Regularization of subsolutions in discrete weak KAM theory

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Abstract

We expose different methods of regularizations of subsolutions in the context of discrete weak KAM theory. They allow to prove the existence and the density of $C^{1,1}$ subsolutions. Moreover, these subsolutions can be made strict and smooth outside of the Aubry set.

1 Introduction

We consider a smooth connected Riemannian manifold M endowed with the distance $\mathrm{d}(.,.)$ coming from the Riemannian metric. Fixing a cost function $c : M \times M \to \mathbb{R}$ we study the functions $u : M \to \mathbb{R}$ which satisfy

$$\forall (x,y) \in M \times M, \quad u(y) - u(x) \leqslant c(x,y),$$

we call them subsolutions, by analogy with those appearing in Weak KAM theory (see [FS04, Ber07] for example). Defining, as usual, the discrete Lax–Oleinik operators

$$T_c^- u(x) = \inf_{y\in M} u(y) + c(y,x), \qquad T_c^+ u(x) = \sup_{y \in M} u(y) - c(x,y),$$

we see that a function u is a subsolution if and only if one of the equivalent relations is verified:

$$u \leqslant T_c^- u \quad \text{ or } \quad T_c^+ u \leqslant u.$$

Our goal is to extend and simplify the results of [Zav10]. Our first result uses the following hypothesis on c. We will give later more concrete hypotheses which imply this one.

Hypothesis 1. For each subsolution u, the functions $T_c^- u$ and $-T_c^+ u$ are locally semiconcave [1].

Under this hypothesis, we have:

Theorem 1. *The set of locally $C^{1,1}$ subsolutions is dense in the set of continuous subsolutions for the strong topology.*

We recall that the strong (or Whitney) topology on $C^0(M,\mathbb{R})$ is induced by the basis of open sets:

$$O_{\epsilon,f} = \{g \in C^0(M,\mathbb{R}), \quad \forall x \in M, \quad |f(x) - g(x)| < \epsilon(x)\}$$

where $f \in C^0(M,\mathbb{R})$ and ϵ is a continuous positive valued function on M. For further precisions on this topology, see [Hir94, Chapter 2]. The existence of $C^{1,1}$ subsolutions was proved in [Zav12], but the density is new. In [Zav12], the existence of $C^{1,1}$ subsolutions is deduced from the following result of Ilmanen (see [Ilm93, Car01, FZ10, Ber10]):

Theorem 2. *Let f and g be locally semiconcave functions on M such that $f + g \geqslant 0$. Then there exists a locally $C^{1,1}$ function u such that $-g \leqslant u \leqslant f$.*

[1] Throughout the paper, we call semiconcave what is sometimes called semiconcave with a linear modulus.

We will offer a direct proof of Theorem 1, which is inspired from the proof of Ilmanen's Lemma given in [Ber10]. Note that Theorem 1 implies Theorem 2. This follows immediately from the equivalence, for a given function u, between the following properties:

- the function $g+u$ is bounded from below and $-g \leqslant u - \inf(g+u) \leqslant f$;
- the function u is a subsolution for the cost $c(x,y) = g(x) + f(y)$.

Our next result uses either of the following stronger hypotheses on c, closer to the setting of [Zav10]:

Hypothesis 2. The function c satisfies the following properties:

- **uniform super-linearity**: for every $k \geqslant 0$, there exists $C(k) \in \mathbb{R}$ such that

$$\forall (x,y) \in M \times M, \quad c(x,y) \geqslant k\mathrm{d}(x,y) - C(k);$$

- **uniform boundedness**: for every $R \in \mathbb{R}$, there exists $A(R) \in \mathbb{R}$ such that

$$\forall (x,y) \in M \times M, \quad \mathrm{d}(x,y) \leqslant R \Rightarrow c(x,y) \leqslant A(R);$$

- **local semiconcavity**: for each point (x_0, y_0) there is a domain of chart containing (x_0, y_0) and a smooth function $f(x,y)$ such that $c - f$ is concave in the chart.

Hypothesis 3. The function c is locally bi–semiconcave:
for all $(x,y) \in M \times M$ we can find the following:

- neighborhoods U and V of respectively x and y,
- diffeomorphisms φ_1 and φ_2 from B_n to respectively U and V (B_n is the unit ball in $\mathbb{R}^n$),
- smooth functions f and g from B_n to $\mathbb{R}$,

such that for each $x \in M$, the function $z \mapsto c\big(x, \varphi_2(z)\big) - g(z)$ is concave and for all $y \in M$, the function $z \mapsto c\big(\varphi_1(z), y\big) - f(z)$ is concave.

It is proved in [Zav10, Proposition 4.6] that Hypothesis 2 implies Hypothesis 1. It is easy to prove in a similar way that Hypothesis 3 also implies Hypothesis 1.

We need to introduce more definitions before we state our second result. The subsolution u is called strict at (x,y) if $u(y) - u(x) < c(x,y)$. We denote by $\overline{\mathcal{A}}_u$ the set of pairs (x,y) at which u is not strict,

$$\overline{\mathcal{A}}_u = \{(x,y) \in M^2, \quad u(y) - u(x) = c(x,y)\} \subset M \times M.$$

We define the Aubry set as

$$\overline{\mathcal{A}} = \bigcap_u \{(x,y) \in M^2, \quad u(y) - u(x) = c(x,y)\} \subset M \times M,$$

where the intersection is taken on the set of continuous subsolutions. A pair (x,y) belongs to the Aubry set if and only if no continuous subsolution is strict at (x,y). The Aubry set is a closed, possibly empty, subset of $M \times M$. We will also use the projection $\mathcal{A}$ of $\overline{\mathcal{A}}$ on the first factor (which, as we will see later, is equal to its projection on the second factor under hypothesis 2). We also introduce

$$\mathcal{A}^* = \bigcap_u \{x \in M, \quad T_c^- u(x) = u(x)\}.$$

Notice that $\mathcal{A}^* \supset \mathcal{A}$. Moreover, as proved in [Zav12], these two sets are equal if Hypothesis 2 is verified, see Appendix A.

Theorem 3. *Assume that c verifies either hypothesis 2 or 3. If there exists a continuous subsolution, then there exists a locally $C^{1,1}$ subsolution strict in the complement of $\overline{\mathcal{A}}$. Moreover, this subsolution may be taken C^∞ in the complement of $\mathcal{A}^*$.*

Strict $C^{1,1}$ subsolutions were obtained in [Zav10] under an additional twist assumption. We will use a simple trick of [Ber07] to obtain easily the general result from Theorem 1. That the subsolutions can be made smooth outside of $\mathcal{A}^*$ is well-known. It will certainly not be a surprise to specialists that this can be done without destroying the global $C^{1,1}$ regularity, although we do not know any reference for this statement. We prove it using a regularization procedure due to De Rham [dR73]. This proof also applies to the "classical" (as opposed to discrete) weak KAM theory. We will often use the following criterion for subsolutions, taken from [Zav12]:

Lemma 1.1. *Let u be a subsolution and let us consider a function v such that*

$$u \leqslant v \leqslant T_c^- u,$$

then v itself is a subsolution.

