# POISSON PROCESSES. RETAKE EXAM (2H).
# 2026, JULY 1ST.

**Before you start.** No document. No electronic device. Answers need to be justified. You may write in English or French. The quality of writing is taken into account. Exercises are independent from each other.

**Exercise 1.** Let $N = (N_t)_{t\geq 0}$ be a Poisson process with parameter one. For every $u, v \geq 0$, let $\delta(u,v) = \mathbb{1}\{N_u = N_v\}$ be the (indicator) random variable that is equal to one if $N_u = N_v$ and zero otherwise. Define the collision time with horizon $t \geq 0$ as

$$C(t) := \int_{0\leq u,v\leq t} \delta(u,v)\mathrm{d}u\mathrm{d}v.$$

(1) Compute $\mathbb{P}(N_u = N_v)$ for every $u, v \geq 0$. Let us first assume that $u \leq v$. Then,

$$\mathbb{P}(N_u = N_v) = \mathbb{P}(N_v - N_u = 0) = \mathbb{P}(\mathrm{Poi}(v-u) = 0) = e^{-(v-u)}.$$

Since this probability is invariant under permutation of $u$ and $v$, we deduce that for every $u, v \geq 0$,

$$\mathbb{P}(N_u = N_v) = e^{-|v-u|}.$$

(2) Compute $m(t) = \mathbb{E}(C(t))$ and determine the limit of $m(t)/t$ as $t \to \infty$. We compute:

$$\begin{aligned} m(t) = \mathbb{E}(C(t)) &= \int_{0\leq u,v\leq t} \mathbb{P}(N_u = N_v)\mathrm{d}u\mathrm{d}v \\ &= \int_{0\leq u,v\leq t} e^{-|v-u|}\mathrm{d}u\mathrm{d}v \\ &= 2\int_{0\leq u<v\leq t} e^{-(v-u)}\mathrm{d}u\mathrm{d}v \\ &= 2\int_0^t e^u(e^{-u} - e^{-t})\mathrm{d}u \\ &= 2\int_0^t (1 - e^{u-t})\mathrm{d}u \\ &= 2(t - 1 + e^{-t}). \end{aligned}$$

Therefore,

$$\lim_{t\to\infty} m(t)/t = 2.$$

(3) Let $\lambda > 0$. What is the law of the process $(N_{\lambda t})_{t\geq 0}$? Justify your answer. Since $N$ is a Poisson process with parameter one, it is a counting process and its inter-arrival times $(T_i - T_{i-1})_{i\geq 1}$ are i.i.d. exponential random variables with parameter one. One can check that $(N_{\lambda t})_{t\geq 0}$ is also a counting process, whose inter-arrival times are $\lambda T_i - \lambda T_{i-1} = \lambda(T_i - T_{i-1})$, for $i \geq 1$. Those are i.i.d. exponential random variables with parameter $\lambda$. Therefore, $(N_{\lambda t})_{t\geq 0}$ is a Poisson process with parameter $\lambda$.

(4) Using your answer to the previous question, answer Question (2) in the general case of a Poisson process with parameter $\lambda > 0$. Let $m(\lambda, t)$ be the expected collision time with time horizon $t \geq 0$ for a Poisson process with parameter $\lambda$. From what precedes, we get

$$\begin{aligned} m(\lambda, t) &= \int_{0 \leq u, v \leq t} \mathbb{P}(N_{\lambda u} = N_{\lambda v}) \mathrm{d}u \mathrm{d}v \\ &= \frac{1}{\lambda^2} \int_{0 \leq u, v \leq \lambda t} \mathbb{P}(N_u = N_v) \mathrm{d}u \mathrm{d}v \\ &= \frac{m(1, \lambda t)}{\lambda^2} \\ &= \frac{2}{\lambda^2} (\lambda t - 1 + e^{-\lambda t}). \end{aligned}$$

We have used the previous question on the first line, a linear change of variable on the second line and our answer to the first question on the last line. Finally,

$$\lim_{t \to \infty} m(\lambda, t)/t = \frac{2}{\lambda}.$$

**Exercise 2.** Let $X = (X_t)_{t \geq 0}$ be a continuous-time Markov process on $\mathbb{N} = \{0, 1, 2, \ldots\}$ (the set of non-negative integers) with generator $Q$. We assume that $Q(k-1, k) = \lambda_k > 0$ and $Q(k, k-1) = \mu_k > 0$ for every positive integer $k \geq 1$, and $Q(i, j) = 0$ for every other couple $(i, j) \in \mathbb{N}$ such that $i \neq j$. We also define

$$V_k = \prod_{1 \leq i \leq k} \left( \frac{\lambda_i}{\mu_i} \right), \qquad k \geq 1.$$

