Exercise sheet n°4 : Continuous-time Markov processes and examples.

Exercice 1. Let X be a continuous-time Markov process on E. We say that X is reversible with respect to a measure μ if for all $x \neq y \in E$ we have

$$\mu(x)Q(x,y) = \mu(y)Q(y,x) \ .$$

We often say that μ is reversible.

a) Show if μ is reversible then μ is invariant.

b) Find an example of a Markov process that admits an invariant measure which is not reversible.

Exercice 2. Let X be the continuous-time Markov process on $E = \mathbb{N}$ with generator Q given by

$$Q(0,1) = 1 \ , \quad Q(n,n+1) = n(1+n) \ , \quad Q(n,0) = 1+n \ , \qquad n \geq 1 \ ,$$

and $Q(x,y) = 0$ for any other values of $x \neq y$. Let Y be the embedded Markov chain and define

$$R_0^Y := \inf\{n \geq 1 : Y_n = 0\} \ .$$

a) Determine λ_n, $\Pi(n,k)$ and $Q(n,n)$ for all values n, k.

b) Show that

$$\mathbb{E}_0[R_0^Y] = \sum_{n\geq 0}(n+1)\mathbb{P}_0(Y_1 = 1, \ldots, Y_n = n, Y_{n+1} = 0) \ .$$

c) Show that for every $n \geq 1$

$$\mathbb{P}_0(Y_1 = 1, \ldots, Y_n = n, Y_{n+1} = 0) = \frac{1}{1+n}\prod_{k=1}^{n-1}\frac{k}{1+k} \ .$$

d) Show that this last term is equivalent to c/n^2 when $n \to \infty$, for some constant $c > 0$.

e) Deduce that 0 is null recurrent for the embedded Markov chain Y.

f) Show that for every $n \geq 1$

$$\mathbb{E}_0[T_{n+1} \,|\, Y_1 = 1, \ldots, Y_n = n, Y_{n+1} = 0] = \sum_{k=0}^{n}\frac{1}{(1+k)^2} \ .$$

g) Show that 0 is positive recurrent for X.

h) Determine the invariant probability measure of X.

Exercice 3. Let Y be an irreducible, discrete-time Markov chain on E. Fix $x \in E$ and define the first return time to x :

$$R_x^Y := \inf\{n \geq 1 : Y_n = x\} .$$

We introduce the measure $\nu^{(x)}$ by setting

$$\nu^{(x)}(y) := \mathbb{E}_x \Big[\sum_{n=1}^{R_x^Y} \mathbf{1}_{\{Y_n = y\}} \Big] , \quad y \in E .$$

The goal of this exercise is to show that if a measure ν' satisfies $\nu'\Pi = \nu'$ and $\nu'(x) = 1$ then $\nu' \geq \nu^{(x)}$. (This property will be used in the course).

a) Show that for every $y \in E$

$$\begin{aligned} \nu^{(x)}(y) &= \sum_{n \geq 1} \mathbb{P}_x(Y_n = y; R_x^Y \geq n) \\ &= \sum_{n \geq 1} \sum_{y_{n-1},\ldots,y_1 \neq x} \mathbb{P}_x(Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}, Y_n = y) . \end{aligned}$$

b) Show that for every $y \in E$

$$\nu'(y) = \Pi(x, y) + \sum_{z_1 \neq x} \nu'(z_1)\Pi(z_1, y) .$$

c) Iterating the previous argument, show that for every $y \in E$

$$\nu'(y) = \Pi(x, y) + \sum_{z_1 \neq x} \Pi(x, z_1)\Pi(z_1, y) + \sum_{z_1, z_2 \neq x} \nu'(z_2)\Pi(z_2, z_1)\Pi(z_1, y) .$$