PROOF. The statement follows from the inequalities $u \leqslant v \leqslant T_c^- u \leqslant T_c^- v$. □

2 The uniform case on $\mathbb{R}^n$ and the Jensen transforms

In this section we work on $M = \mathbb{R}^n$. A function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ is called k–semiconcave if $u(x) - k\|x\|^2$ is concave. We make the following assumption on the cost c:

There exists a constant K such that the function $x \longmapsto c(x, y)$ is K–semiconcave for each y and the function $y \longmapsto c(x, y)$ is K–semiconcave for each x.

We will use the Jensen transforms. If $u : \mathbb{R}^n \to \mathbb{R}$ is a function and $t \in \mathbb{R}_+$ then

$$\forall x \in \mathbb{R}^n, \qquad J^{-t}u(x) = \inf_{y\in\mathbb{R}^n} \Big(u(y) + \frac{1}{t}\|y - x\|^2\Big),$$

$$\forall x \in \mathbb{R}^n, \qquad J^{+t}u(x) = \sup_{y\in\mathbb{R}^n} \Big(u(y) - \frac{1}{t}\|y - x\|^2\Big).$$

Theorem 4. *Let u be a uniformly continuous subsolution. The function $J^{-t} \circ J^{+2t} \circ J^{-t}u$ is finite and, for t small enough, it is a $C^{1,1}$ subsolution. Moreover, it converges uniformly to u as $t \to 0$. More precisely, if u is a uniformly continuous subsolution then for $t, s < K^{-1}$ the functions $J^{-t} \circ J^{t+s} \circ J^{-s}u$ and $J^{+t} \circ J^{-(t+s)} \circ J^{+s}u$ are $C^{1,1}$ subsolutions which converge uniformly to u as $t, s \to 0$.*

We recall a few properties of the Jensen transforms, most of which are proved in [Ber10] or [AD00]. Both families of operators J^- and J^+ are semigroups. They are monotonous in the following way:

$$\forall s > t > 0, \qquad \inf u \leqslant J^{-s}u \leqslant J^{-t}u \leqslant u \leqslant J^{+t}u \leqslant J^{+s}u \leqslant \sup u$$

and in the following one:

$$u \leqslant v \Rightarrow \{\forall t \geqslant 0, \quad J^{-t}u \leqslant J^{-t}v \text{ and } J^{+t}u \leqslant J^{+t}v\}.$$

We call modulus of continuity a continuous function $\rho : [0, \infty) \longrightarrow [0, \infty)$ such that $\rho(0) = 0$. A function f is said ρ–continuous if $|f(y) - f(x)| \leqslant \rho(\|y - x\|)$ for all x and y. Given a modulus of continuity ρ, there exists a modulus of continuity ϵ such that, for each ρ–continuous function u, the following properties hold:

- the functions $J^{-t}u$ and $J^{+t}u$ are finite-valued and ρ–continuous for each $t \geqslant 0$,
- $J^{-t}u$ is t^{-1}–semiconcave and $J^{+t}u$ is t^{-1}–semiconvex,

- $\|J^{-t}u - u\|_\infty + \|J^{+t}u - u\|_\infty \leqslant \epsilon(t)$,
- $J^{-t} \circ J^{+t}u \geqslant u$ and $J^{+t} \circ J^{-t}u \leqslant u$,
- the equality $J^{-t} \circ J^{+t}u = u$ (resp. $J^{+t} \circ J^{-t}u = u$) holds if and only if u is t^{-1}–semiconcave (resp. t^{-1}–semiconvex),
- if u is semiconvex (resp. semiconcave) then $J^{-t} \circ J^{+t}u$ (resp. $J^{+t} \circ J^{-t}u$) is $C^{1,1}$ (and finite valued).

Using these properties, we now prove Theorem 4. Let u be a uniformly continuous subsolution, with modulus ρ. Since the function u is a subsolution, we have $u \leqslant T_c^- u$ hence $T_c^- u$ is finite–valued. Our hypothesis on the cost c implies that the function $T_c^- u$ is K–semiconcave, being a finite infimum of K–semiconcave functions. For $s < K^{-1}$, we have

$$u \leqslant J^{-s} \circ J^{+s}u \leqslant J^{-s} \circ J^{+s}(T_c^- u) = T_c^- u,$$

where the last inequality follows from the K–semiconcavity of $T_c^- u$ and the properties of $J^- \circ J^+$ listed above. We conclude that the function $J^{-s} \circ J^{+s}u$ is a ρ–continuous, s^{-1}–semiconcave subsolution. Similarly, if u is ρ–continuous and $t < K^{-1}$, then the function $J^{+t} \circ J^{-t}u$ is a ρ–continuous, t^{-1}–semiconvex subsolution. Applying this observation to the function $J^{-s} \circ J^{+s}u$, we conclude that $J^{+t} \circ J^{-t} \circ J^{-s} \circ J^{+s}u$ is a ρ–continuous subsolution. This subsolution is $C^{1,1}$ since $J^{-s} \circ J^{+s}u$ is semiconcave. Finally, we observe that

$$u - \epsilon(t) \leqslant J^{+t} \circ J^{-(t+s)} \circ J^{+s}u \leqslant u + \epsilon(s),$$

where ϵ is the modulus associated to ρ in the list of properties of J, which ends the proof. □

3 The general case

In this section, we come back to the general setting and prove Theorem 1. We derive it from the uniform version using partitions of unity, as was done in [Ber10] for Ilmanen's Lemma. We fix a locally finite atlas $(\phi_i)_{i\in I}$ constituted of smooth maps $\phi_i : B_n \to M$, where B_n is the open unit ball. We assume that all the images $\phi_i(B_n)$, for $i \in I$, are relatively compact in M. Moreover, we consider a smooth partition of unity $(g_i)_{i\in I}$ subordinated to the locally finite open covering $\big(\phi_i(B_n)\big)_{i\in I}$. Given positive numbers $a_i, b_i,\ i \in I$, we define the operators

$$\forall x \in M, \qquad Su(x) = \sum_{i\in I} [J^{-a_i} \circ J^{+a_i}(g_i u \circ \phi_i)] \circ \phi_i^{-1}(x), \tag{1}$$

$$\forall x \in M, \qquad \check{S}u(x) = \sum_{i\in I} [J^{+b_i} \circ J^{-b_i}(g_i u \circ \phi_i)] \circ \phi_i^{-1}(x). \tag{2}$$

