GRANDES DÉVIATIONS.
EXAMEN FINAL (3H).

(English version on the back of the sheet)

Documents et appareils électroniques non autorisés. Toute réponse doit être justifiée et lisible. La qualité de la rédaction pourra être prise en compte dans la notation. Vous pouvez répondre en français ou en anglais. La terminologie utilisée est celle du cours.

Exercice 1. (Principe de grandes déviations et produit). Soient $\mathcal{X}$ et $\mathcal{Y}$ deux espaces métriques. Soient $\{X_n\}_{n\geq 1}$ et $\{Y_n\}_{n\geq 1}$ deux suites de variables aléatoires à valeurs dans $\mathcal{X}$ et $\mathcal{Y}$ respectivement. On suppose que ces deux suites sont définies sur un même espace de probabilité $(\Omega, \mathcal{F}, \mathrm{P})$, qu'elles sont indépendantes entre elles (en tant que suites), et qu'elles vérifient un principe de grandes déviations (PGD) de vitesse n et de fonction de taux $I_X : \mathcal{X} \to [0, \infty]$ et $I_Y : \mathcal{Y} \to [0, \infty]$ respectivement.

(1) Montrer que la suite de variables aléatoires $\{(X_n, Y_n)\}_{n\geq 1}$ définies sur l'espace métrique produit $\mathcal{X}\times\mathcal{Y}$ (muni de la topologie produit) vérifie un PGD dont on notera la fonction de taux $I_{X,Y}$. Identifier $I_{X,Y}$ en pensant à vérifier qu'il s'agit bel et bien d'une fonction de taux. On rappelle que les intersections finies de *cylindres* (ensembles de la forme $A \times B$ où A est un ouvert de $\mathcal{X}$ et B un ouvert de $\mathcal{Y}$) forment une base d'ouverts pour la topologie produit (i.e. tout ouvert de la topologie produit s'écrit comme réunion d'ouverts de cette base).

(2) Montrer que si I_X et I_Y sont des *bonnes* fonctions de taux, alors $I_{X,Y}$ est également une *bonne* fonction de taux.

(3) On suppose dans cette question que $\mathcal{X} = \mathcal{Y} = \mathbb{R}$ (muni de la distance euclidienne) et que I_X et I_Y sont des *bonnes* fonctions de taux. Montrer que la suite $\{X_nY_n\}_{n\geq 1}$ vérifie un PGD sur $\mathbb{R}$ dont on déterminera la fonction de taux.

Exercice 2. (Modèle de Curie-Weiss avec champ extérieur). Soit $\sigma = (\sigma_i)_{i\geq 1}$ une suite de variables aléatoires indépendantes et identiquement distribuées (i.i.d.) telles que $\mathrm{P}(\sigma_1 = 1) = \mathrm{P}(\sigma_1 = -1) = 1/2$. La fonction de partition du modèle de Curie-Weiss pour un système de taille $n \geq 1$, de température inverse $\beta > 0$ et de champ extérieur $h \in \mathbb{R}$ est

$$Z_n^{\beta,h} = \mathrm{E}[\exp(H_n^{\beta,h})], \qquad \text{où} \quad H_n^{\beta,h} := \frac{\beta}{n} \sum_{1\leq i<j\leq n} \sigma_i\sigma_j + h \sum_{1\leq i\leq n} \sigma_i.$$

(1) Montrer que la magnétisation moyenne $M_n := \frac{1}{n}\sum_{i=1}^n \sigma_i$ satisfait un principe de grandes déviations (PGD) sur $[-1, 1]$, de vitesse n et de fonction de taux:

$$I(m) := \frac{1}{2}\Big[(1 + m)\log(1 + m) + (1 - m)\log(1 - m)\Big].$$

(2) Énoncer le lemme de Varadhan.

(3) Montrer que

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n^{\beta,h} = \sup_{m\in[-1,1]} \Big\{\tfrac{1}{2}\beta m^2 + hm - I(m)\Big\},$$

(4) Rechercher les maximiseurs (existence, nombre) dans la formule variationnelle ci-dessus. Que remarquez-vous ?

Date: Vendredi 9 janvier 2026.

LARGE DEVIATIONS.
FINAL EXAM (3H).

(Version française au recto)

Documents and electronic devices are not allowed. Answers must be justified and readable. The quality of writing may be taken into account in the grading. You may answer in French or English. We use the same terminology as in class.

