POISSON PROCESSES. MID-TERM EXAM (2H).
2026, MARCH 12.

Notation. We denote by $\mathcal{E}(\lambda)$ the exponential law on the positive half-line $(0,\infty)$ with parameter $\lambda > 0$ (i.e. with mean $1/\lambda$). For every $p \in (0,1)$, we denote by $\mathrm{Geo}(p)$ the law on positive integers with point probabilities $k \geq 1 \mapsto (1-p)^{k-1}p$.

Part (I). Let us first recall that the law of a non-negative real random variable V is characterized by its Laplace transform $\phi_V : t \geq 0 \mapsto \mathbb{E}(e^{-tV})$ (no proof needed). Let $(Y_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with common distribution $\mathcal{E}(\lambda)$.

(1) Compute the Laplace transform of $Y_1 + \ldots + Y_n$ for every $n \geq 1$ (start with $n = 1$). For $n = 1$, we have for every $t \geq 0$, $\phi_{Y_1}(t) = \int_0^\infty \lambda e^{-(t+\lambda)x}\mathrm{d}x = \frac{\lambda}{t+\lambda}$. Letting $S_n = Y_1 + \ldots + Y_n$, we have in general,

$$\phi_{S_n}(t) = \prod_{1\leq i\leq n} \mathbb{E}(e^{-tY_i}) = \Big(\frac{\lambda}{t+\lambda}\Big)^n, \qquad t \geq 0.$$

The first equality uses the fact that the Y_i's are independent and the second equality uses the fact that they are identically distributed.

(2) Compute the Laplace transform of $Y_1 + \ldots + Y_G$, where G is a random integer that is independent of the sequence $(Y_i)_{i\geq 1}$ and distributed as $\mathrm{Geo}(p)$, where $p \in (0,1)$. For every $t \geq 0$, we have

$$\begin{aligned}
\phi_{S_G}(t) &= \mathbb{E}\Big(e^{-t[Y_1+\ldots+Y_G]}\Big) \\
&= \sum_{n\geq 1} \mathbb{E}\Big(e^{-t[Y_1+\ldots+Y_n]}\mathbf{1}_{\{G=n\}}\Big) \\
&= \sum_{n\geq 1} \mathbb{E}\Big(e^{-t[Y_1+\ldots+Y_n]}\Big)\mathbb{P}(G=n) \\
&= \sum_{n\geq 1} \Big(\frac{\lambda}{t+\lambda}\Big)^n (1-p)^{n-1}p \\
&= \frac{\lambda p}{t+\lambda p}.
\end{aligned}$$

The third line uses the independence between G and the sequence $(Y_i)_{i\geq 1}$. The fourth line uses our answer to the previous question. The last line is a straightforward computation (geometric sum).

(3) Deduce thereof the distribution of $Y_1 + \ldots + Y_G$. We deduce thereof that $Y_1 + \ldots + Y_G$ follows the law $\mathcal{E}(\lambda p)$.

Part (II). Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$. Let $0 = T_0 < T_1 < T_2 < \ldots$ be the associated jump times and $\tau_i := T_i - T_{i-1}$ ($i \geq 1$) the inter-arrival times.

(4) Using a theorem from the course, what can we say about the law of the sequence of inter-arrival times? (No proof needed.) We know from the course that $(\tau_i)_{i\geq 1}$ is

a sequence of independent and identically distributed (i.i.d) random variables with common distribution $\mathcal{E}(\lambda)$.

Let $X = (X_i)_{i\geq 1}$ be a sequence of i.i.d. Bernoulli (zero or one) random variables with parameter $p \in (0,1)$, that is independent of the Poisson process N. The goal is to determine the law of the process* $Z = (Z_t)_{t\geq 0}$ defined by:

$$Z_t := \sum_{1\leq i\leq N_t} X_i, \qquad t \geq 0.$$

(5) Show that Z is a standard counting process. The process Z is a counting process. Indeed:
- $Z_0 = 0$ since $N_0 = 0$ (actually $Z_t = 0$ as soon as $N_t = 0$ as the sum defining Z_t is then empty).
- Z_t is a non-negative integer (or infinity) as a (random) sum of non-negative integers.
- For every $t \geq 0$ and $s > 0$, $Z_{t+s} - Z_t = \sum_{N_t < i \leq N_{t+s}} X_i$ (since N is non-decreasing). This is a non-negative random variable, so the process Z is itself non-decreasing. Moreover, when s is smaller than some (random) threshold, the latter sum is empty (by right-continuity of N) which means that $Z_{t+s} - Z_t = 0$. This means that Z is right-continuous.

