Heat Equations on Vector Bundles: Application to Images Regularization

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PDEs devoted to nD image regularization

\[
\frac{\partial I^i}{\partial t} = \sum_{j,k=1}^2 f_{jk} \frac{\partial^2 I^i}{\partial j \partial k} + \text{(part of order } \leq 1) 
\]

where \( I|_{t=0} = I_0 : \Omega \to \mathbb{R}^n \) nD image, and \( f_{jk} \) functions on \( \Omega \).

Geometric interpretation: \( E \) vector bundle of rank \( n \) over \( (\Omega, g) \) well-chosen \( \implies \) Differential operator \( H \) of order 2 acting on \( I \in \Gamma(E) \) is a Generalized Laplacian.

\( \implies \) Solution of \( \partial I/\partial t - HI = 0 \) given by

\[
(e^{-tH}I_0)(x) = \int_{\Omega} K_t(x, y, -H)I_0(y) dy 
\]
Motivation

- PDEs devoted to nD image regularization

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\frac{\partial I^i}{\partial t} = \sum_{j,k=1}^{2} f_{jk} \frac{\partial^2 I^i}{\partial j \partial k} + \text{(part of order } \leq 1)\]

where \( I|_{t=0} = I_0 : \Omega \rightarrow \mathbb{R}^n \) nD image, and \( f_{jk} \) functions on \( \Omega \).

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Let $v, v' \in \Gamma(TX)$ be two tangent vector fields on $X$ and $f, g \in C^\infty(X)$.

**Definition**

A connection $\nabla$ is a map $\Gamma(TX) \times \Gamma(E) \longrightarrow \Gamma(E)$ satisfying

- $\nabla_{fv'+gv'}S = f\nabla_vS + g\nabla_{v'}S$
- $\nabla_v(S + S') = \nabla_vS + \nabla_{v'}S'$
- $\nabla_v fS = f\, \nabla_v S + (d_v f)S$

The operator $\nabla_v$ is called the covariant derivative along $v$.

**Definition**

Let $E$ be a vector bundle over a manifold $X$ equipped with a connection $\nabla^E$, $y \in X$, and $Y_0 \in E_y$. Let $\gamma$ be a $C^1$ curve in $X$ such that $\gamma(0) = y$. The parallel transport of $Y_0$ along $\gamma$ is the solution $Y(t) \in E_{\gamma(t)}$ of the differential equation

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\begin{cases}
\nabla^E_{\gamma'} Y(t) = 0 \\
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Definition (Generalized Laplacian)

Let $E$ vector bundle over $(X, g)$. A **generalized Laplacian** on $E$ is a differential operator of order 2 $H : \Gamma(E) \longrightarrow \Gamma(E)$ that may be written

$$H = - \sum_{ij} g^{ij} \partial_i \partial_j + \text{part of order } \leq 1$$

in any local coordinates system.

Solution $e^{-tH}u_0$ of the heat equation:

$$\frac{\partial u}{\partial t} + Hu = 0, \quad u|_{t=0} = u_0$$

given by

$$e^{-tH}u_0(x) = \int_X K_t(x, y, H)u_0(y) \, dy$$
**Definition (Connection Laplacian)**

Let $\nabla^E$ a connection on $E$. The connection Laplacian associated to $\nabla^E$ is a generalized Laplacian on $E$ defined by

$$\Delta^E = - \sum_{ij} g^{ij} \left( \nabla^E_{\partial_i} \nabla^E_{\partial_j} - \sum_k \Gamma^E_{ij} \nabla^E_{\partial_k} \right)$$

in a local coordinates system.

**Proposition (Berline et al., *Heat kernels and Dirac operators*)**

Given $H$ a generalized Laplacian on $E$, we may equip $E$ with a connection $\nabla^E$ such that $H = \Delta^E + F$ for some $F \in \Gamma(\text{End}(E))$. 
Approximations of heat equations solutions

- Approximation of the heat kernel $K_t(x, y, H)$:
  \[
  K_t^0(x, y, H) = \left( \frac{1}{4\pi t} \right)^{m/2} e^{-d(x, y)^2/4t} \Psi(d(x, y)^2) \tau(x, y) J(x, y)^{-1/2}
  \]
  where
  - $d$ geodesic distance on $X$
  - $\Psi$ cut-off function
  - $\tau : E_y \rightarrow E_x$ transport parallel map wrt $\nabla^E$ along the unique geodesic joining $x$ and $y$.
  - $J(x, y)$ jacobian of the coordinates change.

- Approximation of the solution $e^{-tH} u_0$ of the heat equation:
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  k_t^0 u_0(x) = \int_X K_t^0(x, y, H) u_0(y) dy
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Discrete approximations of heat equations solutions

- Discretization of $k_t^0 u_0$ by discrete convolutions.

