \int_{\mathbb{R}^n} \rho_\varepsilon \, d\mathscr{L}^n \\ &= \operatorname{var} \int_{\mathbb{R}^n} \delta_z \rho_\varepsilon(z) \, d\mathscr{L}^n(z) - \operatorname{var} \int_{\mathbb{R}^n} \operatorname{div}(\varphi_z) + \rho_\varepsilon(z) \, d\mathscr{L}^n(z) \\ &= \operatorname{var} \operatorname{div} \int_{\mathbb{R}^n} (\varphi_z) + \rho_\varepsilon(z) \, d\mathscr{L}^n(z) = \operatorname{var} \operatorname{div}(T_\varepsilon')_+ < \operatorname{var} T\end{aligned}$$

(the latter inequality was proved in 2.6.2). □

2.7.6. With every $a \subset \mathbb{R}$ one can associate the shift $\tau_a : \widehat{S} \to \widehat{S}$. Namely, if $\Phi \in \operatorname{Slow}(\mathbb{R}^{n+1})$, $\Phi(t) \equiv (f(t), t)$, and $\tau_a \Phi_\infty := \varphi_\infty$, then $(\tau_a \Phi)(t) := (f(t+a), t)$ $(t \in \mathbb{R})$. We shall prove that *the measure* λ_ε *defined by* (2.15) *is shift invariant.*

First of all, if $\dot{f}(t) = \tau^*(f(t))$, then $\dot{f}(t+a) \equiv \tau^*(f(t+a))$ (see (2.11) for the definition of τ^*). Hence the set $\Phi(\mathbb{R}^n) \subset \operatorname{Slow}(\mathbb{R}^{n+1})$ is shift invariant. We denote

$$g^a(z) := f(z, a)$$

and take a Borel set $\mathscr{E} \subset \Phi(\mathbb{R}^n)$. This set consists of motions φ_f, $\varphi_f(t) \equiv (f(t), t)$, where $\dot{f} = \tau^*(f)$ (see (2.11)). If

$$E := \Phi^{-1}(\mathscr{E}) = \{ \mathscr{P}\varphi(0) : \varphi \in \mathscr{E} \} = \{ f(0) : \dot{f} = \tau^*(f), \varphi_f \in \mathscr{E} \},$$

then $\nu(\mathscr{E}) = \int_E \rho \, d\mathscr{L}^n$. Clearly, $\Phi^{-1}(\tau_a(E)) = \{ f(a) : \dot{f} = \tau^*(f), \ \Phi_f \in \mathscr{E} \} = g^a(E)$, and $\nu(\tau_a(\mathscr{E})) = \int_{g^a E} \rho \, d\mathscr{L}^n =: I(a, E)$.

So, we must prove that $I(a, E)$ does not depend on a. To do this, we look at ρ $(= \rho_\varepsilon)$ more carefully. We have

$$\rho(z) = [(\mathbf{D}G)(H(z, t)) \cdot \tau(H(z, t))]_{n+1} \cdot |(\mathbf{J}H)(z, t)| \quad (z \in \mathbb{R}^n, \ t \in \mathbb{R}), \tag{2.13}$$

where $[w]_{n+1}$ is the $(n+1)$-th coordinate of $w \in \mathbb{R}^{n+1}$ But since G preserves the $(n+1)$-th coordinate, the expression in the square brackets in (2.13) is equal to $[\tau(H(z, t))]_{n+1}$, and $\tau(H(z, 0)) \equiv \tau(z)$ (since $H(z, 0) \equiv z$). Moreover, $H(z, t) = (f(z, t), t)$,

$$\mathbf{D}H(z, t) = \begin{pmatrix} f_z'(z, t) & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{J}H(z, t) \equiv \det f_z'(z, t).$$

But $f(z, 0) = z$ and $\mathbf{J}H(z, 0) = 1$. Taking $t = 0$ in (2.13) yields $\rho = \tau_{n+1}$. Multiplying the field τ^* (which determines the group g) by τ_{n+1}, we obtain $\mathscr{P}\tau$, which is a solenoidal field in $\mathbb{R}^n$. Hence $\operatorname{div} \tau_{n+1}\tau^* = 0$, and the Liouville theorem ([13] or [14]) implies that the function $a \mapsto I(a, E)$ is constant.

2.8. The final step: the weak limit of the measures ν_ε. We have a family $\{\nu_\varepsilon\}_{\varepsilon>0}$ of Borel measures on $\widehat{S}$, which are shift invariant and uniformly bounded (see 2.7.5 and 2.7.6). Now our aim is to pass to the limit in (2.11′) as ε tends to zero along a suitable sequence.

