

Exercise Sheet 3 – Convex Functions (Basics)

In all the exercises, E designates a real normed vector space.

Exercise 1. Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$.

1. Show that if f is convex then for any $\ell \in \mathbb{R}$ the sub-level set $\{x : f(x) \leq \ell\}$ is convex.
2. Find a non-convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ whose sub-levels are all convex.

Exercise 2. Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function and $A \subseteq E$. Show that $\sup_A f = \sup_{\text{conv}(A)} f$.

Exercise 3.

1. Show that a sublinear function is convex.
2. Show that a convex and positively homogeneous function is sublinear.
3. Show that if f is sublinear, with value in $\mathbb{R}$, and verifies $f(-x) = -f(x)$, then f is linear.
4. For any subset X of E , show that the map $g_X : \mathcal{F}(E, \mathbb{R}) \rightarrow \mathbb{R}$ defined on the space of real functions on E by

$$g_X(x^*) = \sup_{x \in X} \langle x^*, x \rangle$$

is sublinear.

5. Let f be convex and x in $\text{dom}(f)$. Show that $f^+(x, \cdot)$ defined on E by

$$f^+(x, v) = \lim_{\varepsilon \rightarrow 0^+} \frac{f(x + \varepsilon v) - f(x)}{\varepsilon}$$

is sublinear whenever it does not take the value $-\infty$.

Exercise 4. For any non-empty set C we define on E the function distance to this set by $d_C(x) = \inf_{z \in C} \|x - z\|$. Show that if C is convex, then d_C is convex, and that the converse is true when C is closed.

Exercise 5. Let α be a real number. Recall that function $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying

$$f((1 - \theta)x + \theta y) \leq (1 - \theta)f(x) + \theta f(y) - \frac{\alpha}{2} t(1 - t) \|x - y\|^2$$

is said to be α -strongly convex if $\alpha > 0$, and $(-\alpha)$ -semiconvex if $\alpha < 0$.

1. What is the relationship between convexity, strong convexity and semi-convexity?

From now on, assume that E is a Hilbert space.

2. Show that $x \mapsto \frac{\alpha}{2} \|x\|^2$ is α strongly convex if $\alpha > 0$, and $(-\alpha)$ -semi-convex if $\alpha < 0$, and that in both cases the inequality in the definition is in fact an equality.
3. Deduce that f is α -strongly convex if and only if $f - \frac{\alpha}{2} \|\cdot\|^2$ is convex; and that f is β -semi convex if and only if $f + \frac{\beta}{2} \|\cdot\|^2$ is convex.
4. Let X be a subset of E . Show that $\frac{d_X^2}{2}$ is 1-semi concave (i.e. its opposite is 1-semi convex).

Exercise 6. Show that a quadratic function $Q(x) = B(x, x)$, with B bilinear, is convex if and only if it is non-negative.

Exercise 7. Let f be a convex function on E . Show that $\phi(x, t) := tf(x/t)$ is convex on $E \times (0, +\infty)$.

Exercise 8. Let E, F be two normed vector spaces and f a convex function on $E \times F$.

1. Let C be a convex subset of F . We define $g(x) = \inf_{y \in C} \{f(x, y)\} \in \mathbb{R} \cup \{-\infty, +\infty\}$. Prove that $\{g < +\infty\}$ is convex.
2. Prove that g is convex on E if it does not take the value $-\infty$.
3. Assuming that $g(0) \in \mathbb{R}$ and that 0 belongs to the core of $\{g < \infty\}$, prove that g does not take the value $-\infty$.
4. Let C be a convex subset of E . Deduce that the function $d_C(x) := \inf_{y \in C} \|x - y\|$ is convex.

Exercise 9. A convex function $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is said *strictly convex* if $f((x + y)/2) < f(x) + f(y)/2$ for each $x \neq y$ in $\text{dom } f$.

1. Show that if f is strictly convex, then

$$\forall x \neq y \in \text{dom } f, \forall t \in (0, 1), \quad f(tx + (1 - t)y) < tf(x) + (1 - t)f(y).$$

2. Show that a strictly convex function has at most one minimum.
3. Under what condition on K is χ_K strictly convex?
4. If f is strictly convex, show that $f\chi_K$ also is, for each convex K .
5. Show that f is strictly convex if and only if the points of its graph are extremal points of its epigraph.
6. Can a norm be strictly convex?

Exercise 10 (Strictly convex norm). We say that a norm ν is strictly convex (so the definition is slightly different from the general definition of strictly convex function) if $\nu(x + y) < 2$ when $\nu(x) \leq 1, \nu(y) \leq 1, x \neq y$.

1. Show that this is equivalent to saying that points of norm 1 are extremal points of the unit ball.
2. Show that a pre-hilbertian norm (coming from a scalar product) is strictly convex.
3. On $\mathbb{R}^d$, denote $\|x\|_\infty = \max_{1 \leq i \leq d} |x_i|$ and $\|x\|_1 = \sum_{i=1}^d |x_i|$. Show that these norms are not strictly convex and give the extremal points of the unit balls.