LECTURE NOTES ON JUMP PROCESSES

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Contents

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These lecture notes were written for the students of the MATH and MASEF master programs at Université Paris-Dauphine. They contain an introductory course on the topic of *jump* processes, by which we mean (possibly multi-dimensional) real-valued stochastic processes with càdlàg (right-continuous with left-limits*) sample paths. In short, this is an extension of the stochastic calculus course (stochastic integration with respect to Brownian motion and Itô processes that have continuous sample paths) to the world of *jump processes*, or at least a particular class of them. Such processes appear in several areas of mathematical modeling and arise as the scaling limits of *heavy-tailed* random walks, in the same way as Brownian motion arises as the scaling limit of random walks with finite variance step distributions (see Section 4.5 on this matter). Section 1 is devoted to compound Poisson processes, that are one of the most elementary (yet satisfying key properties) jump processes one can start with. Those are extended in Section 2 by considering more general random Poisson measures, leading to the key notion of *compensated* Poisson measures. The other important notion of *infinitely divisible* distributions is covered in Section 3. Then, Section 4 wraps up Brownian motion and the (compensated/compound) Poisson processes considered in Sections 1 and 2 into the more general class of Lévy processes. From Section 5 to Section 7, the structure of the course parallels that of a standard course on stochastic calculus. Finally, an application to option pricing is given in Section 8 while a brief and informal excursion outside of the realm of Markov processes is provided in Section 9.

Notation:

- x being positive (negative) means that $x > 0$ ($x > 0$);
- x being non-negative (non-positive) means that $x \geq 0$ ($x \leq 0$);
- $f\colon \mathbb{R} \mapsto \mathbb{R}$ is increasing (decreasing) if $x > y$ implies $f(x) > f(y)$ ($f(x) < f(y)$).
- $f\colon \mathbb{R} \mapsto \mathbb{R}$ is non-decreasing (non-increasing) if $x > y$ implies $f(x) \geq f(y)$ ($f(x) \leq f(y)$).
- $\mathbb{N}$ is the set of positive integers $\{1, 2, \ldots\}$;
- $\mathbb{N}_0$ is the set of non-negative integers $\{0, 1, 2, \ldots\}$;
- $a \wedge b = \min(a, b)$;
- $a \vee b = \max(a, b)$;
- unless stated otherwise, $\mathrm{d}u$, $\mathrm{d}s$ and $\mathrm{d}t$ are notations for Lebesgue measure on $\mathbb{R}$ (time) and $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ are notations for Lebesgue measure on $\mathbb{R}^d$ (space);
- the symbol $|\cdot|$ stands for absolute value, complex modulus or Euclidean norm according to context.

* *continue à droite, limite à gauche* in French

- $\mathcal{M}(E)$ and $\mathcal{M}_1(E)$ are the sets of σ-finite and probability measures on E, respectively;
- unless stated otherwise, $B = (B_t)$ is a standard Brownian motion.

1. Poisson processes and compound Poisson processes

1.1. **Poisson processes.** Let $(T_n)_{n\in\mathbb{N}}$ be an increasing sequence of random variables on the positive half-line $(0,\infty)$ (the timeline). The counting process $N = (N_t)_{t\geq 0}$ associated to the sequence (T_n) is the $\mathbb{N}_0$-valued process defined by

$$N_t = \sum_{n\geq 1} 1_{\{T_n \leq t\}} = \operatorname{card}\{n \geq 1 \colon T_n \leq t\}. \tag{1.1}$$

Let $T_0 := 0$ and denote by $\Delta T_n = T_n - T_{n-1}$ (for $n \in \mathbb{N}$) the increments of (T_n) (we shall keep this notation for the increments of discrete-time processes). The increments are also called *inter-arrival times.*

These counting processes are our first basic examples of stochastic processes that increase by jumps. Among theses processes, one class is of particular interest.

Definition 1.1 (Poisson counting process on the half-line)**.** *If the increments $(\Delta T_n)_{n\in\mathbb{N}}$ are independent and identically distributed (i.i.d.) with common law $\mathcal{E}(\lambda)$ (the exponential law with parameter $\lambda > 0$) then the associated counting process is called Poisson counting process with intensity λ.*

The next exercise explains why it is called a *Poisson* process.

Exercise 1. *Let (T_i) be the sequence associated to a Poisson counting process (N_t) with intensity λ. Let $n \in \mathbb{N}$.*

1. *Find the joint density of $(T_1, \dots, T_n)$.*
2. *Show that T_n is distributed as the $\Gamma(n,\lambda)$ law (Gamma distribution).*
3. *Prove that for all $t \geq 0$, N_t is distributed as the $\mathcal{P}(\lambda t)$ law (Poisson distribution).*
4. *Prove that, conditionally on $\{N_t = n\}$, the random vector $(T_1, \dots, T_n)$ follows the order-statistics of n i.i.d. random variables uniformly distributed on $[0,t]$.*
5. *Show that $\mathrm{P}(N_t < \infty, \forall t \geq 0) = 1$.*

Exercise 2 (Scaling property)**.** *Check that if (N_t) is a Poisson counting process with intensity λ then for all $c > 0$ the process (N_{ct}) is a Poisson counting process with intensity $c\lambda$.*

Here are some important properties of the Poisson counting process.

Proposition 1.1. *Let $N = (N_t)$ be a Poisson point process with intensity λ.*

1. *N has càdlàg trajectories (right-continuous with left limits) with $N_0 = 0$.*
2. *N has independent increments: for all $s,t \geq 0$, $N_{t+s} - N_t$ is independent of $\sigma(N_u,\ 0 \leq u \leq t)$.*
3. *N has stationary increments: for all $s,t \geq 0$, $N_{t+s} - N_t$ has the same law as N_s.*

Remark 1.1. *In this case, the collection of jump times $\{T_n\}_{n\in\mathbb{N}}$ (seen as a random collection of points on the positive half-line) is distributed as a Poisson point process with intensity λ, see Exercice 5 below.*

We will come back to stochastic processes with independent and stationary increments in Section 4 (Lévy processes), from a larger perspective.

For all $t > 0$, we define

$$\Delta N_t = N_t - \lim_{s \to t^-} N_s \in \{0, 1\}. \tag{1.2}$$

We shall keep this notation for continuous-time processes. There is a jump at time t if $\Delta N_t = 1$ and no jump otherwise.

Exercise 3. *Check that* $\mathrm{P}(\Delta N_t = 0) = 1$ *for all* $t > 0$*, whereas* $\mathrm{P}(\Delta N_t = 0, \forall t > 0) = 0$.

We end this section with the Law of Large Numbers and the Central Limit Theorem.

Theorem 1.1 (Asymptotic properties)**.** *Let* $N = (N_t)$ *be a Poisson counting process with intensity* λ*. The following statements hold true:*

(1) *The process* $(\frac{N_t}{t})$ *converges a.s. to* λ *as* $t \to \infty$*.*
(2) *The process* $\sqrt{\frac{t}{\lambda}}(\frac{N_t}{t} - \lambda)$ *converges in law to* $\mathcal{N}(0,1)$ *as* $t \to \infty$*.*

Proof of Theorem 1.1. To prove the first statement, we first show thanks to Proposition 1.1 and the (strong) Law of Large Numbers that $(\frac{N_n}{n})_{n \in \mathbb{N}}$ converges a.s. to λ and then use that $N_{\lfloor t \rfloor} \le N_t \le N_{\lfloor t \rfloor + 1}$. The second statement may be proven by convergence of characteristic functions. ∎

1.2. **Compound Poisson processes.** The Poisson counting process that was introduced in the last section makes jumps of size one only. In this section we generalize the idea by allowing random $\mathbb{R}^d$-valued jump sizes.

Definition 1.2 (Compound Poisson process)**.** *Let* $N = (N_t)$ *be a Poisson counting process with intensity* λ *and* $(T_n)_{n \in \mathbb{N}}$ *the associated sequence of jump times. Let* $\nu \in \mathcal{M}_1(\mathbb{R}^d)$ *(the space of probability measures on* $\mathbb{R}^d$*) and* $(Z_n)_{n \in \mathbb{N}}$ *be a sequence of i.i.d. random variables with common law* ν*, independent from* N*. The process* $X = (X_t)_{t \ge 0}$ *defined by*

$$X_t = \sum_{1 \le n \le N_t} Z_n = \sum_{n \ge 1} Z_n \mathbf{1}_{\{T_n \le t\}} \tag{1.3}$$

is called a (d-dimensional) compound Poisson process with intensity λ *and jump distribution* ν*. The law of such process shall be denoted by* $\mathrm{CPP}(\lambda, \nu)$*.*

Example 1.1 (Continuous-time random walk on $\mathbb{Z}$)**.** *Suppose* $\mathrm{P}(Z_1 = 1) = \mathrm{P}(Z_1 = 1) = 1/2$ *and define*

$$S_0 = 0, \qquad S_n = Z_1 + \ldots + Z_n, \quad n \in \mathbb{N}. \tag{1.4}$$

Then $S = (S_n)$ *is a discrete-time simple random walk on* $\mathbb{Z}$ *whereas* $(X_t) = (S(N_t))$ *is its continuous-time counterpart.*

Proposition 1.2. *A compound Poisson process has càdlàg trajectories with independent and stationary increments.*

Exercise 4 (Characteristic functions)**.** *Let* $X = (X_t)$ *be a compound Poisson process as in Definition 1.2. Prove that for any* $t > 0$ *and* $u \in \mathbb{R}^d$,

$$\phi_{X_t}(u) := \mathrm{E}\Big[e^{i\langle u, X_t \rangle}\Big] = \exp(\lambda(\phi_\nu(u) - 1)t), \tag{1.5}$$

where

$$\phi_\nu(u) := \int_{\mathbb{R}^d} e^{i\langle u,z\rangle}\nu(\mathrm{d}z). \tag{1.6}$$

Compute the characteristic function of the recentered variable $\bar{X}_t := X_t - \mathrm{E}(X_t)$.

We shall see in Section 4 that similar formulas hold for the more general class of Lévy processes. The exponent $\lambda(\phi_\nu(u)-1)$ sitting in (1.5) will be referred to as a Lévy exponent.

2. Integration with respect to random Poisson measures

Let us give another viewpoint on the Poisson processes defined in the previous section. Let (T_n) be the sequence of jump times associated to a Poisson counting measure. We may think of this sequence as a random subset of points in $\mathbb{R}^+$ and, if a Dirac mass is attached to each of these points, we get an (infinite) *random measure* on $\mathbb{R}^+$. Then, N_t is nothing else but the integral over $[0,t]$ of the constant one (the jump size) with respect to this measure (see Exercise 5). A similar observation can be made for compound Poisson processes if one replaces the point measure on $\mathbb{R}^+$ by a point measure on $\mathbb{R}^+ \times \mathbb{R}^d$ in order to keep track of the values of the jumps (see Exercise 8). This viewpoint may seem quite artificial at first but it will allow us to go beyond the elementary processes of the first section. In particular, the compound Poisson processes of Section 1 have finitely many jumps in any bounded set of the positive half-line. In what follows we will allow accumulation of infinitely many small jumps.

2.1. **Random measures.** Let $E = \mathbb{R}^+ \times \mathbb{R}^d$ (product of time and space) be equipped with its Borel σ-algebra $\mathcal{B}(E)$. Let $\mathcal{M}(E)$ be the set of (non-negative) σ-finite measures on E. We equip $\mathcal{M}(E)$ with the smallest σ-algebra that makes all mappings

$$\varphi_B\colon m \in \mathcal{M}(E) \mapsto m(B) \qquad (B \in \mathcal{B}(E)) \tag{2.1}$$

measurable.

Definition 2.1 (Random measure). *A random measure is a $\mathcal{M}(E)$-valued random variable (with respect to the σ-algebra defined above).*

Proposition 2.1. *$\mathcal{N}\colon \Omega \mapsto \mathcal{M}(E)$ is a random measure iff for all $B \in \mathcal{B}(E)$, $\varphi_B \circ \mathcal{N} =: \mathcal{N}(B)$ is a real-valued non-negative random variable.*

Proposition 2.2 (Independence). *Let $(\mathcal{N}_n)_{n\in\mathbb{N}}$ be a sequence of random measures. They are independent iff for all $m \in \mathbb{N}$, for all Borel sets $B_{n,1},\dots,B_{n,m}$, the random vectors*

$$(\mathcal{N}_n(B_{n,1}),\dots,\mathcal{N}_n(B_{n,m})), \qquad n \in \mathbb{N}, \tag{2.2}$$

are independent.

Remark 2.1. *One can check the Borel sets $B_{n,1},\dots,B_{n,m}$ (for a given $n \in \mathbb{N}$) may be chosen disjoint.*

2.2. **Random Poisson measures.**

Definition 2.2 (Random Poisson measure). *Let $m \in \mathcal{M}(E)$. A random measure $\mathcal{N}$ is called a Poisson measure random measure with intensity measure m and denoted by* $\mathrm{RPM}(m)$ *if the two following conditions hold:*

(1) *For all $B \in \mathcal{B}(E)$, $\mathcal{N}(B)$ is a Poisson random variable with parameter $m(B)$.*

(2) *For all $k \in \mathbb{N}$ and all Borel sets $B_1, \dots, B_k$ that are pairwise disjoint, the random variables $\mathcal{N}(B_1), \dots, \mathcal{N}(B_k)$ are independent.*

Remark 2.2. *By convention, $\mathcal{N}(B) = 0$ a.s. if $m(B) = 0$ and $\mathcal{N}(B) = +\infty$ a.s. if $m(B) = +\infty$.*

Exercise 5. *Let $(T_n)_{n\in\mathbb{N}}$ be the random sequence of jump times associated to a Poisson counting measure $N = (N_t)$ with intensity $\lambda > 0$. Define the random measure*

$$\mathcal{N} = \sum_{n\in\mathbb{N}} \delta_{T_n}, \tag{2.3}$$

where δ_x is a notation for the Dirac mass at x. Check that $N_t = \mathcal{N}((0,t])$ for all $t > 0$. Check that $\mathcal{N}$ is a RPM *on $\mathbb{R}_+$ and identify its intensity measure.*

Exercise 6 (Same exercise with compound Poisson processes)**.** *Let $(T_n, Z_n)_{n\in\mathbb{N}}$ be the random sequence of jump times and jumps sizes associated to a real-valued ($d = 1$) compound Poisson counting process $X = (X_t)$ with intensity $\lambda > 0$ and jump distribution $\nu \in \mathcal{M}_1(\mathbb{R})$. Define the random measure*

$$\mathcal{N} = \sum_{n\in\mathbb{N}} \delta_{(T_n,Z_n)}, \tag{2.4}$$

Check that $\mathcal{N}$ is a RPM *and identify its intensity measure.*

Let us prove the existence of RPM's (in general):

- If $m(E) = 0$, we simply set $\mathcal{N} = 0$ (null measure).
- If $m(E) \in (0, +\infty)$, we consider a sequence of i.i.d. random variables $(X_n)_{n\in\mathbb{N}}$ with common law $\bar{m} := m/m(E)$ and an independent random variable N with law $\mathcal{P}(m(E))$. Then, one can check that the random measure
$$\mathcal{N} := \sum_{1\le i\le N} \delta_{X_i} \tag{2.5}$$
is a RPM(m) (same idea as in Exercises 5 and 6).
- If $m(E) = +\infty$, we let $(E_n)_{n\in\mathbb{N}}$ be a countable partition of E such that $m(E_n) < +\infty$. Such a partition exists because m is σ-finite. Using the previous case, we may now construct $(\mathcal{N}_n)_{n\in\mathbb{N}}$ a sequence of independent RPM with respective intensity measures $m_n := m(\cdot \cap E_n)$. By the superposition property (see Proposition 2.3) the sum of these measures
$$\mathcal{N} := \sum_{n\in\mathbb{N}} \mathcal{N}_n \tag{2.6}$$
is a RPM(m).

