JUMP PROCESSES - M2 MASEF/MATH
EXAM 13/01/2020 (3H)

No phone and no document is allowed. No indication is given during the exam. You can answer in French or English. The grading scheme given below is approximative. The quality of writing, justification and presentation is taken into account.

Notation. Open intervals of $\mathbb{R}$ are denoted by (a, b). Closed intervals are denoted by $[a, b]$. The interval $(a, b]$ is open at a and closed at b. Lebesgue measure is denoted by symbols $\mathrm{d}s$, $\mathrm{d}t$ or $\mathrm{d}z$. All Lévy processes below are one-dimensional and start from zero. The small jumps and large jumps in the Lévy-Itô decomposition are chosen as the sets $\{z \in \mathbb{R}\colon |z| < 1\}$ and $\{z \in \mathbb{R}\colon |z| \geq 1\}$, respectively. If $\{x(t)\}_{t\geq 0}$ is a càd-làg function, then the value of the jump at $t > 0$ is denoted by $\Delta x(t) := x(t) - x(t^-)$. The minimum between two real numbers a and b is denoted by $a \wedge b$.

Questions

(1) What is a Lévy process? Give one example with continuous sample paths and one example that is only made of jumps. A Lévy process is a process with (i) independent and stationary increments, and (ii) càdlàg sample paths (starting from the origin). Brownian motion is one example with continuous sample paths. Any Poisson counting process is an example with jumps only.

(2) What is a Lévy measure? Give one example of a Lévy measure on the real line that is not a finite measure. A measure ν on $\mathbb{R}$ is a Lévy measure if $_{\mathbb{R}}\int (1 \wedge z^2)\nu(\mathrm{d}z) < \infty$ (and $\nu(\{0\}) = 0$).

(3) What is an infinitely divisible probability distribution (on the real line)? What is the connection with Lévy processes? See lecture notes.

(4) What is a subordinator? Give one example. See lecture notes.

(5) Let $\alpha \in (0, 2)$ and $\mathcal{N}$ be a random Poisson measure on $(0, \infty)^2$ with intensity measure $\mathrm{d}t \otimes \nu(\mathrm{d}z)$, where

$$\nu(\mathrm{d}z) = z^{-(1+\alpha)}\mathrm{d}z \qquad (z > 0).$$

For which values of $\beta \in \mathbb{R}$ is the process

$$X(t) = \int_{(0,t]\times(0,1)} z^{\beta}\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0$$

well-defined as a centered and square-integrable càd-làg martingale? From the lecture notes, this is true when

$$\int_{(0,t]\times(0,1)} z^{2\beta}\nu(\mathrm{d}z)\mathrm{d}s = t\int_{(0,1)} z^{2\beta-1-\alpha}\mathrm{d}z < +\infty,$$

that is when $\beta > \alpha/2$.

(6) Let P be a probability distribution under which $(N_t)_{0\le t\le 1}$ is a Poisson counting process with intensity one. Identify the law of $(N_t)_{0\le t\le 1}$ under the probability distribution $\widetilde{\mathrm{P}}$, which is defined by

$$\frac{\mathrm{d}\widetilde{\mathrm{P}}}{\mathrm{dP}} = e^{-1}2^{N_1}.$$

Under $\widetilde{\mathrm{P}}$, $(N_t)_{0\le t\le 1}$ is a Poisson counting process with intensity two. Indeed, for all $n \ge 1$, $t_0 := 0 < t_1 < t_2 < \ldots < t_n := 1$ and $k_0 := 0 \le k_1 \le k_2 \le \ldots \le k_n$,

$$\begin{aligned}
\widetilde{\mathrm{P}}(N_{t_1} = k_1, \ldots, N_{t_n} = k_n) &= \mathrm{E}\Big(1\{N_{t_1} = k_1, \ldots, N_{t_n} = k_n\}\frac{\mathrm{d}\widetilde{\mathrm{P}}}{\mathrm{dP}}\Big) \\
&= e^{-1}2^{k_n}\mathrm{P}(N_{t_1} = k_1, \ldots, N_{t_n} = k_n) \\
&= e^{-1}2^{k_n}\prod_{1\le i\le n}\mathrm{P}(N_{t_i - t_{i-1}} = k_i - k_{i-1}) \\
&= e^{-1}2^{k_n}\prod_{1\le i\le n}\frac{e^{-(t_i - t_{i-1})}}{(k_i - k_{i-1})!} \\
&= \prod_{1\le i\le n} 2^{(k_i - k_{i-1})}\frac{e^{-2(t_i - t_{i-1})}}{(k_i - k_{i-1})!}.
\end{aligned}$$

Exercise 1. Solution of a stochastic differential equation (around 6 points). Let X be a Lévy process with Lévy measure ν and triplet $(0, 0, \nu)$. We assume that $\nu(-\infty, -1] = 0$. Let $h(x) = x - \log(1 + x)$ for $x > -1$. We define the process $Y(t) = X(t) - \sum_{0\le s\le t} h(\Delta X(s))$ for $t \ge 0$.

