

Exam - Convergence of probability measures

Thursday 17 th of November 2022

duration : 2h00

The (\star) -question may be more intricate

Difficulty is not an increasing function of the question number.

Title : *Cylinder sets form a convergence-determining class in \mathcal{S} but not in \mathcal{C} .*

Introduction. Let \mathcal{C} be the space of continuous functions on $[0, 1]$ with values in \mathbb{R} . We consider the metric defined by the uniform norm $\|\cdot\|_\infty$ on \mathcal{C} :

$$\|f\|_\infty = \sup\{|f(t)|, t \in [0, 1]\}, \quad f \in \mathcal{C}.$$

Recall that $(\mathcal{C}, \|\cdot\|_\infty)$ is a polish space. We define the class \mathcal{C}_f of *finite-dimensional sets* (or *cylinder sets*) to be the subsets of \mathcal{C} that write

$$\{f \in \mathcal{C}; f(t_i) \in B_i \text{ for all } i \leq k\},$$

where $k \geq 1$ is an integer, $t_i \in [0, 1]$ and $B_i \in \mathcal{B}(\mathbb{R})$ for all $i = 1, \dots, k$.

1. Prove that \mathcal{C}_f is a *separating class*, that is if P and Q are two probability measures on \mathcal{C} such that $P(A) = Q(A)$ for all $A \in \mathcal{C}_f$ then $P = Q$.
2. Prove with a simple counter-example that if $(P_n)_{n \geq 1}$ and P are probability measures on \mathcal{C} such that $P_n(A) \rightarrow P(A)$ for all $A \in \mathcal{C}_f$, this does not imply that necessarily $P_n \implies P$ (for the uniform topology).

Problem. Let (E, d) be a metric space. We denote by \mathcal{T} the topology endowed by d and by $\mathcal{F} := \sigma(\mathcal{T})$, the Borel sigma algebra. We consider a class $\mathcal{A} \subset \mathcal{F}$ such that $P_n(A) \rightarrow P(A)$ for all $A \in \mathcal{A}$. We first assume that

- (i) \mathcal{A} is closed under finite intersections;
 - (ii) each open set of E writes as a countable union of sets in \mathcal{A} .
3. Prove that if a subset $B \subset E$ writes as finite union of sets in \mathcal{A} , it satisfies

$$P_n(B) \rightarrow P(B).$$

4. Remind the characterization of weak convergence using open sets in the Portmanteau's Theorem and prove that $P_n \implies P$ (where " \implies " denotes the weak convergence with respect to \mathcal{T}).

Instead of (ii) we now assume :

- (ii') E is separable and, for every $x \in E$ and $\varepsilon > 0$, there exists $A \in \mathcal{A}$ that satisfies

$$x \in \overset{\circ}{A} \subset A \subset B(x, \varepsilon). \quad (1)$$

5. (\star) Use that E is separable to prove that, from any covering of an open set B by open sets, one can extract a countable covering (*this is known as Lindelöf's lemma*).

6. Prove that (ii') implies (ii).

A class $\mathcal{G} \subset \mathcal{F}$ is called a *convergence-determining class* if :

For every sequence of probability measures $(P_n)_{n \geq 1}$ and every probability measure P : convergence $P_n(A) \rightarrow P(A)$ for all $A \in \mathcal{G}$ such that $P(\partial A) = 0$ implies $P_n \implies P$.

7. Prove that if \mathcal{G} is a convergence-determining class then it is a separating class, that is if P and Q are probability measures such that $P(A) = Q(A)$ for all $A \in \mathcal{G}$ then $P = Q$.

Given a class $\mathcal{G} \subset \mathcal{F}$ we consider the following notations : For $x \in E$ and $\varepsilon > 0$, we consider the subclass $\mathcal{G}_{x,\varepsilon} \subset \mathcal{G}$ of sets $A \in \mathcal{G}$ that satisfy $x \in A^\circ \subset A \subset B(x, \varepsilon)$. We also denote by $\partial\mathcal{G}_{x,\varepsilon}$ the class of their boundaries that is

$$\partial\mathcal{G}_{x,\varepsilon} = \{\partial A, A \in \mathcal{G}_{x,\varepsilon}\}.$$

Our next goal is to prove the following

Theorem. Let $\mathcal{G} \subset \mathcal{F}$ be a class that satisfies

- (i) \mathcal{G} is closed under finite intersections ;
- (iii) E is separable and for each $x \in E$ and $\varepsilon > 0$, $\partial\mathcal{G}_{x,\varepsilon}$ either contains \emptyset or contains uncountably many disjoint sets.

Then \mathcal{G} is a convergence-determining class.

From now on we fix a class $\mathcal{G} \subset \mathcal{F}$ that satisfies the assumptions of the above theorem, a probability measure P and a sequence of probability measures $(P_n)_{n \geq 1}$.

- 8. Prove that if a class $\mathcal{G} \subset \mathcal{F}$ is stable under finite intersections then the same happens for the class \mathcal{G}_P of sets A in \mathcal{G} that satisfy $P(\partial A) = 0$.
- 9. Prove that, for each $x \in E$ and $\varepsilon > 0$, $\mathcal{G}_{x,\varepsilon}$ contains a set in \mathcal{G}_P and conclude the proof of the Theorem.

Application. Let \mathcal{S} be the space of all real valued sequences $x = (x_i)_{i \geq 1}$. We consider the metric b on \mathbb{R} defined by

$$b(x, y) = |x - y| \wedge 1, \quad x, y \in \mathbb{R}.$$

Note that b defines the same topology as $|\cdot|$ and that (\mathbb{R}, b) is a polish space. We define ρ on \mathcal{S} by

$$\rho(x, y) = \sum_{i=1}^{+\infty} \frac{b(x_i, y_i)}{2^i}.$$

It is easy to check that ρ is a distance on \mathcal{S} .

- 10. Prove that a sequence $(x^n)_{n \geq 1}$ in \mathcal{S} ρ -converges to $x \in \mathcal{S}$ if and only if $(x^n)_{n \geq 1}$ converges pointwise to x , that is for all $i \geq 1$, $x_i^n \rightarrow x_i$ when n goes to ∞ .
- 11. Prove that (\mathcal{S}, ρ) is a polish space :
 - (a) *separable* : prove that the family of sequences having only finitely many non zero coordinates, each of them rational, is dense in \mathcal{S} ;
 - (b) prove that (\mathcal{S}, ρ) is *complete*.

We define the class \mathcal{S}_k of *finite-dimensional sets* to be the sets in \mathcal{S} that write

$$\{x \in \mathcal{S}; x_i \in B_i \text{ for all } i \leq k\},$$

where $k \geq 1$ is an integer and $B_i \in \mathcal{B}(\mathbb{R})$ for all $i = 1, \dots, k$.

12. Use the sets

$$A(x, \eta, k) = \{y \in S; |y_i - x_i| < \eta \text{ for all } i \leq k\}, \quad x \in S, \eta > 0, k \geq 1,$$

to prove that \mathcal{S}_f is a convergence-determining class.

13. In the Introduction we proved that \mathcal{C}_f is not a convergence-determining class for the weak convergence in $(\mathcal{C}, \mathcal{T})$. Why is it clear that \mathcal{C}_f does not satisfy the assumptions of our Theorem?