Université Paris-Dauphine
Master 2 MATH

Exam - Convergence of probability measures
Thursday 17 th of November 2022
duration : 2h00
The $(\star)$-*question may be more intricate*
Difficulty is not an increasing function of the question number.

Title : *Cylinder sets form a convergence-determining class in $\mathcal{S}$ but not in $\mathcal{C}$.*

This problem is inspired mainly by [1]. Instead of a true correction I simply indicate the places where these questions are addressed.

Introduction. Let $\mathcal{C}$ be the space of continuous functions on $[0,1]$ with values in $\mathbb{R}$. We consider the metric defined by the uniform norm $||\cdot||_\infty$ on $\mathcal{C}$:

$$||f||_\infty = \sup\{|f(t)|,\ t \in [0,1]\}, \qquad f \in \mathcal{C}.$$

Recall that $(\mathcal{C}, ||\cdot||_\infty)$ is a polish space. We define the class $\mathcal{C}_\mathrm{f}$ of *finite-dimensional sets* (or *cylinder sets*) to be the subsets of $\mathcal{C}$ that write

$$\{f \in \mathcal{C};\ f(t_i) \in B_i \text{ for all } i \leq k\},$$

where $k \geq 1$ is an integer, $t_i \in [0,1]$ and $B_i \in \mathcal{B}(\mathbb{R})$ for all $i = 1, \cdots, k$.

1. Prove that $\mathcal{C}_\mathrm{f}$ is a *separating class*, that is if P and Q are two probability measures on $\mathcal{C}$ such that $\mathrm{P}(A) = \mathrm{Q}(A)$ for all $A \in \mathcal{C}_\mathrm{f}$ then $\mathrm{P} = \mathrm{Q}$. See the notes (this is a consequence of : monotone class lemma, stability of $\mathcal{C}_\mathrm{f}$ under finite intersection and $\sigma(\mathcal{C}_\mathrm{f}) = \mathcal{F}$)
2. Prove with a simple counter-example that if $(\mathrm{P}_n)_{n\geq 1}$ and P are probability measures on $\mathcal{C}$ such that $\mathrm{P}_n(A) \to \mathrm{P}(A)$ for all $A \in \mathcal{C}_\mathrm{f}$, this does not imply that necessarily $\mathrm{P}_n \implies \mathrm{P}$ (for the uniform topology). See the notes for a counter-example

Problem. Let (E, d) be a metric space. We denote by $\mathcal{T}$ the topology endowed by d and by $\mathcal{F} := \sigma(\mathcal{T})$, the Borel sigma algebra. We consider a class $\mathcal{A} \subset \mathcal{F}$ such that $\mathrm{P}_n(A) \to \mathrm{P}(A)$ for all $A \in \mathcal{A}$. We first assume that

(i) $\mathcal{A}$ is closed under finite intersections ;

(ii) each open set of E writes as a countable union of sets in $\mathcal{A}$.

3. Prove that if a subset $B \subset E$ writes as finite union of sets in $\mathcal{A}$, it satisfies
$$\mathrm{P}_n(B) \to \mathrm{P}(B).$$
See [1, Theorem 2.2 p.17]
4. Remind the characterization of weak convergence using open sets in the Portmanteau's Theorem and prove that $\mathrm{P}_n \implies \mathrm{P}$ *(where "$\implies$" denotes the weak convergence with respect to $\mathcal{T}$)*. See [1, Theorem 2.2 p.17]

Instead of (ii) we now assume :

(ii') E is separable and, for every $x \in E$ and $\varepsilon > 0$, there exists $A \in \mathcal{A}$ that satisfies

$$x \in \mathring{A} \subset A \subset B(x, \varepsilon). \quad (1)$$

5. ($\star$) Use that E is separable to prove that, from any covering of an open set B by open sets, one can extract a countable covering *(this is known as Lindelöf's lemma)*. See https ://en.wikipedia.org/wiki/Lindelöf's lemma
6. Prove that (ii') implies (ii). See [1, Theorem 2.3 p.17]

A class $\mathcal{G} \subset \mathcal{F}$ is called a *convergence-determining class* if :
For every sequence of probability measures $(\mathrm{P}_n)_{n\geq 1}$ *and every probability measure* P *: convergence* $\mathrm{P}_n(A) \to \mathrm{P}(A)$ *for all* $A \in \mathcal{G}$ *such that* $\mathrm{P}(\partial A) = 0$ *implies* $\mathrm{P}_n \implies \mathrm{P}$.

