Exercise sheet n°1 : Poisson processes

Exercise 1. Let $(\tau_n, n \geq 1)$ be a sequence of IID (independent and identically distributed) non-negative random variables. Set $T_0 = 0$,

$$T_n = \tau_1 + ... + \tau_n, \quad n \geq 1,$$

and

$$N_t = \#\{i \geq 1 : T_i \leq t\}, \quad t \geq 0.$$

1. Give a necessary and sufficient condition for having

$$\mathbb{P}(N \text{ only makes jumps of size } 1) > 0.$$

2. Show that under this condition

$$\mathbb{P}(N \text{ only makes jumps of size } 1) = 1.$$

3. Is it possible that $(T_n, n \geq 0)$ converges to a finite limit with positive [1] probability ?
4. Compute the probability of the event $\{\exists t \geq 0 : N(t) = \infty\}$.

Exercise 2. Let N be a Poisson process with intensity $\lambda > 0$. Prove and give an interpretation of the following properties

1. $\mathbb{P}(N_h = 1) = \lambda h + o(h)$ $(h \to 0)$
2. $\mathbb{P}(N_h \geq 2) = o(h)$ $(h \to 0)$
3. $\mathbb{P}(N_h = 0) = 1 - \lambda h + o(h)$ $(h \to 0)$.
4. $\forall t \geq 0$, $\mathbb{P}(N \text{ jumps at time } t) = 0$.
5. Compute $\mathrm{Cov}(N_s, N_t)$, $\forall s, t \geq 0$.

Exercise 3. Let N be a counting process with stationary and independent increments. Assume that there exists $\lambda > 0$ such that

$$\mathbb{P}(N_h = 1) = \lambda h + o(h), \;\; \mathbb{P}(N_h \geq 2) = o(h).$$

For $u \in \mathbb{R}$, let $g_t(u) = \mathbb{E}[e^{iuN_t}]$.

1. En anglais le mot *positive* signifie strictement positif. Pour dire positif au sens large on dit *non-negative*. De même les termes *negative, bigger, smaller* sont à prendre au sens strict.

1. Prove that $g_{t+h}(u) = g_t(u)g_h(u)$ for every $t, h \geq 0$.
2. Prove that
$$\tfrac{d}{dt}g_t(u) = \lambda(e^{iu} - 1)g_t, \;\; g_0(u) = 1.$$
3. Conclude.

Exercise 4. Let N be a Poisson process with intensity $\lambda > 0$, modelling the arrival times of the claims for an insurance company. Let T_1 denote the arrival time of the first claim. Show that the conditional law of T_1 given $N_t = 1$ is uniformly distributed over $[0, t]$.

Exercise 5. Let $(T_n, n \geq 0)$ $(T_0 = 0)$ be a renewal process and N its associated counting process. Assume that N has independent and stationary increments.

1. Show that
$$\mathbb{P}(T_1 > s + t) = \mathbb{P}(T_1 > t)\mathbb{P}(T_1 > s), \quad \forall s, t \geq 0.$$
2. Derive that N is a Poisson process.

Exercise 6.

1. Show that two independent Poisson processes cannot jump simultaneously a.s.
2. Let N^1 and N^2 be two independent Poisson processes with parameters $\lambda_1 > 0$ and λ_2 respectively. Show that the process
$$N_t = N_t^1 + N_t^2, \;\; t \geq 0$$
is a Poisson process and give its intensity.
3. Derive that the sum of n independent Poisson processes with respective intensities $\lambda_1 > 0, ..., \lambda_n > 0$ is a Poisson process and give its intensity.

Exercise 7. Liver transplants arrive at a hospital according to a Poisson process N with intensity $\lambda > 0$. Two patients are waiting for a transplant. The first patient has lifetime T (before the transplant) according to an exponential distribution with parameter μ_1. The second one has lifetime T'(before the transplant) according to an exponential distribution with parameter μ_2. The rule is that the first transplant that arrives at the hospital is given to the first patient if he/she is still alive, and to the second patient otherwise. Assume that T, T' and N are independent.

1. Compute the probability that the first patient is transplanted.
2. Compute the probability that the second patient is transplanted.
3. Let X denote the number of transplants arrived at the hospital during $[0, T]$. Compute the law of X.

Exercise 8. [The bus paradox] Buses arrive at a given bus stop according to a Poisson process with intensity $\lambda > 0$. You arrive at the bus stop at time t.

1. Give a first guess for the value of the average waiting time before the following bus arrives ?
2. Let $B_t = T_{N_t+1} - t$ be the waiting time before the next bus, and let $A_t = t - T_{N_t}$ denote the elapsed time since the last bus arrival. Compute the joint distribution of (A_t, B_t) (hint : compute first $\mathbb{P}(A_t \geq x_1, B_t \geq x_2)$ for $x_1, x_2 \geq 0$).
3. Derive that the random variables A_t and B_t are independent. What are their distributions ?
4. In particular, compute $\mathbb{E}[B_t]$. Compare with your initial first guess.

Exercise 9. [Law of large numbers and central limit theorem.]

