

Exercise Sheet 4 – Convex Functions (Regularity)

In all the exercises, E designates a real normed vector space.

Exercise 1. Show that a convex function $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ is continuous at a point of its domain if and only if its epigraph has non-empty interior.

Exercise 2. *The goal of this exercise is to show the following result from the course : a convex function on a finite-dimensional normed vector space is Gâteaux-differentiable at a point if and only if it is Fréchet-differentiable at this point.*

Let f be a convex function on a normed vector space E and let $x \in \text{dom } f$.

1. Assume that $B[x, r_0] \subseteq \text{dom } f$. For $r \in [0, r_0]$, consider the maps f_r defined on $B[0, 1]$ by

$$f_r(v) = \frac{f(x + rv) - f(x)}{r}.$$

Show that f is Fréchet-differentiable at x if and only if f_r converges uniformly to $f'(x)$ as $r \rightarrow 0^+$.

2. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of maps $g_n : X \rightarrow Y$ between metric spaces X, Y which are L -Lipschitz for some $L \geq 0$. Show that if X is compact and $g_n \rightarrow g$ pointwise, then $g_n \rightarrow g$ uniformly.
3. Assume that E has finite dimension and f is Lipschitz on a neighborhood of x . Deduce that f is Gâteaux-differentiable at x if and only if it is Fréchet-differentiable at x .
4. Conclude.

Exercise 3 (The one-dimensional case). Let $f : \mathbb{R} \rightarrow (-\infty, \infty]$ be a function of domain D .

1. Show that the following conditions are equivalent :

(i) f is convex,

(ii) for every triple $a < b < c$,

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(a)}{c - a} \quad \text{if } a \in D$$

and

$$\frac{f(c) - f(a)}{c - a} \leq \frac{f(c) - f(b)}{c - b} \quad \text{if } c \in D,$$

(iii) for each $s \in D$, the function $\mathbb{R} - \{s\} \ni t \mapsto \frac{f(t) - f(s)}{t - s}$ is non-decreasing.

2. Deduce that $f'_r(s) = \lim_{t \rightarrow s^+} \frac{f(t) - f(s)}{t - s}$ and $f'_l(s) = \lim_{t \rightarrow s^-} \frac{f(t) - f(s)}{t - s}$ exist and are respectively equal to $\inf_{t > s} \frac{f(t) - f(s)}{t - s}$ and $\sup_{t < s} \frac{f(t) - f(s)}{t - s}$, that $f'_l \leq f'_r$, and that the functions f'_l and f'_r are non-decreasing on D .
3. Show that $f'_r(s)$ and $f'_l(s)$ are reals when s belongs to the interior I of D .
4. Show that f is continuous on I .
5. Show that, for each $p \in [f'_l(s), f'_r(s)]$, and each $t \in \mathbb{R}$, we have

$$f(t) \geq f(s) + (t - s)p.$$

6. Show that the function f is differentiable except on an at most countable subset of I .
7. Show that if f is differentiable and f' is nondecreasing then f is convex.
8. Show the Jensen inequality : if $\mathbf{P}$ is a probability measure on Ω , and g is an integrable real-valued function on Ω , then

$$f\left(\int_{\Omega} g \, d\mathbf{P}\right) \leq \int_{\Omega} f \circ g \, d\mathbf{P}.$$

9. (★) Can f be made lower semi-continuous by changing its values on the boundaries of D ? Can it be made upper semi-continuous?

Exercise 4 (Smooth characterizations of convexity). Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function with open convex domain. Assume that f is Gâteaux-differentiable on its domain.

1. Prove that the following assertions are equivalent,
 - (i) f is convex,
 - (ii) $\forall x, y \in \text{dom } f, \quad f(y) \geq f(x) + \langle f'(x), y - x \rangle,$
 - (iii) $\forall x, y \in \text{dom } f, \quad \langle f'(y) - f'(x), y - x \rangle \geq 0.$

Hint : for (ii) $\implies$ (iii), combine (ii) with different pairs of points; for (iii) $\implies$ (i) study the function $g(t) = f((1-t)x + ty)$ on $[0, 1]$.
2. Prove that when f is twice differentiable on $\text{dom } f$, it is equivalent to
 - (iv) for every $x \in \text{dom } f$, $D^2f(x)$ is positive semi-definite, i.e. $\langle D^2f(x)v, v \rangle \geq 0$ for every v .
3. For $\lambda > 0$, in the case when $(E, \|\cdot\|)$ is a Hilbert space, prove that the following assertions are equivalent,
 - (i) f is λ -strongly convex,
 - (ii) $f - \lambda \frac{\|\cdot\|^2}{2}$ is convex,
 - (iii) $\forall x, y \in \text{dom } f, \quad f(y) \geq f(x) + \langle f'(x), y - x \rangle + \lambda \frac{\|y - x\|^2}{2},$
 - (iv) $\forall x, y \in \text{dom } f, \quad \langle f'(y) - f'(x), y - x \rangle \geq \lambda \|y - x\|^2,$

and that if moreover f is twice differentiable on $\text{dom } f$, it is equivalent to

 - (v) for every $x \in \text{dom } f$, $D^2f(x) \succeq \lambda I$, i.e. $\langle D^2f(x)v, v \rangle \geq \lambda \|v\|^2$ for every v .