Random Poisson measures have two convenient properties:

Proposition 2.3 (Superposition property)**.** *Let $(\mathcal{N}_n)$ be a sequence of independent* RPM*'s with respective intensity measures (m_n). If $m := \sum_{n\ge1} m_n$ is σ-finite then $\mathcal{N} := \sum_{n\ge1} \mathcal{N}_n$ is a* RPM(m).

Proposition 2.4 (Thinning property)**.** *Let (E_n) be a countable partition of E and $\mathcal{N}$ be a* RPM(m)*. For all $n \ge 1$, define the restricted measure $\mathcal{N}_n := \mathcal{N}(\cdot \cap E_n)$. The $\mathcal{N}_n$'s are independent* RPM*'s with respective intensity measures $m_n := m(\cdot \cap E_n)$.*

2.3. **Integration.** Let $\mathcal{N}$ be a random measure on E. For all $\omega \in \Omega$ (the underlying probability space), we may define $\int_E f \mathrm{d}\mathcal{N}_\omega$ (or $\int_E f(x)\mathcal{N}_\omega(\mathrm{d}x)$) as the integral of a suitable function $f \colon E \to \mathbb{R}$ with respect to the mesure $\mathcal{N}_\omega$. This gives rise to a random variable $\int_E f \mathrm{d}\mathcal{N}$ (by the way, why is it a random variable?).

Exercise 7. *Find an alternative expression (at least formally) of the integral $\int_E f \mathrm{d}\mathcal{N}$ when $\mathcal{N}$ is a* RPM(m) *and m is a finite measure. Hint: use the explicit construction given in Section 2.2.*

In the following we give some properties of the integral when $\mathcal{N}$ is a random Poisson measure.

Proposition 2.5. *Let $\mathcal{N}$ be a* RPM(m)*.*

(i: Expectation) If $f \in L^1(E,m)$ then $\int f \mathrm{d}\mathcal{N}$ is integrable and

$$\mathrm{E}\Big(\int f \mathrm{d}\mathcal{N}\Big) = \int f \mathrm{d}m. \tag{2.7}$$

(ii: Variance) If $f \in L^2(E,m)$ then $\int f \mathrm{d}\mathcal{N}$ is square-integrable and

$$\mathrm{Var}\left(\int f \mathrm{d}\mathcal{N}\right) = \int f^2 \mathrm{d}m. \tag{2.8}$$

(iii: Laplace transform) If $f \colon E \to \mathbb{R}^+$ is measurable then

$$\mathrm{E}\Big(e^{-\int f \mathrm{d}\mathcal{N}}\Big) = \exp\Big(\int (e^{-f} - 1)\mathrm{d}m\Big). \tag{2.9}$$

(iv: Characteristic function) If $f \in L^1(E,m)$ then

$$\mathrm{E}\Big(e^{i\int f \mathrm{d}\mathcal{N}}\Big) = \exp\Big(\int (e^{if} - 1)\mathrm{d}m\Big). \tag{2.10}$$

Exercise 8 (Connection with compound Poisson processes)**.** *Let $X = (X_t)_{t\geq 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump distribution $\nu \in \mathcal{M}_1(\mathbb{R}^d)$. Show that we may write*

$$X_t = \int_{(0,t]\times\mathbb{R}^d} f \mathrm{d}\mathcal{N} \tag{2.11}$$

for an appropriate function f and a random Poisson measure $\mathcal{N}$ to identify. Check that we can replace the condition $\nu(\mathbb{R}^d) = 1$ by $\nu(\mathbb{R}^d) < \infty$.

Proposition 2.6. *Let $f \colon E \to \mathbb{R}_+$ be measurable. The three following statements are equivalent:*

(1) $\int f \mathrm{d}\mathcal{N} < +\infty$ *a.s.*,
(2) $\int (1 \wedge f)\mathrm{d}m < +\infty$,
(3) $\int (1 - e^{-f})\mathrm{d}m < +\infty$.

Exercise 9 (Continuation of Exercise 8)**.** *Show that we may define a compound Poisson process with jump measure (instead of jump distribution) ν satisfying $\int_{\mathbb{R}^d}(1\wedge|z|)\nu(\mathrm{d}z) < \infty$.*

Exercise 9 shows that it is possible to define a process which has infinitely many (small) jumps over bounded time intervals, provided that the *small jumps* are *small enough* (in the sense of the condition in Exercise 9). We will come back to this dichotomy between *small* and *large* jumps in Theorem 3.1 and Section 4.

2.4. **Compensated Poisson measures.** In what follows, we introduce a recentered version of the integral $\int f \mathrm{d}\mathcal{N}$, which will allow us to weaken Condition (2) in Proposition 2.6 thanks to an approximation argument.

Let $m \in \mathcal{M}(E)$. Since m is σ-finite, there exists $(E_n)_{n\geq 1}$ a countable partition of E such that $m(E_n) < \infty$ for all $n \geq 1$. Let $f\colon E \to \mathbb{R}$ be a measurable function such that $f \in L^1(E_n, m)$ for all $n \in \mathbb{N}$. Define

$$\int_{E_n} f \mathrm{d}(\mathcal{N} - m) := \int_{E_n} f \mathrm{d}\mathcal{N} - \int_{E_n} f \mathrm{d}m, \tag{2.12}$$

and

$$I_n := \sum_{1\leq k\leq n} \int_{E_k} f \mathrm{d}(\mathcal{N} - m). \tag{2.13}$$

Proposition 2.7. *Suppose $\int(|f| \wedge f^2)\mathrm{d}m < \infty$. The random sequence (I_n) converges a.s. and in L^2 to a limit which does not depend on the choice of the partition.*

Definition 2.3. *The limit above is denoted by $\int_E f \mathrm{d}\widetilde{\mathcal{N}}$, where $\widetilde{\mathcal{N}} := \mathcal{N} - m$ is called compensated Poisson measure.*

Again, let us slightly anticipate on Section 4 by noting that Proposition 2.7 will allow us to weaken the condition of Exercise 9 on the jump measure and consider even more general jump processes.

The functions we have integrated so far are *deterministic*. We will treat the case of *random* functions in Section 5 on stochastic integration.

3. Infinite divisibility

3.1. **Infinitely divisible probability distributions.** If μ_1 and μ_2 are two probability distributions on $\mathbb{R}^d$ then the convolution of μ_1 and μ_2, denoted by $\mu_1 * \mu_2$, is defined by

$$(\mu_1 * \mu_2)(A) = (\mu_1 \otimes \mu_2)(\{(x_1, x_2)\colon x_1 + x_2 \in A\}), \qquad A \in \mathcal{B}(\mathbb{R}^d). \tag{3.1}$$

In other words, $\mu_1 * \mu_2$ is the law of the sum of two independent random variables respectively distributed as μ_1 and μ_2. The convolution operation is associative and commutative. We denote by μ^{*n} the n-th fold convolution of μ with itself.

Definition 3.1. *A probability distribution $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is* infinitely divisible *(I.D.) if for all $n \geq 1$ there exists $\mu_n \in \mathcal{M}_1(\mathbb{R}^d)$ such that $\mu = \mu_n^{*n}$.*

Definition 3.2. *An $\mathbb{R}^d$-valued random variable Y is* infinitely divisible *(I.D.) if for all $n \geq 1$ there exists a collection of i.i.d. random variables $Y_{1,n}, \dots, Y_{n,n}$ such that $Y = Y_{1,n} + \ldots + Y_{n,n}$ (in law).*

One can check that the two definitions are consistent with each other in the sense that a random variable is infinitely divisible iff its law is infinitely divisible.

Exercise 10. *Prove that the following probability distributions (or random variables) are I.D.*

(1) $\mathcal{N}(m, \sigma^2)$;
(2) $\mathcal{P}(\lambda)$;

(3) X_1, *where* $X = (X_t)$ *is a compound Poisson process with intensity* λ *and jump distribution* $\nu \in \mathcal{M}_1(\mathbb{R}^d)$*;*

(4) $\mathrm{Gamma}(a, b)$ *with density* $\frac{b^a t^{a-1} e^{-bt}}{\Gamma(a)} 1_{\{t>0\}}$*;*

(5) *the constant* $a \in \mathcal{M}(\mathbb{R}^d)$*.*

Exercise 11. *Prove that* $\mathrm{Ber}(p)$ *is not I.D. when* $p \in (0, 1)$*.*

Proposition 3.1 (Necessary conditions)**.** *Let* $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ *be I.D. and* ϕ_μ *its characteristic function. Then,*

(1) μ *is a Dirac measure or it has an unbounded support;*

(2) *for all* $u \in \mathbb{R}^d$, $\phi_\mu(u) \neq 0$*;*

(3) *there exists a unique continuous function* $\Psi \colon \mathbb{R}^d \to \mathbb{C}$ *such that* $\Psi(0) = 0$ *and* $\phi_\mu(u) = \exp(\Psi(u))$*.*

Exercise 12. *Find the function* Ψ *corresponding to each example in Exercise 10.*

Note that the n-th root probability distribution of an I.D. law is unique. Indeed (with the same notation as in Proposition 3.1) if ϕ_n and $\widetilde{\phi}_n$ are two characteristic functions satisfying

$$\phi_n(u)^n = \widetilde{\phi}_n(u)^n = \phi_\mu(u), \qquad \forall u \in \mathbb{R}^d, \tag{3.2}$$

then there exists $k \colon \mathbb{R}^d \mapsto \{0, \dots, n-1\}$ such that $\phi_n(u) = \widetilde{\phi}_n(u) \exp(2i\pi k(u)/n)$. Since k is $\mathbb{Z}$-valued, continuous on a connected set and $k(0) = 0$, we get $k = 0$. Therefore, $\phi_n = \widetilde{\phi}_n$. One can easily check that the unique solution is $u \in \mathbb{R}^d \mapsto \exp(\frac{1}{n}\Psi(u))$.

We end this section with two useful stability properties.

Proposition 3.2. *Let* Y_1 *and* Y_2 *be two independent I.D. random variables in* $\mathbb{R}^d$ *and* $(\alpha_1, \alpha_2) \in \mathbb{R}^2$*. The random variable* $\alpha_1 Y_1 + \alpha_2 Y_2$ *is also I.D.*

Proposition 3.3. *Any weak limit (i.e. any limit in the sense of weak convergence) of a sequence of I.D. random variables (or probability distributions) is I.D.*

3.2. **Lévy-Khintchine Theorem.** In this section we give a characterization of I.D. distributions in terms of their characteristic functions.

Definition 3.3 (Lévy measure)**.** *A measure* ν *on* $\mathbb{R}^d$ *is called a* Lévy measure *if* $\nu(\{0\}) = 0$ *and* $\int_{\mathbb{R}^d} (1 \wedge |z|^2) \nu(\mathrm{d}z) < \infty$*.*

Remark 3.1. *A probability measure* ν *such that* $\nu(\{0\}) = 0$ *is a Lévy measure. A Lévy measure is necessarily a* σ*-finite measure. The converse statements are false.*

Theorem 3.1 (Lévy-Khintchine theorem)**.** *A probability distribution on* $\mathbb{R}^d$ *is infinitely divisible if and only if there exist* $b \in \mathbb{R}^d$*, a non-negative symmetric* $d \times d$ *matrix* A *and a Lévy measure* ν *on* $\mathbb{R}^d$ *such that its characteristic function writes* $u \in \mathbb{R}^d \mapsto \exp(\Psi(u))$*, where*

$$\Psi(u) = i\langle b, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d} \Big(e^{i\langle u, z\rangle} - 1 - i\langle u, z\rangle 1_{\{|z|\le 1\}}\Big) \nu(\mathrm{d}z). \tag{3.3}$$

The cut-off at the value $|z| = 1$ in the integral above is only a matter of convention. Before proving the theorem, we advise to treat the following two exercices.

Exercise 13. *Explain why the last integral in the equation above is well-defined.*

Exercise 14 (Continuation of Exercise 12)**.** *In the simpler case where $\nu(\mathbb{R}^d) < \infty$ (finite measure) prove that a probability distribution with a characteristic function as in (3.3) coincides with the law of $b + c + Y + \widetilde{Y}_1$ where*

$$c := -\int_{|z|\le 1} z\nu(\mathrm{d}z), \qquad Y \sim \mathcal{N}_d(0_d, A), \qquad \widetilde{Y} \sim \mathrm{CPP}\Big(\nu(\mathbb{R}^d), \frac{\nu}{\nu(\mathbb{R}^d)}\Big), \tag{3.4}$$

and Y and $\widetilde{Y}$ are independent.

4. Lévy processes

We have seen in Proposition 1.2 that compound Poisson processes have independent and stationary increments (with càdlàg trajectories). These are the properties we could expect from a continuous-time counterpart of the discrete random walk (sum of i.i.d. random variables). But they are not the only processes to fulfill such requirements, Brownian motion being another well-known example. In Section 4.1 we will call such processes Lévy processes and state a few important properties such as the Markov property. The most important result of this section is the Lévy-Itô decomposition, which essentially states that any Lévy process may be decomposed into a deterministic drift, a Brownian part and a (possibly compensated) jump component. This decomposition directly parallels that of the Lévy-Khintchine theorem (Theorem 3.1). We will focus on the jump component in Section 4.3 and then treat the special case of subordinators in Sections 4.4. Finally, in Section 4.5 we explain the relevance of Lévy processes in the study of scaling limits of random walks.

Throughout the section $(\Omega, \mathcal{A}, \mathrm{P})$ is a generic probability space.

