

Exercise Sheet 6 – Convex Functions (Fenchel-Legendre conjugate)

In all the exercises, E designates a real normed vector space.

Exercise 1. Compute the Legendre-Fenchel conjugate of the following functions.

1. $f(x) = x$ on $\mathbb{R}$.
2. $f(x) = \exp(x)$ on $\mathbb{R}$.
3. $f(x) = -\log(x)$ with domain $\mathbb{R}_+^*$.
4. $f(x) = x \log(x)$ on $\mathbb{R}_+^*$, $f(0) = 0$ and domain $\mathbb{R}_+$.
5. $f(x) = \frac{1}{p}|x|^p$ for $p \in [1, +\infty)$ on $\mathbb{R}$.
6. $f = \chi_A$ where A is a convex subset of E . Specify when $A = B(0, 1)$ is the unit ball of E , and then when E is $(\mathbb{R}^d, \|\cdot\|_p)$ for $p \in [1, +\infty]$.
7. $f = \|\cdot\|$ on E . Specify when E is $(\mathbb{R}^d, \|\cdot\|_p)$, and $E = (\mathbb{R}, |\cdot|)$.
8. $f = \langle p^*, \cdot \rangle$ where $p^* \in E^*$.
9. $f(x) = \phi(\|x\|)$ where $\phi : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ is even and convex.
10. $f(x) = \frac{1}{2}\langle x, Ax \rangle$ on $E = (\mathbb{R}^n, \|\cdot\|_2)$, where A is a positive definite matrix A .
11. $f(x) = \log\left(\sum_{i=1}^d e^{x_i}\right)$ on $\mathbb{R}^d$.
12. $f(A) = -\log \det A$ where $\text{dom } f = S_d^{++}(\mathbb{R}) \subseteq M_d(\mathbb{R})$.

Exercise 2 (Young and Hölder inequalities). Show that for all $p, q \in (1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\forall x, y \in \mathbb{R}, \quad |xy| \leq \frac{1}{p}|x|^p + \frac{1}{q}|y|^q.$$

Deduce Hölder inequality :

$$\forall x, y \in \mathbb{R}^d, \quad \langle x, y \rangle \leq \|x\|_p \|y\|_q.$$

(Hint : start with the case $\|x\|_p = \|y\|_q = 1$.)

Exercise 3. Let $f : E \rightarrow \mathbb{R}$ be convex and L -Lipschitz. Show that $\text{dom}(f^*)$ is included in $B[0, L]$. Deduce the fact that for every $x \in E$, $\partial f(x) \subseteq B[0, L]$.

Exercise 4. Let $f : E = (\mathbb{R}^d, \|\cdot\|_2) \rightarrow \mathbb{R}$ be continuous and convex. Show that f is affine if and only if $\text{dom}(f^*)$ is a singleton.

Exercise 5. In $E = (\mathbb{R}^n, \|\cdot\|_2)$, find all functions $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$ which are their own conjugate.

Exercise 6 (Strictly convex functions).

1. Show that $x^* \in \partial f(x)$ if and only if x is a global minimum of $f - \langle x^*, \cdot \rangle$.
2. We assume that f is lsc and strictly convex. Show that for every $x^* \in \text{dom } f^*$, $\partial f^*(x^*)$ has at most one point. Deduce that ∂f is injective in the following sense : for every $x \neq y$, $\partial f(x) \cap \partial f(y) = \emptyset$.
3. We assume that f is strictly convex and E is finite-dimensional. Show that f^* is differentiable (in the sense of Fréchet) on the interior of its domain.

Exercise 7. Let f be a C^2 function on $\mathbb{R}^d$, such that the Hessian $\text{Hess } f$ of f satisfies $\text{Hess } f(x) \geq \lambda \text{Id}$ for all $x \in \mathbb{R}^d$, for some fixed $\lambda > 0$.

1. Show that the map $x \mapsto F(x) := f'(x)$ is a diffeomorphism of $\mathbb{R}^d$.
2. Show that f^* is of class C^2 , and express $(f^*)'$ and $\text{Hess } f^*$ in terms of f' and $\text{Hess } f$.