2.8.1. We take a sequence (ε_j), $\varepsilon_j \to 0$, $\varepsilon > 0$ and denote $\nu_j := \nu_{\varepsilon_j}$. The weak compactness of the unit ball in the space of real Borel measures on $\widehat{S}$ allows us to assume that

$$\nu_j \to \nu \quad \text{weakly}, \tag{2.14}$$

where ν is a Borel measure on $\widehat{S}$ (since $\nu_\varepsilon(\{\varphi_\infty\}) = 0$, this measure is supported on Slow). The limit relation (2.14) means that

$$\lim_{j\to\infty} \int_{\widehat{S}} \alpha\, d\nu_j = \int_{\widehat{S}} \alpha\, d\nu \quad \text{for all } \alpha \in C(\widehat{S}).$$

Every ν_j is shift invariant; therefore,

$$\int_{\widehat{S}} \alpha(\tau_a\gamma)\, d\nu_j(\gamma) = \int_{\widehat{S}} \alpha\, d\nu_j, \quad j = 1, 2, \dots, \ \alpha \in C(\widehat{S}).$$

By (2.14),

$$\int_{\widehat{S}} \alpha(\tau_a\gamma)\, d\nu(\gamma) = \int_{\widehat{S}} \alpha\, d\nu, \quad \alpha \in C(\widehat{S}),$$

whence ν is also shift invariant. In accordance with (2.12),

$$\text{(2.15)} \qquad \operatorname{var}\nu \le \liminf_{j\to\infty} \operatorname{var}\nu_j \le \operatorname{var} T.$$

To pass to the limit in (2.11′), we take a vector field $\varphi \in \mathscr{D}^1(\mathbb{R}^n)$ and put $\alpha(\gamma) := R_\gamma(\varphi)(= B_\gamma(\varphi))$, $\alpha(\varphi_\infty) = 0$. Clearly, $\alpha \in C(\widehat{S})$; by (2.14) and (2.11′) we have

$$\text{(2.16)} \qquad T'_{\varepsilon_j}(\varphi) = \int_{\widehat{S}} \alpha\, d\nu_j \xrightarrow[j\to\infty]{} \int_{\widehat{S}} \alpha\, d\nu.$$

But in 2.6.1 we proved that $T'_{\varepsilon_j,\Delta} \xrightarrow{\mathscr{M}} T'_\Delta$ for every segment $\Delta = [a, b]$, whence $T'_{\varepsilon_j} \xrightarrow{\mathscr{M}} T'$. Combining this with (2.15), we conclude that

$$\text{(2.17)} \qquad T' = \int_{\widehat{S}} R_\gamma\, d\nu(\gamma).$$

2.8.2. Now we finish the proof of Theorem A.

Having fixed $l > 0$ and a segment $[a, b] = \Delta$ of length l, we put $T'_\Delta := T' \llcorner S_\Delta$; $S_\Delta := \mathbb{R}^n \times \Delta$. Formula (2.17) means that

$$\text{(2.18)} \qquad T'(\varphi) = \int_{\widehat{S}} R_\gamma(\varphi)\, d\nu(\gamma)$$

for every test field $\varphi \in \mathscr{D}^1(\mathbb{R}^{n+1})$. Since T' is a local charge, (2.18) holds for every Borel measurable bounded vector field φ on $\mathbb{R}^{n+1}$. Hence, we can apply (2.18) to $\chi_\Delta \cdot \varphi$ (instead of $\varphi \in \mathscr{D}^1(\mathbb{R}^{n+1})$), where χ_Δ is the characteristic function of the strip S_Δ. We get

$$\text{(2.19)} \qquad T'_\Delta = \int_{\widehat{S}} (R_\gamma)_\Delta\, d\nu(\gamma) = \int_{\widehat{S}} R_{\gamma_\Delta}\, d\nu(\gamma),$$

where $(R_\gamma)_\Delta := R_\gamma \llcorner \Delta$, $\gamma_\Delta := \gamma|_\Delta$, and R_{γ_Δ} is the curve $\gamma_\sharp\overrightarrow{[a; b]}$ in $\mathbb{R}^{n+1}$. We define a mapping $\xi_\Delta : \widehat{S} \to \widehat{\mathscr{A}_\Delta}$ by

$$\xi_\Delta(\gamma) = R_{\gamma_\Delta} \quad (\gamma \in \operatorname{Slow}(\mathbb{R}^{n+1})), \qquad \xi_\Delta(\varphi_\infty) = 0.$$