4.1. Definition and strong Markov property.

Definition 4.1 (Lévy process)**.** *Let $X = (X_t)_{t\ge 0}$ be an $\mathbb{R}^d$-valued stochastic process. We say X is a Lévy process if it satisfies the following conditions:*

(1) *$X_0 = 0$ a.s.;*
(2) *X has càdlàg trajectories a.s.;*
(3) *X has independent and stationary increments.*

The third item in the definition above means that, for all $n \in \mathbb{N}$ and $0 = t_0 < t_1 < \ldots < t_n$, the random variables $(X_{t_i} - X_{t_{i-1}})_{1\le i\le n}$ are independent and for all $h \ge 0$, the random vectors $(X_{t_i} - X_{t_{i-1}})_{1\le i\le n}$ and $(X_{t_i+h} - X_{t_{i-1}+h})_{1\le i\le n}$ have the same law.

Example 4.1. *Brownian motions (with constant drift and standard deviation) and compound Poisson processes (see Proposition 1.2) are Lévy processes.*

Let $\mathcal{F} = (\mathcal{F}_t)_{t\ge 0}$ be the natural filtration associated to X. We recall that a $[0, +\infty]$-valued random variable T is a stopping time with respect to (w.r.t.) $\mathcal{F}$ if for all $t \ge 0$, the event $\{T \le t\}$ belongs to $\mathcal{F}_t$. We also denote by

$$\mathcal{F}_T := \{A \in \mathcal{A} \colon A \cap \{T \le t\} \in \mathcal{F}_t, \forall t \ge 0\} \tag{4.1}$$

the σ-algebra of events prior to this stopping time. We may now state the following proposition.

Proposition 4.1 (Strong Markov property)**.** *Let $X = (X_t)_{t\geq 0}$ be a Lévy process and T a stopping time such that $T < \infty$ a.s. The process $(X_{T+t} - X_T)_{t\geq 0}$ is a Lévy process independent of $\mathcal{F}_T$ and distributed as X.*

Of course, the strong Markov property implies the simple version of it, when T is a deterministic time.

4.2. **Lévy-Itô decomposition.** It turns out that Lévy processes are strongly connected to infinitely divisible distributions (Section 3).

Proposition 4.2. *Let $X = (X_t)_{t\geq 0}$ be a Lévy process. Then X_t is I.D. for all $t \geq 0$ and there exists a function $\Psi \colon \mathbb{R}^d \mapsto \mathbb{C}$ such that*

$$\phi_{X_t}(u) := \mathrm{E}(e^{i\langle X_t, u\rangle}) = \exp(t\Psi(u)), \qquad (t \geq 0, u \in \mathbb{R}^d). \tag{4.2}$$

The function Ψ is called *Lévy (or characteristic) exponent* of the process.

Exercise 15. *Find the Lévy exponent of a Brownian motion with constant drift $b \in \mathbb{R}^d$ and covariance matrix A. Find the Lévy exponent of a $\mathrm{CPP}(\lambda, \nu)$, where $\lambda > 0$ and $\nu \in \mathcal{M}_1(\mathbb{R}^d)$.*

Theorem 4.1. *The function Ψ is a Lévy exponent iff there exist $b \in \mathbb{R}^d$, a non-negative symmetric $d \times d$ matrix A and a Lévy measure ν on $\mathbb{R}^d$ such that*

$$\Psi(u) = i\langle b, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{\mathbb{R}^d} \Big(e^{i\langle u,z\rangle} - 1 - i\langle u, z\rangle \mathbf{1}_{\{|z|\leq 1\}}\Big)\nu(\mathrm{d}z). \tag{4.3}$$

Remark 4.1. *The eigenvalues of A are not necessarily positive as one may show by considering the $\mathbb{R}^2$-valued process $(t, B_t)_{t\geq 0}$, where B is a one-dimensional standard Brownian motion.*

The following proposition comes as a corollary of the proof of Theorem 4.1.

Proposition 4.3 (Lévy-Itô decomposition)**.** *Suppose $X = (X_t)_{t\geq 0}$ is a Lévy process with a Lévy exponent as in (4.3). Then X has the same law as the process*

$$bt + \sqrt{A}B_t + \int_{(0,t]\times\{z\colon |z|>1\}} z\mathcal{N}(\mathrm{d}s, \mathrm{d}z) + \int_{(0,t]\times\{z\colon |z|\leq 1\}} z\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z) \qquad (t \geq 0), \tag{4.4}$$

where $B = (B_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion and $\mathcal{N}$ is a $\mathrm{RPM}(\mathrm{d}t \otimes \nu)$ independent of B ($\mathrm{d}t$ stands for Lebesgue measure).

Hence, any Lévy process may be written as the sum of a deterministic drift (first term), a Brownian part (second term), a large jump component (third term) and a *compensated* small jump component (fourth part). Let us insist on the term *compensated*: the fourth part does not consist merely of jumps! Of course, some of these components might be equal to zero.

The triplet (b, A, ν) is called *Lévy triplet* of the process. Let us stress that the Lévy triplet depends on our (arbitrary) cut-off between small and large jumps. If we stick to our convention (inherited from Theorem 3.1) then the Lévy triplet associated to the most elementary Poisson counting process, that is $\mathrm{CPP}(1, \delta_1)$, is $(1, 0, \delta_1)$. If we decide that jumps of size one are large instead of small, we get $(0, 0, \delta_1)$ instead (only the first component changes).

4.3. **Jump measure of a Lévy process.** Let $X = (X_t)_{t\geq 0}$ be a Lévy process with triplet (b, A, ν). We recall that the jump (if any) at time $t > 0$ is denoted by

$$\Delta X_t := X_t - X_{t^-} = X_t - \lim_{s\to t^-} X_s. \tag{4.5}$$

Definition 4.2. *The* jump measure *of X is the random measure on $(0,\infty)\times\mathbb{R}^d$ defined by*

$$\mathcal{J} := \sum_{\substack{t>0\\ \Delta X_t\neq 0}} \delta_{(t,\Delta X_t)}. \tag{4.6}$$

Exercise 16. *Check that $\mathcal{J}$ has indeed a countable number of atoms, a.s.*

The next proposition may be deduced from Proposition 4.3:

Proposition 4.4. *The jump measure of a Lévy process with triplet (b, A, ν) is a* $\mathrm{RPM}(\mathrm{d}t\otimes \nu)$*.*

Corollary 4.1. *The sample paths of an $\mathbb{R}^d$-valued Lévy process X are a.s. continuous if and only if X is a Brownian motion with drift (i.e. $X_t = bt + AB_t$ for some $b\in\mathbb{R}^d$, A a positive semi-definite matrix and B a d-dimensional standard Brownian motion).*

Proposition 4.5. *The random variable $\sum_{s\in(0,t]}|\Delta X_s|$ is a.s. finite iff $\int(1\wedge|z|)\nu(\mathrm{d}z)$ is finite.*

Proposition 4.6. *For all $t > 0$, the random variable $\sum_{s\in(0,t]}|\Delta X_s|^2$ is a.s. finite.*

Therefore, we may have in some cases $\sum_{s\in(0,t]}|\Delta X_s|^2 < \infty$ a.s. but $\sum_{s\in(0,t]}|\Delta X_s| = \infty$ a.s. This phenomenon is due to the presence of small jumps.

Exercise 17. *Let X be a pure-jump (i.e. $X_t = \sum_{s\in(0,t]}\Delta X_s$) one-dimensional ($d = 1$) Lévy process which makes only positive jumps. Prove that $\int(1\wedge|z|)\nu(\mathrm{d}z) < \infty$.*

We will focus on this special case further in the chapter.

4.4. **Subordinators.**

Definition 4.3 (Subordinator)**.** *A one-dimensional Lévy process is called a* subordinator *if $X_t \geq 0$ for all $t\geq 0$, a.s.*

Exercise 18. *Give a simple example of a subordinator.*

Exercise 19. *Prove that a subordinator is a.s. non-decreasing.*

Theorem 4.2. *The function $\Psi\colon\mathbb{R}\to\mathbb{C}$ is the Lévy exponent of a subordinator iff it writes*

$$\Psi(u) = i\beta u + \int_{\mathbb{R}}(e^{iuz}-1)\nu(\mathrm{d}z), \qquad (u\in\mathbb{R}), \tag{4.7}$$

with $\beta\geq 0$ and ν a Lévy measure that satisfies

$$\nu((-\infty,0]) = 0 \qquad \textit{and} \qquad \int_0^\infty(1\wedge z)\nu(\mathrm{d}z) < \infty. \tag{4.8}$$

We refer to Exercise 17 for an explanation of the second condition on the Lévy measure.

Since a subordinator is non-negative, we may work with its Laplace transform:

Proposition 4.7. *If X is a subordinator with Lévy exponent Ψ, then*

$$\mathrm{E}(e^{-rX_t}) = e^{t\Psi(ir)}, \qquad (r \geq 0). \tag{4.9}$$

Exercise 20. *Find the Lévy-Itô decomposition corresponding to a subordinator.*

Proposition 4.8 (Asymptotic behaviour)**.** *Let X be a subordinator with a Lévy exponent as in (4.7). The following limits hold a.s. :*

$$\begin{aligned} &(i) \qquad \lim_{t\to\infty} \frac{X_t}{t} = \mathrm{E}(X_1) = \beta + \int_0^\infty z\nu(\mathrm{d}z) \in [0,\infty], \\ &(ii) \qquad \lim_{t\to 0} \frac{X_t}{t} = \beta. \end{aligned} \tag{4.10}$$

Exercise 21 (Brownian ladder times)**.** *Let B be a one-dimensional standard Brownian motion and define*

$$T_a = \inf\{t \geq 0 \colon B_t = a\}, \qquad a \geq 0. \tag{4.11}$$

We denote by $\mathcal{F} = (\mathcal{F}_t)$ the filtration associated to B.

1. *Prove that for all $a \geq 0$, T_a is an a.s. finite $\mathcal{F}$-stopping time and that the process $(T_a)_{a\geq 0}$ is a subordinator.*
2. *Does this process have a.s. continuous paths?*
3. *Prove by a martingale argument that $\mathrm{E}[e^{-uT_a}] = e^{-a\sqrt{2u}}$ for all $u \geq 0$.*
4. *Deduce thereof that the density of T_a is given by*

$$f_a(x) := \frac{a}{(2\pi x^3)^{1/2}} \exp\Big(-\frac{a^2}{2x}\Big), \qquad x \geq 0. \tag{4.12}$$

Hint: define $\mathcal{L}(u) = \int_0^\infty e^{-ux} f_a(x)\mathrm{d}x$ for all $u \geq 0$, use the change of variable $ux = \frac{a^2}{2y}$ to find an ODE satisfied by $\mathcal{L}$.

Exercise 22 (Supremum of Brownian motion)**.** *Let B be a standard Brownian motion and define*

$$M_t = \sup_{0\leq s\leq t} B_s, \qquad t \geq 0. \tag{4.13}$$

Is $M = (M_t)_{t\geq 0}$ a subordinator? Hint: use Corollary 4.1.

Exercise 23. *Let $\alpha \in (0,1)$ and X be a subordinator with Lévy triplet $(b, 0, \nu)$, where*

$$\nu(\mathrm{d}z) = \frac{\alpha \ \mathrm{d}z}{\Gamma(1-\alpha) z^{1+\alpha}}. \tag{4.14}$$

Compute the Laplace transform $\mathcal{L}(u) = \mathrm{E}(e^{-uX_1})$, for $u \geq 0$, and find b such that $\mathcal{L}(u) = \exp(-u^\alpha)$.

Exercise 24. *With the help of Exercise 23, find the jump measure of the process of Brownian ladder times from Exercise 21.*

4.5. **Limits of heavy-tailed random walks.** This section contains a short and informal discussion on how some Lévy processes arise as the limits of random walks (i.e. sums of independent random variables). Let $(X_i)_{i\geq 1}$ be a sequence of real random variables and for all $n \geq 1$, let $S_n := X_1 + \ldots + X_n$, with $S_0 := 0$. Donsker's theorem says that if $\sigma^2 := \mathrm{E}(X_1^2) < \infty$ then the sequence of processes

$$n^{-1/2}(S_{\lfloor nt \rfloor} - \lfloor nt \rfloor \mathrm{E}(X_1)), \qquad t \in [0,1], \tag{4.15}$$

converges weakly to standard Brownian motion as $n \to \infty$, for the uniform topology. What happens when there is no second moment? To simplify, suppose that the X_i's are nonnegative, with

$$\mathrm{P}(X_1 > x) \overset{+\infty}{\sim} Cx^{-\alpha}, \qquad \alpha \in (0,1) \cup (1,2). \tag{4.16}$$

(The case $\alpha = 1$ turns out to be critical and is dismissed at this level of discussion.) Let us consider

$$Z_t^{(n)} := \frac{S_{\lfloor nt \rfloor}}{n^{1/\alpha}}, \qquad t \geq 0, \tag{4.17}$$

which we write as

$$Z_t^{(n)} = \int_{(0,t]\times(0,\infty)} z\mathcal{N}_n(\mathrm{d}s, \mathrm{d}z), \qquad \text{where} \quad \mathcal{N}_n := \sum_{i\in\mathbb{N}} \delta_{\left(\frac{i}{n}, \frac{X_i}{n^{1/\alpha}}\right)}. \tag{4.18}$$

The renormalization by $n^{1/\alpha}$ will be justified *a posteriori* but it is not completely surprising as $n^{1/\alpha}$ is the right order of magnitude for $\max(X_1, \ldots, X_n)$, when $n \to \infty$. Then, one can show that the sequence of random measures $\mathcal{N}_n$ converges (in some sense) to a random Poisson measure that shall be denoted by $\mathcal{N}$, with intensity measure:

$$\mathrm{Leb} \otimes C\alpha z^{-(1+\alpha)} \mathbf{1}_{(0,\infty)}(z)\mathrm{d}z. \tag{4.19}$$

One is then tempted to interchange the limit as $n \to \infty$ and the integral sign in (4.18). There are however two cases:

- If $\alpha \in (0,1)$ then $\int_0^{+\infty} (1 \wedge z) z^{-(1+\alpha)} \mathrm{d}z < \infty$, so $\int_{(0,t]\times(0,\infty)} z\mathcal{N}(\mathrm{d}s, \mathrm{d}z)$ is well-defined as an a.s finite random variable. No centering of (4.17) is needed.
- If $\alpha \in (1,2)$ then $\int_0^{+\infty} (1 \wedge z^2) z^{-(1+\alpha)} \mathrm{d}z < \infty$ and only the compensated integral $\int_{(0,t]\times(0,\infty)} z\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z)$ is well-defined. This means one has to recenter (4.17).

