JUMP PROCESSES - M2 MASEF/MATH
EXAM 13/01/2020 (3H)

No phone and no document is allowed. No indication is given during the exam. You can answer in French or English. The grading scheme given below is approximative. The quality of writing, justification and presentation is taken into account.

Notation. Open intervals of $\mathbb{R}$ are denoted by (a, b). Closed intervals are denoted by $[a, b]$. The interval $(a, b]$ is open at a and closed at b. Lebesgue measure is denoted by symbols $\mathrm{d}s$, $\mathrm{d}t$ or $\mathrm{d}z$. All Lévy processes below are one-dimensional and start from zero. The small jumps and large jumps in the Lévy-Itô decomposition are chosen as the sets $\{z \in \mathbb{R} \colon |z| < 1\}$ and $\{z \in \mathbb{R} \colon |z| \geq 1\}$, respectively. If $\{x(t)\}_{t \geq 0}$ is a càd-làg function, then the value of the jump at $t > 0$ is denoted by $\Delta x(t) := x(t) - x(t^-)$. The minimum between two real numbers a and b is denoted by $a \wedge b$.

Questions

(1) What is a Lévy process? Give one example with continuous sample paths and one example that is only made of jumps.

(2) What is a Lévy measure? Give one example of a Lévy measure on the real line that is not a finite measure.

(3) What is an infinitely divisible probability distribution (on the real line)? What is the connection with Lévy processes?

(4) What is a subordinator? Give one example.

(5) Let $\alpha \in (0, 2)$ and $\mathcal{N}$ be a random Poisson measure on $(0, \infty)^2$ with intensity measure $\mathrm{d}t \otimes \nu(\mathrm{d}z)$, where

$$\nu(\mathrm{d}z) = z^{-(1+\alpha)}\mathrm{d}z \qquad (z > 0).$$

For which values of $\beta \in \mathbb{R}$ is the process

$$X(t) = \int_{(0,t] \times (0,1)} z^{\beta} \widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z), \qquad t \geq 0$$

well-defined as a centered and square-integrable càd-làg martingale?

(6) Let P be a probability distribution under which $(N_t)_{0 \leq t \leq 1}$ is a Poisson counting process with intensity one. Identify the law of $(N_t)_{0 \leq t \leq 1}$ under the probability distribution $\widetilde{\mathrm{P}}$, which is defined by

$$\frac{\mathrm{d}\widetilde{\mathrm{P}}}{\mathrm{d}\mathrm{P}} = e^{-1} 2^{N_1}.$$

Exercise 1. Solution of a stochastic differential equation (around 6 points). Let X be a Lévy process with Lévy measure ν and triplet $(0, 0, \nu)$. We assume that $\nu(-\infty, -1] = 0$. Let $h(x) = x - \log(1 + x)$ for $x \in \mathbb{R}$. We define the process $Y(t) = X(t) - \sum_{0 \leq s \leq t} h(\Delta X(s))$ for $t \geq 0$.

(1) Explain why Y is well defined and write it as a Lévy-type stochastic integral.

(2) Apply the Lévy-Itô formula and compute the stochastic differential of $f(Y(t))$, where $f \in \mathcal{C}^2(\mathbb{R})$, the space of twice continuously differentiable functions on the real line.

(3) Prove that the process $\{\exp(Y(t))\}_{t\geq 0}$ solves the stochastic differential equation $\mathrm{d}S(t) = S(t^-)\mathrm{d}X(t)$. What is the name usually given to this solution?

(4) Find a simple expression for this solution when ν is the Dirac mass at $u > -1$.

Exercise 2. Integrability of a Lévy process (around 6 points). We consider a Lévy process X with bounded jumps, meaning that there exists $C > 0$ such that (for all realizations of the process) $|\Delta X(t)| \leq C$ for all $t \geq 0$. We define the sequence of stopping times $(T_n)_{n\geq 0}$ by $T_0 = 0$ and

$$T_n = \inf\{t > T_{n-1} \colon |X(t) - X(T_{n-1})| > C\} \qquad (n \geq 1).$$

(The candidate is not required to prove that those are stopping times.)

(1) Prove that $|X(t \wedge T_n)| \leq 2nC$ for all $n \geq 1$ and $t \geq 0$.

(2) Prove that $\mathrm{P}(|X(t)| \geq 2nC) \leq e^t \mathrm{E}(e^{-T_1})^n$ for all $n \geq 1$ and $t \geq 0$.

(3) Prove that $\mathrm{E}(|X(t)|) < \infty$ for all $t \geq 0$.

We now assume X is a compound Poisson process with jump probability measure ν.

(4) Prove that $\mathrm{E}(|X(t)|) < \infty$ for all $t \geq 0$ if and only if $\int_{\mathbb{R}} |z|\nu(\mathrm{d}z) < \infty$.

Finally, let X be a Lévy process with Lévy measure ν (with no further assumption).

(5) Prove that $\mathrm{E}(|X(t)|) < \infty$ for all $t \geq 0$ if and only if $\int_{|z|\geq 1} |z|\nu(\mathrm{d}z) < \infty$.