Theorem 1 follows from:

Theorem 5. *Let u be a continuous subsolution and $\epsilon : M \to \mathbb{R}_+^*$ be a continuous function. For suitably chosen positive constants $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$, the function $\check{S} \circ S(u)$ is a locally $C^{1,1}$ subsolution such that $|u - \check{S} \circ Su| \leqslant \epsilon$.*

Proof. Since the image $\phi_i(B_n)$ is relatively compact and since the atlas is locally finite the set $A_i = \{j \in I,\ \ \phi_j(B_n) \cap \phi_i(B_n) \neq \varnothing\}$ is finite, let us denote by e_i its cardinal. Setting

$$\epsilon_i := \frac{\min\limits_{j\in A_i} \inf\limits_{x\in B_n} \epsilon\big(\phi_i(x)\big)}{2 \max\limits_{j\in A_i} e_j},$$

we observe that

$$\forall i \in I, \sum_{j \in A_i} \epsilon_j \leqslant \frac{1}{2} \inf_{x \in B_n} \epsilon\big(\phi_i(x)\big). \tag{3}$$

For each i, we choose a positive constant a_i such that

$$\big\|(g_i u) \circ \phi_i - J^{-a_i} \circ J^{+a_i}\big((g_i u) \circ \phi_i\big)\big\|_\infty < \epsilon_i. \tag{4}$$

Such a constant exists because the function $(g_i u) \circ \phi_i$ is uniformly continuous. Since the functions $(\sup_{\phi_i(B_n)} u) g_i \circ \phi_i$ and $(\inf_{\phi_i(B_n)} u) g_i \circ \phi_i$ (extended by 0 outside of B_n) are C^2 and compactly supported, they are semiconcave, hence we can choose the positive constants a_i such that, in addition,

$$J^{-a_i} \circ J^{+a_i}\Big(\Big(\inf_{\phi_i(B_n)} u\Big) g_i \circ \phi_i\Big) = \Big(\inf_{\phi_i(B_n)} u\Big) g_i \circ \phi_i,$$
$$J^{-a_i} \circ J^{+a_i}\Big(\Big(\sup_{\phi_i(B_n)} u\Big) g_i \circ \phi_i\Big) = \Big(\sup_{\phi_i(B_n)} u\Big) g_i \circ \phi_i.$$

Then,

$$\begin{aligned}\Big(\inf_{\phi_i(B_n)} u\Big) g_i \circ \phi_i =& J^{-a_i} \circ J^{+a_i}\Big(\Big(\inf_{\phi_i(B_n)} u\Big) g_i \circ \phi_i\Big)\\ \leqslant& J^{-a_i} \circ J^{+a_i}(g_i u \circ \phi_i)\\ \leqslant& J^{-a_i} \circ J^{+a_i}\Big(\Big(\sup_{\phi_i(B_n)} u\Big) g_i \circ \phi_i\Big) = \Big(\sup_{\phi_i(B_n)} u\Big) g_i \circ \phi_i,\end{aligned}$$

hence the function $J^{-a_i} \circ J^{+a_i}(g_i u \circ \phi_i)$ is supported in B_n.

Finally, since $T_c^- u$ is locally semiconcave, the function $(g_i T_c^- u) \circ \phi_i$ is semiconcave (see [Ber10]), and we can assume by taking $a_i > 0$ small enough that it is a_i^{-1}–semiconcave. Then, we have

$$(g_i u) \circ \phi_i \leqslant J^{-a_i} \circ J^{+a_i}(g_i u \circ \phi_i) \leqslant J^{-a_i} \circ J^{+a_i}\big([g_i T_c^- u] \circ \phi_i\big) = \big[g_i T_c^- u\big] \circ \phi_i,$$

hence

$$u = \sum_{i \in I} (g_i u) \circ \phi_i \circ \phi_i^{-1} \leqslant Su \leqslant \sum_{i \in I} \big[g_i T_c^- u\big] \circ \phi_i \circ \phi_i^{-1} = T_c^- u,$$

which, by Lemma 1.1, implies that Su is a subsolution. It is locally semiconcave, as a locally finite sum of locally semiconcave functions, and $|u - Su| < \epsilon/2$ everywhere. Similarly, we can choose positive constants b_i such that:

- $\big\|(g_i Su) \circ \phi_i - J^{+b_i} \circ J^{-b_i}\big((g_i Su) \circ \phi_i\big)\big\|_\infty < \epsilon_i$,
- $J^{+b_i} \circ J^{-b_i}\big((g_i Su) \circ \phi_i\big)$ is $C^{1,1}$ and supported in B_n,
- $(g_i T_c^+ Su) \circ \phi_i$ is b_i^{-1}–semiconvex.

Then, $\check{S} Su$ is a locally $C^{1,1}$ subsolution such that $|u - \check{S} Su| \leqslant \epsilon$. □

4 Existence of strict subsolutions

In this section, we make the additional assumption that **there exists a continuous subsolution.**

Lemma 4.1. *There exists a continuous subsolution w_0 such that $\overline{\mathcal{A}}_{w_0} = \overline{\mathcal{A}}$.*

Proof. Since M is separable, the set of continuous subsolutions is also separable (for the compact–open topology), and we consider a dense subsequence $(u_n)_{n\in\mathbb{N}}$. Set

$$w_0 = \sum_{n\in\mathbb{N}} a_n u_n \tag{5}$$

where the a_n are positive real numbers such that $\sum a_n = 1$ and the sum (5) is uniformly convergent on each compact subset. The function w_0 is a subsolution since it is a convex combination of subsolutions. If now $(x, y) \in \overline{\mathcal{A}}_{w_0}$, summing the inequalities

$$\forall n \in \mathbb{N}, \quad a_n\big(u_n(y) - u_n(x)\big) \leqslant a_n c(x, y),$$

gives an equality, therefore all inequalities are equalities and

$$\forall n \in \mathbb{N}, \quad (x, y) \in \overline{\mathcal{A}}_{u_n}.$$

By density of the sequence u_n, we deduce that $(x, y) \in \overline{\mathcal{A}}_u$ for each continuous solution u and therefore $\overline{\mathcal{A}}_{w_0} \subset \overline{\mathcal{A}}$. The reverse inequality follows from the definition of $\overline{\mathcal{A}}$. □

We now prove the main result of this section.

