

Exercise Sheet 5 – Convex Functions (Subdifferential)

In all the exercises, E designates a real normed vector space.

Exercise 1. Let $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex function and let x be a point in the domain of f .

1. Show that $\partial f(x)$ is always convex and closed.
2. Show that $\partial f(x) = \{x^* : f'_+(x, \cdot) \geq \langle x^*, \cdot \rangle\}$.
3. Show that ∂f is monotone, in the sense that

$$\forall x^* \in \partial f(x), \forall y^* \in \partial f(y), \quad \langle y^* - x^*, y - x \rangle \geq 0.$$

4. Show that if f is continuous at x , $x_n \rightarrow x$, and $x_n^* \rightarrow x^*$ where $x_n^* \in \partial f(x_n)$ for every n , then $x^* \in \partial f(x)$.
5. Show that if f is continuous at x then $\partial f(x)$ is bounded.
6. Deduce that if E has finite dimension and $x \in \text{int dom } f$ then $\partial f(x)$ is bounded.

Exercise 2 (Examples of subdifferentials). Compute the subdifferentials of the following functions :

- (i) $f_1(x) = |x|$ on $\mathbb{R}$,
- (ii) $f_2(x) = \sum_{i=1}^d g(x_i)$ on $\mathbb{R}^d$ with $g : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ convex,
- (iii) $f_3(x) = \|x\|_1$ on $\mathbb{R}^d$,
- (iv) $f_4(x) = \|x\|_2$ on $\mathbb{R}^d$,
- (v) $f_5(x) = -\sqrt{x}$ with domain $\mathbb{R}_+$.

Exercise 3. Let C be a convex subset of E . Show that for every $x \in E$,

$$\mathbf{N}_x C = \partial \chi_C(x).$$

Exercise 4 (Subdifferential inclusions).

1. (a) Let $f, g : E \rightarrow [-\infty, +\infty]$. Show that $\partial f(x) + \partial g(x) \subset \partial(f+g)(x)$.
 (b) Show that the converse inclusion is false for the convex (and lower semicontinuous) functions $f = \chi_{(-\infty, 0]}$ and $g(x) = -\sqrt{x}$ with $\text{dom } g = \mathbb{R}_+$.
2. Recall that the adjoint of a continuous linear operator $A : E \rightarrow F$ is the continuous linear operator $A^* : F^* \rightarrow E^*$ defined by $\langle Ax^*, x \rangle = \langle x^*, Ax \rangle$ for every $x^* \in F^*$. Let $A : F \rightarrow E$ a continuous linear operator and $f : E \rightarrow [-\infty, +\infty]$. Show that $\partial(f \circ A)(x) \supseteq A^* \partial f(Ax)$ for every $x \in F$.
3. Let $(f_1, \dots, f_n)$ be functions on E and $f = \max_{1 \leq i \leq n} f_i$. For all $x \in E$, let $I(x) = \{i : f(x) = f_i(x)\}$. Show that

$$\overline{\text{conv}} \left(\bigcup_{i \in I(x)} \partial f_i(x) \right) \subseteq \partial f(x).$$

Exercise 5 (Optimality conditions). Establish the optimality conditions of the following minimization problems.

1. (Fermat's problem)

$$\inf_{x \in \mathbb{R}^d} F(x) \quad \text{where} \quad F(x) =: \sum_{i=1}^n \lambda_i \|x - x_i\|$$

$\lambda_1, \dots, \lambda_n \geq 0$, $x_1, \dots, x_n \in \mathbb{R}^d$ are distinct points and $\|\cdot\|$ denotes the Euclidean norm.

2. (LASSO problem)

$$\inf_{x \in \mathbb{R}^d} G(x) \quad \text{where} \quad G(x) := \frac{1}{2} \|Ax - b\|_2^2 + \gamma \|x\|_1,$$

$A \in M_{m,d}(\mathbb{R})$ is a matrix, $b \in \mathbb{R}^m$ and $\gamma > 0$.

Exercise 6. Express the subdifferential of the norm $f : x \mapsto \|x\|$ at 0 in terms of the dual norm.

Exercise 7. Let $f : E \rightarrow [-\infty, +\infty]$ be a strictly convex function. Show that ∂f is injective in the following sense : if $x \neq y$, then $\partial f(x)$ and $\partial f(y)$ are disjoint.

Exercise 8 (Gradient flow). Let f be a convex function on $\mathbb{R}^d$ and let $x(t)$ and $y(t)$ be two C^1 maps from an interval $I \subseteq \mathbb{R}$ into $\mathbb{R}^d$ such that $x'(t) \in -\partial f(x(t))$ and $y'(t) \in -\partial f(y(t))$ for all t .

1. Show that $t \mapsto |x(t) - y(t)|$ is nonincreasing (where $|\cdot|$ denotes the Euclidean norm).
2. Show that $t \mapsto |x'(t)|$ is nonincreasing.
3. Show that $\limsup_{s \rightarrow t, s < t} \frac{f(x(t)) - f(x(s))}{t - s} \leq -|x'(t)|^2$.

We now assume that f is continuous and strictly convex on $\mathbb{R}^d$, and that it reaches a minimum at x_0 .

4. Show that, for $s < t$, $f(x(t)) - f(x(s)) \leq (s - t)|x'(t)|^2$.
5. Show that $x(t)$ tends to x_0 in $+\infty$.