Exercise 1. (Large deviation principles and product). Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces. Let $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$ be two sequences of random variables respectively valued in $\mathcal{X}$ and $\mathcal{Y}$. We assume these two sequences to be defined on a common probability space $(\Omega, \mathcal{F}, \mathrm{P})$, to be mutually independent (as sequences), and to satisfy a large deviation principle (LDP) with speed n and respective rate functions $I_X : \mathcal{X} \to [0, \infty]$ and $I_Y : \mathcal{Y} \to [0, \infty]$.

(1) Show that the sequence of random variables $\{(X_n, Y_n)\}_{n\geq 1}$ defined on the product metric space $\mathcal{X}\times\mathcal{Y}$ (equipped with the product topology) satisfies an LDP, the rate function of which shall be denoted $I_{X,Y}$. Identify $I_{X,Y}$ and check that this is indeed a rate function. We remind that finite intersections of *cylinder sets* (sets of the form $A \times B$ where A is an open set of $\mathcal{X}$ and B is an open set of $\mathcal{Y}$) constitute a topological basis for the product topology (i.e. every open set in the product topology writes as a union of open sets in that basis).
We will show that $\{(X_n, Y_n)\}_{n\geq 1}$ satisfies an LDP with speed n and rate function

$$I_{X,Y}(x, y) := I_X(x) + I_Y(y), \qquad (x, y) \in \mathcal{X} \times \mathcal{Y}.$$

(i) Lower bound. If A is an open set of $\mathcal{X}$ and B is an open set of $\mathcal{Y}$ then, using the independence of X and Y,

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}((X_n, Y_n) \in A \times B) &\geq \liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}(X_n \in A) + \liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}(Y_n \in B) \\ &\geq - \inf_{x\in A} I_X(x) - \inf_{y\in B} I_Y(y) = - \inf_{(x,y)\in A\times B} I_{X,Y}(x, y).\end{aligned}$$

In the general case, we may write any open set $O \subseteq \mathcal{X} \times \mathcal{Y}$ as

$$O = \cup_{i\in I} \cap_{k\in J_i} (A_{i,k} \times B_{i,k}),$$

where the $J_i's$ are all finite and the $A_{i,k}$'s and $B_{i,k}$'s are open sets of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then, for every $i \in I$,

$$\begin{aligned}\liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}((X_n, Y_n) \in O) &\geq \liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}((X_n, Y_n) \in \cap_{k\in J_i}(A_{i,k} \times B_{i,k})) \\ &= \liminf_{n\to\infty} \frac{1}{n} \log \mathrm{P}((X_n, Y_n) \in (\cap_{k\in J_i} A_{i,k}) \times (\cap_{k\in J_i} B_{i,k})) \\ &\geq - \inf\{I_{X,Y}(x, y),\ (x, y) \in (\cap_{k\in J_i} A_{i,k}) \times (\cap_{k\in J_i} B_{i,k})\},\end{aligned}$$

from what precedes. It remains to maximize over $i \in I$ to get the lower bound.
(ii) Upper bound. Using the same notation as in the previous step, any closed set $F \subseteq \mathcal{X} \times \mathcal{Y}$ may be written as

$$F = \cap_{i\in I} \cup_{k\in J_i} (A_{i,k} \times B_{i,k})^c = \cap_{i\in I} \cup_{k\in J_i} (A^c_{i,k} \times \mathcal{Y}) \cup (\mathcal{X} \times B^c_{i,k}).$$

Therefore, we may write

$$F = \cap_{i\in I} \cup_{k\in K_i} (C_{i,k} \times D_{i,k}),$$

where the K_i's are finite and the $C_{i,k}$'s and $D_{i,k}$'s are closed sets of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Let us first suppose, for simplicity, that I is finite and write $I = \{1, \ldots, n\}$ for some $n \geq 1$. Then,

$$F = \cup_{\bar{k}\in K_1\times\ldots\times K_n} (\cap_{1\leq i\leq n} C_{i,k_i}) \times (\cap_{1\leq i\leq n} D_{i,k_i}) =: \cup_{\bar{k}\in K} \bar{C}_{\bar{k}} \times \bar{D}_{\bar{k}},$$