Theorem 6. *Assume the cost c verifies one of the Hypotheses 2 or 3. If u is a continuous subsolution, then there exists a locally $C^{1,1}$ subsolution u' such that u and u' coincide on $\overline{\mathcal{A}}_u$ and u' is strict outside of $\overline{\mathcal{A}}_u$. There exists a $C^{1,1}$ subsolution which is strict outside of $\overline{\mathcal{A}}$.*

Proof. Let u be a continuous subsolution. The function

$$F : (x, y) \mapsto c(x, y) + u(x) - u(y)$$

is therefore continuous, non-negative on M and positive on the complement of $\overline{\mathcal{A}}_u$. Let now $\varphi : M \times M \to \mathbb{R}$ be a C^∞ function which is bounded, non-negative, positive on the complement of $\overline{\mathcal{A}}_u$ and verifies $\varphi \leqslant F$. In the case where c verifies hypothesis 3, we require in addition that φ be locally bi–semiconcave (see lemma C.1 in the appendix for the construction of such a function).

We introduce the modified cost $c_\varphi = c - \varphi$. It verifies the same hypothesis as c. By definition, u is a c_φ–subsolution. Using Theorem 5, applied with the cost c_φ, we can choose the constants a_i and b_i such that $w = \check{S}Sw_0$ is a $C^{1,1}$ subsolution for the cost c_φ, hence a subsolution for the cost c which is strict outside of $\overline{\mathcal{A}}_u$.

Finally, to prove the second part of the theorem, we apply the preceding construction to the function w_0 obtained in Lemma 4.1. □

The proof also yields:

Corollary 4.2. *The set of locally $C^{1,1}$ subsolutions strict outside $\overline{\mathcal{A}}$ is dense for the open compact topology in the set of continuous subsolutions.*

5 Smoothness

We regularize strict subsolutions outside of the Aubry set and prove Theorem 3. We start with:

Definition 5.1. The leverage function $\lambda_u : M \longrightarrow [0, \infty)$ of the subsolution u is defined by:

$$\lambda_u(x) := \frac{1}{3} \min\big(T_c^- u(x) - u(x),\ u(x) - T_c^+ u(x)\big).$$

The following lemma justifies the denomination of leverage function:

Lemma 5.2. *Let u be a continuous subsolution and let v be another function such that*

$$\forall x \in M, \qquad |u(x) - v(x)| \leqslant \lambda_u(x),$$

then v is itself a subsolution. Moreover, if u is strict at (x, y) then so is v.

Proof. By definition, we have

$$3\max\{\lambda_u(x),\ \lambda_u(y)\} \leqslant \max\{u(x) - T_c^+ u(x),\ T_c^- u(y) - u(y)\} \leqslant \ c(x,y) - u(y) + u(x).$$

Therefore, the following inequalities hold:

$$\begin{aligned} v(y) - v(x) &\leqslant u(y) - u(x) + \lambda_u(x) + \lambda_u(y) \leqslant u(y) - u(x) + \frac{2}{3}\big(c(x,y) - u(y) + u(x)\big) \\ &\leqslant \frac{2}{3}c(x,y) + \frac{1}{3}\big(u(y) - u(x)\big) \leqslant c(x,y), \end{aligned}$$

which proves that v is a subsolution and that it is strict at (x, y) whenever u is. □

The previous proposition shows that it is possible to regularize subsolutions where their leverage function does not vanish. We will prove in Proposition 5.6 that there exists a subsolution whose leverage function is positive outside of $\mathcal{A}^*$. We need some preparation.

Definition 5.3. Let u be a subsolution. Define $\mathcal{A}^*_u \subset M$ as

$$\mathcal{A}^*_u = \{x \in M, \quad T_c^- u(x) = u(x)\}.$$

We then obtain $\mathcal{A}^* = \cap \mathcal{A}^*_u$, where the intersection is taken over continuous subsolution.

Lemma 5.4. *There exists a continuous subsolution w_1 such that $\mathcal{A}^*_{w_1} = \mathcal{A}^*$ and $\overline{\mathcal{A}}_{w_1} = \overline{\mathcal{A}}$.*

Proof. If u is a continuous subsolution, then $\mathcal{A}^*_u$ is closed, hence so is $\mathcal{A}^*$. Let us consider a point $x \notin \mathcal{A}^*$. By definition, there exists a subsolution u_x such that $T_c^- u_x(x) > u_x(x)$. Moreover, by continuity of u_x and $T_c^- u_x$, we may consider a positive number ϵ_x and an open neighborhood of x, O_x, on which the following holds:

$$\forall y \in O_x, \qquad T_c^- u_x(y) > u_x(y) + \epsilon_x.$$

The set $M \setminus \mathcal{A}^*$ satisfies the Lindelöf property (it is a separable metric space). We can thus extract a countable cover O_n, for $n \in \mathbb{N}$ of the cover O_x, where $x \in M \setminus \mathcal{A}^*$. We will denote by u_n and ϵ_n the continuous subsolution and positive real number associated to O_n. As in Lemma 4.1, we consider a convex combination

$$w = \sum_{n\in\mathbb{N}} a_n u_n.$$

The function w is then a continuous subsolution. For each $x \notin \mathcal{A}^*$, there exists $n_0 \in \mathbb{N}$ such that $x \in O_{n_0}$, and we have

$$T_c^- w(x) = T_c^-\Big(\sum_{n\in\mathbb{N}} a_n u_n\Big)(x) \geqslant \sum_{n\in\mathbb{N}} a_n T_c^- u_n(x) \geqslant \sum_{n\in\mathbb{N}} a_n u_n + a_{n_0}\epsilon_{n_0} = w(x) + a_{n_0}\epsilon_{n_0}.$$

This proves that $\mathcal{A}^*_w \subset \mathcal{A}^*$ and then that in fact $\mathcal{A}^*_w = \mathcal{A}^*$ (the reverse inequality falls from the definition of $\mathcal{A}^*$). Finally, setting $w_1 = (w + w_0)/2$, where w_0 is given by Lemma 4.1 proves the result.

□

Lemma 5.5. *If $x_0 \in \mathcal{A}^*$, then for any continuous subsolution u, we have $u(x_0) = T_c^+ u(x_0)$.*

Proof. We will use the following general fact: if f is any function, then $T_c^- T_c^+ f \geqslant f$. Indeed,

$$\forall x \in M, \quad T_c^- T_c^+ f(x) = \inf_{y\in M} \sup_{z\in M} f(z) - c(y,z) + c(y,x) \geqslant f(x),$$

the inequality being found by taking $z = x$. Since u is a subsolution, we have $T_c^- T_c^+ u \geqslant u \geqslant T_c^+ u$. Evaluating at x_0 gives

$$T_c^+ u(x_0) = T_c^- T_c^+ u(x_0) \geqslant u(x_0) \geqslant T_c^+ u(x_0),$$

where the first equality comes from the definition of $\mathcal{A}^*$, since $T_c^+ u$ is a subsolution. □

Proposition 5.6. *The set $\mathcal{A}^*$ verifies*

$$\mathcal{A}^* = \{x \in M, \quad \text{for any continuous subsolution } u, \quad u(x) = T_c^- u(x) = T_c^+ u(x)\}.$$

Moreover, there exists a locally $C^{1,1}$ subsolution w such that $T_c^+ w < w < T_c^- w$, on $M \setminus \mathcal{A}^*$.

