JUMP PROCESSES - M2 MASEF/MATH
EXAM 07/01/2020 (3H)

No phone and no document is allowed. No indication is given during the exam. You can answer in French or English. The grading scheme given below is approximative. The quality of writing, justification and presentation is taken into account.

Notation. Open intervals of $\mathbb{R}$ are denoted by (a,b). Closed intervals are denoted by $[a,b]$. The interval $(a,b]$ is open at a and closed at b. Lebesgue measure is denoted by symbols $\mathrm{d}s$, $\mathrm{d}t$ or $\mathrm{d}z$.

Questions (Around 8 points)

(1) Recall the definition of an infinitely divisible (I.D) probability measure on $\mathbb{R}^d$ ($d \in \mathbb{N}$) and give three examples.

(2) Recall the definition of a Lévy process in $\mathbb{R}^d$. Give two examples: one with a.s. continuous sample paths and one which is not continuous. Explain why any marginal of a Lévy process is I.D.

(3) Recall (without proof) the Lévy-Itô decomposition of a Lévy process in $\mathbb{R}^d$. Explain what are the different terms in this decomposition.

(4) Let $X = (X_t)_{t\geq 0}$ be a one-dimensional Lévy process with Lévy triplet $(0,0,\nu)$, where ν is a Lévy measure. We assume moreover that $\Delta X_t > -1$ almost-surely and we recall that $\Delta X_t = X_t - \lim_{s\to t^-} X_s$. Prove that the process $S = (S_t)_{t\geq 0}$ defined by

$$S_t = e^{X_t} \prod_{0<s\leq t} (1+\Delta X_s)e^{-\Delta X_s}, \qquad t \geq 0,$$

solves the stochastic differential equation $\mathrm{d}S_t = S_{t^-}\mathrm{d}X_t$ with initial condition $S_0 = 1$. What is the name given to this solution? Illustrate with one example the fact that in general it does not coincide with the usual exponential.

Exercise 1 (Around 6 points) For all $c > 0$ and $\alpha \in \mathbb{R}$ we define the measure

$$\nu_{c,\alpha}(\mathrm{d}z) = c\, z^{-(1+\alpha)} 1_{(0,+\infty)}(z)\mathrm{d}z.$$

We recall that the Gamma function is defined on $\{\omega \in \mathbb{C} \colon \mathrm{Re}(\omega) > 0\}$ by

$$\Gamma(\omega) = \int_0^\infty x^{\omega-1}e^{-x}\mathrm{d}x.$$

(1) Give a sufficient and necessary condition on α for $\nu_{c,\alpha}$ to be the jump measure of a subordinator. We assume henceforth that this condition is satisfied.

(2) Let $X = (X_t)_{t\geq 0}$ be a subordinator defined by

$$X_t = \int_{(0,t]\times(0,+\infty)} z\mathcal{N}(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0,$$

where $\mathcal{N}$ is a random Poisson measure on $(0,+\infty)^2$ with intensity $\mathrm{d}t \otimes \nu_{c,\alpha}$. Prove that

$$\mathrm{E}[e^{-uX_1}] = \exp(-\kappa u^\alpha) \qquad u \geq 0,$$

for some constant κ (which may depend on (c,α)) that must be determined. Deduce thereof the value of $\mathrm{E}[e^{-uX_t}]$ for all $u \geq 0$ and $t \geq 0$.

(3) Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion and $T_a = \inf\{t \geq 0 \colon B_t = a\}$ for all $a \geq 0$. We admit that T_a is an almost-surely (a.s.) finite stopping time and that the sample paths of the process (T_a) are right-continuous. Prove that (T_a) is a subordinator.

(4) Show that $\mathrm{E}[e^{-uT_a}] = \exp(-a\sqrt{2u})$ $(u \geq 0)$. *Hint: consider the exponential of Brownian motion.*

(5) Deduce thereof the jump measure of (T_a).

Exercise 2. (Around 6 points) Let ν be a Lévy measure supported by the interval $(0,1)$. For all $\varepsilon \in (0,1)$, let

$$\nu_\varepsilon(\mathrm{d}z) = \mathbf{1}_{(\varepsilon,1)}(z)\nu(\mathrm{d}z)$$

be its restriction to $(\varepsilon, 1)$.

(1) Prove that ν_ε is a finite measure for all $\varepsilon \in (0,1)$.

(2) Let $\mathcal{N}_\varepsilon$ be a random Poisson measure on $(0,+\infty) \times (0,1)$ with intensity measure $\mathrm{d}t \otimes \nu_\varepsilon$ and

$$Z_t^{(\varepsilon)} = \int_{(0,t]\times(0,1)} z\mathcal{N}_\varepsilon(\mathrm{d}s,\mathrm{d}z), \qquad t \geq 0.$$

What kind of process is $Z^{(\varepsilon)} = (Z_t^{(\varepsilon)})_{t\geq 0}$?

(3) Let us define

$$\bar{Z}_t^{(\varepsilon)} = Z_t^{(\varepsilon)} - tc(\varepsilon), \qquad t \geq 0.$$

Find $c(\varepsilon)$ such that $\bar{Z}_t^{(\varepsilon)}$ is a square-integrable centered martingale (for the canonical filtration). Justify your choice. Does $c(\varepsilon)$ converge or diverge as ε tends to zero?

(4) Let $t \geq 0$. Prove that $\bar{Z}_t^{(\varepsilon)}$ converges in L^2 (as $\varepsilon \to 0$) to some random variable which we will denote by $\bar{Z}_t$.

(5) Prove that $(\bar{Z}_t)_{t\geq 0}$ is a Lévy process.