

Advanced Convex Analysis – Partial Exam – March 2025

Duration : 2h. Documents and electronic devices are forbidden.
The scoring system is based on 23 points. It is given as an indication only.

Exercise 1 (5 points). — Let E be a normed vector space and C a non-empty convex subset of E .

- Recall the proof of the following fact : if $w \in \text{int } C$ and $x \in C$ then the point $x_\theta = (1 - \theta)w + \theta x$ belongs to $\text{int } C$ for every $\theta \in [0, 1)$.
- Extend this proof to show the following fact : if $w \in \text{int } C$ and $y \in \overline{C}$ then the point $y_\theta = (1 - \theta)w + \theta y$ belongs to $\text{int } C$ for every $\theta \in [0, 1)$. (*Hint : consider a point $x \in C$ which is close to y and show that $\|x_\theta - y_\theta\| \leq \|y - x\|$.)*)
- Using the previous question, show that if $\text{int } C \neq \emptyset$ then $\text{int } \overline{C} = \text{int } C$.

Exercise 2 (5 points). — We recall the following facts, that may be used without justification : 1) if C is a convex subset of $\mathbb{R}^d$ with empty interior then $\text{aff } C$ is an affine subspace of dimension $k < d$; 2) if C is convex then so is $\text{int } C$.

You may use all versions of the Hahn-Banach theorems that you have seen in class.

- Show that an affine subspace of $\mathbb{R}^d$ is necessarily closed.
- Let $C \subseteq \mathbb{R}^d$ be a non-empty convex set and $x \in \mathbb{R}^d \setminus C$.
 - Show that if $\text{int } C = \emptyset$ and $x \notin \text{aff } C$, then C and x can be linearly separated.
 - Show that if $\text{int } C \neq \emptyset$, then C and x can be linearly separated.
- Using the previous question, show that a convex subset C of a finite-dimensional vector space E and a point $x \notin C$ can be linearly separated. (*Hint : procede by induction on the dimension of E , and assume that $0 \in C$.)*)
- Deduce that two non-empty disjoint convex subsets C_1, C_2 of $\mathbb{R}^d$ can be linearly separated.

Exercise 3 (5 points). —

- Show that the convex hull of a compact subset K of $\mathbb{R}^d$ is still compact.

In the following questions the ambient space is ℓ^∞ , the set of bounded real sequences $x = (x_n)_{n \in \mathbb{N}}$, endowed with the norm $\|x\| = \sup_{n \in \mathbb{N}} |x_n|$.

- For every $k \geq 0$, we denote by $e^k \in \ell^\infty$ the bounded sequence $(0, \dots, 0, \underbrace{1}_{k\text{-th element}}, 0, \dots)$, i.e. $e_n^k = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise} \end{cases}$ for every n . Is the sequence $(e^k/k)_{k \in \mathbb{N}^*} \in (\ell^\infty)^{\mathbb{N}^*}$ convergent? If so, compute its limit.
- Justify that the set $K = \{e^k/k : k \in \mathbb{N}^*\} \cup \{0\}$ is compact.
- Consider a (strictly) positive sequence $(\theta_k)_{k \geq 1}$ such that $\sum_{k=1}^{+\infty} \theta_k = 1$, and set $x^\ell = \sum_{k=1}^{\ell} \theta_k \frac{e^k}{k}$ for every $\ell \in \mathbb{N}^*$. Show that $x^\ell \in \text{conv } K$ for every ℓ and that $(x^\ell)_{\ell \in \mathbb{N}^*}$ converges in ℓ^∞ to a limit x to be determined.
- Conclude that $\text{conv}(K)$ is not compact.

Exercise 4 (8 points). — We consider $\mathcal{M}_n(\mathbb{R})$ the vector space of $n \times n$ matrices, and the subset $\mathcal{B}$ of matrices $M = (m_{ij})_{1 \leq i, j \leq n}$ such that

$$\forall (i, j) \in \{1, \dots, n\}^2, m_{ij} \geq 0, \quad \forall i \in \{1, \dots, n\}, \sum_{j=1}^n m_{ij} = 1, \quad \text{and} \quad \forall j \in \{1, \dots, n\}, \sum_{i=1}^n m_{ij} = 1.$$

For every bijective map $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$, we denote by $M^\sigma = (m_{ij}^\sigma)_{1 \leq i, j \leq n}$ the matrix defined by

$$m_{ij}^\sigma = \delta_{j\sigma(i)} = \begin{cases} 1 & \text{if } j = \sigma(i), \\ 0 & \text{if not,} \end{cases}$$

and we denote by $\mathcal{P}$ the set of such matrices.

1. Justify that $\mathcal{B}$ is convex.
2. Let $M \in \mathcal{M}_n(\mathbb{R})$. Show that $M \in \mathcal{P}$ if and only if it has exactly one entry equal to 1 in each row and each column, with all other entries 0.
3. Show that every matrix $M^\sigma \in \mathcal{P}$ is an extremal point of $\mathcal{B}$.
4. Let $M \in \mathcal{B} \setminus \mathcal{P}$, and define $S = \{(i, j) \in \{1, \dots, n\}^2 : 0 < m_{ij} < 1\}$. We introduce some vocabulary :
 - Consider two pairs of indices $((i, j), (i', j')) \in S^2$. If $i = i'$ and $j \neq j'$ we call it a *horizontal move on S* and denote it by $(i, j) \xrightarrow{h} (i', j')$. If $j = j'$ and $i \neq i'$ we call it a *vertical move on S* and denote it by $(i, j) \xrightarrow{v} (i', j')$.
 - Consider $(i_0, j_0) \rightarrow (i_1, j_1) \dots \rightarrow (i_K, j_K)$ a sequence of moves on S . If any two consecutive moves are of different types (vertical then horizontal or the converse), we call it a *path on S*. Moreover, if $(i_0, j_0) = (i_K, j_K)$ and if the first and last moves are of different types, we call it a *cycle on S*.
 - (a) Show that for every $(i, j) \in S$, there exists horizontal and vertical moves of the form $(i, j) \xrightarrow{h} (i, j')$ and $(i, j) \xrightarrow{v} (i', j)$.
 - (b) Show that there exists a path on S , $(i_0, j_0) \rightarrow (i_1, j_1) \dots \rightarrow (i_{K-1}, j_{K-1}) \rightarrow (i_K, j_K)$, satisfying $(i_K, j_K) = (i_0, j_0)$ and $K > 1$.

We consider such a path $(i_0, j_0) \rightarrow (i_1, j_1) \dots \rightarrow (i_{K-1}, j_{K-1}) \rightarrow (i_0, j_0)$ with minimal size $K > 1$.

- (c) Show that it is a cycle.
- (d) Justify that we can define a matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ by setting

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) = (i_k, j_k) \xrightarrow{h} (i_{k+1}, j_{k+1}) \text{ for some } k < K, \\ -1 & \text{if } (i, j) = (i_k, j_k) \xrightarrow{v} (i_{k+1}, j_{k+1}) \text{ for some } k < K, \\ 0 & \text{otherwise,} \end{cases}$$

and show that each line and each column has zero sum.

- (e) Deduce that $M \notin \text{ext } \mathcal{B}$.
5. Conclude that $\mathcal{B} = \text{conv}(\mathcal{P})$.