

Exercise Sheet 2 – Convex Sets (Sequel)

In all the exercises, E designates a real vector space.

Exercise 1. Let $p : E \rightarrow \mathbb{R}$ be a sublinear function (i.e. subadditive and positively homogeneous). Show that the following assertions are equivalent :

- (i) p is Lipschitz,
- (ii) p is continuous,
- (iii) p is continuous at 0,
- (iv) $\sup_{x \in E: \|x\| \leq 1} |p(x)| = \sup_{x \in E \setminus \{0\}} \frac{|p(x)|}{\|x\|} < +\infty$.

Exercise 2. Let $X \subset E$ be a convex set. We recall that the gauge of X , $g_X : E \rightarrow \mathbb{R} \cup \{+\infty\}$, is defined by

$$g_X(v) = \inf \{t > 0 : v \in tX\}.$$

- 1. Determine the gauge of the closed unit ball.
- 2. Show that $g_X(\lambda v) = \lambda g_X(v)$ for all $\lambda > 0$.
- 3. Show that $g_X(v + v') \leq g_X(v) + g_X(v')$.

From now on, we assume that $0 \in C$.

- 4. Show that $g_X(0) = 0$ and $\{g_X < 1\} \subseteq X$.
- 5. Show that $g_X(v) < +\infty$ for all v if and only if $0 \in \text{core}(X)$.
- 6. Assume $0 \in \text{int } X$. Show that g_X is Lipschitz. *Hint : start by showing that $\frac{g_X(v)}{\|v\|}$ is bounded, then use a previous question.*
- 7. Assume that $0 \in \text{core}(X)$, $X = -X$, and that for any non-zero v there exists $t > 0$ such that $tv \notin X$. Show that the gauge of X is a norm.

Exercise 3 (Hahn-Banach in a Hilbert space). *The aim of this exercise is to provide another proof of the Hahn-Banach separation theorem, in the case of Hilbert spaces.*

Let C be a non-empty closed convex subset of a real Hilbert space $(H, \langle \cdot, \cdot \rangle)$ with $C \neq H$.

- 1. Let $x \in H \setminus C$. Prove that there exists $u \in H$ such that

$$\sup_{y \in C} \langle u, y \rangle < \langle u, x \rangle.$$

Hint : introduce the projection of x on C .

- 2. For every $u \in H$ and $\alpha \in \mathbb{R}$, we set $H(u, \alpha) = \{x \in H : \langle u, x \rangle \leq \alpha\}$. Show that

$$C = \bigcap_{(u, \alpha) : C \subseteq H(u, \alpha)} H(u, \alpha).$$

- 3. Show that every closed strict vector subspace of H is the intersection of a family of closed vector hyperplanes. *Hint : Notice that for every vector subspace F , if $x \in F$ then $\lambda x \in F$ for all $\lambda \in \mathbb{R}$. Then argue as in the preceding question.*
- 4. Let C' be another nonempty convex subset. Assume that C' is compact and $C \cap C' = \emptyset$. We define $\tilde{C} = C' - C$.
 - a) Show that $0 \notin \tilde{C}$ and that $\tilde{C}$ is a closed convex subset of H .

b) Deduce that there exists $u \in H$ such that

$$\sup_{x' \in C'} \langle u, x' \rangle < \inf_{x \in C} \langle u, x \rangle.$$

c) Does this property remain true if C' is only closed and not compact?

d) Give a geometric interpretation of this property.

Exercise 4 (Normal cones in finite dimensional spaces). Let $C \subseteq \mathbb{R}^d$ be a nonempty convex set.

1. We want to show that if C has empty interior, then $\dim C < d$.

(a) Let $(p_0, \dots, p_n)$ be n vectors in $\mathbb{R}^n$ and pick $j \in \{0, \dots, n\}$. Show that $(p_i - p_j)_{i:i \neq j}$ is linearly independent if and only if $(p_0, \dots, p_n)$ is affinely independent, meaning :

$$\left(\sum_{i=1}^n \lambda_i = 1 \text{ and } \sum_{i=0}^n \lambda_i x_i = 0 \right) \implies \forall i, \lambda_i = 0.$$

(b) Show that $(p_0, \dots, p_d)$ is affinely independent if and only if the following matrix $P \in M_{d+1}(\mathbb{R})$ is invertible :

$$P = \begin{pmatrix} p_0 & \cdots & p_n \\ 1 & \cdots & 1 \end{pmatrix}.$$

In that case, deduce that for any $x \in \mathbb{R}^d$, $\Lambda x := P^{-1} \begin{pmatrix} x \\ 1 \end{pmatrix}$ is the unique vector $\lambda = (\lambda_0, \dots, \lambda_d) \in \mathbb{R}^{d+1}$ such that

$$\sum_i \lambda_i = 1 \text{ and } \sum_i \lambda_i p_i = x.$$

The λ_i 's are called the *barycentric coordinates* of x .

(c) Let $(p_0, \dots, p_d)$ be affinely independent. Considering the point $\bar{p} = \frac{1}{n+1} \sum_{i=0}^n p_i$ and the previous questions, show that $\text{conv}\{p_0, \dots, p_n\}$ has non-empty interior.

(d) Conclude.

2. Show that if $x \in \text{int } C$ then $\mathbf{N}_x C = \{0\}$.

3. Show that $x \in \text{rb } C$, then $\mathbf{N}_x C$ contains a nonzero vector. (Start with the case $x \in \partial C$.)

Exercise 5 (External representation of the closed convex hull). Show that the closed convex hull of a set A consists in the intersection of all half-spaces that contain A .

Exercise 6 (A proof of Minkowski Theorem). The goal is to give a proof of the Minkowski Theorem, i.e. Krein-Milman theorem in the finite-dimensional setting.

1. Let $C \subseteq \mathbb{R}^d$ be a compact convex set with non-empty interior and take $x \in C$.

a) Show that if $x \in \partial C$, then there exists a proper face F of C which contains x .

b) Show that if $x \in \text{int } C$, then $x \in \text{conv}(\partial C)$. Hint : For a fixed $u \in \mathbb{R}^d \setminus \{0\}$, consider the intersection of the line $L = \{x + tu : t \in \mathbb{R}\}$ with C .

2. Proceeding by induction, show Minkowski Theorem : any d -dimensional compact convex set is the convex hull of its extremal points.

Exercise 7 (Convex polytopes). Let $C \subseteq \mathbb{R}^d$ be a convex set.

1. Show that the following are equivalent :

- (i) C is a finite intersection of closed half-spaces ;
- (ii) C is the convex hull of finitely many points.

In this case we say that C is *convex polytope*.

2. A face F of C is said *exposed* if there exists a support hyperplane $H = \{\langle x^*, \cdot \rangle = \sigma_C(x^*)\}$ such that

$$F = C \cap H.$$

Show that if C is a convex polytope, then every face is exposed.

Exercise 8. Recall that Carathéodory's theorem implies that in finite dimension, if X is compact then so is $\text{conv}(X)$. The aim of this exercise is to give a counter-example in infinite dimension.

Let E be the set of real sequences such that $\|x\|_1 = \sum_{i=1}^{+\infty} |x_i| < +\infty$ and equipped with the norm $\|\cdot\|_1$. Let e_k be the sequence whose k -th term is 1 and all the others 0. Let $X = \{\frac{1}{n}e_n : n \in \mathbb{N}^*\} \cup \{0\}$.

1. Show that X is compact.
2. Show that $\sum_{i=1}^{+\infty} \frac{1}{n2^n} e_n \in \overline{\text{conv}(X)}$.
3. Deduce that $\text{conv}(X)$ is not closed and conclude.