JUMP PROCESSES - M2 MASEF/MATH. EXAM 8/01/2024 (3H)

Problem: Generator of a Lévy process via the Fourier transform.

Part 1. Let $X = (X_t)_{t\geq 0}$ be a one-dimensional Lévy process with triplet (b, σ^2, ν), where $b \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν a Lévy measure on $\mathbb{R}$. We recall that such a process always starts at the origin, by definition.

(1) Recall the other properties defining a Lévy process (as in the course).

(2) Recall the definition of a Lévy measure on $\mathbb{R}$.

We recall that any Lévy process is a Markov process and we define for all $0 < s \leq t$, $x \in \mathbb{R}$ and bounded measurable $f : \mathbb{R} \to \mathbb{R}$,

$$(T_{s,t}f)(x) = \mathrm{E}(f(X_t)|X_s = x).$$

We admit that the conditioning above is well-defined.

(2) Show that $T_{s,t}$ depends on s and t via the difference $t - s$ only.

$$(T_{s,t}f)(x) = \mathrm{E}(f(X_t)|X_s = x) = \mathrm{E}(f(X_t - X_s + x)|X_s = x).$$

Since $X_t - X_s$ is distributed as X_{t-s} and independent of X_s, we get $(T_{s,t}f)(x) = \mathrm{E}(f(X_{t-s} + x))$.

For the rest of the problem, and due to the time-homogeneity proven above, we consider the linear operator T_t defined by

$$(T_tf)(x) = \mathrm{E}(f(X_t + x))$$

for all $t \geq 0$, $x \in \mathbb{R}$ and bounded measurable $f : \mathbb{R} \to \mathbb{R}$.

(3) Show that $(T_t)_{t\geq 0}$ has the semigroup property, i.e. $T_{t+s} = T_t \circ T_s$ for all $s, t \geq 0$.

$$(T_{t+s}f)(x) = \mathrm{E}[f(X_{t+s} + x)] = \mathrm{E}[\mathrm{E}(f(X_{t+s} - X_t + X_t + x)|X_t)]$$

Since $X_{t+s} - X_t$ is distributed as X_s and independent of X_t, we get

$$\mathrm{E}(f(X_{t+s} - X_t + X_t + x)|X_t) = (T_sf)(X_t + x).$$

Then,

$$(T_{t+s}f)(x) = \mathrm{E}[(T_sf)(X_t + x)] = (T_t(T_sf))(x).$$

Let $\mathcal{C}_0$ be the space of real-valued continuous functions defined on $\mathbb{R}$ and vanishing (i.e. converging to zero) at infinity ($\pm\infty$). Such functions are bounded, so we may equip $\mathcal{C}_0$ with the supremum norm, that is $\|f\|_\infty := \sup\{|f(x)| : x \in \mathbb{R}\}$ for $f \in \mathcal{C}_0$.

(4) Prove that $T_t(\mathcal{C}_0) \subseteq \mathcal{C}_0$. This follows from the Dominated Convergence Theorem.

(5) Prove that $\lim_{t\to 0} \|T_tf - f\|_\infty = 0$ for all $f \in \mathcal{C}_0$. Let $f \in \mathcal{C}_0$ and $\varepsilon > 0$. For all $x \in \mathbb{R}$, $(T_tf - f)(x) = \mathrm{E}[f(X_t + x) - f(x)]$. One can show that f is actually *uniformly*

continuous on the real line. Therefore, there exists $\delta > 0$ such that $|a-b| < \delta$ implies $|f(a) - f(b)| < \varepsilon$. Then,

$$\sup_{x\in\mathbb{R}} |\mathrm{E}[f(X_t + x) - f(X_t)]| \leq \varepsilon + 2\|f\|_\infty \mathrm{P}(|X_t| \geq \delta).$$

It now remains to prove that $\lim_{t\to 0} \mathrm{P}(|X_t| \geq \delta) = 0$. This follows from the fact that X_t a.s. converges to $X_0 := 0$ as $t \to 0$.

(The last two items show that $(T_t)_{t\geq 0}$ is a Feller semigroup.)

Part 2. When f is integrable (w.r.t. Lebesgue measure) we may define its Fourier transform by

$$\hat{f}(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iux} f(x)\mathrm{d}x, \qquad u \in \mathbb{R}.$$

Let $\mathcal{S}$ be the Schwartz space, that is the space of functions $f\colon \mathbb{R} \to \mathbb{R}$ that are infinitely differentiable and such that f and all its derivatives decay faster than any power at infinity. In other words, for every $n \geq 0$ and $k \geq 1$, $\sup\{|x^k f^{(n)}(x)|,\ x \in \mathbb{R}\} < +\infty$ for such functions. We will admit that if $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$ and the Fourier inversion formula holds:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux} \hat{f}(u)\mathrm{d}u.$$

Let η be the Lévy symbol of X.