(5) Determine (explicitly) the transition matrix of the embedded Markov chain. Let $\Pi$ be the transition matrix of the embedded Markov chain. First we compute

$$Q(k, k) = -\sum_{\ell \neq k} Q(k, \ell) = \begin{cases} -\lambda_1 & (k = 0) \\ -(\lambda_{k+1} + \mu_k) & (k \geq 1), \end{cases}$$

and we recall that for every $k \neq \ell$,

$$\Pi(k, \ell) = \frac{Q(k, \ell)}{-Q(k, k)}.$$

Therefore, $\Pi(0, 1) = 1$ and for every $k \geq 1$,

$$\Pi(k, k+1) = \frac{\lambda_{k+1}}{\lambda_{k+1} + \mu_k}, \qquad \Pi(k, k-1) = \frac{\mu_k}{\lambda_{k+1} + \mu_k}.$$

(6) Show that $X$ is irreducible. By the course, it is enough to show that the embedded Markov chain $Y$ is irreducible. Indeed, the state 0 communicates with any other state $k \geq 1$, as

$$\mathrm{P}(Y_k = k | Y_0 = 0) \geq \prod_{1 \leq i \leq k} \Pi(i-1, i) > 0$$

$$\mathrm{P}(Y_k = 0 | Y_0 = k) \geq \prod_{1 \leq i \leq k} \Pi(k-i+1, k-i) > 0.$$

(7) Determine the invariant measures of $X$ using the sequence $(V_k)_{k\geq 1}$. We first recall that a (non-negative) measure $\nu = (\nu_k)_{k\geq 1}$ is invariant for $Q$ if and only if $\nu Q = 0$. This equation writes

$$\begin{cases} -\lambda_1\nu_0 + \mu_1\nu_1 = 0 \\ \nu_{k-1}\lambda_k - (\lambda_{k+1} + \mu_k)\nu_k + \mu_{k+1}\nu_{k+1} = 0 \quad (k \geq 1). \end{cases}$$

We can then show by iteration that this is equivalent to the condition

$$\nu_k = \nu_0 V_k, \qquad k \geq 1.$$

(8) Show that $X$ is positive recurrent if and only if $\sum_{k\geq 1} V_k$ is finite. The process $X$ is irreducible. By the course, it is positive recurrent if and only if it has a unique invariant *probability* measure. From the previous question, this is equivalent to $\sum_{k\geq 1} V_k$ being finite.

(9) We now assume that $\mu_k = \mu > 0$ and $\lambda_k = \lambda > 0$ for every $k \geq 1$ (case of constant rates). What does the condition of Question (8) become in this special case? If $X$ is positive recurrent, what is its invariant probability measure? Give (in a few lines) an interpretation inspired from queueing theory. In that special case, we have $V_k = (\lambda/\mu)^k$ for every $k \geq 1$ and the condition of Question (8) becomes $\rho := \lambda/\mu < 1$. Under that condition, the invariant probability measure of $X$ is the geometric distribution $(1-\rho)\rho^k$, $k \geq 0$. The process $X$ can be interpreted as the evolution in time of the number of customers in an M/M/1 queue, that is a queue with Markov arrival and service times, and a single server unit. When the average customer arrival time ($\lambda$) is (strictly) less than the average service time ($\mu$) then the process is stable and admits an invariant probability distribution (which also describes the long-time behavior of the queue).

**Exercise 3.** For each question, justify your answer, i.e. carefully show that your example meets all the requirements.

(10) Give an example of an irreducible continuous-time Markov process which is *positive* recurrent but whose embedded Markov chain is *null* recurrent. (Short answer) Continuous-time random walk on the Euclidean lattice in dimension $d = 1$ or $d = 2$ with appropriate holding times.

(11) Give an example of an irreducible continuous-time Markov process which is transient and possesses an invariant measure. (Short answer) Continuous-time random walk on the Euclidean lattice with unit holding times in dimension $d \geq 3$.