EXAM M2 - LIMIT THEOREMS - 2023 - 1H30
THE ARCSINE LAW

No document, no electronic device allowed. You may skip some questions if necessary (Q1 and Q2 for instance) and come back to it later, provided that you clearly label your answers. Clarity of exposition will be taken into account. You may answer in French or English. *Bon courage !*

Let B be a standard one-dimensional Brownian motion starting from the origin. The goal of the problem is to determine the law of the random variable

$$T = \sup\{t \in [0,1] \colon B(t) = 0\}.$$

Let $\mathcal{C}$ be the space of real-valued continuous functions defined on the compact interval $[0,1]$. We equip $\mathcal{C}$ with the uniform norm $\|\cdot\|_\infty$ and define

$$h(x) = \sup\{t \in [0,1] \colon x(t) = 0\}, \quad \forall x \in \mathcal{C},$$

with the convention $\sup \emptyset = 0$.

(1) Is h a continuous function (with respect to the uniform topology)? If not, is it lower semi-continuous*? Is it upper semi-continuous?
Consider $x_n(t) = t/n$ for $t \in [0,1]$. Then x_n converges uniformly to the zero function. The fact that $h(0_\mathcal{C}) = 1 > \liminf h(x_n) = 0$ shows that h fails to be lower semi-continuous. However, we can prove h is upper semi-continuous. Let $x \in \mathcal{C}$. Suppose $h(x) < 1$ otherwise there is nothing to prove. By continuity of x, $h(x)$ is actually a maximum, i.e. $x(h(x)) = 0$ and $x(t) \neq 0$ (say $x(t) > 0$ w.l.o.g.) for $t \in (h(x), 1)$. Let δ be any positive number such that $h(x)+\delta < 1$. Note that $\min\{x(t), t \in [t+\delta, 1]\} > 0$. Therefore, if $x_n \to x$ uniformly then for n large enough, any zero of x_n is smaller than $h(x) + \delta$. Since δ is arbitrary, we get $\limsup h(x_n) \leq h(x)$.

(2) Give a sufficient condition for x to be a continuity point of h. From our answer to the previous question, a sufficient condition for lower semi-continuity of h at x is enough. Suppose that $h(x) > 0$ (otherwise there is nothing to prove) and that there exists an increasing sequence $t_k \in [0, h(x))$ converging to $h(x)$ such that $x(t_k) > 0$ for odd k's and $x(t_k) < 0$ for even k's. Let us now consider a sequence x_n converging uniformly to x. Pick an (arbitrarily small) $\delta > 0$. From our assumption, there exists k such that $h(x) - \delta < t_k < t_{k+1}$ and $x(t_k) < 0 < x(t_{k+1})$. When n is large enough, we get $x_n(t_k) < 0 < x_n(t_{k+1})$. By the intermediate value theorem, this implies that x_n has a zero in (t_k, t_{k+1}), from which we obtain $h(x_n) > t_k > h(x) - \delta$. Hence $\liminf h(x_n) \geq h(x)$.

Let $\mathrm{Disc}(h)$ be the set of discontinuity points of h. We admit for now that

$$\mathrm{P}(B \in \mathrm{Disc}(h)) = 0.$$

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables uniformly distributed on -1 and 1. Let S be the associated random walk, that is $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$ for $n \in \mathbb{N}$. Define

$$T_n = \max\{0 \leq k \leq n \colon S_k = 0\}, \qquad n \in \mathbb{N},$$

*h is lower semi-continuous at x if $h(x) \leq \liminf h(x_n)$ for any sequence (x_n) converging to x

and

$$u(k) = \mathrm{P}(S_k = 0), \qquad k \in \mathbb{N} \cup \{0\},$$
$$v(k) = \mathrm{P}(S_i \neq 0,\ 1 \le i \le k), \qquad k \in \mathbb{N}.$$

(3) Give one formulation of weak convergence. See lecture notes.

(4) Cite Donsker's theorem. See lecture notes.

(5) Prove that T_n/n converges in law to T as $n \to \infty$.
Let us write $B^{(n)}$ the continuous process that linearly interpolates $S_{k/n}/\sqrt{n}$ for $0 \le k \le n$. Note that

$$\frac{T_n}{n} = \sup\left\{t \in [0,1] \cap \frac{\mathbb{N}}{n} \colon \frac{S_{nt}}{\sqrt{n}} = 0\right\} = h(B^{(n)}).$$

By Donsker's theorem, $B^{(n)}$ converges weakly to B and since $\mathrm{P}(B \in \mathrm{Disc}(h)) = 0$ we from the mapping theorem that $\frac{T_n}{n}$ converges weakly to $h(B) = T$.

(6) For every $0 \le k \le n$, write $\mathrm{P}(T_n = k)$ in terms of u and v.
By using the Markov property at time index k, we get

$$\mathrm{P}(T_n = k) = \mathrm{P}(S_k = 0, S_i \neq 0, k < i \le n) = u(k)v(n-k).$$

Note that this is zero when k is odd.

We admit for now that there exist two positive constants C_1 and C_2 such that

$$u(2k) \sim \frac{C_1}{\sqrt{2k}} \qquad \text{and} \qquad v(k) \sim \frac{C_2}{\sqrt{k}}, \qquad k \to \infty.$$

(7) Deduce thereof the density of T and draw its graph.
Let ϕ be a continuous and bounded function.

$$\begin{aligned}\mathrm{E}(\phi(\tfrac{T_n}{n})) &= \sum_{k=0}^{n} \phi(\tfrac{k}{n}) u(k) v(n-k) \\ &= \tfrac{1}{n} \sum_{k=0}^{n} \phi(\tfrac{k}{n}) [\sqrt{n} u(k)][\sqrt{n} v(n-k)],\end{aligned}$$

By a Riemann sum approximation, we get that

$$\mathrm{E}(\phi(T)) = \int_0^1 \phi(s) \frac{C\ \mathrm{d}s}{\sqrt{s(1-s)}}$$

for some constant $C > 0$ that we will compute in the next question.

(8) Compute and draw the cumulative distribution function [†] of T (mind the title of the problem).
For all $t \in [0,1]$,

$$\int_0^t \frac{\mathrm{d}s}{\sqrt{s(1-s)}} = 2\int_0^t \frac{\mathrm{d}x}{\sqrt{1-x^2}} = 2\arcsin(\sqrt{t}).$$

Since $\arcsin(1) = \pi/2$ we get $C = 1/\pi$ and

$$\mathrm{P}(T \le t) = \frac{2}{\pi}\arcsin(\sqrt{t}).$$

[†] fonction de répartition

Bonus questions

(9) Prove the asymptotic behaviours of u (doable) and v (quite difficult) given above.
Note that $u(2k) = \frac{1}{2^{2k}}\binom{2k}{k}$. Stirling's formula gives the asymptotic for u. For v, see for instance Exercice 1.15 in"Exercices de probabilités" (Cottrell et al).

(10) Prove that $\mathrm{P}(B \in \mathrm{Disc}(h)) = 0$.
It is enough to prove that a.s., B satisfies our sufficient condition for continuity as in Question 2, i.e. B changes sign on any left neighborhood of T. This can be proven to be true on any *right* neighbourhood of the *first* zero of a Brownian motion. It is then enough to notice that $T = \sup\{t \in (0,1] \colon \frac{1}{t}B(t) = 0\} = \inf\{s \geq 1 \colon sB(1/s) = 0\}$, and that $sB(1/s)$ is also a Brownian motion.