1. Recall and prove a law of large numbers for a Poisson process with intensity $\lambda > 0$.
2. Prove that N satisfies the following central limit theorem

$$\frac{N_t - \lambda t}{\sqrt{\lambda t}} \xrightarrow{\text{law}} \mathcal{N}(0,1) \text{ as } t \to \infty,$$

   (a) by using characteristic functions

   (b) by showing first that $(N_n - \lambda n)/\sqrt{\lambda n}$ converges in distribution as $n \to \infty$ and then $\max_{t \in [n,n+1)} (N_t - N_n)/\sqrt{n} \to 0$ in probability.

Exercise 10.

1. Give an expression for the density function of the conditional distribution of

$$(T_1, ..., T_n) \text{ given } N_t = n$$

   when N is a Poisson process with intensity λ and $0 < T_1 < ... < T_n < ...$ are its jump times.
2. Derive an expression for the density of T_i given $N_t = n$, $\forall 1 \leq i \leq n$ and similarly for (T_i, T_j) given $N_t = n$, $\forall 1 \leq i < j \leq n$.
3. Set $U_{i,j} = T_j - T_i$, $1 \leq i < j \leq n$. Give an expression for the density of $U_{i,j}$ given $N_t = n$. Derive an expression for the density of $T_n - T_{n-1}$ given $N_t = n$.

Exercise 11. Let $X = (X_t, t \geq 0)$ be a continuous time process and $\mathcal{F} = (\mathcal{F}_t, t \geq 0)$ a *filtration*, i.e. a nested family of sigma-fields $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{A}$ $\forall s \leq t$, where $\mathcal{A}$ is the sigma-field on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ over which X is defined. The process X is a *martingale* with respect to the filtration $\mathcal{F}$ if X_t is $\mathcal{F}_t$-measurable and integrable $\forall t$ and

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s, \quad \forall 0 \leq s \leq t.$$

Let $N = (N_t, t \geq 0)$ be a Poisson process with intensity $\lambda > 0$. Show that the three processes

1. $(N_t - \lambda t, t \geq 0)$;
2. $((N_t - \lambda t)^2 - \lambda t), t \geq 0)$;
3. $(\exp(uN_t + \lambda t(1 - e^u)), t \geq 0)$ (for a given real number u) ;

are martingales with respect to the filtration generated by N, *i.e.* $\mathcal{F}_t^N = \sigma(N_s, s \leq t)$.

Exercise 12. Let N be a Poisson process with intensity $\lambda > 0$ and let $0 < T_1 < \ldots < T_n < \ldots$ denote its jump times.

1. Show that T_n/n converges almost surely as $n \to \infty$ and identify its limit.
2. Show that $\sum_{i\geq 1} T_i^{-2}$ converges almost surely. Let X denote its limit.
3. Show that $X_{N_t} = \sum_{i=1}^{N(t)} T_i^{-2} \to X$ a.s. as $t \to \infty$.
4. Let $(U_i, i \geq 1)$ denote a sequence of independent uniform random variables on $[0, 1]$. We admit the following result

$$n^{-2} \sum_{i=1}^{n} U_i^{-2} \underset{n \longrightarrow \infty}{\overset{\text{law}}{\to}} Z,$$

where Z is a positive random variable, whose Laplace transform is given by $\mathbb{E}[\exp(-sZ)] = \exp(-c\sqrt{s}), \forall s \geq 0$, for some $c > 0$. **The goal is to show that X and $c'Z$ have same law for some c' that we will explicitly compute**.

We asssume moreover that $(U_i, i \geq 1)$ is independent of N.

(a) Show that for every $n \geq 1$ and every $t > 0$, the law of X_{N_t} given $N_t = n$ is the same as the law of $t^{-2} \sum_{i=1}^{n} U_i^{-2}$.

(b) Derive that $X_{N(t)}$ has same distribution as $t^{-2} \sum_{i=1}^{N(t)} U_i^{-2}$.

(c) Prove that

$$N(t)^{-2} \sum_{i=1}^{N(t)} U_i^{-2} \overset{\text{law}}{\longrightarrow} Z \quad \text{as } t \to \infty.$$

(d) Recall the law of large numbers for Poisson processes and conclude.

5. Derive $\mathbb{E}[X] = \infty$.

Exercise 13. Let $N = (N_t, t \geq 0)$ be a standard Poisson process with intensity $\lambda > 0$. Let $f : [0, \infty) \to [0, \infty)$ be a locally bounded Borel function. Set

$$N(f)_t = \sum_{i \geq 1} f(T_i) \mathbf{1}_{\{T_i \leq t\}} \text{ for } t \geq 0,$$

where the $(T_i)_{i\geq 1}$ are the jump times of N.

1. Show that for all $t \geq 0$, we have $N(f)_t < \infty$ almost-surely.

2. If $f(s) = \mathbf{1}_{(a,b]}(s)$ where $[a,b] \subset [0,t]$, what is the distribution of $N(\mathbf{1}_{(a,b]})_t$?
3. Show that for $u \geq 0$, we have

$$\mathbb{E}\left[e^{-uN(f)_t} \middle| N_t = n\right] = \frac{1}{t^n}\left(\int_0^t e^{-uf(s)} ds\right)^n.$$

4. Derive $\mathbb{E}\left[e^{-uN(f)_t}\right]$ and find back the result of Question 2.
5. Compute $\mathbb{E}\left[N(f)_t\right]$ and $\mathrm{Var}[N(f)_t]$.
6. Prove that $N(f)_t - \lambda \int_0^t f(s)ds$ is a martingale.