Assuming that the previous argument can be made rigorous, we obtain thereof that for all $t \geq 0$,

$$\begin{aligned} Z_t^{(n)} &\overset{\text{(law)}}{\longrightarrow} \int_{(0,t]\times(0,\infty)} z\mathcal{N}(\mathrm{d}s, \mathrm{d}z) \quad (0 < \alpha < 1) \\ Z_t^{(n)} - \mathrm{E}Z_t^{(n)} &\overset{\text{(law)}}{\longrightarrow} \int_{(0,t]\times(0,\infty)} z\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z) \quad (1 < \alpha < 2). \end{aligned} \tag{4.20}$$

Finally, one may upgrade the above to a weak convergence statement at the level of *processes*, provided we equip the space of càdlàg functions with a suitable topology. For a complete treatment, we refer the reader to [11, Exercise 4.4.2.8].

5. Stochastic integration

Our goal in this section is to make sense of integration and differentiation of Lévy processes (or stochastic processes based on them). As we have seen, Lévy processes in general have jumps, hence differentiation must be thought of in a broader sense than usual. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function (in the usual sense). Then $f(t) - f(0) = \int_0^t f'(s)\mathrm{d}s$ or, equivalently, $f(t) - f(0)$ is the value on $[0, t]$ of a (signed) measure that is absolutely continuous w.r.t. to Lebesgue measure and the Radon-Nikodym derivative of the former w.r.t to the latter is a.e. equal to f'. Now, consider the Heaviside function g defined by $g(t) = 0$ if $t < 0$ and $g(t) = 1$ if $t \geq 0$. This function is càdlàg but it is not differentiable in the usual sense because of the jump at the origin. Yet, we may write that for all $t \in \mathbb{R}$, $g(t)$ is the value on $(-\infty, t]$ of the measure δ_0. Hence, the Dirac measure at 0 is in some sense a *weak* derivative of the Heaviside function. This lays the path to the notion of (random) distributions, which we will not pursue, but it is useful to keep this example in mind.

In what follows, $\mathcal{N}$ is a RPM on $E = \mathbb{R}^+ \times \mathbb{R}^d$ with intensity measure $\mathrm{d}t \otimes \nu$ (ν a Lévy measure) and B is a standard Brownian motion independent from $\mathcal{N}$. We denote by $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by $\mathcal{N}$ and B. In other words, $\mathcal{F}_t = \sigma(B_s,\ \mathcal{N}((0, s] \times A),\ s \leq t,\ A \in \mathcal{B}(\mathbb{R}^d))$.

5.1. **Stochastic integral with respect to a random Poisson measure.** In Section 2 we learnt how to integrate deterministic functions w.r.t. random Poisson measures (but the reader should keep in mind that even in this case the integral is a *random variable*). We will now treat the case of *random integrands.*

Definition 5.1. *Let $T > 0$. The* predictable σ-algebra[†]*, denoted by $\mathcal{P}$, is the smallest σ-algebra w.r.t. which all mappings $F\colon [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ that satisfy the two conditions below are measurable.*

(1) *$\forall t \in [0, T]$, the mapping $(z, \omega) \mapsto F(t, z, \omega)$ is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$-measurable.*

(2) *$\forall z \in \mathbb{R}^d, \omega \in \Omega$, the mapping $t \mapsto F(t, z, \omega)$ is left-continuous.*

A process $F\colon [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ is said to be predictable *if it is $\mathcal{P}$-measurable.*

Let us now introduce the space of random functions that we want to integrate. We define $\mathcal{H}^2(T)$ as the space of predictable processes that are square integrable on $[0, T] \times \mathbb{R}^d \times \Omega$ with respect to $\mathrm{d}t \otimes \nu \otimes \mathrm{P}$.

Proposition 5.1. *The space $\mathcal{H}^2(T)$ is a Hilbert space equipped with the scalar product:*

$$\langle F, G\rangle_{\mathcal{H}^2(T)} := \int_0^T \int_{\mathbb{R}^d} \mathrm{E}\Big(F(t, z)G(t, z)\Big)\mathrm{d}t\nu(\mathrm{d}z). \tag{5.1}$$

We now aim at defining a notion of integral of $F \in \mathcal{H}^2(T)$ on $[0, T] \times \mathbb{R}^d$ w.r.t a compensated RPM denoted by $\widetilde{\mathcal{N}}$. The main idea is to define this integral for a set of *simple* functions that are dense in $\mathcal{H}^2(T)$. This defines an isometry between this space of simple functions and the space of square-integrable random variables, which can be extended to the whole space $\mathcal{H}^2(T)$ thanks to a density argument.

[†] *tribu prévisible* in French

Let $\mathcal{S}$ be the space of *simple* predictable functions, which satisfy

$$F(t,z) = \sum_{1\le j\le m}\sum_{1\le i\le n} c_i Y_j \mathbf{1}_{(t_j,t_{j+1}]}(t)\mathbf{1}_{A_i}(z), \tag{5.2}$$

where

- $m, n \in \mathbb{N}$;
- $0 < t_1 < \ldots < t_m < t_{m+1} := T$;
- $A_1, \ldots, A_n$ are disjoint Borel sets such that $\nu(A_i) < \infty$ for all $1 \le i \le n$;
- the c_i's are real numbers;
- Y_j is a bounded $\mathcal{F}_{t_j}$-measurable random variable.

We then define the mapping

$$I_T \colon F \in \mathcal{S} \mapsto \sum_{1\le j\le m}\sum_{1\le i\le n} c_i Y_j \widetilde{\mathcal{N}}((t_j,t_{j+1}]\times A_i). \tag{5.3}$$

Exercise 25. *The goal of this exercise is to define the stochastic integral $I_T(F)$ for all functions $F \in \mathcal{H}^2(T)$.*

(1) *Check that for all $F \in \mathcal{S}$, $I_T(F)$ is a centered square-integrable random variable.*
(2) *Prove that I_T is an isometry from $\mathcal{S}$ to $L^2(\Omega)$.*
(3) *Check that $\mathcal{S}$ is a dense subset of $\mathcal{H}^2(T)$.*
(4) *Conclude.*

Proposition 5.2 (Martingale property)**.** *For all $F \in \mathcal{H}^2(T)$, the stochastic process $(I_t(F))_{0\le t\le T}$ is a square-integrable $\mathcal{F}$-adapted and centered càdlàg martingale.*

We will henceforth write

$$I_T(F) = \int_0^T \int_{\mathbb{R}^d} F(t,z)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z), \qquad (F \in \mathcal{H}^2(T)). \tag{5.4}$$

Remark 5.1. *The stochastic integral in (5.4) may be extended to the case where the predictable process F satisfies the weaker assumption $\int_0^T \int F(t,z)^2 \nu(\mathrm{d}z)\mathrm{d}t < \infty$ a.s., in which case the process $(I_t(F))_{0\le t\le T}$ is a local martingale that has a càdlàg modification, see* [1, Theorem 4.2.12].

5.2. **Itô's formula.** To simplify, we restrict from now on to dimension one ($d = 1$). Let $T > 0$ (deterministic time horizon), $b, \sigma \colon [0,T] \times \Omega \to \mathbb{R}$ be predictable processes[‡] such that

$$\mathrm{E}\Big(\int_0^T |b(t)|\mathrm{d}t\Big) < \infty, \qquad \mathrm{E}\Big(\int_0^T \sigma(t)^2\mathrm{d}t\Big) < \infty, \tag{5.5}$$

$K \colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ predictable and $H \in \mathcal{H}^2(T)$. The process $X = (X_t)_{0\le t\le T}$ defined by

$$X_t = X_0 + \int_0^t b(s)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}B_s + \int_0^t \int_{|z|\le 1} H(s,z)\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z) + \int_0^t \int_{|z|>1} K(s,z)\mathcal{N}(\mathrm{d}s,\mathrm{d}z) \tag{5.6}$$

[‡]Here, b and σ do not depend on the space (z) variable so we only ask that both functions are left-continuous and adapted.

is called a *Lévy-type stochastic integral*, which we may also write, formally,

$$\mathrm{d}X_t = b(t)\mathrm{d}t + \sigma(t)\mathrm{d}B_t + \int_{|z|\le 1} H(t,z)\widetilde{\mathcal{N}}(\mathrm{d}t,\mathrm{d}z) + \int_{|z|>1} K(t,z)\mathcal{N}(\mathrm{d}t,\mathrm{d}z). \tag{5.7}$$

Remark 5.2. *The last term in (5.6) is well-defined as ν is finite on $[-1,1]^c$, so that it may be written as*

$$\int_0^t \int_{|z|>1} K(s,z)\mathcal{N}(\mathrm{d}s,\mathrm{d}z) = \sum_{0<s\le t} K(s,\Delta X_s)1_{\{|\Delta X_s|>1\}}. \tag{5.8}$$

The sum above has an a.s. finite number of nonzero terms.

Exercise 26. *Check that a Lévy process is a Lévy-type stochastic integral.*

Theorem 5.1 (Itô's formula)**.** *Let $X = (X_t)_{0\le t\le T}$ be a Lévy-type stochastic integral of the form (5.6) and $f \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R})$. Then, for all $0 \le t \le T$, we have a.s.*

$$f(t,X_t) - f(0,X_0) = \text{(I)} + \text{(II)} + \text{(III)}, \tag{5.9}$$

with

$$\text{(I)} = \int_0^t \partial_t f(s,X_s)\mathrm{d}s + \int_0^t \partial_x f(s,X_s)b(s)\mathrm{d}s, \tag{5.10}$$

$$\text{(II)} = \int_0^t \partial_x f(s,X_s)\sigma(s)\mathrm{d}B_s + \frac{1}{2}\int_0^t \partial^2_{xx} f(s,X_s)\sigma^2(s)\mathrm{d}s, \tag{5.11}$$

and

$$\begin{aligned}\text{(III)} = &\int_0^t \int_{|z|>1} [f(s^-,X_{s^-}+K(s,z)) - f(s^-,X_{s^-})]\mathcal{N}(\mathrm{d}s,\mathrm{d}z)\\ &+ \int_0^t \int_{|z|\le 1} [f(s^-,X_{s^-}+H(s,z)) - f(s^-,X_{s^-})]\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z)\\ &+ \int_0^t \int_{|z|\le 1} [f(s^-,X_{s^-}+H(s,z)) - f(s^-,X_{s^-}) - H(s,z)\partial_x f(s^-,X_{s^-})]\mathrm{d}s\nu(\mathrm{d}z).\end{aligned} \tag{5.12}$$

The first term is the deterministic part of the differentiation while the second term comes from the standard Itô's formula for stochastic calculus based on Brownian motion. We will thus focus on the third term, which is specific to this course. This last term itself is decomposed in three parts : the first part is due to the large jumps while the second and third ones, which can be somehow put in parallel with the two terms in (II) (first-order and second-order variations), are due to the small jumps.

5.3. A bit of practice: integration with respect to a Poisson counting measure.

Exercise 27 (Integration with respect to a Poisson counting measure)**.** *Let $N = (N_t)_{t\ge 0}$ be a Poisson counting measure with intensity $\lambda > 0$ and $B = (B_t)_{t\ge 0}$ a standard Brownian motion.*

(1) *Give a meaning to $\mathrm{d}N_s$ and $\mathrm{d}\widetilde{N}_s$.*
(2) *Let $s \mapsto f(s)$ be a (deterministic) function. Give simple alternative expressions for the stochastic integrals*

$$\int_0^t f(s)\mathrm{d}N_s \quad \text{and} \quad \int_0^t f(s)\mathrm{d}\widetilde{N}_s. \tag{5.13}$$

(3) *Give a simple description of the stochastic process* $\int_0^t B_s \mathrm{d}N_s$ $(t \geq 0)$.

(4) *Compute explicitely* $X_t := \int_0^t N_{s^-} \mathrm{d}N_s$. *Same question for the compensated version of the integral. Why do we need to write* s^- *instead of* s?

(5) *Use Itô's formula to write* $Y_t = N_t^2$ *as a Lévy-type stochastic integral and check that* $\mathrm{d}Y_t \neq 2N_{t^-}\mathrm{d}N_t$.

5.4. **A quick look at infinitesimal generators.** Infinitesimal generators are tools to describe the evolution of a Markov process $X = (X_t)_{t\geq 0}$. Let us assume that for functions f in certain space (called domain of the generator), the following limit exists:

$$\mathcal{L}f(x) := \lim_{h\to 0} \mathrm{E}\Big[\frac{f(X_h) - f(X_0)}{h}\Big| X_0 = x\Big]. \tag{5.14}$$

Then $\mathcal{L}$ is a linear functional defined on this domain, called generator of the process. Suppose X is a one-dimensional Lévy process with triplet (b, σ^2, ν). An application of Itô's formula shows that for smooth enough functions f we have

$$\mathcal{L}f(x) = bf'(x) + \tfrac{1}{2}\sigma^2 f''(x) + \int_{\mathbb{R}} [f(x+z) - f(x) - zf'(x)\mathbf{1}_{\{|z|\leq 1\}}]\nu(\mathrm{d}z). \tag{5.15}$$

Exercise 28. *Identify the drift, Brownian and jump components. Identify the local and non-local parts of the operator* $\mathcal{L}$. *Find an analogy between the infinitesimal generator and the Lévy-Khintchine formula in Theorem 3.1.*

6. Stochastic differential equations

6.1. **Existence and uniqueness of solutions.** Let $\sigma, b \colon \mathbb{R} \to \mathbb{R}$ and $F, G \colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be measurable (deterministic) functions. In this section, a *stochastic differential equation* (SDE) is an equation of the form

$$\begin{aligned} Y_t = Y_0 + \int_0^t b(Y_{s^-})\mathrm{d}s + \int_0^t \sigma(Y_{s^-})\mathrm{d}B_s + \int_0^t \int_{\{|z|\leq 1\}} F(Y_{s^-}, z)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z) \\ + \int_0^t \int_{\{|z|>1\}} G(Y_{s^-}, z)\mathcal{N}(\mathrm{d}s, \mathrm{d}z), \end{aligned} \tag{6.1}$$

where the stochastic process $Y = (Y_t)_{t\geq 0}$ is the unknown process and Y_0 is assumed to be a square-integrable $\mathcal{F}_0$-measurable random variable (hence it is independent of B and $\mathcal{N}$).

Definition 6.1. *A solution of the SDE above, if it exists, is any adapted càdlàg process* Y *such that for all* $t \geq 0$, *the equality in* (6.1) *holds a.s. A solution is said to be unique if, for any pair of solution* $Y^{(1)}$ *and* $Y^{(2)}$, *we have* $\mathrm{P}(\forall t \geq 0,\ Y_t^{(1)} = Y_t^{(2)}) = 1$.

Assumption 6.1. *There exists* $K \in (0, \infty)$ *such that, for all* $y, y' \in \mathbb{R}$,

(i) $|b(y) - b(y')|^2 + |\sigma(y) - \sigma(y')|^2 + \int_{\{|z|\leq 1\}} |F(y,z) - F(y',z)|^2 \nu(\mathrm{d}z) \leq K|y-y'|^2$;

(ii) $|b(y)|^2 + |\sigma(y)|^2 + \int_{\{|z|\leq 1\}} |F(y,z)|^2\nu(\mathrm{d}z) \leq K(1 + |y|^2)$

and for all $|z| > 1$,

(iii) $y \mapsto G(y,z)$ *is continuous.*

Theorem 6.1. *There exists a unique solution to the SDE in* (6.1) *under Assumption* (6.1).

