Poisson processes. Academic year 2025-2026.

Exercise sheet n°2 : Continuous-time Markov processes.

Exercice 1. Let (T_n) be a sequence of independent random variables with exponential laws of respective parameters $\lambda_n > 0$.

a) Prove the equivalence between
   — $\sum \lambda_n < \infty$,
   — $\mathbb{P}(\inf T_n > 0) > 0$,
   — $\mathbb{P}(\inf T_n > 0) = 1$,

   and give the law of the variable $S := \inf T_n$ under this assumption, which is also assumed in the next questions.

b) What is the probability that $T_n = S$?

c) Prove that almost surely there is one and only one index n such that $T_n = S$.

Exercice 2. Let N be a Poisson process of parameter $\lambda > 0$. Show that N is a Markov process on $E = \mathbb{N}$ and determine the matrix $P_s(x, y) = \mathbb{P}[N_{t+s} = y \mid N_t = x]$ for every $s \geq 0$ and every $x, y \in E$.

Exercice 3. Let A be a stochastic $d \times d$ matrix, meaning that it has non-negative coefficients and that $\sum_j A(i, j) = 1$, and let $\mu > 0$ be given. Le N_t be a Poisson process of parameter μ, and let Y_n be a Markov chain (on the set of d states) generated by A, independent from N_t. Define $X_t = Y_{N_t}$.

a) Show that X_t is a Markov process, and that the matrix P_t defined by

$$P_t(i, j) := \mathbb{P}(X_{s+t} = j | X_s = i)$$

   is equal to $P_t = e^{tQ}$, with $Q = \mu(A - I)$.

b) Is the first jump time of X equal to the first jump time of N ? Show that the first jump time S of X_t has an exponential law conditionally to $X_0 = i$ and compute its parameter.

c) Compute the matrix $\Pi(i, j)$ of the embedded markov chain.

Exercice 4. Let X be the size of a population that evolves in continuous-time as follows : at time 0 there is only one individual in the population, at any time t if there are n individuals in the population then after an exponential time of parameter $\lambda_n > 0$ a new individual is born.

a) Check that $(X_t - 1, t \geq 0)$ is a counting process and determine the law of its inter-arrival times τ_n.

We denote by $T_n = \sup\{t : X_t = n\}$ the jump times.

b) Compute $\mathbb{E}[T_n]$ and $\mathbb{E}(e^{-T_n})$ for all $n \geq 1$.

c) Deduce the equivalence between the following properties :
   — $\sum \lambda_k^{-1} = \infty$.
   — $\Pi(1 + \lambda_k^{-1}) = \infty$.
   — $P(T_n \to \infty) = 1$.

— $P(T_n \to \infty) > 0$.

Exercice 5. Let X be a continuous-time Markov process and set $P_t(x,y) = \mathbb{P}_x(X_t = y)$ for all $t \geq 0$ and all $x, y \in E$. Show that $(P_t, t \geq 0)$ is a semigroup, that is,

a) $P_0 = \mathrm{Id}$,

b) For every $t \geq 0$, P_t is a stochastic matrix, that is, for all $x, y \in E$ we have $P_t(x,y) \geq 0$ and $\sum_{y \in E} P_t(x,y) = 1$,

c) For every $t, s \geq 0$, we have $P_{t+s} = P_t P_s$, that is, for all x, y

$$P_{t+s}(x,y) = \sum_{z \in E} P_t(x,z) P_s(z,y) \ .$$

Exercice 6. Let X be the continuous-time Markov process on $E = \{0,1\}$ with generator Q given by

$$Q(0,1) = 1 \ , \quad Q(1,0) = 2 \ .$$

Explain heuristically the dynamic of the process. Determine λ_x and $\Pi(x,y)$ for all $x, y \in E$.

Exercice 7. Let X be a continuous-time Markov process on E and let Y be the embedded Markov chain. The goal of this exercise is to show that the following properties are equivalent

a) X is irreducible, that is, for all $t > 0$ and all $x \neq y$, $\mathbb{P}_x(X_t = y) > 0$.

b) Y is irreducible, that is, for all $x \neq y$, there exists $n \geq 1$ such that $\mathbb{P}_x(Y_n = y) > 0$.

Fix $x \neq y$.

a) Suppose there exists $t > 0$ such that $\mathbb{P}_x(X_t = y) > 0$ and show that there exists $n \geq 1$ such that $\mathbb{P}_x(Y_n = y) > 0$.

b) Deduce that if X is irreducible then Y is.

c) Suppose there exists $n \geq 1$ such that $\mathbb{P}_x(Y_n = y) > 0$ and show that there exists $x =: z_0 \neq z_1 \neq \ldots \neq z_{n-1} \neq z_n := y$ such that

$$\mathbb{P}_x(Y_1 = z_1, \ldots, Y_{n-1} = z_{n-1}, Y_n = y) > 0 \ .$$

d) Show that conditionally given $\{Y_1 = z_1, \ldots, Y_{n-1} = z_{n-1}, Y_n = y\}$, the r.v. τ_k, $k = 1, \ldots, n+1$ are independent $\mathcal{E}(z_{k-1})$ r.v.

e) We recall that the sum of independent exponential r.v. admits a density which is positive on $(0, \infty)$. Deduce that for all $t > 0$

$$\mathbb{P}_x(X_t = y) > 0 \ .$$

f) Deduce that if Y is irreducible then X is.

Exercice 8. Let X be a continuous-time Markov process on a finite state-space E with generator Q. We set

$$e^{tQ} = \sum_{n \geq 0} \frac{t^n}{n!} Q^n \ , \quad t \geq 0 \ .$$

a) Check that there exists $M > 0$ such that for all $n \geq 0$, $\max_{x,y} |Q^n(x,y)| \leq M^n$.

b) Deduce that e^{tQ} is well-defined.

c) Check that e^{tQ} satisfies the Chapman-Kolmogorov equations. Deduce that $P_t = e^{tQ}$.