CONVERGENCE OF PROBABILITY MEASURES : ADDENDUM

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This is an addendum to the lectures notes written by François Simenhaus for the course "Limit theorems" as part of the master program MATH at Université Paris-Dauphine. These extra notes extend some results of the course to càdlàg processes. The main result is Theorem 6. The content is largely taken from [1]. When the proofs are only sketched the students are encouraged to fill in the gaps by themselves as an exercise. Detailed proofs can be found in the relevant sections of [1]. In the following, we denote by D the space of real-valued *càdlàg* (right-continuous with left limits) functions defined on $[0,1]$.

1. Properties of càdlàg functions

Let us start with basic properties of càdlàg functions. In the following, we note

$$w(f,I) := \sup\{|f(s)-f(t)| \colon s,t \in I\}, \qquad I \subseteq [0,1]. \tag{1}$$

Lemma 1. *Let $f \in D$ and $\varepsilon > 0$. There exists $k \in \mathbb{N}$ and a subdivision $0 = t_0 < t_1 < \ldots < t_k = 1$ such that*

$$\max_{1\le i\le k} w(f,[t_{i-1},t_i)) < \varepsilon. \tag{2}$$

Proof. Fix $f \in D$ and $\varepsilon > 0$. Define $\mathcal{T}_\varepsilon$ as the supremum of those t's between 0 and 1 such that $[0,t)$ may be decomposed into finitely many such subintervals. Let $\underline{t}$ be the supremum value of $\mathcal{T}_\varepsilon$. Check that (i) $\underline{t} > 0$ (ii) $\underline{t} \in \mathcal{T}_\varepsilon$ and (iii) assuming $\underline{t} < 1$ leads to a contradiction. ■

Corollary 2. *For every $f \in D$,*

1. $\forall \delta > 0$, *the set $\{t \in (0,1] \colon |f(t)-f(t^-)| \ge \delta\}$ is finite.*
2. *The set of discontinuity points* $\mathrm{Disc}(f)$ *is at most countable.*
3. $\|f\|_\infty := \sup_{0\le t\le 1} |f(t)| < +\infty$.
4. $\lim_{\delta\to 0} w'(f,\delta) = 0$ *(non-increasing limit), where*

$$w'(f,\delta) := \inf_{\substack{0=t_0<t_1<\ldots<t_k=1\\ \min t_i - t_{i-1} > \delta}} \max_{1\le i\le k} w(f,[t_{i-1},t_i)) \tag{3}$$

2. The Skorokhod metric

In this section we introduce a new metric, called Skorokhod metric, and investigate how projection maps interact with the induced topology and Borel σ-field. Let Λ be the space of increasing continuous functions $\lambda\colon [0,1] \to [0,1]$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$ (the time parametrizations). One can check that the following map $d_S \colon D \times D \mapsto \mathbb{R}_+$:

$$d_S(f,g) := \inf_{\lambda\in\Lambda} \max(\|\lambda - \mathrm{id}\|_\infty, \|f - g\circ\lambda\|_\infty) \tag{4}$$

defines a metric that is called the Skorokhod metric.

Lemma 3. *Let $(f_n), f$ in D such that (f_n) converges to D w.r.t. the Skorokhod metric. Then,*

1. $f_n \to f$ *Lebesgue-a.e. on* $[0,1]$.
2. (f_n) *is uniformly bounded (for the uniform metric).*

Proof. (1) Show that $f_n(t)$ converges to $f(t)$ for every $t \notin \mathrm{Disc}(f)$.

(2) Use that $\|\lambda_n - \mathrm{id}\|_\infty \to 0$ and $\|f \circ \lambda_n - f_n\|_\infty \to 0$ for some sequence (λ_n) in Λ along with Item (3) from Corollary 2.

■

In the following we denote by $\mathcal{D}$ the Borel σ-algebra on D induced by the Skorokhod metric (generated by the Skorokhod-open sets). Let us now investigate the continuity and measurability properties of the projection maps.