6.2. **Proof without the large jumps.** For simplicity, we first assume in this section that $G = 0$ (no large jumps). We shall only focus on the jump component and hereby assume that b and σ are identically zero. The main idea of the proof is Picard iteration. Define, for all $t \geq 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} Y_t^{(0)} &= Y_0 \\ Y_t^{(n)} &= Y_0 + \int_0^t \int_{|z|\leq 1} F(Y_{s^-}^{(n-1)}, z)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z). \end{aligned} \tag{6.2}$$

Here are the main steps of the proof.

(1) Note that, for all $n \in \mathbb{N}$, $(Y_t^{(n)})$ is a square-integrable càdlàg martingale.

(2) Define

$$y_n(t) := \mathrm{E}\Big(\sup_{0\leq s\leq t} |Y_s^{(n)} - Y_s^{(n-1)}|^2\Big), \tag{6.3}$$

and prove that

$$\begin{aligned} y_1(t) &\leq 4t \int_{|z|\leq 1} \mathrm{E}\Big(F(Y_0, z)^2\Big)\nu(\mathrm{d}z), \\ y_{n+1}(t) &\leq C_1(t)K \int_0^t y_n(s)\mathrm{d}s. \end{aligned} \tag{6.4}$$

(3) Deduce thereof that for all t, $(Y_t^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence in L^2 and denote by Y_t the limit (in L^2).

(4) Prove that the convergence holds a.s. and that $t \mapsto Y_t$ is adapted and càdlàg. Hint: prove that for some constant $C = C(t)$,

$$\mathrm{P}\Big(\sup_{0\leq s\leq t} |Y_s^{(n)} - Y_s^{(n-1)}| \geq 2^{-n}\Big) \leq \frac{C^n}{n!}, \tag{6.5}$$

and deduce thereof that $Y^{(n)}$ converges a.s. and uniformly to Y on compact sets.

(5) Define

$$\widetilde{Y}_t := Y_0 + \int_0^t \int_{|z|\leq 1} F(Y_{s^-}, z)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z) \tag{6.6}$$

and check that, for all t, $Y_t^{(n)}$ converges to $\widetilde{Y}_t$ in L^2. Deduce thereof that $\widetilde{Y}_t = Y_t$ a.s. (This concludes the existence part of the proof)

(6) Prove uniqueness of solutions.

6.3. **Proof with the large jumps.** For simplicity, we stick to the case where $b = \sigma = 0$. We now briefly explain how to incorporate the large jumps to the solution. Let $\hat{Y}$ be the solution of the SDE with $G = 0$. Let $(T_i, Z_i)_{i\in\mathbb{N}}$ be the atoms of the RPM $\mathcal{N}$ on $\mathbb{R}^+ \times \{z\colon |z| > 1\}$. Define

$$\begin{aligned} Y_t &= \hat{Y}_t && (0 < t < T_1) \\ Y_{T_1} &= Y_{T_1^-} + G(Y_{T_1^-}, Z_1) && \\ Y_t &= Y_{T_1} + \hat{Y}_t^{(1)}(t - T_1) && (T_1 < t < T_2) \\ Y_{T_2} &= Y_{T_2^-} + G(Y_{T_2^-}, Z_2) && \end{aligned} \tag{6.7}$$

where $\hat{Y}^{(1)}$ is the unique solution of the SDE with $G = 0$ and initial condition Y_{T_1}, and so on, recursively. One can check that Y solves the SDE.

7. Exponential martingales and change of measures

Throughout this section, $X = (X_t)_{t\geq 0}$ is a one-dimensional Lévy process with Lévy triplet (b, σ^2, ν).

7.1. **Doléans-Dade exponential.** We consider the following SDE

$$\mathrm{d}S_t = S_{t-}\mathrm{d}X_t, \tag{7.1}$$

which describes for instance the evolution of the price of a risked asset.

Exercise 29. *Check that (7.1) has a unique solution (given some initial condition).*

We remind from standard stochastic calculus (see [8, Section 8.4.2]) that if $X_t = \sigma B_t$ (i.e. $b = 0$ and $\nu = 0$) then

$$S_t = S_0 \exp\Big(\sigma B_t - \frac{1}{2}\sigma^2 t\Big) \tag{7.2}$$

and we note that (i) the solution is a martingale and (ii) the solution *is not the usual exponential*. Let us now come back to the general case, to which we add the following:

Assumption 7.1. $\nu((-\infty, -1]) = 0$.

Under this assumption (which guarantees that the process remains positive, as we may expect from the price of an asset for instance) we get the following:

Proposition 7.1. *The process*

$$S_t := S_0 \exp\Big(X_t - \frac{1}{2}\sigma^2 t\Big) \prod_{0<s\leq t} (1 + \Delta X_s) e^{-\Delta X_s}, \qquad (t \geq 0) \tag{7.3}$$

is well-defined and solves the SDE in (7.1). Moreover, there exists a process $L = (L_t)_{t\geq 0}$ such that $S_t = S_0 \exp(L_t)$.

The process $S = (S_t)$ is called a *Doléans-Dade exponential.*

Remark 7.1. *The product in (7.3) may be rewritten as the exponential of*

$$\sum_{0<s\leq t} \log(1 + \Delta X_s) - \Delta X_s, \tag{7.4}$$

and $\log(1 + \delta) - \delta$ is equivalent to $-\frac{1}{2}\delta^2$ as $\delta \to 0$ (compare to the Brownian part). This sum is well-defined, by Proposition 4.6.

Exercise 30. *The goal of this exercise is to prove Proposition 7.1.*

1. *Determine $L = (L_t)$ and write it as a Lévy-type stochastic integral.*
2. *Prove that L is actually a Lévy process and identify its Lévy triplet.*
3. *Apply Itô's formula to $f(L_t)$, where $f \in \mathcal{C}^2(\mathbb{R})$.*
4. *Pick $f = \exp$ and conclude.*

Exercise 31. *Let (N_t) be a Poisson counting process with unit rate. Prove that $t \in [0, \infty) \to 2^{N_t}$ is the unique solution of the stochastic differential equation $\mathrm{d}S_t = S_{t-}\mathrm{d}N_t$ with initial condition $S_0 = 1$, using (i) Itô's formula, or (ii) Proposition 7.1, or (iii) a more elementary approach, where $\mathrm{d}S_t$ and $\mathrm{d}N_t$ are seen as random measures.*

7.2. **Exponential martingales.** In this section, X is a Lévy process with triplet (b, σ^2, ν). We now want to find a condition under which the exponential of Lévy process (such as the Doléans-Dade exponential from Section 7.1) is a martingale. An application will be given in Section 8 (determination of a risk-neutral measure in option pricing).

Proposition 7.2. *Assume $\int_1^\infty e^z \nu(\mathrm{d}z) < \infty$. The process $(e^{X_t})_{t\geq 0}$ is a martingale iff*

$$b + \tfrac{1}{2}\sigma^2 + \int \Big(e^z - 1 - z\mathbf{1}_{\{|z|\leq 1\}}\Big)\nu(\mathrm{d}z) = 0. \tag{7.5}$$

Exercise 32. *Using Proposition 7.2 and Question (2) in Exercise 30, give a condition for the Doléans-Dade exponential defined in (7.3) to be a martingale.*

7.3. **Change of measures.** Let us first recall a few definitions. Let m_1 and m_2 be two measures on a measurable space $(\mathcal{X}, \mathcal{A})$. We say that m_1 is absolutely continuous with respect to m_2, which is denoted by $m_1 \ll m_2$, if for all $A \in \mathcal{A}$, $m_2(A) = 0$ implies that $m_1(A) = 0$. The two measures are said to be equivalent if $m_1 \ll m_2$ and $m_2 \ll m_1$. In this section we will inspect conditions under which probability measures corresponding to Lévy processes with different Lévy triplets are equivalent (on a finite time horizon).

Let us start with the simpler case of Brownian motions with drifts (no jump component).

Proposition 7.3. *Let P_1 (resp. P_2) be the law of a Brownian motion with drift b_1 (resp. b_2) and standard deviation $\sigma_1 > 0$ (resp. $\sigma_2 > 0$). The probability measures P_1 and P_2 restricted to $\mathcal{F}_t$ are equivalent iff $\sigma_1 = \sigma_2$, in which case*

$$\frac{\mathrm{dP}_1}{\mathrm{dP}_2}\Big|_{\mathcal{F}_t} = \exp\Big(\frac{b_1 - b_2}{\sigma^2}B_t - \frac{1}{2}\frac{b_1^2 - b_2^2}{\sigma^2}t\Big), \qquad (\sigma := \sigma_1 = \sigma_2). \tag{7.6}$$

Let us now treat the case of two Poisson counting processes (jumps have size one) with different parameters.

Proposition 7.4. *Let P_1 (resp. P_2) be the law of a Poisson counting process with intensity λ_1 (resp. λ_2). The probability measures P_1 and P_2 restricted to $\mathcal{F}_t$ are equivalent and*

$$\frac{\mathrm{dP}_1}{\mathrm{dP}_2}\Big|_{\mathcal{F}_t} = \exp\Big(\log(\lambda_1/\lambda_2)N_t - (\lambda_1 - \lambda_2)t\Big). \tag{7.7}$$

We now provide a more general formula for compound Poisson process.

Proposition 7.5. *Let P_1 (resp. P_2) be the law of a* CPP *with* finite *jump measure ν_1 (resp. ν_2). The probability measures P_1 and P_2 restricted to $\mathcal{F}_t$ are equivalent iff ν_1 and ν_2 are equivalent, in which case*

$$\frac{\mathrm{dP}_1}{\mathrm{dP}_2}\Big|_{\mathcal{F}_t} = \exp\Big(\sum_{\substack{0<s\leq t:\\ \Delta X_s \neq 0}} \phi(\Delta X_s) - (\nu_1(\mathbb{R}) - \nu_2(\mathbb{R}))t\Big), \qquad \phi(z) := \log\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2}(z). \tag{7.8}$$

Exercise 33. *Check that (7.8) is consistent with (7.7).*

Exercise 34. *Using Proposition 7.2, check that the processes appearing on the right-hand sides of (7.6), (7.7) and (7.8) are martingales w.r.t. P_2.*

8. Application to finance

Let $S = (S_t)$ and $A = (A_t)$ be two stochastic processes that respectively describe the evolution in time of the values of a *risk asset* (like stock) and a *risk-free asset* (like bank account). Unless stated otherwise, the risk-free asset follows the simple deterministic evolution $A_t = A_0 e^{rt}$, where r is the *interest rate*. A portfolio (U, V) is a couple of processes such that U_t and V_t are the amounts of units of risk and risk-free assets (respectively) at time t. It is such that (U_t, V_t) only depends on the past, that is $(S_s)_{0 \leq s < t}$, and not on the future $(S_s)_{s \geq t}$. The *value of the portfolio* at time t shall be denoted by

$$W_t := U_t S_t + V_t A_t. \tag{8.1}$$

We will first recall the notions of arbitrage opportunities and complete markets by way of simple discrete models. Then, we remind the Black-Scholes pricing formula in the case when the stock value evolves according to Brownian motion. Finally we consider a model where the stock value evolves according to a Lévy process (with possible jumps) and we will see how it may lead to *incomplete markets*.

8.1. **Reminders on arbitrage opportunities, contingent claims and complete markets.**

8.1.1. *Arbitrage opportunities.*

Definition 8.1 (S.F.P). *A portfolio is* self-financing *(S.F.P.) when any risk asset bought is paid for from the risk free asset holdings, and vice-versa.*

Definition 8.2 (Arbitrage opportunity). *There is an* arbitrage opportunity *if there exists a S.F.P. $W = (W_t)$ such that $W_0 = 0$ (a.s.) and, for some $t > 0$,*

(1) $\mathrm{P}(W_t \geq 0) = 1$,
(2) $\mathrm{P}(W_t > 0) > 0$.

In other words, there is an arbitrage opportunity when there exists a portfolio that guarantees a positive chance of making profit without any risk of loss. Let us illustrate this notion with a simple example.

Example 8.1 (One-period model, see [15]). *Let us consider a simple time evolution with only an initial time ($t = 0$) and a final time ($t = 1$). We assume that between these two times, the stock value may jump from its initial state S_0 to two possible distinct final states $S_1(-)$ and $S_1(+)$, depending on two possible scenarii. For instance, S_0 is the initial stock value of an ice-cream company, $S_1(-)$ its value after a cold summer (bad scenario) and $S_1(+)$ its value after a hot summer (good scenario), with of course $S_1(-) < S_1(+)$. The initial portfolio value is assumed to be zero: if $U_0 < 0 < V_0$ this means selling U_0 shares of risk asset to buy V_0 units of risk-free asset; if $V_0 < 0 < U_0$, this means the opposite. There is arbitrage opportunity in the two following cases:*

Case 1: risk-free asset is uniformly preferable to risk asset. *This happens when*

$$\frac{S_0}{A_0} > \frac{S_1(+)}{A_1}. \tag{8.2}$$

Indeed, in this case we pick $U_0 < 0 < V_0$ and get:

$$W_1 \geq U_0 S_1(+) + V_0 A_1 = V_0 A_0 \Big(\frac{A_1}{A_0} - \frac{S_1(+)}{S_0} \Big) > 0. \tag{8.3}$$

We get the equality above from the initial condition $W_0 = U_0 S_0 + V_0 A_0 = 0$.

Case 2: risk asset is uniformly preferable to risk-free asset. *This happens when*

$$\frac{S_1(-)}{A_1} > \frac{S_0}{A_0}. \tag{8.4}$$

Indeed, in this case we pick $V_0 < 0 < U_0$ and get

$$W_1 \geq U_0 S_1(-) + V_0 A_1 = U_0 S_0 \Big(\frac{S_1(-)}{S_0} - \frac{A_1}{A_0} \Big) > 0. \tag{8.5}$$

A first lesson is to be learned from this elementary example: to investigate the existence of arbitrage opportunities, we should not consider the arithmetic increase of the risk asset but rather its geometric (or logarithmic) increase with respect to that of the risk-free asset. Equivalently, we may consider the logarithmic increase of the *discounted* risk asset, defined and denoted by

$$\widetilde{S}_t := A_t^{-1} S_t, \qquad t \geq 0. \tag{8.6}$$

Let us now turn to the notion of *arbitrage-free price* of a *contingent claim.*

Definition 8.3. *A* contingent claim *(or* derivative security*) with maturity $T > 0$ is a contract that provides the owner with a payoff dependent on the performance of $(S_s)_{0 \leq s \leq T}$.*

Example 8.2 (European call option)**.** *This contingent claim (bought at some initial time $t = 0$) gives the option to buy stock at a fixed later time T (expiration time) at a given price k (strike price or exercise price). The corresponding payoff equals $(S_T - k)_+$, where $(\cdot)_+$ is the positive part of some real number.*

The question that motivates the rest of the section is the following: what should be the fair price of such contingent claim? We first give some basic definitions, which we shall then illustrate with a continuation of Example 8.1.