- $J(x, y) \approx 1$ near $x$. Approximate $J$ by $1 \Rightarrow$ convolutions with small masks.

- Construction of $V_x$:
  - for each $i, j \in \{-1, 0, 1\}$, construct the geodesic from $x$ with tangent vector $(i, j)$ at $x$, by the use of the transport parallel map on $TX$ associated to L-C connection.
  - for each $y$, set $y \in V_x$ if two geodesics from $x$ do not intersect at $y$, and $y$ does not follow such a point on a geodesic from $x$.

- Discrete convolution of $\tau(x, y) u_0(y)$ with a mask whose input in $y$ is the geodesic distance to $x$ if $y \in V_x$ and 0 otherwise.
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The scalar Laplacian $\Delta$

**Definition**

Let $(X, g)$ be a Riemannian manifold. The scalar Laplacian on $(X, g)$ is the connection Laplacian on a vector bundle of rank 1, endowed with the connection $\nabla^E$ defined by $\nabla^E e_1 = 0$ in a frame $e_1$ of $E$. In a local coordinates system, it is defined by

$$\Delta(fe_1) = - \sum_{ij} g^{ij} \left( \partial_i \partial_j - \sum_k \Gamma_{ij}^k \partial_k \right) f e_1$$

**Proposition (Parallel transport associated to $\nabla^E$)**

Let $\gamma$ be a $C^1$ curve in $X$ such that $\gamma(0) = y$. The parallel transport $Y$ of the vector $Y_0 = Y_0 e_1(y)$ along $\gamma$ is $Y(t) = Y_0 e_1(\gamma(t))$.

**Consequence:** $\tau(x, y)(Y_0 e_1(y)) = Y_0 e_1(x)$
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$$\Delta(f e_1) = -\sum_{ij} g^{ij} \left( \partial_i \partial_j - \sum_k \Gamma_{ij}^k \partial_k \right) f \, e_1$$

**Proposition (Parallel transport associated to $\nabla^E$)**

Let $\gamma$ be a $C^1$ curve in $X$ such that $\gamma(0) = y$. The parallel transport $Y$ of the vector $Y_0 = Y_0 \, e_1(y)$ along $\gamma$ is $Y(t) = Y_0 \, e_1(\gamma(t))$.

**Consequence:** $\tau(x, y)(Y_0 \, e_1(y)) = Y_0 \, e_1(x)$
Remark: Identify $C^\infty(X)$ with a vector bundle $E$ of rank 1 of frame $e_1$ equipped with the connection $\nabla^E$ defined by $\nabla^E e_1 = 0$, and equip $(X, g)$ with the L-C connection $\implies \Delta = -\Delta_g$.

Consequence: PDEs $\partial u/\partial t = \Delta_g u$ and $\partial u/\partial t + \Delta u = 0$ are equivalent.

$\implies$ Beltrami flow [Sochen et al.]

$\implies$ The kernel $K_t^0(x, y, \Delta)$ corresponds to the 'short time Beltrami kernel'.
Remark: Identify $C^\infty(X)$ with a vector bundle $E$ of rank 1 of frame $e_1$ equipped with the connection $\nabla^E$ defined by $\nabla^E e_1 = 0$, and equip $(X, g)$ with the L-C connection $\nabla^E \Rightarrow \Delta = -\Delta_g$.

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$\Rightarrow$ Beltrami flow [Sochen et al.]
$\Rightarrow$ The kernel $K_t^0(x, y, \Delta)$ corresponds to the 'short time Beltrami kernel'.
Let $X$ be a manifold of dimension 2, and $f \in C^\infty(X)$. We consider the differential operator $\Delta$ of order 2 given by

$$\Delta(f) = -c_1 \, d^2_{\xi,\xi} f - c_2 \, d^2_{\eta,\eta} f$$

where $(\xi, \eta)$ is a mobile frame of $TX$, and $c_1, c_2 \in C^\infty(X)$.

$\Rightarrow \Delta$: generalized Laplacian $H$ on a vector bundle of rank 1.
Two order differential operator wrt mobile frame as generalized Laplacian

**Proposition**

Let \((X, g)\) be a Riemannian manifold of dim 2, with

\[
g = \frac{1}{c_1 c_2 (\xi_1 \eta_2 - \eta_1 \xi_2)^2} \begin{pmatrix}
  c_1 \xi_2^2 + c_2 \eta_2^2 & -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) \\
  -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) & c_1 \xi_1^2 + c_2 \eta_1^2
\end{pmatrix}
\]

in a local coordinates system \((x_1, x_2)\).