Moreover, it is a *standard* counting process. Indeed,
- $Z_t \to \infty$ as $t \to \infty$ since $N_t \to \infty$ and the sequence X contains (almost-surely) an infinite number of ones.
- Since $X_i \in \{0,1\}$ for every $i \geq 1$, a discontinuity of Z necessarily corresponds to a jump of size one.
- Since N_t is finite for every $t \geq 0$, so is Z_t.

(We omitted the "almost-sure" statements several times).

Let $\sigma_0 := 0$ and $\sigma_k := \inf\{j > \sigma_{k-1} \colon X_j = 1\}$ for every $k \geq 1$. Finally, we denote by $0 = \hat{T}_0 < \hat{T}_1 < \hat{T}_2 < \dots$ the jump times associated to the counting process Z.

(6) Prove that $\mathbb{P}(\sigma_1 = k) = (1-p)^{k-1}p$ for every $k \geq 1$, i.e. σ_1 is distributed as Geo(p). For every $k \geq 1$, we have

$$\begin{aligned}\mathbb{P}(\sigma_1 = k) = \mathbb{P}(X_i = 0, \forall 1 \leq i < k, X_k = 1) &= \Big[\prod_{1\leq i<k} \mathbb{P}(X_i = 0)\Big]\mathbb{P}(X_k = 1) \\ &= (1-p)^{k-1}p.\end{aligned}$$

We have used the independence of the X_i's at the second equality, and the Bernoulli distribution at the last equality.

(7) Show that the sequence $(\sigma_k - \sigma_{k-1})_{k\geq 1}$ is a sequence of i.i.d. random variables. It is enough to prove that for every $K \geq 1$,

$$\mathbb{P}(\sigma_k = n_k,\ 1 \leq i \leq K) = \prod_{1\leq k\leq K} \mathbb{P}(\sigma_1 = n_k - n_{k-1}), \quad (0 = n_0 < n_1 < \dots < n_K).$$

We prove it by iteration. This is true for $K = 1$ (see previous question). Assume that this is true for some $K \geq 1$ and let us prove it for $K+1$. We write

$$\mathbb{P}(\sigma_k = n_k,\ 1 \leq i \leq K+1) = \mathbb{P}(E_1 \cap E_2),$$

*without any prior knowledge on *compound* Poisson processes!

where $E_1 = \{\sigma_k = n_k,\ 1 \le i \le K\}$ and $E_2 = \{X_i = 0,\ n_K < i < n_{K+1},\ X_{n_{K+1}} = 1\}$. The events E_1 and E_2 are independent since $E_1 \in \sigma(X_i, i \le n_K)$, $E_2 \in \sigma(X_i, i > n_K)$ and the X_i's are independent. Moreover, one can easily check that $\mathbb{P}(E_2) = \mathbb{P}(\sigma_1 = n_{K+1} - n_K)$, which allows to conclude:

$$\begin{aligned}\mathbb{P}(\sigma_k = n_k, 1 \le i \le K+1) &= \mathbb{P}(E_1)\mathbb{P}(E_2)\\ &= \Big[\prod_{1\le k\le K} \mathbb{P}(\sigma_1 = n_k - n_{k-1})\Big]\mathbb{P}(\sigma_1 = n_{K+1} - n_K)\\ &= \prod_{1\le k\le K+1} \mathbb{P}(\sigma_1 = n_k - n_{k-1}).\end{aligned}$$

(8) Write the sequence $\hat{T}$ in terms of the sequences T and σ. The process Z jumps at $t > 0$ if and only if $t = T_j$ for some (unique) $j \ge 1$ (i.e. if t is a jump time for N) and $X_j = 1$. Therefore, $\hat{T}_k = T_{\sigma_k}$ for every integer $k \ge 0$.

(9) Using the result from **Part 1**, determine the probability distribution of $\hat{T}_1$. We have $\hat{T}_1 = T_{\sigma_1} = \sum_{1\le i\le\sigma_1} \tau_i$. Combining (3), (4) and (6) we get that $\hat{T}_1$ follows the law $\mathcal{E}(\lambda p)$.

