

Advanced Convex Analysis – Final Exam – May 2025

Duration : 2h. Documents and electronic devices are forbidden.

You are expected to demonstrate both mathematical rigor and clarity in writing.

The scoring system is based on 25 points. It is given as an indication only.

We advise focusing on the first 4 exercises and tackling the last one, more difficult, if time permits.

Exercice 1 (5 points). — Show that the following functions are convex, and for each f_i , compute : its continuity points, its (Gâteaux/Fréchet)-differentiability points, the subdifferential at every point, and the Fenchel-Legendre conjugate.

1. $f_1(x) = \frac{1}{x}$ for $x > 0$ and $f_1(x) = +\infty$ for $x \leq 0$;

2. $f_2(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$ defined on $\mathbb{R}$;

3. $f_3 = \chi_{[0,1]}$ on $\mathbb{R}$.

Exercice 2 (7 points). — Let E be a nontrivial real normed vector space and C a nonempty convex subset of E .

1. Show that if C has nonempty interior, then for every $x \in \partial C$, C admits a supporting hyperplane at x , i.e. there exists $x^* \in E^* \setminus \{0\}$ such that

$$\sup_{y \in C} \langle x^*, y \rangle \leq \langle x^*, x \rangle.$$

2. We assume that $E = \mathbb{R}^d$ and we admit the following fact : $\text{int } \overline{C} = \text{int } C$. Let $x \in \partial C$.

- a. Show that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \notin \overline{C}$ for every n and $x_n \rightarrow x$.
b. Deduce that C admits a supporting hyperplane at x .

3. We assume that $E = \ell^2(\mathbb{N}, \mathbb{R})$, the Hilbert space of square-integrable real sequences, i.e. $E = \{(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{n \geq 0} x_n^2 < +\infty\}$ equipped with the scalar product $\langle x, y \rangle = \sum_{n \geq 0} x_n y_n$. We consider the set :

$$C = \{(x_n)_{n \in \mathbb{N}} \in E : x_n \geq 0 \quad (\forall n)\}.$$

- a. Show that C is a closed convex cone.
b. Show that $C = \partial C$.
Hint : you may start by noticing that given $x \in C$, for every $\varepsilon > 0$, $|x_N| < \varepsilon$ for some suitable N .
c. Consider $x \in C$ such that there exists $n_0 \in \mathbb{N}$ satisfying $x_{n_0} = 0$. Show that C admits a supporting hyperplane at x .
d. Consider $x \in C$ such that $x_n > 0$ for every n . Does C admit a supporting hyperplane at x ?

Exercice 3 (4 points). — We consider the convex function $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by $f(x) = \max\{0, x\}$, and fix some $\tau > 0$. For every $y \in \mathbb{R}$, we define

$$h_y(x) = f(x) + \frac{1}{2\tau}(y - x)^2.$$

1. Show that h_y is strictly convex and that if it admits a minimizer on $\mathbb{R}$, then such a minimizer is unique.
2. Write, with justifications, the optimality conditions characterizing minimizers of h_y .
3. Deduce that for every $y \in \mathbb{R}$, h_y has a unique minimizer $p(y)$, and give an explicit expression of $p(y)$. Draw the graph of p .

Exercice 4 (3 points). — Given $\bar{x} \in \mathbb{R}^d$, we consider the minimization problem

$$\text{minimize}_{x=(x_1, \dots, x_d) \in \mathbb{R}^d} \quad \{\|x - \bar{x}\|_2^2 : \forall i \in \{1, \dots, d\}, 0 \leq x_i - x_{i-1} \leq 1\}$$

where in the above expression we have set $x_0 = 0$. We define $f(x) = \|x - \bar{x}\|_2^2$ for every $x \in \mathbb{R}^d$.

1. Rewrite the minimization problem in the form

$$(P) \quad \text{minimize}_{x \in \mathbb{R}^d} \quad \{f(x) + g(Ax)\} \quad \text{where} \quad \forall y \in \mathbb{R}^d, \quad g(y) = \sum_{i=1}^d \phi(y_i),$$

for a suitable convex function $\phi : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ and a suitable square matrix $A \in M_d(\mathbb{R})$ to write explicitly.

2. Write the dual problem of (P) explicitly, giving a precise expression for all functions and matrices involved. State the strong duality theorem seen in class, and justify that strong duality holds in this case.

Exercise 5 (★ – 6 points). — The goal of this exercise is to give a proof of strong duality for the following minimization problem :

$$\text{minimize}_{x \in \mathbb{R}} \quad \{f(x) + g(x)\}$$

where $f(x) = \log(1 + e^x)$ for every $x \in \mathbb{R}$ and $g = \chi_{[0,1]}$. Therefore, you *are not allowed* to use the strong duality result from the course in this exercise.

We define a function $H : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\forall z \in \mathbb{R}, \quad H(z) = \inf_{x \in \mathbb{R}} \{f(x) + g(x - z)\}.$$

1. Show that H is convex.
2. Show that for every z , the infimum in the definition of $H(z)$ is attained, i.e. there exists x' such that $H(z) = f(x') + g(x' - z)$.
3. Show that H is lower semicontinuous.
Hint : do not hesitate to extract subsequences when you can, and start by reducing to the case $\liminf_n H(z_n) = \lim_n H(z_n) < +\infty$.
4. Show that strong duality holds (you are not expected to compute the dual problem explicitly).
Hint : consider H^{**} .