Then $\xi_\Delta(\operatorname{Slow}(\mathbb{R}^{n+1})) = \mathscr{A}_\Delta$ (see 2.4.3). The mapping ξ_Δ is continuous and transforms ν into a Borel measure $\lambda_\Delta := (\xi_\Delta)_\sharp\nu$ on $\widehat{\mathscr{A}_\Delta}$, ($\lambda_\Delta(\{\varphi_\infty\}) = 0$, $\operatorname{var}\lambda_\Delta = \operatorname{var}\nu \le \operatorname{var} T$). Now, (2.19) becomes

$$\text{(2.20)} \qquad T'_\Delta = \int_{\mathscr{A}_\Delta} R\, d\lambda_\Delta(R).$$

Taking assertion 5) in 2.5 into account, we obtain

$$lT = \mathscr{P}_{\|}T'_\Delta = \int_{\mathscr{A}_\Delta} \mathscr{P}_{\|}R\, d\lambda_\Delta(R). \tag{2.21}$$

Let $\mu (= \mu_\Delta) := l^{-1}(\mathscr{P}_{\|})_{\|}\lambda_\Delta$ (i.e., μ is the image of $l^{-1}\lambda_\Delta$ under $\mathscr{P}_{\|}: \mathscr{A}_\Delta \to B_l$, where $B_l := \{ U \subset \mathrm{Ch}(\mathbb{R}^n) : \operatorname{var} U \le l \}$; we have $\mathscr{P}_{\|}\mathscr{A}_\Delta \subset B_l$ by virtue of (b) in (2.5)). It is clear that

$$T = \int_{B_l} R\, d\mu(R), \qquad l \operatorname{var}\mu = \operatorname{var}\lambda_\Delta = \operatorname{var}\lambda \le \operatorname{var} T. \tag{2.22}$$

But then

$$\operatorname{var} T \le \int_{B_l} \operatorname{var} R\, d\mu(R) \le \int_{B_l} l\, d\mu(R) = l \operatorname{var}\mu \le \operatorname{var} T,$$

whence $\int_{B_l} \operatorname{var} R\, d\mu(R) = \int_{B_l} l\, d\mu(R)$, $l \operatorname{var}\mu = \operatorname{var} T$, and

$$\operatorname{var} R = l, \qquad \nu\text{-a.e. on } B_l.$$

It follows that μ_Δ is supported on $\mathfrak{C}_l$ (this is the set of all curves of length l, which is a Borel set). Therefore,

$$T = \int_{\mathfrak{C}_l} R\, d\mu(R), \qquad \operatorname{var} T = \int_{\mathfrak{C}_l} \operatorname{var} R\, d\mu(R),$$

and we have (1.9) and (1.10) (see the statement of Theorem A). It remains to prove (1.21) and (1.22). In accordance with a property of T'_Δ (see 3) in 2.5),

$$\|\operatorname{div} T'_\Delta\| = \|T\| \times \delta_a + \|T\| \times \delta_b, \quad \mathscr{P}_{\|}\|\operatorname{div} T'_\Delta\| = 2\|T\|.$$

By (2.20),

$$\begin{aligned} \|\operatorname{div} T'_\Delta\| &= \left\| \int_{\mathscr{A}_\Delta} \operatorname{div} R\, d\lambda(R) \right\| \\ &= \left\| \int_{\mathscr{A}_\Delta} \delta_{b(R)}\, d\lambda(R) - \int_{\mathscr{A}_\Delta} \delta_{e(R)}\, d\lambda(R) \right\| \\ &= \int_{\mathscr{A}_\Delta} \delta_{b(R)}\, d\lambda(R) + \int_{\mathscr{A}_\Delta} \delta_{e(R)} d\lambda(R) \end{aligned} \tag{2.23}$$

(we have used the fact that the measures $\int_{\mathscr{A}_\Delta} \delta_{b(R)}\, d\lambda(R)$ and $\int_{\mathscr{A}_\Delta} \delta_{e(R)}\, d\lambda(R)$ are supported on nonintersecting hyperplanes of $\mathbb{R}^{n+1}$, because $\mathscr{C}b(R) \equiv a$, $\mathscr{C}e(R) \equiv b$ for $R \in \mathscr{A}_\Delta$).