Definition 8.4. *Let us consider a contingent claim with maturity $T > 0$ and payoff Z_T. If there exists a S.F.P with value $W_T = Z_T$ at time T then such a portfolio is called a* hedging portfolio *process replicating the contingent claim. The value $W_0 = U_0 S_0 + V_0 A_0$ of the hedging portfolio at time 0 is called the* arbitrage-free price *of the derivative security.*

If the option sells at $P > W_0$ there is an opportunity of risk-free profit for the seller. If the option sells at $P < W_0$ there is an opportunity of risk-free profit for the buyer.

Example 8.3 (Continuation of Example 8.1)**.** *Consider a contingent claim with payoff $W_1(-)$ and $W_1(+)$ (we choose this notation for the payoff by anticipation of the hedging portfolio). To compute the hedging portfolio (U_0, V_0) that replicates this contingent claim, we must solve the linear equation:*

$$\begin{aligned} U_0 S_1(+) + V_0 A_1 &= W_1(+) \\ U_0 S_1(-) + V_0 A_1 &= W_1(-). \end{aligned} \tag{8.7}$$

The solution is

$$\begin{pmatrix} U_0 \\ V_0 \end{pmatrix} = \begin{pmatrix} S_1(+) & A_1 \\ S_1(-) & A_1 \end{pmatrix}^{-1} \begin{pmatrix} W_1(+) \\ W_1(-) \end{pmatrix}, \tag{8.8}$$

hence

$$W_0 = U_0 S_0 + V_0 A_0 = \frac{A_0}{A_1} \Big(q W_1(+) + (1-q) W_1(-) \Big), \tag{8.9}$$

where

$$q := \frac{S_0 \frac{A_1}{A_0} - S_1(-)}{S_1(+) - S_1(-)} \in (0,1), \tag{8.10}$$

provided we are in the regime free of any arbitrage opportunity. In conclusion the arbitrage-free price may be written

$$W_0 = \frac{A_0}{A_1} \mathrm{E}_q(W_1), \tag{8.11}$$

where P_q *is the measure that puts probability* q *to the plus scenario and* $1-q$ *to the minus scenario. The ratio* A_0/A_1 *is a discount (see discussion below Example 8.1). Note that* q *has nothing to do in general with the actual probability (say* p*) of plus scenario happening. However, one should keep in mind that the probability measures* P_p *and* P_q *are equivalent, provided* $p \in (0,1)$*.*

In the example above, any contingent claim can be hedged by a S.F.P. This leads to the following definition:

Definition 8.5 (Complete model)**.** *A market model is* complete *if every contingent claim can be hedged by a S.F.P.*

Let us end this section with a slightly more elaborate model, which turns out to be complete as well, and will serve as a transition with the continuous-time stochastic models below.

Exercise 35 (n-period model – Continuation of Examples 8.1 and 8.3)**.** *Suppose time is discrete, with index* $k \in \{0,1,\dots,n\}$*. The risk asset evolves as the exponential of a random walk, that is* $S_k = e^{R_k}$*, where*

$$R_0 = 0 \qquad \text{and} \qquad R_k = \sum_{1 \le i \le k} X_i, \tag{8.12}$$

with $(X_i)_{1\le i\le n}$ *a collection of i.i.d. random variables with common law*

$$\mathrm{P}_p(X_1 = 1) = 1 - \mathrm{P}_p(X_1 = -1) = p \in (0,1). \tag{8.13}$$

Let $W_n \colon \{-1,1\}^n \to \mathbb{R}$ *be a contingent claim, where each element of* $\{-1,+1\}^n$ *corresponds to a realization of the random vector* $(X_i)_{1\le i\le n}$*.*

(1) *Suppose first that* $A_0 = A_1 = \ldots = A_n = 1$ *(no interest rate). Let* $q := 1/(1+e)$*. Prove that the value of the contingent claim at time* $k \in \{0,1,\dots,n\}$ *is given by*

$$W_k = \mathrm{E}_q(W_n|\mathcal{F}_k), \qquad \text{where } \mathcal{F}_k = \sigma(X_1,\dots,X_k) \text{ if } k>0 \text{ and } \mathcal{F}_0 = \{\emptyset, \Omega\}. \tag{8.14}$$

(2) *We now consider the case* $A_0 = 1$ *and* $r \in (-1,1)$ *(arbitrage-free model with interest rate). Check that the value of the contingent claim at time* $k \in \{0,1,\dots,n\}$ *is now given by*

$$W_k = e^{-r(n-k)} \mathrm{E}_q(W_n|\mathcal{F}_k), \tag{8.15}$$

for a new value of $q \in (0,1)$*.*

(3) *Check that the discounted process* $(\widetilde{S}_k) := (e^{-rk} S_k)$ *is a martingale under* P_q*.*

We proved in the exercise above that the n-period model is complete as well. Moreover, we showed that the value of the contingent claim at any time may be written as the conditional expectation of its value at maturity. The corresponding probability measure is equivalent (but not necessarily equal) to the probability measure underlying the evolution of the risk asset value. The fact that there exists such an equivalent probability measure (called *risk-neutral measure*) under which the discounted risk asset value is a martingale, is actually a general feature of market models that are free of arbitrage opportunities. The fact that it is unique is a feature of complete markets. This is the *fundamental theorem of asset pricing*, see Propositions 9.1 to 9.3 in [4].

8.2. **Reminders on the Black-Scholes pricing formula.** Let us illustrate once more the fundamental theorem of asset pricing, this time with a continuous-time model such that the stock (risk asset) value evolves according to Brownian motion (no jump). This will eventually lead us to the famous Black-Scholes pricing formula. An example of a market model with jumps shall be treated in the next section. To simplify, we assume $A_0 = 1$.

Throughout this section, $B = (B_t)$ is a standard Brownian motion and the corresponding Wiener measure is denoted by P. We assume that the stock process satisfies the following S.D.E

$$\mathrm{d}S_t = S_{t^-}(\sigma \mathrm{d}B_t + \mu \mathrm{d}t), \tag{8.16}$$

the solution of which writes

$$S_t = S_0 \exp(\sigma B_t + [\mu - \tfrac{1}{2}\sigma^2]t). \tag{8.17}$$

We shall first seek a probability measure equivalent to P under which the discounted process is a martingale. Then we use it to find the arbitrage-free value (as well as a replicating portfolio) for any contingent claim.

8.2.1. *Girsanov transform.* Let $Y = (Y_t)$ be a stochastic integral of the form

$$Y_t = \int_0^t G(s)\mathrm{d}s + \int_0^t F(s)\mathrm{d}B_s \tag{8.18}$$

(with F and G to be determined later) and such that (e^{Y_t}) is a P-martingale.

Exercise 36. *Check that the martingale above defines a probability measure Q equivalent to P (if restricted to $[0,t]$ for any t) such that*

$$\frac{\mathrm{d}Q}{\mathrm{d}P}\Big|_{\mathcal{F}_t} = e^{Y_t}, \qquad (t \geq 0). \tag{8.19}$$

Hint: use Kolmogorov's extension theorem, see [8, Theorem 6.3] *or* [7, Section 2.2]. *See also* [6, Chap. VII] *for the extension theorem in a general case.*

By Girsanov's theorem (see Justin Salez's lecture notes or [8, Theorem 5.22] for a general formulation), the process

$$B_Q(t) := B_t - \int_0^t F(s)\mathrm{d}s \tag{8.20}$$

is a Q-martingale.

Exercise 37. *Assume $\sigma > 0$. Compute $\mathrm{d}\widetilde{S}_t$ and find the only choice of F that guarantees $(\widetilde{S}_t)$ to be a Q-martingale. Find the corresponding value of G. In this case, check that we may write*

$$\mathrm{d}\widetilde{S}_t = \sigma \widetilde{S}_{t^-}\mathrm{d}B_Q(t). \tag{8.21}$$

8.2.2. *Derivation of the replicating portfolio and pricing.* Let Z be a contingent claim with expiration time $T > 0$. Let us define the process

$$Z_t := A_T^{-1} \mathrm{E}_Q(Z|\mathcal{F}_t) = e^{-rT} \mathrm{E}_Q(Z|\mathcal{F}_t), \qquad t \in [0, T]. \tag{8.22}$$

One may check that this process is a Q-martingale. Hence, by the *martingale representation theorem* (see Djalil Chafaï's lecture notes or [8, Theorem 5.18]), there exists a process (δ_t) such that

$$\mathrm{d}Z_t = \delta_t \mathrm{d}B_Q(t). \tag{8.23}$$

Combining the equality above with (8.21), we obtain

$$\mathrm{d}Z_t = \gamma_t \mathrm{d}\widetilde{S}_t, \qquad \gamma_t := \frac{\delta_t}{\sigma \widetilde{S}_t}. \tag{8.24}$$

This allows us to define the following portfolio:

$$\begin{aligned} U_t &:= \gamma_t \\ V_t &:= Z_t - \gamma_t \widetilde{S}_t, \qquad 0 \le t \le T. \end{aligned} \tag{8.25}$$

Exercise 38. *The goal of the present exercise is to prove that the portfolio we have just defined is a hedging portfolio.*

(1) *For all $t \in [0, T]$, compute the value of the portfolio $W_t = U_t S_t + V_t A_t$.*
(2) *Deduce thereof that the portfolio is replicating.*
(3) *Prove that $\mathrm{d}W_t = U_t \mathrm{d}S_t + V_t \mathrm{d}A_t$, hence (U, V) is a S.R.P.*
(4) *Compute the arbitrage-free value of the contingent claim.*

We conclude this section with

Exercise 39 (Black-Scholes formula)**.** *Apply Question (4) of Exercise 38 in the case of a European call option (see Example 8.2) and deduce the Black-Scholes pricing formula.*

8.3. **Merton's model.** Robert Merton first applied jump processes to option pricing in 1976. On a probability space $(\Omega, \mathcal{A}, \mathrm{P})$, let $X = (X_t)$ be the following Lévy process:

$$X_t = \mu t + \sigma B_t + \sum_{1 \le i \le N_t} Y_i, \tag{8.26}$$

where $B = (B_t)$ is a standard Brownian motion, $N = (N_t)$ is a Poisson counting process with intensity $\lambda > 0$ and $Y = (Y_i)_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables with distribution $\mathcal{N}(m, v^2)$. The processes B, N and Y are independent. Let us start with a warm-up exercise.

Exercise 40. *Let (b, σ^2, ν) be the Lévy triplet of X. Find the values of b and ν.*

In the present model we assume that the risk asset value is given by

$$S_t = S_0 \exp(X_t), \qquad t \ge 0. \tag{8.27}$$

Our goal is to show that there may exist several risk-neutral measures, hence such model is (in general) not complete.

8.3.1. *Playing with the drift.* Suppose $\sigma > 0$. By using Proposition 7.3 we may find a measure Q_1 equivalent to P such that, under Q_1, X has Lévy triplet (b', σ^2, ν), with

$$\frac{b'-b}{\sigma^2} =: \eta \in \mathbb{R}. \tag{8.28}$$

By Proposition 7.2, we get that $\widetilde{S}_t = S_0 \exp(X_t - rt)$ is a Q_1-martingale provided

$$b' - r + \tfrac{1}{2}\sigma^2 + \int \Big(e^z - 1 - z\mathbf{1}_{\{|z|\le 1\}}\Big)\nu(\mathrm{d}z) = 0. \tag{8.29}$$

This allows us to find the value of b' (or equivalently, of η) such that Q_1 is a risk-neutral measure.

8.3.2. *Exponential tilting of the jump measure.* The technique of the previous section does not apply if $\sigma = 0$. Another solution consists in changing the jump measure. More precisely, we consider the one-parameter family of measures defined by

$$\nu_\theta(\mathrm{d}z) := e^{\theta z}\nu(\mathrm{d}z), \qquad \theta \in \mathbb{R}. \tag{8.30}$$

Exercise 41. *Check that for all $\theta \in \mathbb{R}$, ν_θ is a Lévy measure.*

This technique of *exponential tilting* is standard in Large Deviation Theory. In this context, it is also known as the *Esscher transform*. By applying the change of measure in Proposition 7.5 with $(\nu_1, \nu_2) = (\nu_\theta, \nu)$, we get a new measure Q_2 equivalent to P (on finite intervals) under which X has Lévy triplet $(b', \sigma^2, \nu_\theta)$, with

$$b' := b + \int_{-1}^{1} z(e^{\theta z} - 1)\nu(\mathrm{d}z). \tag{8.31}$$

Therefore, $(\widetilde{S}_t)$ is a Q_2-martingale if θ solves the equation

$$b - r + \tfrac{1}{2}\sigma^2 + f(\theta) = 0, \tag{8.32}$$

where

$$f(\theta) := \int_{-\infty}^{\infty} (e^z - 1 - z\mathbf{1}_{\{|z|\le 1\}})e^{\theta z}\nu(\mathrm{d}z) + \int_{-1}^{1} z(e^{\theta z} - 1)\nu(\mathrm{d}z). \tag{8.33}$$

We conclude with the following exercise:

Exercise 42. *The aim of this exercise is to prove that (8.32) has a unique solution.*

1. *Compute f' and prove that f is nondecreasing.*
2. *Using that $\nu((0,\infty))$ and $\nu((-\infty,0))$ are both positive, prove that there exists a constant $M \in (0,\infty)$ such that $f'(\theta) \ge M$ for all $\theta \in \mathbb{R}$.*
3. *Conclude.*

9. Beyond the Markov property : Hawkes processes

We have treated so far the case of jump processes that have independent stationary increments and satisfy the Markov property. In this section we give an example of a jump process the evolution of which may depend on the past history of the process.

Let $N = (N_t)_{t\ge 0}$ be a counting process, that is a nondecreasing process starting at $N_0 := 0$, with jumps of size one. Let $\mu > 0$ and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a locally bounded function.

Definition 9.1 (Hawkes process)**.** *Suppose that $t \mapsto N_t - \lambda(t)$ is a local martingale, where*

$$\lambda(t) := \mu t + \int_0^t \int_0^s \varphi(s-u)\mathrm{d}N_u \mathrm{d}s. \tag{9.1}$$

The process (N_t) is called a Hawkes process*.*

If $\varphi = 0$ then (N_t) is actually a Poisson counting measure with intensity λ, so dependence on the past is encoded in φ, which is typically chosen as a decreasing function (close past events have a stronger influence on the evolution of the process than remote ones).