Let \(E\) be a vector bundle of rank 1 over \(X\), of global frame \(e_1\), equipped with a connection \(\nabla^E\) defined by \(\nabla^E_{\partial_{x_1}} e_1 = \gamma_1 e_1\) et \(\nabla^E_{\partial_{x_2}} e_1 = \gamma_2 e_1\) where

\[
\gamma_1 = 0.5(g_{11} a + g_{12} b) \quad \gamma_2 = 0.5(g_{12} a + g_{22} b)
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and

\[
a = c_1 \frac{\partial \xi_1}{\partial x_1} \xi_1 + c_1 \frac{\partial \xi_1}{\partial x_2} \xi_2 + c_2 \frac{\partial \eta_1}{\partial x_1} \eta_1 + c_2 \frac{\partial \eta_1}{\partial x_2} \eta_2 + 2g_{12} \Gamma^1_{12} + g_{11} \Gamma^1_{11} + g_{22} \Gamma^1_{22}
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Proposition

Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be a $C^1$ curve in $X$ such that $\gamma(0) = y$, and $Y_0 = Y_0 e_1(y) \in E_y$. The parallel transport $Y$ of the vector $Y_0$ along $\gamma$ is defined by

$$Y(t) = Y_0 \exp \left(-\int_0^t \dot{\gamma}_1(s) \gamma_1(\gamma(s)) + \dot{\gamma}_2(s) \gamma_2(\gamma(s)) \ ds\right) e_1(\gamma(t))$$

Proof: The parallel transport of $Y_0$ along $\gamma$ is the solution $Y(t) = Y(t) e_1(\gamma(t))$ of the differential equation

$$\begin{cases} \nabla^E_{\dot{\gamma}} Y(t) = 0 \\ Y(0) = Y_0 \end{cases}$$
Assume that functions $c_1, c_2$ and components $\xi_1, \xi_2, \eta_1, \eta_2$ are constant on a neighborhood $D$ of $x \in X$. Then

$$H(\tilde{f})(x) = \text{Trace}(T(x)\text{Hess}(x))e_1$$

where $T(x) = c_1(x)\xi(x)\xi(x)^T + c_2(x)\eta(x)\eta(x)^T$, and $\text{Hess}$ hessian matrix of $f$ at $x$ in the usual coordinates system.

**Consequences:** Symbols $\Gamma^k_{ij}$ et $\Upsilon_i$ vanish on $D$, and the kernel $K^0_t(x, y, H)$ is given on $D \times D$ by

$$(1/4\pi t)e^{-g((x_1-y_1,x_2-y_2),(x_1-y_1,x_2-y_2))/4t}Id(x, y)$$

where $Id(x, y) : E_y \rightarrow E_x$ is the 'Identity' map.

$\implies$ Oriented Gaussian kernel [Tschumperlé et al.]
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$$H(\tilde{f})(x) = \text{Trace}(T(x)\text{Hess}(x))e_1$$

where $T(x) = c_1(x)\xi(x)\xi(x)^T + c_2(x)\eta(x)\eta(x)^T$, and $\text{Hess}$ hessian matrix of $f$ at $x$ in the usual coordinates system.

**Consequences:** Symbols $\Gamma^k_{ij}$ et $\Upsilon_i$ vanish on $D$, and the kernel $K_t^0(x, y, H)$ is given on $D \times D$ by

$$(1/4\pi t)e^{-g((x_1-y_1, x_2-y_2), (x_1-y_1, x_2-y_2))/4t} \text{Id}(x, y)$$

where $\text{Id}(x, y) : E_y \to E_x$ is the 'Identity' map.

$\implies$ Oriented Gaussian kernel [Tschumperlé et al.]
Example: Color image regularization

Original image
Scalar/Beltrami Laplacian kernel
Oriented Gaussian kernel
Laplacian wrt mobile frames kernel
Heat Equations on Vector Bundles: Application to Regularization

- Heat equation associated to the scalar/Beltrami operator on $C^\infty(X)$:
  $\Rightarrow$ Regularize functions on a Riemannian manifold.
  $\Rightarrow$ Regularize nD images.

- Heat equation associated to the Hodge operator on $\text{Cl}(X, g)$:
  $\Rightarrow$ Regularize functions, tangent vector fields and orthonormal frame fields on a Riemannian manifold.
  $\Rightarrow$ Regularize nD images and related fields: vector fields and orientation fields.
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Application 1: Vector fields regularization

Figure: Clifford-Hodge flow on tangent vector fields
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Figure: Clifford-Hodge flow on tangent vector fields
Application 2: Orientation fields regularization

Figure: Clifford-Hodge flow on orthonormal frame fields
Summary/Future works

PDEs in image regularization as heat equations on vector bundles:
- Short time Beltrami kernel.
- Extension of the oriented Gaussian kernel diffusion.
- Extension of the Short time Beltrami kernel diffusion to tangent vector fields and orthonormal frame fields.

- Better continuous/discrete approximations.
- Extension to base manifolds of dimension 3 → Extension to videos.