Proof. The first statement follows immediately from Lemma 5.5. We define, similarly to $\mathcal{A}^*$, the set

$$\mathcal{A}^+ = \bigcap_u \{x \in M, \quad T_c^+ u(x) = u(x)\},$$

where the intersection is once again taken on all continuous subsolutions. By Lemma 5.5, we have $\mathcal{A}^* \subset \mathcal{A}^+$. A symetric version of Lemma 5.5 implies that $\mathcal{A}^* = \mathcal{A}^+$. A variant of Lemma 5.4 gives the existence of a continuous subsolution w_1^+ such that $T_c^+ w_1^+ > w_1^+$ on the complement of $\mathcal{A}^+ = \mathcal{A}^*$. The continuous subsolution $w_2 := (w_1^+ + w_1)/2$ then satisfies the inequalities $T_c^+ w_2 < w_2 < T_c^- w_2$ or equivalentely $\lambda_{w_2} > 0$ on the complement of $\mathcal{A}^*$.

Let ψ be a smooth bounded function such that $0 \leqslant \psi \leqslant \lambda_{w_2}$, with strict inequalities outside of $\mathcal{A}^*$. The function w_2 is a subsolution for the cost $\tilde{c}(x,y) = c(x,y) - \psi(y)$, since

$$T_{\tilde{c}}^- w_2(x) \geqslant T_c^- w_2(x) - \psi(x) \geqslant w_2(x).$$

By applying Theorem 5 to the cost $\tilde{c}$, we obtain the existence of a $C^{1,1}$ subsolution w^- for the cost $\tilde{c}$. This implies that

$$w^-(x) \leqslant T_{\tilde{c}}^- w^-(x) = T_c^- w^-(x) - \psi(x) \leqslant T_c^- w^-(x),$$

with a strict inequality outside of $\mathcal{A}^*$. Similarly, by considering the cost $c(x,y) - \psi(x)$, we obtain the existence of a $C^{1,1}$ subsolution w^+ which satisfies

$$w^+(x) \geqslant T_c^+ w^+(x) + \psi(x) \geqslant T_c^+ w^+(x).$$

The locally $C^{1,1}$ subsolution $w = (w^- + w^+)/2$ then satisfies $T_c^+ w < w < T_c^- w$ outside of $\mathcal{A}^*$. □

Proof of Theorem 3. Let us consider the $C^{1,1}$ subsolution $u = (v + w)/2$, where v is given by Theorem 6 and w is given by Proposition 5.6. This subsolution is $C^{1,1}$, it is strict outside of $\overline{\mathcal{A}}$, and its leverage function λ_u is positive outside of $\mathcal{A}^*$. We can apply Theorem 7 below to the function u with $\epsilon(x) = \lambda_u(x)$. By lemma 5.2, the function we obtain is a subsolution which is strict outside of $\overline{\mathcal{A}}$. It is not hard to prove in addition that its leverage function is positive outside of $\mathcal{A}^*$. □

The key regularization result in the proof is the following theorem (used with $k = 1$), which will be proved in the Appendix using a procedure due to De Rham.

Theorem 7. *Let f be a locally $C^{k,1}$ function on M and let $\epsilon : M \longrightarrow [0, \infty)$ be a continuous function. Then, there exists a locally $C^{k,1}$ function $g : M \to \mathbb{R}$ which is smooth on the open set $\Omega := \epsilon^{-1}(0, +\infty)$ and satisfies, for all $x \in M$,*

$$|f(x) - g(x)| + \|\mathrm{d}_x f - \mathrm{d}_x g\| + \cdots + \|\mathrm{d}_x^k f - \mathrm{d}_x^k g\| \leqslant \epsilon(x).$$

A More on the Aubry set

In this section, we assume c verifies hypothesis 2. We prove the sets $\overline{\mathcal{A}}$ and $\mathcal{A}^*$ introduced in the introduction are actually the usual Aubry set and projected Aubry set introduced in the framework of discrete weak KAM theory in [Zav12] (see also [BB07]). In particular, in this case, the projection of $\overline{\mathcal{A}}$ on either the first or the second component is $\mathcal{A}^*$. As explained in [Zav12], hypothesis 2 ensures that, if u is a continuous subsolution and $x \in M$ then there exists $y \in M$ such that $T_c^- u(x) = u(y) + c(y,x)$.

Proposition A.1. *There exists a set $\widetilde{\mathcal{A}} \subset M^{\mathbb{Z}}$ invariant by both left and right shifts and whose projection on $M \times M$ by $\pi : (x_i)_{i\in\mathbb{Z}} \mapsto (x_0, x_1)$ is $\overline{\mathcal{A}}$.*

In other terms, if $(x_i)_{i\in\mathbb{Z}} \in \widetilde{\mathcal{A}}$ then for any $i \in \mathbb{Z}$, $(x_i, x_{i+1}) \in \overline{\mathcal{A}}$ and conversely, if $(x_0, x_1) \in \overline{\mathcal{A}}$, there exists a sequence $(x_i)_{i\in\mathbb{Z}} \in \widetilde{\mathcal{A}}$ extending (x_0, x_1).

Proof. We will prove that if $(x_0, x_1) \in \overline{\mathcal{A}}$ then there are x_{-1} and x_2 such that (x_{-1}, x_0) and (x_1, x_2) are in $\overline{\mathcal{A}}$. Iterating this process then gives the bi-infinite chain in $\widetilde{\mathcal{A}}$.

Let w_1 be given by lemma 5.4 and x_{-1} verify $T_c^- w_1(x_0) = w_1(x_{-1}) + c(x_{-1}, x_0)$. Existence of such a point can be proved using the continuity of w_0 and the superlinearity of c (see [Zav12]). The following inequalities now hold:

$$T_c^- w_1(x_1) = T_c^- w_1(x_0) + c(x_0, x_1) \geqslant w_1(x_0) + c(x_0, x_1) \geqslant T_c^- w_1(x_1),$$

where the first equality comes from $(x_0, x_1) \in \overline{\mathcal{A}}$, the rest comes from the definition of T_c^- and the fact that w_1 is a subsolution.

This chain of inequalities tells us two things. First, $T_c^- w_1(x_0) = w_1(x_0)$ and also $w_1(x_0) + c(x_0, x_1) = T_c^- w_1(x_1)$. But this yields that

$$w_1(x_0) = T_c^- w_1(x_0) = w_1(x_{-1}) + c(x_{-1}, x_0),$$

hence $(x_{-1}, x_0) \in \overline{\mathcal{A}}_{w_1} = \overline{\mathcal{A}}$. The construction of the point x_2 is similar using T_c^+. □

The following proposition implies that the set $\overline{\mathcal{A}}$ is the Aubry set introduced in [Zav12].

