JUMP PROCESSES - M2 MASEF/MATH. EXAM 8/01/2024 (3H)

Documents, electronic devices and red ink are not allowed. No indication is given during the exam. You may answer in French or English. The quality of writing, justification and presentation may be taken into account. Please label your answers clearly. We follow the convention used in the lecture notes regarding the truncation in the Lévy-Itô decomposition: the set of small *and* large *jumps are* $\{z \in \mathbb{R}^d \colon |z| \leq 1\}$ *and* $\{z \in \mathbb{R}^d \colon |z| > 1\}$ *respectively, where* $d \in \mathbb{N}$ *is the dimension and* $|\cdot|$ *is the Euclidian norm. We use* $(\Omega, \mathcal{A}, \mathrm{P})$ *as a probability space (equipped with a* σ*-algebra and a probability measure) where all the relevant random variables are defined.*

Quizz. Short and precise answers expected.

(1) What is a subordinator? Give one example with continuous sample paths and one example with discontinuous sample paths.

(2) What is a compound Poisson process?

(3) What is the Lévy-Itô decomposition of a *one-dimensional* Lévy process?

(4) What is an infinitely divisible distribution? Give two examples that are different from the Dirac distribution.

Problem: Generator of a Lévy process via the Fourier transform.

Part 1. Let $X = (X_t)_{t\geq 0}$ be a *one-dimensional* Lévy process with triplet (b, σ^2, ν), where $b \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν a Lévy measure on $\mathbb{R}$. We recall that such a process always starts at the origin, by definition.

(1) Recall the other properties defining a Lévy process (as in the course).

(2) Recall the definition of a Lévy measure on $\mathbb{R}$.

We recall that any Lévy process is a Markov process and we define for all $0 < s \leq t$, $x \in \mathbb{R}$ and bounded measurable $f \colon \mathbb{R} \to \mathbb{R}$,

$$(T_{s,t}f)(x) = \mathrm{E}(f(X_t)|X_s = x).$$

We admit that the conditioning above is well-defined.

(3) Show that $T_{s,t}$ depends on s and t via the difference $t - s$ only.

For the rest of the problem, and due to the time-homogeneity proven above, we consider the linear operator T_t defined by

$$(T_t f)(x) = \mathrm{E}(f(X_t + x))$$

for all $t \geq 0$, $x \in \mathbb{R}$ and bounded measurable $f \colon \mathbb{R} \to \mathbb{R}$.

(4) Show that $(T_t)_{t\geq 0}$ has the semigroup property, i.e. $T_{t+s} = T_t \circ T_s$ for all $s, t \geq 0$.

Let $\mathcal{C}_0$ be the space of real-valued *continuous* functions defined on $\mathbb{R}$ and *vanishing* (i.e. converging to zero) at infinity ($\pm\infty$). Such functions are bounded, so we may equip $\mathcal{C}_0$ with the supremum norm, that is $\|f\|_\infty := \sup\{|f(x)| \colon x \in \mathbb{R}\}$ for $f \in \mathcal{C}_0$.

(4) Prove that $T_t(\mathcal{C}_0) \subseteq \mathcal{C}_0$.

(5) Prove that $\lim_{t\to 0} \|T_t f - f\|_\infty = 0$ for all $f \in \mathcal{C}_0$.

(The last two items establish that $(T_t)_{t\geq 0}$ is a Feller semigroup.)

Part 2. When f is integrable (on the real line and w.r.t. Lebesgue measure) we may define its Fourier transform by

$$\hat{f}(u) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iux} f(x)\mathrm{d}x, \qquad u \in \mathbb{R}.$$

Let $\mathcal{S}$ be the Schwartz space, that is the space of functions $f \colon \mathbb{R} \to \mathbb{R}$ that are infinitely differentiable and such that f and all its derivatives decay faster than any positive power at infinity. In other words, for every $n \geq 0$ and $k \geq 1$, $\sup\{|x^k f^{(n)}(x)|,\ x \in \mathbb{R}\} < +\infty$ for such functions. We will admit that if $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$ and the Fourier inversion formula holds:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux} \hat{f}(u)\mathrm{d}u, \qquad x \in \mathbb{R}.$$

Let $\eta \colon \mathbb{R} \to \mathbb{C}$ be the Lévy symbol of X.

(7) Recall (without proof) the explicit expression of η when the triplet of X is (b, σ^2, ν).

(8) Show that there exists a constant C such that $|\eta(u)| \leq C(1 + u^2)$ for all $u \in \mathbb{R}$. The constant C may depend on the triplet of X. We use $|\cdot|$ to write the modulus of a complex number.

(9) Show that for all $f \in \mathcal{S}$, $x \in \mathbb{R}$ and $t \geq 0$,

$$(T_t f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iux} e^{t\eta(u)} \hat{f}(u)\mathrm{d}u.$$

Part 3. Let $\mathcal{D}$ be the set of functions $f \in \mathcal{C}_0$ such that $(T_t f - f)/t$ converges (*pointwise*) to an element of $\mathcal{C}_0$ as $t \to 0$ (from the right). We call Af this limit and note that A is a linear operator on $\mathcal{D}$ called the generator of X.

(10) Using the results of **Part 2**, show that $\mathcal{S} \subseteq \mathcal{D}$ and that for every $f \in \mathcal{S}$ and $x \in \mathbb{R}$,

$$(Af)(x) = \frac{1}{\sqrt{2\pi}} \int e^{iux} \eta(u) \hat{f}(u)\mathrm{d}u.$$

(11) Prove that for all $f \in \mathcal{S}$ and $x \in \mathbb{R}$,

$$(Af)(x) = bf'(x) + \tfrac{1}{2}\sigma^2 f''(x) + \int_{\mathbb{R}} [f(x+y) - f(x) - yf'(x)\ 1_{\{|y|\leq 1\}}]\nu(\mathrm{d}y).$$

(12) What would be the generator of a Lévy process in any dimension $d \in \mathbb{N}$? You may state the (precise) result and only sketch the argument.

(13) **(Bonus question)** Explain how you would find the generator of a one-dimensional Lévy process from Itô's formula instead of the Fourier transform. Hint: start by applying Itô's formula to the process $(f(X_t))_{t\geq 0}$.