Applying $\mathscr{P}_{\|}$, we find

$$\begin{aligned} \mathscr{P}_{\|}\|\operatorname{div} T'_\Delta\| &= \int_{\mathscr{A}_\Delta} \delta_{\mathscr{P}b(R)}\, d\lambda(R) + \int_{\mathscr{A}_\Delta} \delta_{\mathscr{P}e(R)}\, d\lambda(R) \\ &= l \int_{\mathscr{A}_\Delta} \delta_{b(R)}\, d\lambda(R) + l \int_{\mathscr{A}_\Delta} \delta_{e(R)}\, d\lambda(R). \end{aligned} \tag{2.24}$$

Using (2.23) and fact that $\operatorname{div} T = 0$, we get

$$0 = \operatorname{div} T = \int_{C_l} (\delta_{b(R)} - \delta_{e(R)})\, d\mu(R),$$

whence (by (2.23) and (2.24))

$$\|T\| = l \int_{\mathfrak{C}_l} \delta_{b(R)}\, d\mu(R) = l \int_{\mathfrak{C}_l} \delta_{e(R)}\, d\mu(R),$$

and we arrive at (1.22). Finally,

$$(2.25)\quad \begin{aligned} l\int_{\mathfrak{C}_l}\|\operatorname{div}R\|\,d\mu(R) &= \int_{\mathscr{A}_\Delta}\|\operatorname{div}\mathscr{P}_{\mathbf{l}}R\|\,d\lambda(R) \le \int_{\mathscr{A}_\Delta}\|\mathscr{P}_{\mathbf{l}}\operatorname{div}R\|\,d\lambda(R) \\ &= \int_{\mathscr{A}_\Delta}\delta_{\mathscr{P}e(R)}+\delta_{\mathscr{P}b(R)}\,d\lambda(R) = \int_{\mathfrak{C}_l}\delta_{e(R)}+\delta_{b(R)}\,d\lambda(R) = 2\|T\|, \end{aligned}$$

and we have (1.21). (The inequality in (2.25) means that $\mathscr{P}_{\mathbf{l}}R$ may be a closed curve, and then $\operatorname{div}\mathscr{P}_{\mathbf{l}}R = 0$; if $\mathscr{P}_{\mathbf{l}}R$ is not closed, then $\operatorname{div}\mathscr{P}_{\mathbf{l}}R = \mathscr{P}_{\mathbf{l}}\operatorname{div}R$). Theorem A is proved.

2.9. The proof of Theorems B′ and B. We take a solenoid $T \in \mathrm{Sol}(\mathbb{R}^n)$ and consider the corresponding shift invariant measure $\nu = \nu_T$ defined in 2.8 (see (2.13)). Our proof will be based on the Birkhoff–Khinchin ergodic theorem ([15]). We apply it to the measure space (S, ν) with $S := \mathrm{Slow}(\mathbb{R}^{n+1})$. Let τ denote the shift $\tau: S \to S$ defined by

$$\tau(\gamma)(t) := (f_\gamma(t+1), t), \quad \text{where } \gamma(t) = (f_\gamma(t), t),\ \gamma \in S.$$

Since τ is an automorphism of the measure space (S, ν), the ergodic theorem asserts that for every $\theta \in L^1(S, \nu)$ and for ν-almost all $\gamma \in S$ the limit

$$(2.26)\qquad \lim_{k\to\infty}\frac{1}{2k}\sum_{j=-k}^{k-1}\theta(\tau^j\gamma) =: \overline{\theta(\gamma)},$$

exists, and

$$(2.27)\qquad \int_S \theta\,d\nu = \int_S \bar{\theta}\,d\nu.$$

2.9.1. With every motion $\gamma \in \mathrm{Slow}(\mathbb{R}^{n+1})$ we associate its "first hour part" $\gamma_0 := \gamma \mid S_{[0,1]}$ (see 2.8.2) and the corresponding charge $R_0 = B(\gamma_0) \in \mathscr{A}_{[0,1]}$. We take a test field $\varphi \in \mathscr{D}^1(\mathbb{R}^n)$ and put $\theta_\varphi(\gamma) = (\mathscr{P}_{\mathbf{l}}R_0)(\varphi)$ (= the circulation of φ along the curve $\mathscr{P}_{\mathbf{l}}R_0$).

The function θ_φ is continuous and bounded on $\mathrm{Slow}(\mathbb{R}^{n+1})$. Consequently, $\theta_\varphi \in L^1(S, \nu)$, and we can apply the ergodic theorem to $\theta = \theta_\varphi$. Let γ be a slow motion, $\gamma(t) = (f_\gamma(t), t)$, $f_\gamma \in \mathrm{Lip}_1(\mathbb{R}^n)$. Then

$$\theta_\varphi(\tau^j\gamma) = (f_\gamma)_{\mathbf{l}}\overrightarrow{[j;j+1]}(\varphi),$$

and the expression under the limit sign in (2.26) becomes $(2k)^{-1}f_{\mathbf{l}}(\overrightarrow{[-k;k]})(\varphi) =: \overline{R_{\gamma_k}}(\varphi)$. Hence there is a set $N_\varphi \subset S$ such that $\nu(N_\varphi) = 0$ and the limit $\overline{R_\gamma}(\varphi) := \lim_{k\to\infty}\overline{R_{\gamma_k}}(\varphi)$ exists for every $\gamma \in S \setminus N_\varphi$.