Proposition A.2. *Let u be a (not necessarily continuous) subsolution, then $\overline{\mathcal{A}} \subset \overline{\mathcal{A}}_u$.*

Proof. Let $(x_i)_{i\in\mathbb{Z}} \in \widetilde{\mathcal{A}}$. Recall that $T_c^- u$ is continuous. From the inequalities

$$\forall i \in \mathbb{Z}, \quad T_c^- u(x_{i+1}) = T_c^- u(x_i) + c(x_i, x_{i+1}) \geqslant u(x_i) + c(x_i, x_{i+1}) \geqslant T_c^- u(x_{i+1}),$$

we infer that for each i, $T_c^- u(x_i) = u(x_i)$. Since $T_c^- u$ is continuous, we conclude that

$$\forall i \in \mathbb{Z}, \quad u(x_{i+1}) - u(x_i) = T_c^- u(x_{i+1}) - T_c^- u(x_i) = c(x_i, x_{i+1}).$$

Hence $(x_i, x_{i+1}) \in \overline{\mathcal{A}}_u$ and $\overline{\mathcal{A}} \subset \overline{\mathcal{A}}_u$. □

We can now prove that $\mathcal{A}^*$ is the projection of $\overline{\mathcal{A}}$ on the first factor, as well as its projection on the second factor.

Proposition A.3. *Let $y \in \mathcal{A}^*$, then there exist x and z such that (x, y) and (y, z) are in $\overline{\mathcal{A}}$.*

Proof. Let w_1 be the subsolution given by lemma 5.4. Let x be such that $T_c^- w_1(y) = w_1(x) + c(x, y)$. Since $y \in \mathcal{A}^*$ we obtain that $w_1(y) - w_1(x) = c(x, y)$. Hence $(x, y) \in \overline{\mathcal{A}}_{w_1} = \overline{\mathcal{A}}$. The existence of z is proved in the same way, using T_c^+. □

B Proof of Theorem 7

We prove Theorem 7 using a regularization procedure due to De Rham, see [dR73]. The idea of De Rham is to construct an action $\mathfrak{t}$ of $\mathbb{R}^n$ on $\mathbb{R}^n$ by smooth diffeomorphisms supported on the unit sphere B_n, in such a way that the induced action on B_n is conjugated to the standard action of $\mathbb{R}^n$ on itself by translations. More precisely, there exists a diffeomorphism $\mathfrak{h} : B_n \longrightarrow \mathbb{R}^n$ and diffeomorphisms $\mathfrak{t}_y$, $y \in \mathbb{R}^n$, of $\mathbb{R}^n$, equal to the identity outside of the open unit ball B_n, such that the map $(x, y) \longmapsto \mathfrak{t}_y(x)$ is smooth and such that

$$\mathfrak{h} \circ \mathfrak{t}_y = y + \mathfrak{h}$$

on B_n. This implies that $\mathfrak{t}$ is an action of the group $\mathbb{R}^n$ on $\mathbb{R}^n$, which means that $\mathfrak{t}_y \circ \mathfrak{t}_{y'} = \mathfrak{t}_{y+y'}$ for each y, y'. Since $\mathfrak{t}$ is smooth, $\mathfrak{t}_0 = Id$, and $\mathfrak{t}_y = Id$ outside of the unit ball, the maps $\mathfrak{t}_y$ converge uniformly to the identity as $y \longrightarrow 0$, and all their derivatives converge uniformly to the derivatives of the identity.

Let us give some details on the construction of $\mathfrak{h}$ and $\mathfrak{t}$. We set

$$\mathfrak{h}(x) = \frac{h(\|x\|)}{\|x\|} x,$$

where $h : [0, 1[\to \mathbb{R}_+$ is a smooth, strictly increasing ($h' > 0$) function such that

$$\begin{cases} h(r) = r, & 0 \leqslant r \leqslant 1/3, \\ h(r) = \exp\left((r-1)^{-2}\right), & 2/3 \leqslant r \leqslant 1. \end{cases}$$

We then define $\mathfrak{t}_y$, for each $y \in \mathbb{R}^n$ by

$$\begin{cases} \mathfrak{t}_y(x) = \mathfrak{h}^{-1}\big(\mathfrak{h}(x) + y\big) & \text{if } x \in B_n, \\ \mathfrak{t}_y(x) = x & \text{if } x \in \mathbb{R}^n \setminus B_n. \end{cases}$$

It is clear from these formula that $\mathfrak{t}_{y+y'} = \mathfrak{t}_y \circ \mathfrak{t}_{y'}$. The only issue is the smoothness of $\mathfrak{t}$. Differentiating the previous group property with respect to y' and taking $y' = 0$ yields the following relation:

$$\frac{\partial}{\partial y}\mathfrak{t}_y = \frac{\partial}{\partial y}\mathfrak{t}_0 \circ \mathfrak{t}_y.$$

This implies that

$$\mathfrak{t}_y(x) = x + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{t}_{ty}(x)\mathrm{d}t = x + \int_0^1 \Big(\frac{\partial}{\partial y}\mathfrak{t}_{ty}(x)\Big) y \mathrm{d}t = x + \int_0^1 \Big(\frac{\partial}{\partial y}\mathfrak{t}_0\big(\mathfrak{t}_{ty}(x)\big)\Big) y \mathrm{d}t.$$

In other words, the map $\mathfrak{t}_y$ is the time-one flow of the vectorfield $X_y(x) := M(x)y$, where $M(x) = \partial_y \mathfrak{t}_y(x)_{|y=0}$. In order to prove that the map $\mathfrak{t}$ is smooth, it is enough to observe that the matrix $M(x)$ depends smoothly on x. This matrix can be computed, recalling that the gradient of the norm $x \mapsto \|x\|$ is $r_x := x/\|x\|$:

$$M(x) = \mathrm{d}_{\mathfrak{h}(x)}\mathfrak{h}^{-1} = \frac{1}{h'(\|x\|)} r_x {}^t r_x + \frac{\|x\|}{h(\|x\|)}(I_n - r_x {}^t r_x).$$

Since $1/h$, $1/h'$, as well as all their derivatives go to 0 when $\|x\| \to 1$, we conclude that $M(x)$ is smooth.

We have exposed the construction of $\mathfrak{h}$ and $\mathfrak{t}$. They allow to define a local regularization procedure with the help of a smooth kernel $K_1 : \mathbb{R}^n \to [0, \infty)$. We assume that K_1 is supported in the unit ball B_n, and that $\int K_1 = 1$. For $\eta > 0$, we set $K_\eta(x) = \eta^{-n} K_1(\eta^{-1}x)$.