(6) Recall (without proof) the explicit expression of η when the triplet of X is (b, σ^2, ν).

$$\eta(u) = ibu - \tfrac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \Big(e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}\Big)\nu(\mathrm{d}y)$$

(7) Show that there exists a constant C such that $|\eta(u)| \leq C(1 + u^2)$ for all $u \in \mathbb{R}$. The constant C may depend on the triplet of X. We use $|\cdot|$ to write the modulus of a complex number. The only nontrivial part is the third term. We split it in two:

$$\Big|\int_{[-1,1]} \Big(e^{iuy} - 1)\nu(\mathrm{d}y)\Big| \leq 2\nu([-1,1]) < +\infty$$

and

$$\Big|\int_{[-1,1]^c} \Big(e^{iuy} - 1 - iuy\Big)\nu(\mathrm{d}y)\Big| \leq \tfrac{1}{2}u^2 \int_{[-1,1]^c} y^2\nu(\mathrm{d}y) \leq (\mathrm{Cst})u^2.$$

(8) Show that for all $f \in \mathcal{S}$, $x \in \mathbb{R}$ and $t \geq 0$,

$$(T_t f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux} e^{t\eta(u)} \hat{f}(u)\mathrm{d}u.$$

$$\begin{aligned}(T_t f)(x) = \mathrm{E}(f(X_t + x)) &= \mathrm{E}\Big(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iu(X_t+x)} \hat{f}(u)\mathrm{d}u.\Big) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbb{E}(e^{iuX_t}) e^{iux} \hat{f}(u)\mathrm{d}u.\end{aligned}$$

Since $\mathrm{E}(e^{iuX_t}) = e^{t\eta(u)}$, we may conclude. The interchange of integral and expectation, via the Fubini theorem, can be justified from the fact that the integrand is smaller than $|\hat{f}|$ in modulus and that $\hat{f} \in \mathcal{S} \subseteq L^1(\mathbb{R})$.

Part 3. Let $\mathcal{D}$ be the set of functions $f \in \mathcal{C}_0$ such that $(T_t f - f)/t$ converges (pointwise) to an element of $\mathcal{C}_0$ as $t \to 0$. We call Af this limit and note that A is a linear operator on $\mathcal{D}$ called the generator of X.

(9) Using the results of **Part 2**, show that $\mathcal{S} \subseteq \mathcal{D}$ and that for every $f \in \mathcal{S}$ and $x \in \mathbb{R}$,

$$(Af)(x) = \frac{1}{\sqrt{2\pi}} \int e^{iux} \eta(u) \hat{f}(u) \mathrm{d}u.$$

$$\frac{(T_t f - f)(x)}{t} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux} \Big[\frac{e^{t\eta(u)} - 1}{t}\Big] \hat{f}(u) \mathrm{d}u.$$

By the Intermediate Value Theorem, the term in brackets is less than $|\eta(u)|$ in modulus. Recall Question (8). Since $\int |\eta(u)\hat{f}(u)| \mathrm{d}u \le C \int (1+u^2)|\hat{f}(u)| \mathrm{d}u < +\infty$ (recall that $\hat{f} \in \mathcal{S}$), we may let $t \to 0$ inside the integral and get the result.

(10) Prove that for all $f \in \mathcal{S}$ and $x \in \mathbb{R}$,

$$(Af)(x) = bf'(x) + \frac{1}{2}\sigma^2 f''(x) + \int_{\mathbb{R}} [f(x+y) - f(x) - yf'(x)\ 1_{\{|y| \le 1\}}] \nu(\mathrm{d}y).$$

It is enough to use the explicit expression of η and use the fact that for all $f \in \mathcal{S}$, $f^{(n)}(x) = (2\pi)^{-1/2} \int e^{iux} (iu)^n \hat{f}(u) \mathrm{d}u$, as can be checked via the integration by part formula. The interchange of integrals in u and y may be justified along similar lines as above.

(11) What would be the generator of a d-dimensional Lévy process? You may only give the (precise) result and sketch the argument. With shorthand notations:

$$Af = (b, \nabla)f + \tfrac{1}{2}(\nabla, A\nabla)f + \int_{\mathbb{R}} [f(\cdot + y) - f - (y, \nabla) f 1_{\{|y| \le 1\}}] \nu(\mathrm{d}y),$$

where $(\cdot, \cdot)$ is the usual scalar product in $\mathbb{R}^d$.

(12) **(Bonus question)** Explain how you would derive the generator of a Lévy process from the Itô formula instead of Fourier transform.