d) Recursively, show that for every $y \in E$ and every $n \geq 1$

$$\begin{aligned} \nu'(y) = \Big(\Pi(x, y) + \sum_{z_1 \neq x} \Pi(x, z_1)\Pi(z_1, y) + \ldots + \sum_{z_1,\ldots,z_{n-1} \neq x} \Pi(x, z_{n-1})\Pi(z_{n-1}, z_{n-2}) \ldots \Pi(z_1, y)\Big) \\ + \sum_{z_1,\ldots,z_n \neq x} \nu'(z_n)\Pi(z_n, z_{n-1}) \ldots \Pi(z_1, y) . \end{aligned}$$

e) Deduce that $y \in E$ and for every $n \geq 1$

$$\nu'(y) \geq \sum_{k=1}^{n} \sum_{y_{k-1},\ldots,y_1 \neq x} \mathbb{P}_x(Y_1 = y_1, \ldots, Y_{k-1} = y_{k-1}, Y_k = y) .$$

f) Conclude.

Exercice 4. Let X be a continuous-time Markov process on E. The goal of the exercise is to show the following dichotomy :

— If $\lambda_x = 0$ or $\mathbb{P}_x(R_x < \infty) = 1$, then x is recurrent and $\int_0^\infty P_t(x, x)dt = \infty$,

— If $\lambda_x > 0$ and $\mathbb{P}_x(R_x < \infty) < 1$, then x is transient and $\int_0^\infty P_t(x,x)dt < \infty$.

a) Assume that $\lambda_x = 0$ and show that x is recurrent and $\int_0^\infty P_t(x,x)dt = \infty$.

b) Assume from now on that $\lambda_x > 0$. Show that $\mathbb{P}_x(R_x^Y < \infty) = \mathbb{P}_x(R_x < \infty)$.

c) Deduce that x is recurrent for X if and only if $\mathbb{P}_x(R_x < \infty) = 1$.

d) Show that

$$\int_0^\infty P_t(x,x)dt = \frac{1}{\lambda_x}\sum_{n\geq 0}\Pi^n(x,x) \ .$$

e) Deduce that x is recurrent for X if and only if $\int_0^\infty P_t(x,x)dt = \infty$.

f) Conclude.

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Exercice 5. Let X be irreducible and recurrent. Let μ' be a measure on E and fix $t > 0$. Assume that $\mu' P_t = \mu'$. The goal of this exercise is to prove that $\mu' Q = 0$.
By a result of the course, we know that all solutions of $\mu' Q = 0$ are multiple of each other. Furthermore, we proved in the course that any solution of $\mu' Q = 0$ satisfies $\mu' P_t = \mu'$. Consequently it suffices to show that all solutions of $\mu' P_t = \mu'$ are multiple of each other we are done.

a) Check that $\tilde{\Pi} := P_t$ is a transition matrix of a recurrent Markov chain. Let $(Z_n, n \geq 0)$ be a Markov chain with transition matrix $\tilde{\Pi}$.

b) Using the Markov property show that for all $nt \leq s < (n+1)t$ we have for any $x \in E$

$$P_{(n+1)t}(x,x) \geq e^{-\lambda_x t} P_s(x,x) \ .$$

c) Deduce that for any $x \in E$

$$\int_0^\infty P_s(x,x)ds \leq t e^{\lambda_x t}\sum_{n\geq 1}\tilde{\Pi}^n(x,x) \ .$$

d) Using the previous exercise, deduce that $\sum_{n\geq 1}\tilde{\Pi}^n(x,x) = \infty$. Show that this implies that Z is recurrent. (Indication : introduce $N_x := \#\{n \geq 1 : Z_n = x\}$ and compute its expectation).

e) Using a result of the course of the first semester, deduce that all solutions of $\mu' P_t = \mu'$ are multiple of each other.

Exercice 6. Let X be an M/M/∞ queue. We let $\lambda > 0$ and $\gamma > 0$ be the parameters of the exponential r.v. associated with the arrival and the service times of the customers. Set $\rho = \lambda/\gamma$. Show that X admits the measure μ as an invariant measure :

$$\mu(n) = \frac{\rho^n}{n!}e^{-\rho} \ , \quad n \geq 0 \ .$$

Deduce that X is positive recurrent.