Now we choose a countable set $\mathbb{D}^* \subset \mathscr{D}^1(\mathbb{R}^n)$ uniformly dense in $\mathscr{D}^1(\mathbb{R}^n)$ and put $N = \cup_{\varphi\in\mathbb{D}^*}N_\varphi$. Then $\nu(N) = 0$, and for $\gamma \in S \setminus N$ the limit $\overline{R_\gamma}(\varphi)$ exists for every $\varphi \in \mathbb{D}^*$. For a fixed $k = 1, 2, \dots$, we have

$$\operatorname{var}\overline{R_{\gamma_k}} \le \frac{1}{2k}\mathrm{Lip}(f_\gamma)\cdot 2k < 1$$

(see (1.7)). Hence, by the Banach–Steinhaus theorem, for a fixed $\gamma \in S$, the existence of $\overline{R_\gamma}(\varphi)$ for all $\varphi \in \mathbb{D}^*$ implies its existence for all $\varphi \in \mathscr{D}^1(\mathbb{R}^n)$, and $\overline{R_\gamma}$ is a charge in $\mathbb{R}^n$ with $\operatorname{var}\overline{R_\gamma} \le 1$.

We have obtained the following result: for ν-almost all $\gamma \in \mathrm{Slow}(\mathbb{R}^{n+1})$, the limit $\overline{R_\gamma}(\varphi)$ exists for all $\varphi \in \mathscr{D}^1(\mathbb{R}^n)$.

2.9.2. Now we recall formula (2.21):

$$T(\varphi) = (\mathscr{P}_1 T'_{[0,1]})(\varphi) = \int_S (\mathscr{P}_1 R_{\gamma_0})(\varphi)\, d\nu(\gamma) = \int_S (\theta_\varphi)\, d\nu.$$

Combining this with (2.27), we see that

$$T = \int_S \overline{R_\gamma}\, d\nu(\gamma), \quad \nu = \nu_T. \tag{2.28}$$

This is a decomposition of T; indeed,

$$\operatorname{var} T \le \int_S \operatorname{var} \overline{R_\gamma}\, d\nu(\gamma) \le \int_S d\nu = \operatorname{var} \nu = \operatorname{var} T,$$

whence $\operatorname{var} \overline{R_\gamma} = 1$ for ν-almost all γ, and $\operatorname{var} T = \int_S \operatorname{var} \overline{R_\gamma}\, d\nu(\gamma)$.

2.9.3. Now, let T be an extreme point of the unit ball B_{Sol}. We have obtained a decomposition of T into charges R_γ, which, clearly, are also solenoidal. Hence (see [5]) $R_\gamma = T$ for almost every γ.

2.9.4. The preceding statement implies Theorem B′.

Indeed, if $T \in \mathrm{Sol}(\mathbb{R}^n)$, then ν_T-almost all motions $\gamma \in \mathrm{Slow}(\mathbb{R}^{n+1})$ satisfy

$$f_\gamma(\mathbb{R}) \subset \operatorname{spt} T, \quad f_\gamma := \mathscr{P}\gamma.$$

This follows from (2.19) and (2.21), where we can take $\Delta = [-k, k]$, $k = 1, 2, \dots$ (excluding a countable union of exceptional sets of measure 0 afterwards). For a fixed K, the relations $\operatorname{var}\lambda_\Delta = \operatorname{var} T$, (2.21), and (2.19) imply the inclusion $f_{\gamma_\Delta} \subset \operatorname{spt} T$ for ν-almost all $\gamma \in \mathrm{Slow}(\mathbb{R}^{n+1})$; see 1.2.

So, if T is extreme in Sol_1, then, by 2.9.3, $T = \overline{R}_{f_\gamma}$ for some $\gamma \in \mathrm{Slow}(\mathbb{R}^{n+1})$. But we may also assume that (2.29) holds, so that all the conditions (1.16)–(1.19) are fulfilled. Hence $T \in \operatorname{extr} \mathrm{Sol}_1 \implies T \in \mathrm{elem}$. This proves Theorem B′ and hence Theorem B.