Theorem 4. *The following properties hold:*

1. π_0 *and* π_1 *are continuous from* (D, d_S) *to* $(\mathbb{R}, |\cdot|)$.
2. *For every* $t \in [0,1]$, π_t *is continuous at* $f \in D$ *if and only if* f *is continuous at* t.
3. *For every* $n \in \mathbb{N}$ *and* $t_1, \ldots, t_n \in [0,1]$, $\pi_{t_1,\ldots t_k} \colon (D, \mathcal{D}) \mapsto (\mathbb{R}^k, \mathcal{B}(\mathbb{R}^k))$ *is measurable.*
4. *For every* $T \subseteq [0,1]$ *that satisfies* $1 \in T$ *and* $\bar{T} = [0,1]$, $\mathcal{D} = \sigma(\pi_t, t \in T)$.

Proof. (1) This follows from $\lambda(0) = 0$ and $\lambda(1) = 1$ when $\lambda \in \Lambda$.

(2) If f is continuous at t, decompose $f(t) - f_n(t)$ as $[f(t) - f(\lambda_n(t))] + [f(\lambda_n(t)) - f_n(t)]$ for some appropriate sequence (λ_n) and use the triangular inequality. If f is not continuous at t, construct a counterexample.

(3) It is enough to consider $k = 1$ and $t < 1$. Show that for every $\varepsilon > 0$, the map $f \in (D, d_S) \mapsto h_\varepsilon(f) := \frac{1}{\varepsilon}\int_t^{t+\varepsilon} f(s)\mathrm{d}s \in (\mathbb{R}, |\cdot|)$ is continuous. Conclude by noting that π_t is the pointwise limit of $h_{1/m}$ as $m \to \infty$.

(4) The proof is divided into several steps:
- Assume that $0 \in T$ (w.l.o.g.).
- Show that for every $m \in \mathbb{N}$, there exists $k \in \mathbb{N}$, $s_0, \ldots, s_k \in T$ such that $0 = s_0 < \ldots < s_k = 1$ and $\max_{1 \le i \le k}(s_i - s_{i-1}) < 1/m$. Let $\sigma_m := \{s_i\}_{0 \le i \le k}$ in the sequel.
- For all $f \in D$, let $A_m f \in D$ be the step function such that (i) f and $A_m f$ coincide on σ_m and (ii) $A_m f$ is constant on the intervals $[s_{i-1}, s_i)$ for every $1 \le i \le k$. Check that

$$d_S(f, A_m f) \le \max\Big(\tfrac{1}{m}, w'(f, \tfrac{1}{m})\Big). \tag{5}$$

- Deduce thereof that $(A_m f)$ converges to f in the Skorokhod topology.
- Show that A_m is measurable from $(D, \sigma(\pi_t,\ t \in T))$ to $(D, \mathcal{D})$.
- Conclude.

■

3. Weak convergence of càdlàg processes

In this section we explain the general mechanism for proving weak convergence of càdlàg processes. It relies on the following proposition:

Proposition 5. *Let* $\mathrm{P} \in \mathcal{M}_1(D, \mathcal{D})$ *and* $T_\mathrm{P} := \{t \in [0,1] \colon \mathrm{P}(\mathrm{Disc}(\pi_t)) = 0\}$. *Then*

(1) T_{P} *contains* 0 *and* 1.

(2) $[0,1]\setminus T_{\mathrm{P}}$ *is at most countable.*

Proof. The first item is a direct consequence of Item (1) in Theorem 4 so we only need to prove the second one. Let $t\in(0,1)$ and

$$J_t=\{f\in D\colon f(t)\neq f(t^-)\}. \tag{6}$$

Then $t\in T_{\mathrm{P}}$ if and only if $\mathrm{P}(J_t)=0$, by Theorem 4. For every $\varepsilon>0$, let

$$J_t(\varepsilon):=\{f\in D\colon |f(t)-f(t^-)|>\varepsilon\}, \tag{7}$$

so that $\mathrm{P}(J_t)$ is the nondecreasing limit of $\mathrm{P}(J_t(\varepsilon))$ as $\varepsilon\to 0$. Therefore,

$$\mathrm{P}(J_t)>0\Rightarrow \exists\delta,\varepsilon\in\mathbb{Q}\cap(0,1)\colon \mathrm{P}(J_t(\varepsilon))\geq\delta. \tag{8}$$

