

Exercise Sheet 1 – Convex Sets (Basics)

In all the exercises, E designates a real vector space.

Exercise 1. Let X be a subset of E . Assume that for all x and y in X , $\frac{x+y}{2} \in X$.

1. Show that X need not be convex.
2. Show that if E is a normed vector space and X is closed in E , then X is convex.

Exercise 2. Let X and Y be two convex subsets of E .

1. Show that $\text{conv}(X \cup Y) = \{tx + (1-t)y, t \in [0, 1], x \in X, y \in Y\}$.
2. Give a similar formula for the affine envelope of the union of 2 affine sets.
3. Give a similar formula for the convex (resp. affine) envelope of the union of n convex (resp. affine) sets.

Exercise 3. Let A be an affine set and $a \in A$.

1. Show that $A - a$ is a sub-vector space of E .
2. Show that this subspace does not depend on a .

Exercise 4. A set C of E is a cone if for all $x \in C$ and $t \geq 0$, $tx \in C$. If it is also convex, it is said to be a convex cone.

1. Show that C is a convex cone if and only if for all $(x, x') \in C^2$ and all $t, t' \in \mathbb{R}_+$, $tx + t'x' \in C$.
2. For any non-empty set X and $x \in X$, we define the radial cone of X in x by $R_X(x) := \{v \in E, \exists \alpha > 0, [x, x + \alpha v] \subset X\}$. Show that $R_X(x)$ is a cone.
3. If X is convex, show that $R_X(x)$ is a convex cone.
4. If A is affine, show that $R_A(a) = A - a$. You can use results from the previous exercise.

Exercise 5. Let X be a subset of E .

1. Show that if X is finite then $\text{conv}(X)$ is compact.
2. Show that if X is open then $\text{conv}(X)$ is open.
3. For any X show that $\overline{\text{conv}(X)}$ equals $\overline{\text{conv}} X$, the closed convex hull of X , defined as the smallest closed convex set containing X .
4. Let $E = \mathbb{R}^2$ and $X = \{(x, y) : y(1 + x^2) \geq 1\}$. Show that X is closed but not $\text{conv}(X)$.
5. Can the closure and convex hull operations be exchanged in question 3?

Exercise 6. We recall that a point x of a convex C is said to be extremal if for all $y, z \in X^2$ such that $y \neq z$, and for all $t \in (0, 1)$, $ty + (1-t)z \neq x$. In other words, x can only trivially be written as a convex combination of elements of C . Let C be a convex subset of E .

1. Show that x is an extremal point of C if and only if $C \setminus \{x\}$ is convex.
2. Let A be non-empty and x an extremal point of $\text{conv}(A)$. Show that $x \in A$.
3. Give an example of a non-empty set with no extremal point.

Exercise 7 (Faces of faces). We recall that given C convex, a convex subset $F \subseteq C$ is called a face of C if

$$\forall x, y \in C, \quad (x, y) \cap F \neq \emptyset \implies [x, y] \subseteq F.$$

Let $C \subseteq E$ be a convex set and $F \subseteq C$ be a face. Prove that $G \subseteq F$ is a face of F if and only if it is a face of C .

Exercise 8. The closed unit ball $B[0, 1]$ of a normed vector space E is said to be strictly convex if $\|tx + (1 - t)y\| < 1$ for all $x \neq y$ in $B[0, 1]$ and all $t \in (0, 1)$.

1. Show that the following assertions are equivalent :
 - a) $B[0, 1]$ is strictly convex.
 - b) If x and y are two distinct points of E and $t \in (0, 1)$, then $\|tx + (1 - t)y\| < \max(\|x\|, \|y\|)$.
 - c) If $\|x\| = 1$, then x is an extremal point of $B[0, 1]$.
2. If E is a Hilbert space, show that $B[0, 1]$ is strictly convex.
3. Let $E = \mathbb{R}^d$ equipped with the infinite norm. Is $B[0, 1]$ strictly convex? What are its extremal points?
4. Same questions for the norm one defined by $\|x\|_1 := \sum_{k=1}^d |x_k|$.

Exercise 9. In this exercise, E is the set of continuous functions from $[0, 1]$ into $\mathbb{R}$, with infinite norm, and B is the closed unit ball. We want to characterize the extremal points of B .

1. Let f be an extremal point, and suppose there exists x_0 such that $|f(x_0)| < 1$. Show that there exists a non-empty open interval I on which $|f(x)| < 1$.
2. Construct a continuous function g such that $1 \geq g(x) > 0$ on I and $g(x) = 0$ on the complementary of I .
3. Construct two distinct functions f_1 and f_2 in E such that $f = \frac{f_1 + f_2}{2}$.
4. Deduce the set C of extremal points of B . Is B strictly convex?
5. Calculate $\text{conv}(C)$ and comment.

Exercise 10. The aim of this exercise is to show that the convex hull of a precompact set is precompact. Recall that a set X is precompact if for any $\varepsilon > 0$, X is contained in a finite union of (open) balls of radius ε . We therefore assume that X is precompact. Fix $\varepsilon > 0$ and x_i such that $X \subset \bigcup_{i=1}^n B(x_i, \varepsilon)$. We denote $\Lambda_n = \{(\lambda_1, \dots, \lambda_n), \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1\}$ the n -dimensional simplex.

1. Let $F : \Lambda_n \rightarrow E$ be defined by $F(\lambda) = \sum \lambda_i x_i$. Show that Λ_n and then $F(\Lambda_n)$ are compact.
2. Show that for all $x \in \text{conv } X$, $\exists \lambda \in \Lambda_n$, $\|x - F(\lambda)\| < \varepsilon$.
3. From the two previous questions, deduce that $\text{conv } X$ is precompact.
4. Deduce that if E is a Banach space and X is compact, $\overline{\text{conv}} X$ is compact.

Exercise 11. Let $X \subseteq E$, and define the core of X by

$$\text{core}(X) = \{x \in X, \forall v \in E, \exists t > 0, [x, x + tv] \subset X\}.$$

1. Show that for any set X , $\text{int } X \subset \text{core}(X)$
2. Let $X = \{(x, y) : y = 0 \text{ or } |y| \geq x^2\}$. Show that $0 \in \text{core}(X)$ but $0 \notin \text{int } X$.
3. Now assume that X is convex with a non-empty interior and show that $\text{core}(X) \subset \text{int } X$.
Let y be in $\text{int } X$ and x in $\text{core}(X)$. Show that there exists $t > 0$ such that $(1+t)x - ty \in X$.
4. Show that $x \in [y, (1+t)x - ty]$ and conclude.