JUMP PROCESSES - M2 MASEF/MATH. EXAM 18/01/2023 (3H)

No phone and no document is allowed. No indication is given during the exam. You may answer in French or English. The quality of writing, justification and presentation may be taken into account. We follow the convention used in the lecture notes regarding the truncation in the Lévy-Itô decomposition: the set of small *and* large *jumps are* $\{z \in \mathbb{R}^d : |z| \leq 1\}$ *and* $\{z \in \mathbb{R}^d : |z| > 1\}$ *respectively, where* $d \in \mathbb{N}$ *is the dimension and* $|\cdot|$ *is the Euclidian norm. We use* $(\Omega, \mathcal{A}, \mathrm{P})$ *as a generic probability space (equipped with a* σ*-algebra and a probability measure) where all the relevant processes are defined.*

Exercise 1. (Quizz). (See Lecture Notes) Let $d \in \mathbb{N}$ and $\mathcal{N}$ be a random Poisson measure (RPM) with a σ-finite intensity measure m on $E = \mathbb{R}_+ \times \mathbb{R}^d$. Let $f \colon E \to \mathbb{R}$ be (Borel-)measurable.

(1) Give the two properties which characterize $\mathcal{N}$.

(2) Suppose $\int_E |f| \mathrm{d}m < +\infty$. What is the expectation of $\int f \mathrm{d}\mathcal{N}$? (No proof required).

(3) Suppose $\int_E f^2 \mathrm{d}m < +\infty$. What is the variance of $\int f \mathrm{d}\mathcal{N}$? (No proof required).

We henceforth assume that $d = 1$ and $m = \mathrm{d}t \otimes \nu$, where $\mathrm{d}t$ is Lebesgue measure and ν is a Lévy measure on $\mathbb{R}$.

(4) Recall the definition of a Lévy measure on $\mathbb{R}$.

(5) Give an example of a Lévy measure on $\mathbb{R}$ that is *not* a finite measure.

(6) Recall the (one-line formula) *informal* definition of the compensated measure $\widetilde{\mathcal{N}}$.

(7) Give the Lévy-Itô decomposition of a (one-dimensional) Lévy process $X = (X_t)_{t \geq 0}$ with triplet (b, σ^2, ν), where $b \in \mathbb{R}$ and $\sigma^2 \in \mathbb{R}_+$. What can we say about the characteristic function of X_t?

(8) Let $T > 0$ and $F \colon [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a predictable process. Give (without proof) a sufficient integrability condition on F (as seen in the lectures) for the following process:

$$I_t(F) := \int_0^t \int_{\mathbb{R}} F(s, z) \widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z), \qquad 0 \leq t \leq T,$$

to be a well-defined square-integrable centered martingale. Under this condition, what is the variance of $I_t(F)$?

Exercise 2. (Local martingales) Let $\mathcal{N}$ be a random Poisson measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $\mathrm{d}t \otimes \nu$, where ν is a Lévy measure on $\mathbb{R}$. Let $F \colon \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ be a predictable process such that

$$\forall t \geq 0, \quad \int_0^t \int_{\mathbb{R}} F(s, z)^2 \nu(\mathrm{d}z) \mathrm{d}s < \infty, \quad \text{P-a.s.}$$

For every $n \in \mathbb{N}$, we define

$$T_n := \inf \Big\{ t \geq 0 \colon \int_0^t \int_{\mathbb{R}} F(s, z)^2 \nu(\mathrm{d}z) \mathrm{d}s \geq n \Big\}.$$

(1) Assuming that the following process

$$I_t(F) := \int_0^t \int_{\mathbb{R}} F(s,z)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0,$$

is well-defined, prove that it is a *local martingale* w.r.t. the filtration generated by $\mathcal{N}$ (see *reminder* below). From our assumption on F, we get $T_n \to \infty$ as $n \to \infty$. Moreover, $I_{t\wedge T_n}(F) = I_t(F_n)$, if we define $F_n(s,z) := F(s,z)\mathbf{1}_{[0,T_n]}(s)$, which is a predictable process **[details?]**. Since for all $t \geq 0$,

$$\mathrm{E} \int_0^t \int_{\mathbb{R}} F_n(s,z)^2 \nu(\mathrm{d}z)\mathrm{d}s \leq n < +\infty,$$

the process $(I_{t\wedge T_n}(F))_{t\geq 0}$ is a martingale, see Exercise 1, Question (8).