(1) Explain why Y is well defined and write it as a Lévy-type stochastic integral.

(a) If s is a continuity point of X (actually, of a sample path of X) then $\Delta X(s) = 0$ and $h(\Delta X(s)) = 0$. Therefore the sum in the definition of $Y(t)$ is restricted to the countably many jumps of the sample path (that is when $\Delta X(s) \neq 0$). Moreover, one can find a constant $C > 0$ such that $0 \le h(x) \le Cx^2$ for all $x > -1$ (by a Taylor expansion). This gives

$$0 \le \sum_{0\le s\le t} h(\Delta X(s)) \le C\sum_{0\le s\le t}\Delta X(s)^2,$$

and the right-hand side is a.s. finite (see lecture notes).

(b) By the Lévy-Itô decomposition, we may write

$$X(t) = \int_{(0,t]\times(-1,1)} z\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z) + \int_{(0,t]\times[1,\infty)} z\mathcal{N}(\mathrm{d}s, \mathrm{d}z).$$

Also,

$$\sum_{0\le s\le t} h(\Delta X(s)) = \int_{(0,t]\times(-1,\infty)} h(z)\mathcal{N}(\mathrm{d}s, \mathrm{d}z).$$

After straightforward simplifications, we obtain

$$Y(t) = -bt + \int_{(0,t]\times(-1,1)}\log(1+z)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z) + \int_{(0,t]\times[1,\infty)}\log(1+z)\mathcal{N}(\mathrm{d}s, \mathrm{d}z),$$

where $b := \int_{(-1,1)} h(z)\nu(\mathrm{d}z)$.

(2) Apply the Lévy-Itô formula and compute the stochastic differential of $f(Y(t))$, where $f \in \mathcal{C}^2(\mathbb{R})$, the space of twice continuously differentiable functions on the real line.

$$\begin{aligned}\mathrm{d}f(Y(t)) = -bf'(Y(t))\mathrm{d}t + \int_{[1,\infty)} [f(Y(t^-) + \log(1+z)) - f(Y(t^-))]\mathcal{N}(\mathrm{d}t, \mathrm{d}z) \\ + \int_{(-1,1)} [f(Y(t^-) + \log(1+z)) - f(Y(t^-))]\widetilde{\mathcal{N}}(\mathrm{d}t, \mathrm{d}z) \\ + \int_{(-1,1)} [f(Y(t^-) + \log(1+z)) - f(Y(t^-)) - \log(1+z)f'(Y(t^-))]\mathrm{d}t\nu(\mathrm{d}z).\end{aligned}$$

(3) Prove that the process $\{\exp(Y(t))\}_{t\geq 0}$ solves the stochastic differential equation $\mathrm{d}S(t) = S(t^-)\mathrm{d}X(t)$. What is the name usually given to this solution? Pick $f = \exp$ in the above equality and simplify (see lecture notes). This solution is known as the Doléans-Dade exponential.

(4) Find a simple expression for this solution when ν is the Dirac mass at $u > -1$. In this case, we may write $X(t)$ as $uN(t) - ut\mathbf{1}_{(-1,1)}(u)$, where $(N(t))$ is a Poisson counting process with intensity one. Therefore,

$$\sum_{0\leq s\leq t} h(\Delta X(s)) = h(u)N(t),$$

which gives $Y(t) = \log(1+u)N(t) - ut\mathbf{1}_{(-1,1)}(u)$, and

$$\exp(Y(t)) = (1+u)^{N(t)}e^{-ut\mathbf{1}_{(-1,1)}(u)}, \qquad t \geq 0.$$

Exercise 2. Integrability of a Lévy process (around 6 points). We consider a Lévy process X with bounded jumps, meaning that there exists $C > 0$ such that (for all realizations of the process) $|\Delta X(t)| \leq C$ for all $t \geq 0$. We define the sequence of stopping times $(T_n)_{n\geq 0}$ by $T_0 = 0$ and

$$T_n = \inf\{t > T_{n-1} \colon |X(t) - X(T_{n-1})| > C\} \qquad (n \geq 1).$$

(The candidate is not required to prove that those are stopping times.)