Lemma B.1. *Let $O \subset \mathbb{R}^n$ be an open set containing $\overline{B}_n$. Given a locally integrable function $f : O \longrightarrow \mathbb{R}$ and $\eta \in]0, 1[$, we define*

$$f_\eta(x) = \int_{\mathbb{R}^n} f\big(\mathfrak{t}_y(x)\big) K_\eta(-y)\mathrm{d}y.$$

The following assertions hold:

1. *The function f_η is C^∞ in B_n, and equal to f outside of B_n,*

2. *If f is C^k on O, then so are the functions f_η, and $f_\eta \longrightarrow f$ in C^k as $\eta \longrightarrow 0$.*

3. *If f is $C^{k,1}$ on O, then so are the functions f_η, and $\limsup_{\eta\longrightarrow 0} \mathrm{Lip}(\mathrm{d}^k f_\eta) \leqslant \mathrm{Lip}(\mathrm{d}^k f)$.*

4. *If, in some open set $O' \subset O$, f is C^l in O', then so is f_η.*

Proof. On B_n we have

$$f_\eta \circ \mathfrak{h}^{-1} = (f \circ \mathfrak{h}^{-1}) \star K_\eta,$$

where $\star$ is the convolution. Since the functions K_η are smooth, this implies the first claim. Writing

$$f_\eta - f = \int_{B(0,\eta)} (f \circ \mathfrak{t}_y - f) K_\eta(-y) \mathrm{d}y$$

and observing that $f \circ \mathfrak{t}_y - f \longrightarrow 0$ in $C^k(\mathbb{R}^n, \mathbb{R}^n)$ as $y \longrightarrow 0$ $\big($because $\mathfrak{t}_y \longrightarrow Id$ in $C^k(\mathbb{R}^n, \mathbb{R}^n)\big)$ yields the second claim. We will now prove that

$$\limsup_{y\longrightarrow 0} \mathrm{Lip}\big(\mathrm{d}^k(f \circ \mathfrak{t}_y)\big) \leqslant \mathrm{Lip}(\mathrm{d}^k f), \tag{6}$$

which yields the third claim in view of the relation

$$\mathrm{d}^k_x f_\eta = \int_{B(0,\eta)} \mathrm{d}^k_x(f \circ \mathfrak{t}_y) K_\eta(-y) \mathrm{d}y.$$

Let us consider a component $\partial^\alpha_x(f \circ \mathfrak{t}_y)$ of the differential $\mathrm{d}^k(f \circ \mathfrak{t}_y)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index such that $|\alpha| = \sum \alpha_i = k$. By the Faà di Bruno formula, expressed in terms of partial differentials (see [CS96] for example), we have

$$\partial^\alpha_x(f \circ \mathfrak{t}_y) = \sum_{1\leqslant|\lambda|\leqslant|\alpha|} \partial^\lambda_{\mathfrak{t}_y(x)} f \cdot B_{\alpha,\lambda}(\mathrm{d}_x\mathfrak{t}_y, \ldots, \mathrm{d}^{|\alpha|}_x \mathfrak{t}_y),$$

where the $B_{\alpha,\lambda}$ are universal multi-variable polynomials with no constant terms. These polynomials satisfy the equalities

$$B_{\alpha,\alpha}(Id, 0, \cdots, 0) = 1 \quad \text{and} \quad B_{\alpha,\lambda}(Id, 0, \cdots, 0) = 0$$

for all $\lambda \neq \alpha$. Since $\mathfrak{t}_y \longrightarrow Id$ in C^∞, the first of these equalities implies that the function $x \longmapsto B_{\alpha,\alpha}(\mathrm{d}_x\mathfrak{t}_y, \ldots, \mathrm{d}^{|\alpha|}_x \mathfrak{t}_y)$ is converging to 1 in C^∞. Concerning the other factor in this term, we have

$$\mathrm{Lip}\big((\partial^\alpha f) \circ \mathfrak{t}_y\big) \leqslant \mathrm{Lip}(\partial^\alpha f)\mathrm{Lip}(\mathfrak{t}_y) \longrightarrow \mathrm{Lip}(\partial^\alpha f).$$

We deduce that the upper limit of the Lipschitz constants of the term corresponding to $\lambda = \alpha$ is not greater than $\mathrm{Lip}(\partial^\alpha f)$.

On the other hand, for each of the terms with $\lambda \neq \alpha$, the function $x \longmapsto B_{\alpha,\lambda}(\mathrm{d}_x\mathfrak{t}_y, \ldots, \mathrm{d}^{|\alpha|}_x \mathfrak{t}_y)$ is converging to 0 in C^∞ hence the Lipschitz constant of the function

$$x \longmapsto \partial^\lambda_{\mathfrak{t}_y(x)} f \cdot B_{\alpha,\lambda}(\mathrm{d}_x\mathfrak{t}_y, \ldots, \mathrm{d}^{|\alpha|}_x \mathfrak{t}_y)$$

is converging to 0. We conclude that

$$\limsup \mathrm{Lip}\big(\partial^\alpha(f \circ \mathfrak{t}_y)\big) \leqslant \mathrm{Lip}(\partial^\alpha f),$$

which implies (6) hence the third point of the statement.

Regarding the last claim of the statement, we observe that the function f_η is smooth inside B_n and that it is C^l in $O' \setminus \overline{B}_n$. Moreover, it is C^l on the set $\cap_{y\in\overline{B}(0,\eta)} \mathfrak{t}_y^{-1}(O')$. By a standard compactness argument, this set is open, and it contains $O' \setminus B_n$. We have covered O' by two open sets such that the function f_η is C^l on each of them, we conclude that this function is C^l on O'. □

Lemma B.2. *Let O be open subsets of $\mathbb{R}^n$ and let $f : O \to \mathbb{R}$ be a $C^{k,1}$ function. Given a continuous function $\epsilon : O \to [0, \infty)$, there exists a function f_ϵ such that:*

1. *the function f_ϵ is C^∞ in the open set $\{x \in O, \epsilon(x) > 0\} \subset O$,*
2. *$|f_\epsilon(x) - f(x)| + \|\mathrm{d}_x f_\epsilon - \mathrm{d}_x f\| + \cdots + \|\mathrm{d}^k_x f_\epsilon - \mathrm{d}^k_x f\| \leqslant \epsilon(x)$ for each $x \in O$,*
3. *the function f_ϵ is $C^{k,1}$ on O, and $\mathrm{Lip}(\mathrm{d}^k f_\epsilon) \leqslant 1 + \mathrm{Lip}(\mathrm{d}^k f)$.*