(2) Show that every Lévy process is a *semi-martingale*, that is the sum of a local martingale and a process with finite variation (see *reminder* below). To simplify, let us restrict to one-dimensional processes (the idea is the same for in higher dimensions). Any Lévy process X may be decomposed as

$$X_t = bt + \sigma B_t + \int_0^t \int_{[-1,1]^c} z\mathcal{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{[-1,1]} z\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0,$$

where **[complete...]**. The first and third terms are finite variation processes **[explain]** while the second and the fourth ones are martingales. Hence the result.

(3) Let X be a one-dimensional Lévy process. Using Itô's formula, give a sufficient condition on $f\colon \mathbb{R} \to \mathbb{R}$ for $(f(X_t))_{t\geq 0}$ to be a semi-martingale. Identify the finite variation and the local martingale part. Assume that f is twice differentiable on the real line. Then, by Itô's formula,

$$f(X_t) - f(X_0) = A_t + M_t,$$

where

$$\begin{aligned} A_t &= b\int_0^t f'(X_{s^-})\mathrm{d}s + \tfrac{1}{2}\sigma^2 \int_0^t f''(X_{s^-})\mathrm{d}s \\ &\quad + \int_0^t \int_{[-1,1]^c} (f(X_{s^-} + z) - f(X_{s^-}))\mathcal{N}(\mathrm{d}s, \mathrm{d}z) \\ &\quad + \int_0^t \int_{[-1,1]} (f(X_{s^-} + z) - f(X_{s^-}) - zf'(X_{s^-}))\nu(\mathrm{d}z)\mathrm{d}s \end{aligned}$$

is a finite variation process, while

$$M_t = \sigma \int_0^t f'(X_{s^-})\mathrm{d}B_s + \int_0^t \int_{[-1,1]} (f(X_{s^-} + z) - f(X_{s^-}))\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)$$

is a local martingale.

<u>*Reminder 1:*</u> A real-valued stochastic process $(M_t)_{t\geq 0}$ is said to be a *local martingale* w.r.t. a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ if there exists a non-decreasing sequence of $\mathcal{F}$-stopping times $(T_n)_{n\geq 1}$ such that:

- $\lim_{n\to\infty} T_n = +\infty$, a.s.
- For every $n \geq 1$, $(M_{t\wedge T_n})_{t\geq 0}$ is an $\mathcal{F}$-adapted martingale.

Reminder 2: A real-valued function h is said to have *finite variation* on $\mathbb{R}_+$ if $V(h,a,b)$ is finite for all $0 \leq a \leq b$, where $V(h,a,b) := \sup \sum_k |h(t_k) - h(t_{k-1})|$ and the supremum runs over all subdivisions of the interval $[a,b]$.

Exercise 3. (Change of measure for compound Poisson processes) Let ν be a finite measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and let $q\colon \mathbb{R} \to \mathbb{R}$ be a measurable function. We assume that $q(0) = 0$ and $\int_{\mathbb{R}} \exp(q(z))\nu(\mathrm{d}z) < +\infty$. Let $\mathcal{N}$ be a random Poisson measure with intensity measure $\mathrm{d}t \otimes \nu$ and define for every $t \geq 0$,

$$X_t := \int_0^t \int_{\mathbb{R}} z\mathcal{N}(\mathrm{d}s, \mathrm{d}z), \qquad Y_t := \int_0^t \int_{\mathbb{R}} q(z)\mathcal{N}(\mathrm{d}s, \mathrm{d}z).$$

We denote by $(\mathcal{F}_t)$ the filtration generated by the random Poisson measure.

