POISSON PROCESSES. RETAKE EXAM (2H).
2026, JULY 1ST.

Before you start. No document. No electronic device. Answers need to be justified. You may write in English or French. The quality of writing is taken into account. Exercises are independent from each other.

Exercise 1. Let $N = (N_t)_{t\geq 0}$ be a Poisson process with parameter one. For every $u, v \geq 0$, let $\delta(u,v) = \mathbb{1}\{N_u = N_v\}$ be the (indicator) random variable that is equal to one if $N_u = N_v$ and zero otherwise. Define the collision time with horizon $t \geq 0$ as

$$C(t) := \int_{0\leq u,v\leq t} \delta(u,v)\mathrm{d}u\mathrm{d}v.$$

(1) Compute $\mathbb{P}(N_u = N_v)$ for every $u, v \geq 0$.

(2) Compute $m(t) = \mathbb{E}(C(t))$ and determine the limit of $m(t)/t$ as $t \to \infty$.

(3) Let $\lambda > 0$. What is the law of the process $(N_{\lambda t})_{t\geq 0}$? Justify your answer.

(4) Using your answer to the previous question, answer Question (2) in the general case of a Poisson process with parameter $\lambda > 0$.

Exercise 2. Let $X = (X_t)_{t\geq 0}$ be a continuous-time Markov process on $\mathbb{N} = \{0, 1, 2, \ldots\}$ (the set of non-negative integers) with generator Q. We assume that $Q(k-1, k) = \lambda_k > 0$ and $Q(k, k-1) = \mu_k > 0$ for every positive integer $k \geq 1$, and $Q(i,j) = 0$ for every other couple $(i,j) \in \mathbb{N}$ such that $i \neq j$. We also define

$$V_k = \prod_{1\leq i\leq k} \left(\frac{\lambda_i}{\mu_i}\right), \qquad k \geq 1.$$

(5) Determine (explicitly) the transition matrix of the embedded Markov chain.

(6) Show that X is irreducible.

(7) Determine the invariant measures of X using the sequence $(V_k)_{k\geq 1}$.

(8) Show that X is positive recurrent if and only if $\sum_{k\geq 1} V_k$ is finite.

(9) We now assume that $\mu_k = \mu > 0$ and $\lambda_k = \lambda > 0$ for every $k \geq 1$ (case of constant rates). What does the condition of Question (8) become in this special case? If X is positive recurrent, what is its invariant probability measure? Give (in a few lines) an interpretation inspired from queueing theory.

Exercise 3. For each question, justify your answer, i.e. carefully show that your example meets all the requirements.

(10) Give an example of an irreducible continuous-time Markov process which is *positive* recurrent but whose embedded Markov chain is *null* recurrent.

(11) Give an example of an irreducible continuous-time Markov process which is transient and possesses an invariant measure.