Assume that the assertation on the right-hand side holds for some fixed pair $\delta,\varepsilon\in\mathbb{Q}\cap(0,1)$ and infinitely many t_n's, $n\geq 1$. Then, for that pair (ε,δ), one has

$$\mathrm{P}\Big(\limsup_{n\to\infty} J_{t_n}(\varepsilon)\Big)\geq\delta. \tag{9}$$

This is absurd, since any $f\in D$ must have only finitely many jumps larger than ε in absolute value. This means that the set

$$\bigcup_{\delta,\varepsilon\in\mathbb{Q}\cap(0,1)}\{t\in[0,1]\colon \mathrm{P}(J_t(\varepsilon))\geq\delta\} \tag{10}$$

is at most countable, which concludes the proof. ■

Theorem 6. *Let* (P_n) *and* P *be probability measures on* $(D,\mathcal{D})$*. Suppose that the two following conditions hold:*

(1) (P_n) *is tight;*

(2) $(\mathrm{P}_n\pi^{-1}_{t_1,\dots,t_k})$ *converges weakly to* $\mathrm{P}\pi^{-1}_{t_1,\dots,t_k}$ *for every* $t_1,\dots t_k\in T_{\mathrm{P}}$.

Then, (P_n) *converges weakly to* P *w.r.t. the Skorokhod topology.*

Proof. The sequence (P_n) is tight, hence relatively compact by Prokhorov's theorem. Then, it is enough to show that P is the only possible limit for any subsequence that converges weakly. Let $\varphi(n)$ be such a subsequence and Q the limit. By the continuous mapping theorem, for all $t_1,\dots,t_k\in T_{\mathrm{Q}}$,

$$\mathrm{P}_{\varphi(n)}\pi^{-1}_{t_1,\dots,t_k}\to\mathrm{Q}\pi^{-1}_{t_1,\dots,t_k},\qquad n\to\infty,\quad\text{weakly.} \tag{11}$$

From our second assumption, we get for all $t_1,\dots,t_k\in T_{\mathrm{P}}\cap T_{\mathrm{Q}}$,

$$\mathrm{P}\pi^{-1}_{t_1,\dots,t_k}=\mathrm{Q}\pi^{-1}_{t_1,\dots,t_k}. \tag{12}$$

Moreover, by Proposition 5, $T_{\mathrm{P}}\cap T_{\mathrm{Q}}$ contains $\{0,1\}$ and is dense in $[0,1]$ (as its complement is countable). We conclude that $\mathrm{P}=\mathrm{Q}$ by using Item (4) in Theorem 4. ■

4. A tightness criterion

We close this addendum by stating without proof a tightness criterion for càdlàg processes.

Theorem 7. *Let* (P_n) *be a sequence of probability measures on* $(D,\mathcal{D})$*. Assume:*

(1) $\lim_{M\to\infty}\limsup_{n\to\infty}\mathrm{P}_n(\{f\in D\colon \|f\|_\infty\geq M\})=0$.

(2) *For every* $\eta>0$, $\lim_{\delta\to 0}\limsup_{n\to\infty}\mathrm{P}_n(\{f\in D\colon w'(f,\delta)\geq\eta\})=0$.

This criterion relies on the analogue of the Ascoli theorem for càdlàg functions. Let us end these notes with one subtlety: the metric space (D, d_S) is not complete (see Exercise 1) but d_S is topologically equivalent to another metric that makes D a Polish space. Therefore, a single probability measure on $(D, \mathcal{D})$ is tight.

Exercise 1. *Consider* $f_n := 1_{[0,1/2^n)}$.

(1) *Show that* (f_n) *is a Cauchy sequence for* d_S.

(2) *Assume that* $d_S(f_n, f) \overset{n\to\infty}{\longrightarrow} 0$ *for some* $f \in D$. *Show that* $f(t) = 0$, $\forall t \in [0, 1]$.

(3) *Conclude.*

References

[1] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.