We shall now prove that such processes exist by means of an explicit construction. We first iteratively define a sequence of jump times by $T_0 := 0$, T_1 distributed as $\mathcal{E}(\mu)$ and, for all $n \in \mathbb{N}$,

$$\mathrm{P}(T_{n+1} > T_n + s | T_1, \ldots, T_n) = \exp\Big(-\mu s - \int_0^s \sum_{1 \le i \le n} \varphi(u + T_n - T_i)\mathrm{d}u\Big), \qquad (s \ge 0). \tag{9.2}$$

We now simply define (N_t) as the counting process associated to these jump times. Let us prove that this process satisfies the local martingale property, by a formal derivation. To be more precise, we shall prove below that

$$\mathrm{P}((N_t) \text{ makes a jump in } [t, t+\mathrm{d}t] | \mathcal{F}_t) = \lambda'(t)\mathrm{d}t. \tag{9.3}$$

Suppose that $T_n \le t < T_{n+1}$ for some positive integer n (the only other remaining case $n = 0$ can be treated separately). The probability in (9.3) equals

$$1 - \frac{\mathrm{P}(T_{n+1} > t + \mathrm{d}t | T_1, \ldots T_n)}{\mathrm{P}(T_{n+1} > t | T_1, \ldots T_n)}. \tag{9.4}$$

By (9.2), this quantity equals

$$\Big(\mu + \sum_{1 \le i \le n} \varphi(t - T_i)\Big)\mathrm{d}t = \Big(\mu + \int_0^t \varphi(t-s)\mathrm{d}N_s\Big)\mathrm{d}t = \lambda'(t)\mathrm{d}t, \tag{9.5}$$

which concludes the proof.

Appendix A.

A.1. **Hilbert spaces.**

Lemma A.1. *Let $\mathcal{H}$ be a pre-Hilbert space (i.e. equipped with an inner product) and $\mathcal{S}$ a linear subspace of $\mathcal{H}$. Then $\mathcal{S}$ is dense in $\mathcal{H}$ if and only if the orthogonal of $\mathcal{S}$ is reduced to $\{0\}$.*

Proof of Lemma A.1. Use that $(\mathcal{S}^\perp)^\perp = \bar{\mathcal{S}}$. ■

A.2. **Measurability.**

Proposition A.1. *Let $\varphi = (\varphi_i)_{i \in I}$ be a collection of mappings from X to $(Y, \mathcal{A}_Y)$ and $\Psi\colon (Z, \mathcal{A}_Z) \mapsto X$. Then, Ψ is measurable from $\mathcal{A}_Z$ to $\sigma(\varphi_i, i \in I)$ iff for all $i \in I$, $\varphi_i \circ \Psi$ is measurable from $\mathcal{A}_Z$ to $\mathcal{A}_Y$.*

A.3. **Simple functions.** Let $(\Omega, \mathcal{A})$ be a measurable space. We recall that a simple function is a measurable function which can only take finitely many values.

Lemma A.2. *Any $[0, +\infty]$-valued measurable function may be written as the point-wise nondecreasing limit of simple functions.*

Proof of Lemma A.2. From [10, Lemme 4.3.3]. Let us call f the $[0, +\infty]$-valued measurable function under consideration. For all $n \in \mathbb{N}$, define

$$\phi_n(x) = 2^{-n} \lfloor 2^n x \rfloor 1_{[0,n]}(x) \tag{A.1}$$

for $x < +\infty$ and $\phi_n(+\infty) = n$. Check that the sequence of simple functions (f_n) defined by $f_n = \phi_n \circ f$ non-decreasingly converges to f (point-wise convergence). Check that the convergence is actually uniform on the set of points where f is finite. ■

Let μ be a measure on $(\Omega, \mathcal{A})$.

Lemma A.3. *The set of simple functions is dense in any $L^p(\mu)$, for $p \in [1, +\infty)$.*

Proof of Lemma A.3. From [10, Théorème 7.3.1]. Let $f \in L^p(\mu)$, where $p \in [1, +\infty)$. Pick f_n as in the proof of Lemma A.2 and check that $|f_n - f|^p \leq f^p$. Conclude. ■

A.4. **Weak convergence.**

Theorem A.1 (Prokhorov). *Any tight collection of probability measures is relatively compact (for the topology of weak convergence).*

A.5. **Càdlàg martingales.**

Proposition A.2. *Let (f_n) be a sequence of $\mathbb{R}^d$-valued càdlàg functions (defined on a subset of the real line) converging uniformly to f. Then, f is also càdlàg.*

The metric space $\mathbb{R}^d$ may actually be replaced by any complete metric space. The statement is false when uniform convergence is replaced by the weaker point-wise convergence, as one may check by considering the example

$$f(t) = \sin\Big(\frac{1}{t}\Big) 1\{t > 0\} \quad \text{and} \quad f_n(t) = \sin\Big(\frac{1}{t + \frac{1}{n}}\Big) 1\{t > 0\}, \quad n \in \mathbb{N}. \tag{A.2}$$

Proposition A.3 (Doob's inequality in L^p). *Let $(X_t)_{t \geq 0}$ be a martingale with right-continuous sample paths. For $t > 0$ and $p > 1$,*

$$\mathrm{E}\Big[\sup_{0 \leq s \leq t} |X_s|^p\Big] \leq \Big(\frac{p}{p-1}\Big)^p \mathrm{E}(|X_t|^p). \tag{A.3}$$

See [8, Proposition 3.15].

A.6. **Grönwall's lemma.**

Lemma A.4. *Let $(x(t))$ be a non-negative function that is integrable on $[0, T]$ (w.r.t. Lebesgue measure) and satisfies*

$$x(t) \leq \alpha + \beta \int_0^t x(s) \mathrm{d}s, \qquad t \in [0, T], \tag{A.4}$$

for some constants $\alpha, \beta \geq 0$. Then, $x_t \leq \alpha e^{\beta t}$ for all $t \in [0, T]$.

Proofs

Proof of Proposition 1.1. The first statement is quite clear from the definition. In order to prove the other two points we shall prove that for suitable functions f and g,

$$\mathrm{E}[f(N_u,\ u \le t)g(N_{t+s} - N_t)] = \mathrm{E}[f(N_u,\ u \le t)]\mathrm{E}[g(N_s)]. \tag{A.5}$$

We decompose the expectation on the left-hand side according to the countable partition of events $A_n = \{T_n \le t < T_{n+1}\}$, $(n \in \mathbb{N}_0)$. On A_n, we may write

$$N_{t+s} - N_t = \sum_{k\in\mathbb{N}} 1_{\{T_{n+k} \le t+s\}} = \sum_{k\in\mathbb{N}} 1_{\{\widetilde{T}_{n,k} \le s\}}, \tag{A.6}$$

where $\widetilde{T}_{n,k} := T_{n+k} - t$. It is now a matter of checking that conditionally on A_n, the sequence $(\widetilde{T}_{n,k})_{k\ge1}$ is independent of $(N_u)_{u\le t}$ and follows the same law as (T_k) (the assumption that the inter-arrival times are exponentially distributed is here crucial). *Hint* : check that for all $i, n \in \mathbb{N}$ and all bounded measurable functions $\phi\colon \mathbb{R}^n \to \mathbb{R}$, $\psi\colon \mathbb{R}^i \to \mathbb{R}$,

$$\begin{aligned}\mathrm{E}[\phi(T_1,\dots,T_n)1_{\{T_n\le t<T_{n+1}\}}\psi(\widetilde{T}_{n,1},\dots,\widetilde{T}_{n,i})|T_1,\dots T_n]\\ = \phi(T_1,\dots,T_n)1_{\{T_n\le t\}}e^{\lambda(T_n-t)}\mathrm{E}[\psi(T_1,\dots,T_i)]\end{aligned} \tag{A.7}$$

and conclude. ■

Proof of Proposition 2.1. Use Proposition A.1. ■

Proof of Proposition 2.2. This follows from the fact that the law of a random measure $\mathcal{N}$ is characterized by the collection of joint laws:

$$(\mathcal{N}(B_1),\dots,\mathcal{N}(B_k)), \qquad k \in \mathbb{N}, B_1,\dots,B_k \in \mathcal{B}(E), \tag{A.8}$$

which itself comes from the definition of a random measure and Dynkin's theorem. ■

Proof of Proposition 2.3. The first condition of the definition of a RPM is proved by using that the sum of independent Poisson random variables is still a Poisson random variables. Let us check the second condition. Let $B_1,\dots,B_m$ be pairwise distinct Borel sets. Define $X_{i,j} = \mathcal{N}_i(B_j)$ for $1 \le i \le n$ and $1 \le j \le m$. One may check using Proposition 2.2 that these $n\times m$ random variables are (mutually) independent. Therefore, the random variables

$$\Big(\sum_{1\le i\le n}\mathcal{N}_i(B_1),\dots,\sum_{1\le i\le n}\mathcal{N}_i(B_m)\Big) \tag{A.9}$$

are independent. We may finally conclude by letting $n \to \infty$. ■

Proof of Proposition 2.4. The fact that each $\mathcal{N}_n$ is RPM(m_n) is quite clear. To prove the second part of the proposition, we use the characterization of independence given in Proposition 2.2 (see also the remark just below it) which we combine with the second part of the definition of a RPM and the fact that the E_n's are disjoint. ■

Proof of Proposition 2.5. We first check the four relations when f is a simple function, that is, when there exist $k \in \mathbb{N}$, real numbers $a_1,\dots,a_k$ and disjoint Borel sets $B_1,\dots,B_k$ such that

$$f = \sum_{1\le i\le k} a_i 1_{B_i}. \tag{A.10}$$

Suppose now that f is a non-negative measurable function. Then, there exists a sequence of simple functions (f_n) that converge non-decreasingly to f. Thus, we get (i), (iii) and (iv) for such functions thanks to the monotone convergence theorem and dominated convergence, respectively. To complete the proof of (i), let us decompose $f \in L^1(E,m)$ as $f_+ - f_-$, where f_+ and f_- are the positive and negative parts, respectively. From what precedes,

$$\mathrm{E}\Big(\int f_+ \mathrm{d}\mathcal{N}\Big) = \int f_+ \mathrm{d}m < \infty, \qquad \mathrm{E}\Big(\int f_- \mathrm{d}\mathcal{N}\Big) = \int f_- \mathrm{d}m < \infty, \tag{A.11}$$

so

$$\int f \mathrm{d}\mathcal{N} := \int f_+ \mathrm{d}\mathcal{N} - \int f_- \mathrm{d}\mathcal{N} \tag{A.12}$$

is integrable, with

$$\mathrm{E}\Big(\int f \mathrm{d}\mathcal{N}\Big) := \int f_+ \mathrm{d}m - \int f_- \mathrm{d}m = \int f \mathrm{d}m. \tag{A.13}$$

To extend (ii) to the square-integrable functions, we may use the density of simple functions on E in $L^2(E,m)$ (see Lemma A.3). *Hint :* Pick a sequence of simple functions (f_n) converging to f in $L^2(E,m)$ and define $X_n := \int f_n \mathrm{d}\mathcal{N}$. Prove that (X_n) is a Cauchy sequence in the Hilbert space of square integrable random variables and that its limit (in the L^2 sense) necessarily coincides with $\int f \mathrm{d}\mathcal{N}$. Let us now prove that (iv) extends to integrable functions. Let $f \in L^1(E,m)$. There exists a sequence of simple functions (f_n) which converge to f in $L^1(E,m)$. This readily implies that $\int (e^{if_n} - 1)\mathrm{d}m$ converges to $\int (e^{if} - 1)\mathrm{d}m$ as $n \to \infty$. By using (i) we get that the sequence of random variables $(\int f_n \mathrm{d}\mathcal{N})$ converges to $\int f \mathrm{d}\mathcal{N}$ in $L^1(\Omega, \mathcal{P})$. Therefore, the convergence also holds a.s. on a subsequence (use the Markov inequality and the Borel-Cantelli lemma). By the dominated convergence theorem, we get that $\mathrm{E}(e^{i\int f_n \mathrm{d}\mathcal{N}})$ converges to $\mathrm{E}(e^{i\int f \mathrm{d}\mathcal{N}})$ (on a subsequence). We may conclude by uniqueness of the limit. Note: another way to obtain (i) and (ii) from (iii) is by differentiation of the function $\lambda \mapsto \mathrm{E}(e^{-\lambda \int f \mathrm{d}\mathcal{N}})$. ∎

Proof of Proposition 2.6. Statements (2) and (3) are clearly equivalent, since there exists $C > 0$ such that

$$C(1 \wedge x) \le 1 - e^{-x} \le 1 \wedge x \qquad (x \ge 0). \tag{A.14}$$

Let us now prove that (2) implies (1). To this purpose, we make two observations:

(i) if $\int f \mathrm{d}m$ is finite, then $\int f \mathrm{d}\mathcal{N}$ is a.s. finite, by the first item of Proposition 2.5.
(ii) if $m(E)$ is finite then $\int f \mathrm{d}m$ is a.s. finite, see Exercise 7.

We now write $\int f \mathrm{d}\mathcal{N}$ as the sum of the restrictions to $\{|f| \le 1\}$ and $\{|f| > 1\}$, to which we apply (i) and (ii) respectively. This proves the desired implication. We finally prove that (1) implies (3). By the third item in Proposition 2.5, we get

$$\mathrm{E}\Big(e^{-\int f \mathrm{d}\mathcal{N}}\Big) = \exp\Big(\int (e^{-f} - 1)\mathrm{d}m\Big) > 0, \tag{A.15}$$

which implies that $\int (1 - e^{-f})\mathrm{d}m$ is finite. ∎

Proof of Proposition 2.7. We proceed in several steps.
Step 1. We first deal with the part of the integral which corresponds to large values of f. Let $F = \{x \in E \colon |f(x)| > 1\}$. Note that

$$m(F) \le \int_F |f| \mathrm{d}m = \int_F (|f| \wedge f^2)\mathrm{d}m \le \int (|f| \wedge f^2)\mathrm{d}m < \infty. \tag{A.16}$$

Thus we get that $\int_F f \mathrm{d}m$ and $\int_F f \mathrm{d}\mathcal{N}$ are both a.s. finite, hence $\int_F f \mathrm{d}(\mathcal{N} - m)$ is well-defined. Henceforth we shall assume that $|f| \le 1$, so that our assumption becomes

$\int f^2 \mathrm{d}m < \infty$.

Step 2. We now prove convergence of the sequence (I_n). Let

$$\Delta I_k := \int_{E_k} f \mathrm{d}(\mathcal{N} - m)^{\S}, \qquad (k \in \mathbb{N}). \tag{A.17}$$

One can check that

$$\mathrm{E}(\Delta I_k) = 0, \qquad \mathrm{Var}(\Delta I_k) = \int_{E_k} f^2 \mathrm{d}m. \tag{A.18}$$

Moreover, the ΔI_k's are independent, by Proposition 2.4. Therefore, (I_n) is a martingale. Since $\int f^2 \mathrm{d}m < \infty$, it is bounded in L^2 so the sequence converges a.s. and in L^2 to a square-integrable random variable with variance $\int f^2 \mathrm{d}m$.