§3. The proof of Theorem C

3.1. Reduction to the Approximate Decomposition Lemma. We deduce Theorem C from the following assertion.

Lemma 3. *Every charge $T \in \mathrm{N}_1(\mathbb{R}^n)$ can be decomposed into the sum of charges $P, Q \in \mathrm{N}_1(\mathbb{R}^n)$ such that Q is completely decomposable into simple curves and*

$$\operatorname{var}(\operatorname{div} Q) \ge \frac{1}{10} \operatorname{var}(\operatorname{div} T).$$

Suppose the lemma proved; applying it to T, we obtain charges P_1 and Q_1. Now we apply the lemma to P_1 to obtain P_2 and Q_2, then apply the lemma to P_2, and so on. As a result, we obtain two sequences of normal charges $\{P_k\}$ and $\{Q_k\}$ such that for every natural number k

$$Q_1 + Q_2 + \cdots + Q_k + P_k = T, \tag{3.1}$$

$$\|Q_1\| + \|Q_2\| + \cdots + \|Q_k\| + \|P_k\| = \|T\|, \tag{3.2}$$

$$\|\operatorname{div} Q_1\| + \|\operatorname{div} Q_2\| + \cdots + \|\operatorname{div} Q_k\| + \|\operatorname{div} P_k\| = \|\operatorname{div} T\|, \tag{3.3}$$

$$\operatorname{var}(\operatorname{div} P_k) \le \left(\frac{9}{10}\right)^k \operatorname{var}(\operatorname{div} T). \tag{3.4}$$

By (3.1) and (3.2), the series $Q_1 + Q_2 + \dots$ converges (in the variation norm) to a normal charge Q, whereas P_k tends (in the same norm) to a normal charge P satisfying

$$P + Q = T, \qquad \|P\| + \|Q\| = \|T\|. \tag{3.5}$$

By (3.4), $\operatorname{var}(\operatorname{div} P_k) \to 0$ as $k \to \infty$, whence

$$\operatorname{div} P = 0, \qquad \operatorname{div} Q = \operatorname{div} T. \tag{3.6}$$

It follows from (3.1), (3.2), (3.3) that Q is completely decomposable into charges Q_k, $k = 1, 2, \dots$ and, since every Q_k is completely decomposable into simple curves, so is Q. Together with (3.5) and (3.6), this proves Theorem C, and now we must prove our lemma.

3.2. The proof of Lemma 3. We assume that $\operatorname{div} t \neq 0$, since otherwise we can put $P = T$, $Q = 0$. We take a number $l > \frac{20\operatorname{var}(T)}{\operatorname{var}(\operatorname{div} T)}$ and apply Theorem A with this l to the charge $S \in \mathbb{N}(\mathbb{R}^{n+1})$ defined by

$$S := T \times (\delta_0 - \delta_l) + \operatorname{div} T \times \overrightarrow{[0;l]}.$$

It can easily be checked that its divergency is zero. We obtain a decomposition

$$S = \int_{\mathfrak{C}_l} R\, d\mu(R), \tag{3.7}$$

$$\|S\| = \int_{\mathfrak{C}_l} \|R\|\, d\mu(R), \tag{3.8}$$

$$\|S\| = l\int_{\mathfrak{C}_l} \delta_{b(R)}\, d\mu(R) = l\int_{\mathfrak{C}_l} \delta_{e(R)}\, d\mu(R). \tag{3.9}$$

Up to a set of $\|\operatorname{div} T\|$-measure 0, the support of the charge $\operatorname{div} T$ in $\mathbb{R}^n$ splits into two sets E_+ and E_-, where $\overrightarrow{\operatorname{div}} T(x) = +1$ and $\overrightarrow{\operatorname{div}} T(x) = -1$, respectively. Then for $\|S\|$-almost every $X \in E_\pm \times (0;l)$ we have $\mathbb{S}(X) = \pm dX_{n+1}$, and, up to a $\|S\|$-null set,

$$\operatorname{spt} S \cap \mathbb{R}^n \times (0;l) \subset (E_+ \cup E_-) \times (0;l).$$