(10) Show that the sequence $(\hat{T}_k - \hat{T}_{k-1})_{k\ge1}$ is a sequence of i.i.d. random variables. Hint: one may condition on σ. Let $K \ge 1$ and $(\Phi_k)_{1\le k\le K}$ a collection of bounded measurable functions from $\mathbb{R}$ to $\mathbb{R}$. Then, (with the convention $n_0 = 0$),

$$\begin{aligned}&\mathbb{E}\Big[\prod_{1\le k\le K} \Phi_k(\hat{T}_k - \hat{T}_{k-1})\Big] = \mathbb{E}\Big[\prod_{1\le k\le K} \Phi_k(T_{\sigma_k} - T_{\sigma_{k-1}})\Big]\\ &= \sum_{0<n_1<\ldots<n_K} \mathbb{E}\Big[\prod_{1\le k\le K} \Phi_k(T_{n_k} - T_{n_{k-1}})\mathbb{1}_{\{\sigma_i = n_i,\ 1\le i\le K\}}\Big]\\ &= \sum_{0<n_1<\ldots<n_K} \mathbb{E}\Big[\prod_{1\le k\le K} \Phi_k(T_{n_k} - T_{n_{k-1}})\Big]\mathbb{P}(\sigma_i = n_i,\ 1 \le i \le K)\\ &= \sum_{0<n_1<\ldots<n_K} \prod_{1\le k\le K} \mathbb{E}\Big[\Phi_k(T_{n_k} - T_{n_{k-1}})\Big]\mathbb{P}(\sigma_k - \sigma_{k-1} = n_k - n_{k-1})\\ &= \sum_{0<n_1<\ldots<n_K} \prod_{1\le k\le K} \mathbb{E}\Big[\Phi_k(T_{n_k - n_{k-1}})\Big]\mathbb{P}(\sigma_1 = n_k - n_{k-1})\\ &= \prod_{1\le k\le K} \sum_{j\ge1} \mathbb{E}[\Phi_k(T_j)]\mathbb{P}(\sigma_1 = j)\\ &= \prod_{1\le k\le K} \mathbb{E}[\Phi_k(T_{\sigma_1})] = \prod_{1\le k\le K} \mathbb{E}[\Phi_k(\hat{T}_1)].\end{aligned}$$

We have used: independence between X and N (thus σ and T) on lines 3 and 7, independence of the increments of T and σ on line 4, stationarity of the increments of T and σ on line 5.

(11) Deduce from the two previous questions the law of the process $Z = (Z_t)_{t\ge0}$. We have proved that Z is a counting process whose inter-arrival times are i.i.d. with common distribution $\mathcal{E}(p\lambda)$. Therefore, Z is a Poisson process with intensity $p\lambda$.

(12) Using a result from the lecture notes (that you do not need to prove), determine the almost-sure asymptotic behaviour of the random variable Z_t/N_t as $t \to \infty$. We know

from the course that, almost-surely,

$$\frac{N_t}{t} \to \lambda, \qquad \frac{Z_t}{t} \to p\lambda, \qquad t \to \infty.$$

Then, almost-surely

$$\frac{Z_t}{N_t} = \frac{Z_t/t}{N_t/t} \to p, \qquad t \to \infty.$$

(13) Compute the covariance of N_t and Z_t for every $t \geq 0$. The variables N_t and Z_t are Poisson random variables with respective parameters λt and $p\lambda t$, hence $\mathbb{E}(N_t) = \lambda t$ and $\mathbb{E}(Z_t) = p\lambda t$. Moreover,

$$\begin{aligned}\mathbb{E}(N_t Z_t) = \mathbb{E}(N_t \mathbb{E}(Z_t|N_t)) = \mathbb{E}(pN_t^2) = p\mathbb{E}(N_t^2) &= p[\mathbb{V}(N_t) + \mathbb{E}(N_t)^2] \\ &= p[\lambda t + (\lambda t)^2].\end{aligned}$$

We have used in the second equality that conditioned on N_t, Z_t is a binomial random variable with parameters p and N_t, thus $\mathbb{E}(Z_t|N_t) = pN_t$. In conclusion,

$$\text{Cov}(Z_t, N_t) = \mathbb{E}(N_t Z_t) - \mathbb{E}(N_t)\mathbb{E}(Z_t) = p\lambda t.$$