

POISSON PROCESSES. FINAL EXAM (2H).
2026, MAY 20.

Exercise. Let $Z = (Z_k)_{k \geq 0}$ be a discrete-time Markov chain on $\{0, 1\}$, with transition matrix A such that $A(0, 1) = 1 - A(0, 0) = \alpha$ and $A(1, 0) = 1 - A(1, 1) = \beta$, where $\alpha, \beta \in (0, 1)$. Let $N = (N_t)_{t \geq 0}$ be a Poisson process with intensity parameter $\lambda > 0$. We assume that Z and N are defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and that they are independent (as processes). This means that for every $m, n \geq 1$, $0 \leq k_1 < \dots < k_m$ and $0 \leq t_1 < \dots < t_n$, the random vectors $(Z_{k_i})_{1 \leq i \leq m}$ and $(N_{t_j})_{1 \leq j \leq n}$ are independent. Finally, we define the (continuous-time) process $X_t = Z_{N_t} \in \{0, 1\}$ for every $t \geq 0$.

- (1) Recall one of the two definitions of a Poisson process with intensity parameter $\lambda > 0$. A Poisson process with intensity parameter $\lambda > 0$ is a counting process such that the inter-arrival times $(\mathcal{T}_i - \mathcal{T}_{i-1})_{i \geq 1}$ are independent and identically distributed (i.i.d) with the exponential law of parameter $\lambda > 0$ as common distribution. [Additional remarks] Many imprecise (if not wrong) answers. For $0 \leq s \leq t$, the fact that $N_t - N_s$ is independent of N_s is not the same as $N_t - N_s$ being independent of $\sigma(N_u, 0 \leq u \leq s)$. One definition was enough! Giving one correct definition and one incorrect definition count as a mistake.
- (2) Show that X has (almost-surely) piecewise constant (*constant par morceaux* in French) and càdlàg (right-continuous with left-limits) paths. We have $N_t = k$, therefore $X_t = Z_k$, for every $t \in [\mathcal{T}_k, \mathcal{T}_{k+1})$ and $k \in \mathbb{N} = \{0, 1, 2, \dots\}$, with $[0, \infty) = \cup_{k \in \mathbb{N}} [\mathcal{T}_k, \mathcal{T}_{k+1})$ (disjoint union). This shows that X is (a.s.) piecewise constant. Also, the process is (a.s.) càdlàg because the intervals $[\mathcal{T}_k, \mathcal{T}_{k+1})$ are open to the right and closed to the left. [Additional remarks] Many argued that N is itself piecewise constant and càdlàg, but that was (often) not enough to conclude convincingly. Some (more or less loosely) argued that X is the “composition” of Z and N without properly explaining the topology under consideration for the discrete-time process Z . The quality of writing and argumentation was particularly important for this question. Finally, it is not true in general that a function taking values in a finite set is piecewise constant. Counter-example: $t \in \mathbb{R} \mapsto 1_{\{t \in \mathbb{Q}\}}$.

For every $t \geq 0$ and $x, y \in \{0, 1\}$ we define $\mathcal{P}_t(x, y) = \mathbb{P}_x(X_t = y)$, where the subscript in \mathbb{P}_x means that $X_0 = Z_0 = x$.

- (3) Prove that $\mathcal{P}_t = \exp(\lambda t(A - I))$, where I is the identity matrix and for every matrix $M: \{0, 1\}^2 \rightarrow \mathbb{R}$, we define $\exp(M) := \sum_{n \geq 0} \frac{1}{n!} M^n$. We also recall that $\exp(M_1 + M_2) = \exp(M_1) \exp(M_2)$ when the two matrices M_1 and M_2 commute. On the one

side, we have

$$\begin{aligned}
\mathcal{P}_t(x, y) &= \mathbb{P}_x(X_t = y) = \sum_{k \geq 0} \mathbb{P}_x(X_t = y, N_t = k) \\
&= \sum_{k \geq 0} \mathbb{P}_x(Z_k = y, N_t = k) \\
&= \sum_{k \geq 0} \mathbb{P}_x(Z_k = y) \mathbb{P}_x(N_t = k) \\
&= \sum_{k \geq 0} A^k(x, y) \times \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\
&= e^{-\lambda t} \sum_{k \geq 0} \frac{(\lambda t A)^k(x, y)}{k!} \\
&= e^{-\lambda t} [\exp(\lambda t A)](x, y)
\end{aligned}$$

We have used the independence between Z and N from the second to third line, and the fact that N_t has a Poisson distribution (with parameter λt) from the third to fourth line. Since I and A commute, we have on the other side $\exp(\lambda t(A - I)) = \exp(\lambda t A) \exp(-\lambda t I) = e^{-\lambda t} \exp(\lambda t A)$, which concludes the proof. **[Additional remarks]** Too many students replaced $A^k(x, y)$ by $A(x, y)^k$, which is of course not the same! In the same flavor, in general $[\exp(M)](x, y)$ is *not* $\exp(M(x, y))$.