It follows that the restriction of μ-almost every curve R to the "sprip" $\mathbb{R}^n \times (0;l)$ is representable either as an oriented vertical interval

$$\pm\delta_z \times \overrightarrow{[a;b]}$$

(where $z \in E_\pm$ and $a = 0$ or $b = l$) or as the union of two such intervals. Let

$$\mathfrak{N} := \{ R \in \mathfrak{C}_l : b(R) \in E_- \times (0;l) \},$$
$$\mathfrak{M} := \{ R \in \mathfrak{C}_l : b(R) \in E_- \times (0;l),\ e(R) \in \mathbb{R}^n \times (0;l) \};$$

then for μ-almost all $R \in \mathfrak{N}\setminus\mathfrak{M}$

$$R \llcorner \mathbb{R}^n \times (0;l) = \delta_{\mathscr{P}b(R)} \times \overrightarrow{[\mathscr{P}b(R);0]},$$

whence (because $R \in \mathfrak{N}$, $\mathscr{P}b(R) < \frac{l}{2}$)

$$\operatorname{var}(R \llcorner \mathbb{R}^n \times \{0\}) = \operatorname{var}(R) - \operatorname{var}(R \llcorner \mathbb{R}^n \times (0;l]) = l - \mathscr{P}b(R) \geq l - \frac{l}{2} \geq \frac{l}{2}$$

for μ-almost every $R \in \mathfrak{N}$. Hence,

$$\operatorname{var}(T) = \operatorname{var}(S \llcorner \mathbb{R}^n \times \{0\}) - \int_{\mathfrak{C}_l} \operatorname{var}(R \llcorner \mathbb{R}^n \times \{0\})\, d\mu(R)$$
$$\geq \int_{\mathfrak{N} \setminus \mathfrak{M}} \operatorname{var}(R \llcorner \mathbb{R}^n \times \{0\})\, d\mu(R) > \int_{\mathfrak{N} \setminus \mathfrak{M}} \frac{l}{2}\, d\mu(R) = \frac{l}{2} \mu(\mathfrak{N} \setminus \mathfrak{M}).$$

Thus,

$$\mu(\mathfrak{N} \setminus \mathfrak{M}) < \frac{2}{l} \operatorname{var}(T) < \frac{2}{\frac{20 \operatorname{var}(T)}{\operatorname{var}(\operatorname{div} T)}} \operatorname{var}(T) = \frac{1}{10} \operatorname{var}(T) \tag{3.10}$$

(here we have used the lower bound for l stipulated at the beginning of the proof). But, on the other hand, it follows from (3.9) that

$$\frac{l}{4} \operatorname{var}(\operatorname{div} T) = \frac{l}{2} \operatorname{var}(\operatorname{div} T \llcorner E_-) = \operatorname{var}\Big(S \llcorner E_- \times \big(0; l/2\big)\Big)$$
$$- l \int_{\mathfrak{N}} \operatorname{var}(\delta_{b(R)})\, d\mu(R) = l \mu(\mathfrak{N}),$$

i.e., $\mu(\mathfrak{N}) = \frac{1}{4} \operatorname{var}(\operatorname{div} T)$. Subtracting (3.10) from this identity, we obtain

$$\mu(\mathfrak{N}) \geq \mu(\mathfrak{M} \cap \mathfrak{N}) = \mu(\mathfrak{N}) - \mu(\mathfrak{N} \setminus \mathfrak{M}) > \frac{1}{4} \operatorname{var}(\operatorname{div} T) - \frac{1}{10} \operatorname{var}(\operatorname{div} T) > \frac{1}{10} \operatorname{var}(\operatorname{div} T),$$

whence

$$\mu(\mathfrak{N}) > \frac{1}{10} \operatorname{var}(\operatorname{div} T). \tag{3.11}$$

Now we put

$$P := \int_{\mathfrak{C}_l \setminus \mathfrak{M}} (R \llcorner \mathbb{R}^n \times \{0\})\, d\mu(R),$$
$$Q := \int_{\mathfrak{M}} (R \llcorner \mathbb{R}^n \times \{0\})\, d\mu(R),$$

and check the desired properties. Clearly $P, Q \subset \mathrm{N}_1(\mathbb{R}^n)$. Restricting (3.7) and (3.8) to $\mathbb{R}^n \times \{0\}$, we get

$$T = S \llcorner \mathbb{R}^n \times \{0\} = \int_{\mathfrak{C}_l} R \llcorner \mathbb{R}^n \times \{0\}\, d\mu(R),$$
$$\|T\| = \|S \llcorner \mathbb{R}^n \times \{0\}\| = \int_{\mathfrak{C}_l} \|R \llcorner \mathbb{R}^n \times \{0\}\|\, d\mu(R).$$