7. Prove that if $\mathcal{G}$ is a convergence-determining class then it is a separating class, that is if P and Q are probability measures such that $\mathrm{P}(A) = \mathrm{Q}(A)$ for all $A \in \mathcal{G}$ then $\mathrm{P} = \mathrm{Q}$. Use the sequence (P_n) constant equal to P that weakly converges to both P and Q

Given a class $\mathcal{G} \subset \mathcal{F}$ we consider the following notations : For $x \in E$ and $\varepsilon > 0$, we consider the subclass $\mathcal{G}_{x,\varepsilon} \subset \mathcal{G}$ of sets $A \in \mathcal{G}$ that satisfy $x \in A^\circ \subset A \subset B(x, \varepsilon)$. We also denote by $\partial\mathcal{G}_{x,\varepsilon}$ the class of their boundaries that is

$$\partial\mathcal{G}_{x,\varepsilon} = \{\partial A,\ A \in \mathcal{G}_{x,\varepsilon}\}.$$

Our next goal is to prove the following

Theorem. *Let* $\mathcal{G} \subset \mathcal{F}$ *be a class that satisfies*

(i) $\mathcal{G}$ *is closed under finite intersections ;*

(iii) E *is separable and for each* $x \in E$ *and* $\varepsilon > 0$, $\partial\mathcal{G}_{x,\varepsilon}$ *either contains* $\emptyset$ *or contains uncountably many disjoint sets.*

Then $\mathcal{G}$ *is a convergence-determining class.*

From now on we fix a class $\mathcal{G} \subset \mathcal{F}$ that satisfies the assumptions of the above theorem, a probability measure P and a sequence of probability measures $(\mathrm{P}_n)_{n\geq 1}$.

8. Prove that if a class $\mathcal{G} \subset \mathcal{F}$ is stable under finite intersections then the same happens for the class $\mathcal{G}_{\mathrm{P}}$ of sets A in $\mathcal{G}$ that satisfy $\mathrm{P}(\partial A) = 0$. It is just a consequence of $\partial A \cap B \subset \partial A \cup \partial B$ (the proof of that inclusion would have been appreciated)
9. Prove that, for each $x \in E$ and $\varepsilon > 0$, $\mathcal{G}_{x,\varepsilon}$ contains a set in $\mathcal{G}_{\mathrm{P}}$ and conclude the proof of the Theorem. See [1, Theorem 2.4 p.18]

Application. Let $\mathcal{S}$ be the space of all real valued sequences $x = (x_i)_{i\geq 1}$. We consider the metric b on $\mathbb{R}$ defined by

$$b(x, y) = |x - y| \wedge 1, \qquad x, y \in \mathbb{R}.$$

Note that b defines the same topology as $|\cdot|$ and that $(\mathbb{R}, b)$ is a polish space. We define ρ on $\mathcal{S}$ by

$$\rho(x, y) = \sum_{i=1}^{+\infty} \frac{b(x_i, y_i)}{2^i}.$$

It is easy to check that ρ is a distance on $\mathcal{S}$. See [1, Example 2.4 p.19]

10. Prove that a sequence $(x^n)_{n\geq 1}$ in $\mathcal{S}$ $\rho-$converges to $x \in \mathcal{S}$ *if and only if* $(x^n)_{n\geq 1}$ converges pointwise to x, that is for all $i \geq 1$, $x_i^n \to x_i$ when n goes to ∞. The implication $\implies$ is easy. The converse also once noticed that for all $\varepsilon > 0$ there exists $N \geq 1$ so that if $x, y \in \mathcal{S}$ are such that $x_i = y_i$ for all $i \leq N$ then $\rho(x, y) < \varepsilon$.
11. Prove that $(\mathcal{S}, \rho)$ is a polish space :
    (a) *separable :* prove that the family of sequences having only finitely many non zero coordinates, each of them rational, is dense in $\mathcal{S}$; Use the above property together with the density of $\mathbb{Q}$ in $\mathbb{R}$ and the characterisation of the convergence in 10.
    (b) prove that $(\mathcal{S}, \rho)$ is *complete*. For all $i \geq 1$, $(x_i^n)_{n\geq 1}$ is a Cauchy (for b or $|\cdot|$) sequence and thus converges. Use question 10 to conclude.

We define the class $\mathcal{S}_{\mathrm{f}}$ of *finite-dimensional sets* to be the sets in $\mathcal{S}$ that write

$$\{x \in \mathcal{S};\ x_i \in B_i \text{ for all } i \leq k\},$$

where $k \geq 1$ is an integer and $B_i \in \mathcal{B}(\mathbb{R})$ for all $i = 1, \cdots, k$.

12. Use the sets

$$A(x, \eta, k) = \{y \in S;\ |y_i - x_i| < \eta \text{ for all } i \leq k\}, \qquad x \in S,\ \eta > 0,\ k \geq 1,$$

    to prove that $\mathcal{S}_{\mathrm{f}}$ is a convergence-determining class. See [1, Example 2.4 p.19]
13. In the Introduction we proved that $\mathcal{C}_{\mathrm{f}}$ is not a convergence-determining class for the weak convergence in $(\mathcal{C}, \mathcal{T})$. Why is it clear that $\mathcal{C}_{\mathrm{f}}$ does not satisfy the assumptions of our Theorem ? For $f \in \mathcal{C}$ and $\varepsilon > 0$, the subclass $\mathcal{G}_{x,\varepsilon} = \emptyset$ as no cylinder sets is included in $B(x, \varepsilon)$.

Références

[1] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics : Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.