- (4) Show that for every $n \geq 1$, every $(x_i)_{0 \leq i \leq n} \in \{0, 1\}^{n+1}$ and $0 = t_0 < t_1 < \dots < t_n$, we have

$$\mathbb{P}_{x_0}(X_{t_i} = x_i, \forall 1 \leq i \leq n) = \prod_{1 \leq i \leq n} \mathcal{P}_{t_i - t_{i-1}}(x_{i-1}, x_i).$$

Let us prove it by iteration on $n \geq 1$. The assertion is true for $n = 1$ (see previous question). Assume it is true for $n \geq 1$ and let us prove it for $n + 1$. Let $(x_i)_{0 \leq i \leq n+1} \in \{0, 1\}^{n+2}$ and $0 = t_0 < t_1 < \dots < t_{n+1}$. Then, using independence of Z and N at the second line,

$$\begin{aligned}
&\mathbb{P}_{x_0}(X_{t_i} = x_i, \forall 1 \leq i \leq n + 1) \\
&= \sum_{0 \leq k_1 \leq \dots \leq k_n \leq k_{n+1}} \mathbb{P}_{x_0}(Z_{k_1} = x_1, \dots, Z_{k_{n+1}} = x_{n+1}) \mathbb{P}(N_{t_1} = k_1, \dots, N_{t_{n+1}} = k_{n+1}) \\
&= \sum_{\substack{0 \leq k_1 \leq \dots \leq k_n \\ j \geq 0}} \mathbb{P}_{x_0}(Z_{k_1} = x_1, \dots, Z_{k_n} = x_n) \mathbb{P}(N_{t_1} = k_1, \dots, N_{t_n} = k_n) \\
&\quad \times \mathbb{P}_{x_n}(Z_j = x_{n+1}) \mathbb{P}(N_{t_{n+1} - t_n} = j) \\
&= \mathbb{P}_{x_0}(X_{t_i} = x_i, \forall 1 \leq i \leq n) \sum_{j \geq 0} \mathbb{P}_{x_n}(Z_j = x_{n+1}) \mathbb{P}(N_{t_{n+1} - t_n} = j).
\end{aligned}$$

We have also used that N has independent and stationary increments at the third line. From what we saw in the previous question, the sum over $j \geq 0$ in the last line equals $\mathbb{P}_{x_n}(X_{t_{n+1} - t_n} = x_{n+1}) = \mathcal{P}_{t_{n+1} - t_n}(x_n, x_{n+1})$, which completes the proof.

[Additional remarks] Very few students managed to correctly answer this question. It was not possible to use that X is Markov (circular argument, in view of the next question). I also noticed abusive uses of the Markov property for Z .

- (5) We have just proved that X is a (continuous-time) Markov process on $\{0, 1\}$ (no need to elaborate more on that point). Recall the (forward and backward) Chapman-Kolmogorov equations and show that the generator of X , further denoted by Q , equals $\lambda(A - I)$. The Chapman-Kolmogorov equations write $\partial_t \mathcal{P}_t = Q\mathcal{P}_t = \mathcal{P}_t Q$, with $\mathcal{P}_0 = I$. We deduce from (3) that $Q = [\partial_t \mathcal{P}_t]_{t=0} = [\lambda(A - I)\mathcal{P}_t]_{t=0} = \lambda(A - I)$.
 [Additional remarks] Many students correctly stated the Chapman-Kolmogorov equations but failed to prove that $Q = \lambda(A - I)$, using some “identification” step. Under the (abusive) term “identification” actually hides an *injective* map. However, the exponential map for matrices is *not* injective. In the same way, the equality $Q_1 P_t = Q_2 P_t$ alone does not imply that $Q_1 = Q_2$, unless you explicitly add that P_t is invertible. Other students argued that the Chapman-Kolmogorov equations (which bear on the semi-group) have a unique solution. That is irrelevant to the question. For instance, 0 is the unique real solution to the equations $x = 0$ and $x^2 = 0$, even though the polynomials x and x^2 are not the same.
- (6) From the expression of the generator, deduce the probability distribution of the holding (or waiting) time $T_1 := \inf\{t \geq 0 : X_t \neq X_0\}$, when $X_0 = x \in \{0, 1\}$. When $X_0 = x$, we know from the course that T_1 is exponentially distributed with parameter $-Q(x, x) = \lambda(I - A)(x, x) = \lambda(\alpha \mathbf{1}_{\{x=0\}} + \beta \mathbf{1}_{\{x=1\}})$. [Additional remarks] No particular remark here.
- (7) Determine the embedded Markov chain associated to X . Since there are only two states, the embedded Markov chain Y is the trivial Markov chain with transition matrix $\Pi(0, 1) = 1 - \Pi(0, 0) = 1$ and $\Pi(1, 0) = 1 - \Pi(1, 1) = 1$. [Additional remarks] Recalling the definition of the embedded Markov chain was not enough here.
- (8) Show that X is irreducible and recurrent. The embedded chain Y , which alternates deterministically between 0 and 1, is clearly irreducible (it takes one step to move from one state to the other one) and recurrent (return times are equal to two) so we know from the course that the same holds for X . [Additional remarks] Quite often, answers were ill-written, with awkward or unnecessary long arguments. Some confusion between irreducibility and *strong* irreducibility. It was also possible to answer to the question without resorting to Y . Note that the exponential of a *matrix* does not necessarily have positive entries.
- (9) Show that X has a unique invariant probability measure, which you have to determine explicitly. A (non-negative) measure $\mu = (\mu_0, \mu_1)$ is invariant for X if and only if $\mu Q = 0$. The solutions are the measures μ such that $\alpha \mu_0 = \beta \mu_1$. Therefore, the only invariant probability measure is $(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$. [Additional remarks] The “if and only if” is important here. The fact that X is irreducible and recurrent does not imply existence of an invariant probability measure.
- (10) Is X positive recurrent or null recurrent? Justify your answer. Since X is irreducible with an invariant probability measure, we know from the course that X is positive recurrent. [Additional remarks] Specifying that X is irreducible is important for the justification, as there exist reducible Markov processes that possess invariant probability measures without being positive recurrent. Another possibility is showing that the embedded Markov chain Y is positive recurrent and arguing that X has jumping/holding rates bounded from above and below by positive constants.
- (11) Find an explicit value of $\mathcal{P}_t(x, y)$ and determine its limit as $t \rightarrow \infty$. What do you notice? What is the speed of convergence? Hint: diagonalize the matrix $A - I$. Let