Hence, $T = P + Q$ and $\|T\| = \|P\| + \|Q\|$; Q is decomposable into curves $R \llcorner \mathbb{R}^n \times \{0\}$ (not necessarily simple, but we can assume them to be simple, since otherwise we decompose them into simple curves and "move" "extra" closed curves into P). For μ-almost all R, the charge $R \llcorner \mathbb{R}^n \times \{0\}$ is a curve with the origin $\mathscr{P} b(R) \in E_-$ and the end $\mathscr{P} e(R) \in E_+$. Therefore

$$\|\operatorname{div} R\| = \operatorname{div} R \llcorner E_+ - \operatorname{div} R \llcorner E$$

for μ-almost all $R \in \mathfrak{M}$, whence

$$\begin{aligned}\|\operatorname{div} Q\| &= \left\| \int_{\mathfrak{M}} \operatorname{div}(R \llcorner \mathbb{R}^n \times \{0\})\, d\mu(R) \right\| \\ &= \Big(\int_{\mathfrak{M}} \operatorname{div}(R \llcorner \mathbb{R}^n \times \{0\})\, d\mu(R) \Big) \llcorner E_+ \\ &\quad - \Big(\int_{\mathfrak{M}} \operatorname{div}(R \llcorner \mathbb{R}^n \times \{0\})\, d\mu(R) \Big) \llcorner E_- \\ &= \int_{\mathfrak{M}} \Big(\operatorname{div}(R \llcorner E_+ \times \{0\}) - \operatorname{div}(R \llcorner E_- \times \{0\}) \Big)\, d\mu(R) \\ &= \int_{\mathfrak{M}} \|\operatorname{div}(R \llcorner \mathbb{R}^n \times \{0\})\|\, d\mu(R)\end{aligned}$$

(we have used the relation $E_+ \cap E_- = \varnothing$, which implies that the negative and the positive part of the divergence of the curves $R \llcorner \mathbb{R}^n \times \{0\}$ cannot mutually cancel out). We see that Q is not only decomposable, but also completely decomposable into simple curves.

Now it remains to verify that

$$\|\operatorname{div} T\| = \|\operatorname{div} P\| + \|\operatorname{div} Q\|, \qquad \operatorname{var}(\operatorname{div} Q) \geq \frac{1}{10} \operatorname{var}(\operatorname{div} T).$$

The last estimate follows from (3.11):

$$\operatorname{var}(\operatorname{div} Q) = \int_{\mathfrak{M}} \operatorname{var}\big(\operatorname{div}(R \llcorner \mathbb{R}^n \times \{0\})\big)\, d\mu(R).$$

To prove the first one, we introduce

$$\mathfrak{M}_- := \{ R \in \mathfrak{C}_l : b(R) \in E_- \times (0; l) \},$$

$$\mathfrak{M}_+ := \{ R \in \mathfrak{C}_l : e(R) \in E_+ \times (0; l) \}.$$

Clearly, $\mathfrak{M} = \mathfrak{M}_- \cap \mathfrak{M}_+$. We define a new charge $P' \in \mathbb{N}_1(\mathbb{R}^n)$ by the formula

$$P' := \mathscr{P}_1 \Big(\int_{\mathfrak{M}_+ \setminus \mathfrak{M}} \delta_{e(R)}\, d\mu(R) - \int_{\mathfrak{M}_- \setminus \mathfrak{M}} \delta_{b(R)}\, d\mu(R) \Big).$$

By the definition of Q,

$$\operatorname{div} Q = \mathscr{P}_1 \int_{\mathfrak{M}} (\delta_{e(R)} - \delta_{b(R)})\, d\mu(R).$$

Summing the last two identities and recalling (3.10), (3.11), we get

$$\begin{aligned}\operatorname{div} Q + P' &= \mathscr{P}_1 \Big(\int_{\mathfrak{M}_+} \delta_{e(R)}\, d\mu(R) - \int_{\mathfrak{M}_-} \delta_{b(R)}\, d\mu(R) \Big) \\ &= \frac{1}{l} \mathscr{P}_1 \big(\|S\| \llcorner E_+ \times (0; l) - \|S\| \llcorner E_- \times (0; l) \big) \\ &= \|\operatorname{div} T\| \llcorner E_+ - \|\operatorname{div} T\| \llcorner E_- = \operatorname{div} T.\end{aligned} \tag{3.12}$$

Hence $\operatorname{div} Q + P' = \operatorname{div} T$. Combined with $Q + P = T$, this implies that $P' = \operatorname{div} P$. But we can also write an identity similar to (3.12) for variations and obtain

$$\|\operatorname{div} Q\| + \|P'\| = \|\operatorname{div} T\|$$

and

$$\|\operatorname{div} Q\| + \|\operatorname{div} P\| = \|\operatorname{div} T\|.$$

Our lemma is proved, and hence also Theorem C.

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