Proof. Let us denote by F the closed set $\{\epsilon = 0\}$. The complement of F in O is open, and we consider a locally finite covering $(O_i)_{i \in \mathbb{N}^*}$ of $O \setminus F$ by open balls compactly included in $O \setminus F$. Since $\inf\{\epsilon(x), \quad x \in O_i\} > 0$. we can construct inductively, using Lemma B.1 a sequence of functions, $(f_i)_{i \in \mathbb{N}}$ such that

- $f_0 = f$,
- for each $i \in \mathbb{N}$, the function f_{i+1} is C^∞ in $O_1 \cup \cdots \cup O_{i+1}$,
- for each $i \in \mathbb{N}$, the functions f_i and f_{i+1} are equal in $O \setminus O_{i+1}$,
- for each $i \in \mathbb{N}$, the function f_{i+1} is $C^{k,1}$ in O, and $\mathrm{Lip}(\mathrm{d}^k f_{i+1}) \leqslant 2^{-i-1} + \mathrm{Lip}(\mathrm{d}^k f_i)$,
- $|f_{i+1}(x) - f_i(x)| + \|\mathrm{d}_x f_{i+1} - \mathrm{d}_x f_i\| + \cdots + \|\mathrm{d}^k_x f_{i+1} - \mathrm{d}^k_x f_i\| \leqslant 2^{-1-i}\epsilon(x)$ for each $x \in O$, $i \in \mathbb{N}$,

Each point of O has a neighborhood on which the sequence f_i is eventually constant, hence the limit $f_\epsilon := \lim f_i$ is well-defined and smooth on $\cup_i O_i = O \setminus F$. The desired estimates on f_ϵ follow immediately from the inductive estimates by summation. □

Proof of Theorem 7. We fix a locally finite atlas $(\phi_i)_{i \in \mathbb{N}^*}$ constituted of smooth maps $\phi_i : 2B_n \to M$, where B_n is the open unit ball. We assume that all the images $\phi_i(2B_n)$, $i \in \mathbb{N}^*$ are relatively compact in M and that the $\phi_i(B_n)$, $i \in \mathbb{N}^*$ still cover M. By Lemma B.2, it is possible to construct inductively a sequence of functions f_i, by iteratively modifying $f_i \circ \phi_{i+1}$ on B_n, such that

- $f_0 = f$,
- for each $i \in \mathbb{N}$, the function f_{i+1} is C^∞ in $\bigcup_{j \leqslant i+1} \phi_j(B_n) \cap \Omega$,
- for each $i \in \mathbb{N}$, in $M \setminus \phi_{i+1}(B_n)$, the functions f_i and f_{i+1} are equal,
- for each $i \in \mathbb{N}$, the function f_{i+1} is $C^{k,1}$ on M,
- for each $i \in \mathbb{N}$, $x \in M$, $|f_i(x) - f_{i+1}(x)| + \cdots + \|\mathrm{d}^k_x f_i - \mathrm{d}^k_x f_{i+1}\| \leqslant 2^{-i-1}\epsilon(x)$.

Each point $x \in M$ has a neighborhood on which the sequence f_i is eventually constant, hence the limit $g = \lim f_i$ is well defined, locally $C^{k,1}$, and smooth on Ω. The inequality on the differentials follows by summation from the iterative assumptions. □

C Existence of small smooth bi–semiconcave functions

This last section is devoted to finish the proof of theorem 6 by giving explicit ways to construct the function φ used to correct the cost.

Lemma C.1. *Let $F : M \times M \to \mathbb{R}_+$ be a continuous function. There exists a smooth bi–semiconcave function $\varphi : M \times M \to \mathbb{R}_+$ such that $\varphi^{-1}\{0\} = F^{-1}\{0\}$ and $\varphi \leqslant F$.*

Proof. Let $(\phi_i : B_n \to M, \quad i \in \mathbb{N})$ be an atlas of M such that the images $\phi_i(B_n)$ are relatively compact in M, and the diffeomorphisms $\phi_j^{-1} \circ \phi_i$ are C^2-bounded.

We will construct a smooth function $\varphi : M \times M \to \mathbb{R}_+$ such that $\varphi^{-1}\{0\} = F^{-1}\{0\}$, $\varphi \leqslant F$, and such that the following holds:

$$\forall (i,j,x,y) \in \mathbb{N} \times \mathbb{N} \times B_n \times B_n, \quad \left\| D^2_{(x,y)}\big(f \circ (\phi_i, \phi_j)\big) \right\|_\infty < 1,$$

which implies that φ is locally bi-semiconcave. Let us consider a locally finite cover of $M \times M \setminus F^{-1}\{0\} = O$ by open sets of the form $O_i = A_i \times B_i$, $i \in \mathbb{N}^*$. Assume moreover that each O_i is relatively compact in O and that each $\overline{A}_i$ and $\overline{B}_i$ are included in a chart of the atlas.

For each $i \in \mathbb{N}$ we consider a smooth function f_i on $M \times M$ such that $f_i \geqslant 0$ and $f_i^{-1}\{0\} = M \setminus O_i$. We will construct the function φ of the form $\sum \epsilon_i f_i$ with carefully chosen $\epsilon_i > 0$.

First, since O_i is compactly included in O, we will assume that ϵ_i is small enough for

$$0 < \epsilon_i < 2^{-i} \inf_{(x,y) \in O_i} F(x,y) \tag{7}$$

to hold.

Let us fix here $i \in \mathbb{N}^*$ and set j_i and k_i such that $\overline{O_i} \subset \phi_{j_i}(B_n) \times \phi_{k_i}(B_n)$. The function $f_i \circ (\phi_{j_i}, \phi_{k_i})$ is then smooth and bounded along with all its derivatives up to order two. By the hypothesis made on the changes of coordinates, this remains true for all $f_i \circ (\phi_j, \phi_k)$, $(j,k) \in \mathbb{N}^* \times \mathbb{N}^*$. Since the atlas is locally finite and O_i is relatively compact, only a finite number of these functions $f_i \circ (\phi_j, \phi_k)$ are actually non everywhere vanishing. Up to taking ϵ_i smaller, we may therefore also assume

$$\forall (i,j,k,x,y) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times B_n \times B_n, \quad \left\| \epsilon_i D^2_{(x,y)}\big(f_i \circ (\phi_j, \phi_k)\big) \right\|_\infty < 2^{-i}.$$

Finally, by a standard Cantor diagonal argument, up to taking the ϵ_i smaller, we will assume that the sum $\sum \epsilon_i f_i$ is locally uniformly convergent for the open–compact topology on $C^\infty(M, \mathbb{R})$. The function

$$\varphi = \sum_{i \in \mathbb{N}^*} \epsilon_i f_i$$

is then a smooth bi–semiconcave function taking nonnegative values and such that $\varphi^{-1}\{0\} = F^{-1}\{0\}$. Moreover, by (7), we have $\varphi \leqslant F$ which concludes the proof.

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