Exercise 5. Show that

1. $(x, y) \mapsto \frac{x^2}{y}$ is convex on $\mathbb{R} \times (0, +\infty)$;
2. $x \mapsto \log \left(\sum_{i=1}^d e^{x_i} \right)$ is convex on $\mathbb{R}^d$;
3. $x \mapsto \left(\prod_{i=1}^d x_i \right)^{1/d}$ is concave on $(0, +\infty)^d$;
4. The function which associates a $d \times d$ symmetric matrix with its largest eigenvalue is convex;
5. $A \mapsto \log \det A$ is concave on the set of $d \times d$ positive definite symmetric matrices.

Exercise 6. Let E be the set ℓ^1 of summable real sequences $(u_n)_{n \in \mathbb{N}}$, meaning that $\|u\|_1 = \sum_{n \geq 0} |u_n| < +\infty$. The aim of the exercise is to study the differentiability of $f = \|\cdot\|_1$. For all n , let e^n be the sequence whose terms are all 0 except the n -th, which is 1.

1. Let $x \in E$ be such that $x_n = 0$ for some $n \in \mathbb{N}$. Compute $f'_+(x, e^n)$ and $f'_+(x, -e^n)$, and deduce that f is not Gâteaux-differentiable at x .
2. Let $x \in E$ be such that $x_n \neq 0$ for all $n \in \mathbb{N}$. Show that $f'_+(x, v) = \sum_{n \geq 0} v_n \text{sign}(x_n)$. Deduce that f is Gâteaux-differentiable at x .
3. Show that f is not Fréchet-differentiable at any point of E .

Exercise 7. Let K be a compact subset of $E = \mathbb{R}^n$ equipped with the Euclidean norm. Let d_K be the distance function from K defined by $d_K(x) = \min_{z \in K} \|x - z\|$ for every $x \in E$. The set of minimizers (the projections of x onto K) is denoted by $P_K(x)$.

1. Let $g(x) = \frac{1}{2}(\|x\|^2 - d_K(x)^2)$. Show that $g(x) = \max_{z \in K} \langle x | z \rangle - \|x\|^2/2$ and deduce that g is convex.
2. Let $z \in P_K(x)$. Show that $g(x) = \langle x | z \rangle - \|x\|^2/2$. Show that for every $v \in E$, $g(x + tv) - g(x) \geq t \langle v | z \rangle$ and then that $g'_+(x, v) \geq \max_{z \in P_K(x)} \langle v | z \rangle$.
3. We fix a direction $v \in E$ and consider $x_n = x + \frac{1}{n}v$ and $z_n \in P_K(x_n)$ for $n \in \mathbb{N}^*$. Up to extracting a sub-sequence, assume that z_n converges to a point z . Show that $g'_+(x, v) \leq \lim_{n \rightarrow \infty} (g(x_n) - g(x)) \leq \langle v | z \rangle$. Deduce that $g'_+(x, v) \leq \langle v | z \rangle$. Show that $z \in P_K(x)$.
4. Deduce from the two previous questions that $g'_+(x, v) = \max_{z \in P_K(x)} 2 \langle v | z \rangle$. Show that g is Gâteaux-differentiable at x if and only if $P_K(x)$ is a singleton.
5. Find a set K such that the set $\{x : \text{card}(P_K(x)) \geq 2\}$ is uncountable.

Exercise 8. Let α be a real number. For any function $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ and for any $\tau > 0$ we define the Moreau-Yosida τ -regularization of f by

$$f_\tau(x) = \inf_{u \in E} f(u) + \frac{1}{2\tau} \|x - u\|^2$$

1. Compute the Moreau-Yosida regularizations of χ_C where C is a non-empty convex set.
2. If $E = \mathbb{R}$, calculate the Moreau-Yosida regularizations of the absolute value function.

In the following, we assume that f is a proper continuous convex function bounded from below.

3. Show that for all $\tau > 0$, f_τ is convex and everywhere finite.
4. Show that for all x , $f_\tau(x)$ is nonincreasing in τ , and that $\sup_{\tau > 0} f_\tau(x) = \lim_{\tau \rightarrow 0^+} f_\tau(x) \leq f(x)$.
5. Show that for all τ there exists u_τ such that $f(u_\tau) + \frac{1}{2\tau}\|x - u_\tau\|^2 \leq f_\tau(x) + \tau$.
6. Show that u_τ tends to x and deduce that $\lim_{\tau \rightarrow 0} f_\tau(x) = f(x)$.