where $\bar{k} := (k_1, \dots, k_n)$, $K := K_1 \times \dots \times K_n$ (that is finite) and the $\bar{C}_{\bar{k}}$'s and $\bar{D}_{\bar{k}}$'s are closed sets of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{P}((X_n, Y_n) \in F) \le \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{P}((X_n, Y_n) \in \cup_{\bar{k}\in K} \bar{C}_{\bar{k}} \times \bar{D}_{\bar{k}}) \\
&\le \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\bar{k}\in K} \mathrm{P}((X_n, Y_n) \in (\bar{C}_{\bar{k}} \times \bar{D}_{\bar{k}})) \\
&\le \sup_{\bar{k}\in K} \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{P}((X_n, Y_n) \in (\bar{C}_{\bar{k}} \times \bar{D}_{\bar{k}})) \\
&\le \sup_{\bar{k}\in K} \left\{ \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{P}(X_n \in \bar{C}_{\bar{k}}) + \limsup_{n\to\infty} \frac{1}{n} \log \mathrm{P}(Y_n \in \bar{D}_{\bar{k}}) \right\} \\
&\le \sup_{\bar{k}\in K} - \inf_{x\in\bar{C}_{\bar{k}}} I_X(x) - \inf_{y\in\bar{D}_{\bar{k}}} I_Y(y) \\
&\le \sup_{\bar{k}\in K} - \inf_{(x,y)\in\bar{C}_{\bar{k}}\times\bar{D}_{\bar{k}}} I_{X,Y}(x,y) \\
&= - \inf_{\bar{k}\in K} \inf_{(x,y)\in\bar{C}_{\bar{k}}\times\bar{D}_{\bar{k}}} I_{X,Y}(x,y) \\
&= - \inf_{(x,y)\in\cup_{\bar{k}\in K}(\bar{C}_{\bar{k}}\times\bar{D}_{\bar{k}})} I_{X,Y}(x,y) = - \inf_{(x,y)\in F} I_{X,Y}(x,y).
\end{aligned}$$

In the general case, we may optimize over "finite extractions" of I. An additional assumption is actually necessary to treat the general case, like goodness of the rate functions, see for instance Exercise 4.2.7 in Dembo and Zeitouni's monograph on Large Deviations (second edition). Another solution (which some of you wrote) is proving the upper bound for compact sets (using the finite covering property), thus proving a weak LDP.

(iii) Rate function. We check that $I_{X,Y}$ is non-negative, non-trivially infinite and lower semi-continuous.

(2) Show that if I_X and I_Y are *good* rate functions, then $I_{X,Y}$ is a *good* rate function as well. Let us show that the level sets $\{I_{X,Y} \le a\}$ $(a \ge 0)$ are compact. Let $a \ge 0$. First, $\{I_{X,Y} \le a\}$ is a closed set by lower semi-continuity of the rate function. Moreover, since rate functions are non-negative,

$$\{(x,y) \in \mathcal{X} \times \mathcal{Y} \colon I_{X,Y}(x,y) \le a\} \subseteq \{x \in \mathcal{X} \colon I_X(x) \le a\} \times \{y \in \mathcal{X} \colon I_Y(y) \le a\}.$$

Since I_X and I_Y are both *good* rate functions, the set on the right-hand side in the above line is compact (as a product of compact sets). In conclusion, the level set $\{I_{X,Y} \le a\}$ is a closed subset of a compact set, therefore it is compact.

(3) We suppose in this question that $\mathcal{X} = \mathcal{Y} = \mathbb{R}$ (equipped with the Euclidian metric) and that I_X and I_Y are *good* rate functions. Show that the sequence $\{X_n Y_n\}_{n\ge1}$ satisfies an LDP on $\mathbb{R}$ with a rate function to be determined. The map $\Phi\colon (x,y) \in \mathbb{R}^2 \mapsto xy \in \mathbb{R}$ is continuous and from what precedes, the sequence $\{(X_n, Y_n)\}_{n\ge1}$ satisfies an LDP on $\mathbb{R}^2$ with speed n and *good* rate function $I_{X,Y}$. By the contraction principle, we infer that the sequence $\{X_n Y_n\}_{n\ge1} = \{\Phi(X_n, Y_n)\}_{n\ge1}$ satisfies an LDP on $\mathbb{R}$ with speed n and *good* rate function

$$z \in \mathbb{R} \mapsto \inf\{I_X(x) + I_Y(y) \colon x, y \in \mathbb{R},\ xy = z\}.$$