$P = \begin{pmatrix} 1 & \alpha \\ 1 & -\beta \end{pmatrix}$. The columns of P are (right) eigenvectors of $A - I$, with respective eigenvalues 0 and $-(\alpha + \beta)$. Therefore, for every $n \geq 0$, we have $(A - I)^n P = P \text{diag}(0, [-(\alpha + \beta)]^n)$. Consequently, $\mathcal{P}_t P = \exp(\lambda t(A - I))P = P \text{diag}(1, e^{-\lambda t(\alpha + \beta)})$, from which we deduce

$$\mathcal{P}_t = P \text{diag}(1, e^{-\lambda t(\alpha + \beta)}) P^{-1} = \frac{1}{\alpha + \beta} \begin{pmatrix} \beta + \alpha e^{-\lambda t(\alpha + \beta)} & \alpha - \alpha e^{-\lambda t(\alpha + \beta)} \\ \beta - \beta e^{-\lambda t(\alpha + \beta)} & \alpha + \beta e^{-\lambda t(\alpha + \beta)} \end{pmatrix}.$$

As a conclusion, we see that $\lim_{t \rightarrow \infty} \mathcal{P}_t(x, y) = \frac{\beta \mathbf{1}_{\{y=0\}} + \alpha \mathbf{1}_{\{y=1\}}}{\alpha + \beta}$. We remark that the limit coincides with the unique invariant probability measure and does not depend on the starting point x . Moreover, the rate of convergence is exponential with rate $\lambda(\alpha + \beta)$. [Additional remarks] In that question we use (in some way or another) that for every $n \times n$ diagonal matrix, $\exp(\text{diag}(\rho_1, \dots, \rho_n)) = \text{diag}(\exp(\rho_1), \dots, \exp(\rho_n))$. A common mistake was to replace $\exp(0)$ by 0 instead of 1. A good way to check your result is recalling that \mathcal{P}_t must be a stochastic matrix.

- (12) Same question as the previous one in the special case $\alpha = 0$ or $\beta = 0$. Let us treat the case $\alpha = 0$ (the case $\beta = 0$ follows by interchanging the roles of the states 0 and 1). In that case, clearly $\mathcal{P}_t(0, 0) = 1 - \mathcal{P}_t(0, 1) = 1$ for every $t \geq 0$. Moreover, also in that special case,

$$\begin{aligned} 1 - \mathcal{P}_t(1, 0) &= \mathcal{P}_t(1, 1) = \mathbb{P}_1(X_s = 1, 0 \leq s \leq t) \\ &= \sum_{k \geq 0} \mathbb{P}_1(X_s = 1, 0 \leq s \leq t, N_t = k) \\ &= \sum_{k \geq 0} \mathbb{P}_1(Z_i = 1, 1 \leq i \leq k, N_t = k) \\ &= \sum_{k \geq 0} (1 - \beta)^k \mathbb{P}(N_t = k) \\ &= (\dots) = \exp(-\beta \lambda t). \end{aligned}$$

We have used that 0 is an absorbing state for Z at the third line, and the independence of Z and N at the fourth line. Therefore, $\mathcal{P}_t(1, 1)$ is identically equal to 1 if $\beta = 0$ and converges to 0 (exponentially fast) if $\beta > 0$. [Additional remarks] Some students considered instead the case $\alpha = 0$ and $\beta = 0$, which is not the aim of this question. It is possible to skip the calculation above by using the result in Q.6.