(1) Preliminary question: if Z is a one-dimensional Lévy process with Lévy triplet $(b, 0, \mu)$, where $b \in \mathbb{R}$ and μ is a *finite* measure on $\mathbb{R}$ satisfying $\int_1^\infty e^z \mu(\mathrm{d}z) < +\infty$, prove that $(\exp(Z_t))_{t\geq 0}$ is a martingale if and only if

$$b + \int_{\mathbb{R}} [e^z - 1 - z\mathbf{1}_{[-1,1]}(z)]\mu(\mathrm{d}z) = 0.$$

In that case Z is the sum of a drift term and a compound Poisson process with jump distribution $\bar{\mu} := \mu/\mu(\mathbb{R})$ and intensity $\mu(\mathbb{R})$. One can then compute the exponential moment of Z_t explicitly and check that $(\exp(Z_t))_{t\geq 0}$ is a martingale if and only if $\mathrm{E}(\exp(Z_t)) = 1$ for all $t \geq 0$, which is equivalent to the condition above. See Lecture Notes.

(2) Prove that $(Y_t)_{t\geq 0}$ is a compound Poisson process and determine its jump measure. Observe that $Y_t = \sum_{0<s\leq t} q(\Delta X_s)$. Thus, Y is a compound process with measure ν_Y defined as $\nu_Y(A) = \nu(\{x \in \mathbb{R}\colon q(x) \in A\})$ for all $A \in \mathcal{B}(\mathbb{R})$, that is

$$\int_{\mathbb{R}} \phi \ \mathrm{d}\nu_Y = \int_{\mathbb{R}} (\phi \circ q) \ \mathrm{d}\nu$$

for all measurable and bounded functions $\phi\colon \mathbb{R} \to \mathbb{R}$.

(3) Determine $\alpha \in \mathbb{R}$ such that $(\exp(Y_t + \alpha t))_{t\geq 0}$ is a martingale. Using Questions (1) and (2), we see that it is a martingale iff

$$\alpha + \int (e^{q(z)} - 1)\nu(\mathrm{d}z) = 0.$$

(4) For the value of α found above, we define a new probability measure Q by

$$\mathrm{Q}(A) := \mathrm{E}\Big(\exp(Y_t + \alpha t)\mathbf{1}_A\Big), \qquad \forall A \in \mathcal{F}_t, \qquad \forall t \geq 0.$$

Determine the law of the process X under the new law Q. Under Q, the process X is a compound Poisson process with intensity measure $\exp(q(z))\nu(\mathrm{d}z)$ on $\mathbb{R}$ (see Lecture Notes).

(5) Application: let $(N_t)_{t\geq 0}$ be a Poisson counting process with rate $\lambda = 1$. What is the almost-sure limit of N_t/t as $t \to \infty$? Give the best possible upper bound for

$$\limsup -\frac{1}{t} \log \mathrm{P}(N_t \leq ut), \qquad t \to \infty, \quad u \in (0,1).$$

Hint: using the previous question, find a probability measure Q (possibly depending on u) for which $\mathrm{Q}(N_t \leq ut)$ converges to one as $t \to \infty$. It follows by a slight adaptation of the Law of the Large Numbers that (N_t/t) converges to one as $t \to \infty$, almost-surely (see Lecture Notes). Write:

$$\mathrm{P}(N_t \leq ut) = \mathrm{E}_Q\Big(1\{N_t \leq ut\}\, \frac{\mathrm{d}P}{\mathrm{d}Q}\Big|_{\mathcal{F}_t}\Big).$$

One should choose Q in such a way that $\mathrm{Q}(N_t \leq ut)$ converges to one as $t \to \infty$. For instance, choose Q such that under Q, (N_t) is a Poisson counting process with rate $u - \varepsilon$, where $\varepsilon \in (0, u)$. To this end, let $Y_t = qN_t$, where $q := \log(u - \varepsilon)$. It follows that

$$\begin{aligned}
\mathrm{P}(N_t \leq ut) &= \mathrm{E}_Q\Big(1\{N_t \leq ut\} e^{t(e^q - 1) - qN_t}\Big) \\
&\geq Q(N_t \leq ut) \exp([e^q - 1 - qu]t) \\
&\geq [1 + o(1)] \exp([u - \varepsilon - 1 - \log(u - \varepsilon)u]t).
\end{aligned}$$

Since ε may be chosen arbitrarily small, we get, as $t \to \infty$,

$$\limsup -\frac{1}{t} \log \mathrm{P}(N_t \leq ut) \leq u \log(u) - u + 1.$$