Exercise 2. (Curie-Weiss model with external field). Let $\sigma = (\sigma_i)_{i\ge1}$ be a sequence of independent and identically distributed (i.i.d.) random variables such that $\mathrm{P}(\sigma_1 = 1) = \mathrm{P}(\sigma_1 = -1) = 1/2$. The partition function of the Curie-Weiss model for a system of size $n \ge 1$, inverse temperature $\beta > 0$ and external field $h \in \mathbb{R}$ is

$$Z_n^{\beta,h} = \mathrm{E}[\exp(H_n^{\beta,h})], \qquad \text{where} \quad H_n^{\beta,h} := \frac{\beta}{n} \sum_{1\le i<j\le n} \sigma_i\sigma_j + h \sum_{1\le i\le n} \sigma_i.$$

(1) Show that the mean magnetization $M_n := \frac{1}{n}\sum_{i=1}^n \sigma_i$ satisfies a Large Deviation Principle on $[-1,1]$ with speed n and rate function

$$I(m) := \frac{1}{2}\Big[(1+m)\log(1+m) + (1-m)\log(1-m)\Big].$$

Same as in class.

(2) State Varadhan's lemma. See lecture notes.

(3) Prove that
$$\lim_{n\to\infty}\frac{1}{n}\log Z_n^{\beta,h} = \sup_{m\in[-1,1]}\left\{\tfrac{1}{2}\beta m^2 + hm - I(m)\right\},$$
First, note that
$$H_n^{\beta,h} = \frac{\beta}{2n}(n^2M_n^2 - n) + hnM_n = n\Big[\frac{\beta}{2}M_n^2 + hM_n\Big] - \frac{\beta}{2}.$$
By applying Varadhan's lemma to the continuous and bounded (from above) function $m\in[-1,1]\mapsto(\beta/2)m^2+hm$ and the sequence of random variables $(M_n)_{n\geq 1}$, we obtain the desired result.

(4) Investigate the (existence and number of) maximizers in the variational formula above. What do you notice?
First, we compute, for every $m\in(-1,1)$:
$$I'(m) = \frac{1}{2}\log\Big(\frac{1+m}{1-m}\Big), \quad \lim_{m\to-1} I'(m) = -\infty, \quad \lim_{m\to 1} I'(m) = +\infty$$
$$I''(m) = \frac{1}{1-m^2}\in[1,+\infty).$$
Letting $f(m) := \frac{1}{2}\beta m^2 + hm - I(m)$, we obtain for every $m\in(-1,1)$:
$$f'(m) = \beta m + h - \frac{1}{2}\log\Big(\frac{1+m}{1-m}\Big), \quad \lim_{m\to-1} f'(m) = +\infty, \quad \lim_{m\to 1} f'(m) = -\infty$$
$$f''(m) = \beta - \frac{1}{1-m^2}\in(-\infty,\beta-1].$$
We now distinguish two cases:

Case $\beta\leq 1$. In that case, $f''(m)\leq 0$ (so f is concave) with equality if and only if $\beta=1$ and $m=0$. Therefore, f' is (strictly) monotone and the equation $f'(m)=0$ has exactly one solution. It follows that the variational formula has a unique maximiser (which is zero if $h=0$ and has the sign of h otherwise).

Case $\beta>1$. In that case, $f''(m)\geq 0$ if $|m|\leq m(\beta) := \sqrt{1-1/\beta}$ (and $f''(m)\leq 0$ otherwise), with equality if and only if $m=\pm m(\beta)$. We deduce thereof the following:

- If $h=0$ then there are two distinct maximizers $\pm m^*(\beta)$ (note that $f'(0)=0$ and f is symmetric in that case). Let $h(\beta)$ be the value of $f'(m(\beta))$ when $h=0$.
- If $0<h\leq h(\beta)$ then the equation $f'(m)=0$ has two or three solutions (one that is positive and one or two that are negative). However only the positive solution corresponds to a maximizer (note that $f'(m)-f'(-m)=2hm$).
- If $h(\beta)<h$ then the equation $f'(m)=0$ has a unique solution (that is positive) and therefore a unique maximizer.
- A similar reasoning applies to the case $h<0$ (changing m to $-m$).

In conclusion, we observe a phase transition at the critical value $\beta_c=1$ when $h=0$ but no phase transition when $h\neq 0$ (however there remains a local maximizer when $\beta>1$ and $|h|<h(\beta)$).