(1) Prove that $|X(t \wedge T_n)| \leq 2nC$ for all $n \geq 1$ and $t \geq 0$. Define

$$U_i(t) = |X(t \wedge T_i) - X(t \wedge T_{i-1})|, \qquad t \geq 0, \quad i \in \mathbb{N}.$$

Since $X_0 = 0$, we get by the triangular inequality

$$|X(t \wedge T_n)| \leq \sum_{1\leq i\leq n} U_i(t).$$

Let us now prove that for any $1 \leq i \leq n$ and $t \geq 0$, $U_i(t) \leq C$, which is enough to conclude. We use a pathwise approach. Suppose first that $t \leq T_{i-1}$. Then $U_i(t) = 0$. Now, assume that $t > T_{i-1}$. Then

$$U_i(t) \leq \sup_{T_{i-1}<s\leq T_i} |X(s) - X(T_{i-1})|.$$

If $T_{i-1} < s < T_i$ then $|X(s) - X(T_{i-1})| \leq C$ (by definition of T_i). Moreover,

$$|X(T_i) - X(T_{i-1})| \leq |X(T_i) - X(T_{i-1}^-)| + |\Delta X(T_i)| \leq 2C.$$

This concludes the proof.

(2) Prove that $\mathrm{P}(|X(t)| \geq 2nC) \leq e^t \mathrm{E}(e^{-T_1})^n$ for all $n \geq 1$ and $t \geq 0$. By the strong Markov property, $(T_i - T_{i-1})_{i\geq 1}$ is a sequence of i.i.d. random variables. Then, using the Markov inequality, we obtain

$$\mathrm{P}(|X(t)| \geq 2nC) \leq \mathrm{P}(T_n \leq t) = \mathrm{P}(e^{-T_n} \geq e^{-t}) \leq e^t \mathrm{E}(e^{-T_n}) = e^t \mathrm{E}(e^{-T_1})^n.$$

(3) Prove that $\mathrm{E}(|X(t)|) < \infty$ for all $t \geq 0$. We have from the previous question

$$\begin{aligned}
\mathrm{E}(|X(t)|) &= \int_0^{+\infty} \mathrm{P}(|X(t)| \geq x)\mathrm{d}x \\
&= \sum_{n\in\mathbb{N}_0} \int_{2nC}^{2(n+1)C} \mathrm{P}(|X(t)| \geq x)\mathrm{d}x \\
&\leq 2C\Big[1 + \sum_{n\in\mathbb{N}} \mathrm{P}(|X(t)| \geq 2nC)\Big] \\
&\leq 2C\Big[1 + e^t \sum_{n\in\mathbb{N}} \mathrm{E}(e^{-T_1})^n\Big],
\end{aligned}$$

which is finite, since $\mathrm{E}(e^{-T_1}) \in (0,1)$.

We now assume X is a compound Poisson process with jump probability measure ν.

(4) Prove that $\mathrm{E}(|X(t)|) < \infty$ for all $t \geq 0$ if and only if $\int_{\mathbb{R}} |z|\nu(\mathrm{d}z) < \infty$.

Let us write

$$X(t) = \sum_{i\leq N(t)} Z_i = \sum_{i\geq 1} Z_i \mathbf{1}_{\{N(t)\geq i\}},$$

where $(N(t))$ is a Poisson counting process with intensity $\lambda > 0$ and $(Z_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables with law ν. By the triangular inequality, we get on the one side

$$\mathrm{E}(|X(t)|) \leq \mathrm{E}(|Z_1|)\mathrm{E}(N(t)) = \lambda t \mathrm{E}(|Z_1|).$$

On the other side, we have

$$\mathrm{E}(|X(t)|) \geq \mathrm{E}(|X(t)|\mathbf{1}_{\{N_t=1\}}) = \mathrm{E}(|Z_1|)\lambda t e^{-\lambda t}.$$

The two combined inequalities prove our claim.

Finally, let X be a Lévy process with Lévy measure ν (with no further assumption).

(5) Prove that $\mathrm{E}(|X(t)|) < \infty$ for all $t \geq 0$ if and only if $\int_{|z|\geq 1} |z|\nu(\mathrm{d}z) < \infty$.

By the Lévy-Itô decomposition, we may decompose $X(t)$ as

$$X(t) = at + \sigma B_t + X_1(t) + X_2(t),$$

where $a, \sigma \in \mathbb{R}$, B is a Brownian motion, X_1 is a Lévy process with bounded jumps (less than one in absolute values) and X_2 is a compound Poisson process whose jump probability measure is

$$\bar{\nu}(\mathrm{d}z) = \frac{\nu(\mathrm{d}z)\mathbf{1}_{\{|z|\geq 1\}}}{\nu((-\infty,-1] \cup [1,\infty))}.$$

From what precedes, $X(t)$ is integrable if and only if $X_2(t)$ is integrable, that is if and only if $\int_{\mathbb{R}} |z|\bar{\nu}(\mathrm{d}z) < \infty$, which is equivalent to the condition $\int_{|z|\geq 1} |z|\nu(\mathrm{d}z) < \infty$.