Step 3. Let us finally check that the limit does not depend on the choice of the partition. Let $(E_k)_{k\geq 1}$ and $(F_\ell)_{\ell \geq 1}$ two countable partitions. It is enough to show that the corresponding limits coincide with that of the third partition $(E_k \cap F_\ell)_{k,\ell\geq 1}$ (left to the reader). ■

Proof of Proposition 3.1. (1) Let X be a random variable with law μ, which we assume to have bounded support. Therefore there exists $M > 0$ such that $|X| \leq M$ almost surely. Let $n \in \mathbb{N}$. Since μ is I.D. there exists n i.i.d. random variables $Y_{1,n}, \ldots, Y_{n,n}$ such that $Y_{1,n} + \ldots + Y_{n,n}$ equals X in law. Check that $|Y_{1,n}| \leq M/n$ almost surely and deduce thereof that $\mathrm{Var}(X) = 0$, hence μ is a Dirac measure.

(2) Let $\mu_n \in \mathcal{M}_1(\mathbb{R}^d)$ such that $\mu_n^{*n} = \mu$. For simplicity, we write ϕ and ϕ_n for the respective characteristic functions of μ and μ_n. Check that for all $u \in \mathbb{R}^d$,

$$|\phi_n(u)| \longrightarrow 1_{\{\phi(u)\neq 0\}}, \qquad n \to \infty. \tag{A.19}$$

Note that $|\phi_n(\cdot)|^2$ is the characteristic function of $Y - Y'$, where Y and Y' are independent and distributed as μ_n. From what precedes,

$$|\phi_n(u)|^2 \longrightarrow 1_{\{\phi(u)\neq 0\}}, \qquad n \to \infty. \tag{A.20}$$

Since the right-hand side is continuous in a neighborhood of the origin, it is the characteristic function of some probability distribution, by Lévy's theorem [10, Théorème 11.2.2]. Furthermore, this characteristic function is constant equal to one in a neighborhood of the origin. By continuity, it is actually the characteristic function of the Dirac measure at the origin. Consequently, $\phi(u) \neq 0$ for all $u \in \mathbb{R}^d$.

(3) This part is taken from [13, p. 32-34]. Let us start with existence. Let $u \in \mathbb{R}^d$. As $t \in [0,1]$, $\phi_\mu(tu)$ draws a continuous curve in $\mathbb{C}^*$. Let $h_u(t)$ be the unique branch of $\log \phi_\mu(tu)$ (multi-valued complex logarithm) such that $h_u(0) = 0$ and $h_u(t)$ is continuous in t (that is the lifting property w.r.t. the covering map $\exp\colon \mathbb{C} \to \mathbb{C}^*$, see e.g. [9, Chapter 2]). Define $\Psi(u) = h_u(1)$. By definition, $\Psi(0) = 0$ and $\exp(\Psi(u)) = \phi_\mu(u)$. We now sketch the proof of continuity for Ψ. To this end, let $u_0, u \in \mathbb{R}^d$. Let $u\colon t \in [0,3] \mapsto \mathbb{R}^d$ describe a triangle going from $u(0) = 0$ to $u(1) = u_0$ then to $u(2) = u$ and back to $u(3) = 0$. One may check that as $t \in [0,3]$, $\phi_\mu(t)$ describes

§If need be, we may partition space further by using the collection of disjoint sets $\{z \in \mathbb{R}^d \colon \frac{1}{p+1} < |f(z)| \leq \frac{1}{p}\}$, where $p \in \mathbb{N}$. In this way, f is guaranteed to be bounded from below on each E_k, which yields that $f \in L^1(E_k, m)$. Therefore, ΔI_k is well-defined.

a curve whose rotation number around the origin is zero, provided u is close to u_0. Under the latter condition, the imaginary part of $\Psi(u)$ coincides with the value at $t = 2$ of the unique branch of $\arg \phi_\mu(u(t))$ which is continuous in t and equals zero at $t = 0$. Continuity follows. Uniqueness follows from uniqueness of the function h_u defined above.

■

Proof of Proposition 3.2. Let $n \in \mathbb{N}$. There exist two (independent) vectors of i.i.d. random variables $(Y_{1,n}^{(i)})_{1\leq i\leq n}$ and $(Y_{2,n}^{(i)})_{1\leq i\leq n}$ such that the two following equalities hold in law:

$$Y_1 = \sum_{i=1}^{n} Y_{1,n}^{(i)}, \qquad Y_2 = \sum_{i=1}^{n} Y_{2,n}^{(i)}. \tag{A.21}$$

Then, we have the following equality in law:

$$\alpha_1 Y_1 + \alpha_2 Y_2 = \sum_{i=1}^{n} [\alpha_1 Y_{1,n}^{(i)} + \alpha_2 Y_{2,n}^{(i)}]. \tag{A.22}$$

Since the random variables $(\alpha_1 Y_{1,n}^{(i)} + \alpha_2 Y_{2,n}^{(i)})_{1\leq i\leq n}$ are i.i.d, we may conclude the proof. ■

Proof of Proposition 3.3. For simplicity, we assume $d = 1$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real random variables such that for all n, X_n is I.D. and follows the law μ_n. By assumption, X_n converges in law to a real r.v. X, with law μ. Let $p \in \mathbb{N}$, which is fixed throughout the proof. We want to show that there exists $\nu_p \in \mathcal{M}_1(\mathbb{R})$ such that $\mu = \nu_p^{*p}$. By assumption, there exists a sequence of probability measures $(\nu_{n,p})_{n\in\mathbb{N}}$ such that $\mu_n = \nu_{n,p}^{*p}$. The idea of the proof is to show that this sequence is tight (in n, for every fixed p). Then, by Prokhorov's theorem (Theorem A.1), it converges along some subsequence to a limit which we denote by ν_p. From what precedes, this convergence enforces the relation $\mu = \nu_p^{*p}$. Let us now check tightness. Let $K > 0$ and write

$$X_n = \sum_{1\leq i\leq p} X_{n,p}^{(i)}, \tag{A.23}$$

where the $X_{n,p}^{(i)}$'s are independent and distributed as $\nu_{n,p}$. Clearly,

$$\nu_{n,p}(]K, +\infty[) = \mathrm{P}(X_{n,p}^{(1)} > K) \leq \mathrm{P}(X_n > Kp)^{1/p} = \mu_n(]Kp, +\infty[)^{1/p}. \tag{A.24}$$

Since (μ_n) converges weakly, it is tight, hence $\mu_n(]Kp, +\infty[)^{1/p}$ converges to zero as $K \to \infty$ uniformly in n, and the same holds for $\nu_{n,p}(]K, +\infty[)$. A similar argument holds for $\nu_{n,p}(] - \infty, -K[)$. This settles the proof of tightness. ■

Proof of Theorem 3.1. We will only prove one implication, namely that the characteristic function being of the given form implies infinite divisibility[¶]. The case of a finite measure ν has already been treated in Exercise 14 hence we will explain how to treat the case of an infinite Lévy measure ν. The main idea is to use a truncation argument. For all $k \in \mathbb{N}$, define

$$\Psi_k(u) := i\langle b, u\rangle - \frac{1}{2}\langle u, Au\rangle + \int_{|z|>1/k} (\dots)\, \nu(\mathrm{d}z), \qquad (u \in \mathbb{R}^d). \tag{A.25}$$

[¶]To prove the reverse implication, Applebaum [1, Theorem 1.2.14 and bottom of Corollary 2.4.20] uses the construction of a canonical Lévy process from a given I.D. probability measure and the Lévy-Itô decomposition, see Section 4 of these notes. See also [5, Theorem 1, Section 3.18]

By using Exercise 14, we may assert that each Ψ_k corresponds to some (explicit) I.D. probability distribution, which we denote by μ_k. By using the dominated convergence theorem, we may then prove that Ψ_k converges (pointwise) to a function Ψ (defined in the same way as Ψ_k but without truncating the integral) that is continuous at 0. Hence, by Lévy's theorem, the sequence (μ_k) converges weakly to some $\mu \in \mathcal{M}_1(\mathbb{R}^d)$. The limit μ is I.D. by Proposition 3.3. ∎

Proof of Proposition 4.1. Let us define $Y_t = X_{T+t} - X_T$ for all $t \geq 0$. We will show that for all $0 \leq s_1 < s_2 < \ldots < s_n$, $A \in \mathcal{F}_T$ and $F\colon (\mathbb{R}^d)^n \mapsto \mathbb{R}$ continuous and bounded,

$$\mathrm{E}[F(Y_{s_1}, \ldots, Y_{s_n})1_A] = \mathrm{E}[F(X_{s_1}, \ldots, X_{s_n})]\mathrm{P}(A). \tag{A.26}$$

This simultaneously proves that the processes X and Y have the same law (since they have the same finite-dimensional distributions) and that Y is independent from $\mathcal{F}_T$. We proceed in two steps.

Step 1. Let us first assume that T takes its values in a countable set $\{t_m\}_{m\geq 1}$. Then

$$\mathrm{E}[F(Y_{s_1}, \ldots, Y_{s_n})1_A] = \sum_{m\geq 1} \mathrm{E}[F(X_{t_m+s_1} - X_{t_m}, \ldots, X_{t_m+s_n} - X_{t_m})1_{A\cap\{T=t_m\}}]. \tag{A.27}$$

Note that $A \cap \{T = t_m\} \in \mathcal{F}_{t_m}$. By applying the simple Markov property (X has independent increments) we get

$$\mathrm{E}[F(Y_{s_1}, \ldots, Y_{s_n})1_A] = \sum_{m\geq 1} \mathrm{E}[F(X_{t_m+s_1} - X_{t_m}, \ldots, X_{t_m+s_n} - X_{t_m})]\mathrm{P}(A \cap \{T = t_m\}). \tag{A.28}$$

Since X has stationary increments, we obtain

$$\begin{aligned}\mathrm{E}[F(Y_{s_1}, \ldots, Y_{s_n})1_A] &= \sum_{m\geq 1} \mathrm{E}[F(X_{s_1}, \ldots, X_{s_n})]\mathrm{P}(A \cap \{T = t_m\}) \\ &= \mathrm{E}[F(X_{s_1}, \ldots, X_{s_n})]\mathrm{P}(A).\end{aligned} \tag{A.29}$$

Step 2. In general, one can set

$$T_k = \frac{\lfloor 2^k T \rfloor + 1}{2^k}, \qquad k \in \mathbb{N}. \tag{A.30}$$

The random sequence (T_k) converges a.s. and non-increasingly to T. Define $Y_s^{(k)} = X_{T_k+s} - X_{T_k}$ for all $s \geq 0$ and note that $Y_s^{(k)}$ a.s. converges to Y_s as $k \to \infty$, by right-continuity of X. Therefore, we may write

$$\begin{aligned}\mathrm{E}[F(Y_{s_1}, \ldots, Y_{s_n})1_A] &= \lim_{k\to\infty} \mathrm{E}[F(Y_{s_1}^{(k)}, \ldots, Y_{s_n}^{(k)})1_A] \\ &= \lim_{k\to\infty} \mathrm{E}[F(Y_{s_1}^{(k)}, \ldots, Y_{s_n}^{(k)})]\mathrm{P}(A) \\ &= \mathrm{E}[F(Y_{s_1}, \ldots, Y_{s_n})]\mathrm{P}(A).\end{aligned} \tag{A.31}$$

The first and third equalities above follow from the dominated convergence theorem, while the second equality follows from the first step of the proof (T_k has a countable support and one may check that $A \in \mathcal{F}_{T_k}$ for all $k \in \mathbb{N}$). ∎

Proof of Proposition 4.2. To prove that X_t is I.D., write

$$X_t = \sum_{1\leq i\leq n} X_{\frac{it}{n}} - X_{\frac{(i-1)t}{n}} \tag{A.32}$$

and use that X has stationary and independent increments. Let us now prove the second property. Let us fix $u \in \mathbb{R}^d$ and define, for all $t \geq 0$, $\Phi(t) := \phi_{X_t}(u)$. By Proposition 3.1, we may write $\Phi(1) = \exp(\Psi(u))$ for some continuous complex-valued function Ψ. From the definition of a Lévy process, one can check that $t \mapsto \Phi(t)$ is right-continuous and $\Phi(t+s) = \Phi(s)\Phi(t)$. Necessarily, $\Phi(t) = \exp(t\Psi(u))$. ■

Proof of Theorem 4.1. The fact that a Lévy exponent is of the given form directly follows from Proposition 4.2 and Theorem 3.1 (Lévy-Khintchine). To prove the reverse statement, we pick Ψ of the given form and construct a Lévy process with corresponding Lévy exponent ‖. We proceed step by step.

Step 1. The map $u \in \mathbb{R}^d \mapsto i\langle b, u\rangle - \frac{1}{2}\langle u, Au\rangle$ is the characteristic exponent of the Lévy process $t \mapsto bt + \sqrt{A}B_t$ ($\sqrt{A}$ is an abuse of notation for a matrix M such that $A = MM^t$).

Step 2. If ν does not put any mass on $\{z\colon |z| \leq 1\}$ then $u \in \mathbb{R}^d \mapsto \int_{\mathbb{R}^d}[e^{i\langle u,z\rangle} - 1]\nu(\mathrm{d}z)$ is the characteristic exponent of a CPP$(1, \nu)$, which is a Lévy process.

Step 3. Let us now assume that ν only charges $\{z\colon |z| \leq 1\}$ and satisfies $\int |z|^2\nu(\mathrm{d}z) < \infty$. Let $\mathcal{N}$ be a RPM on $\mathbb{R}^+ \times \mathbb{R}^d$ with intensity $\mathrm{d}t \otimes \mathrm{d}\nu$ and define, for all $n \in \mathbb{N}_0$ and $t > 0$,

$$X_t^{(n)} := \int_{(0,t]\times\{z\colon |z|>1/(n+1)\}} z \, \mathrm{d}\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z). \tag{A.33}$$

By Proposition 2.7, this sequence of random variables converges a.s. and in $L^2(\Omega, \mathrm{P})$, as $n \to \infty$, to

$$X_t := \int_{(0,t]\times\mathbb{R}^d} z \, \mathrm{d}\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z). \tag{A.34}$$

It is not too difficult to check that (X_t) has independent and stationary increments and that it has the following characteristic exponent:

$$u \in \mathbb{R}^d \mapsto + \int_{\mathbb{R}^d} \left(e^{i\langle u,z\rangle} - 1 - i\langle u, z\rangle\right)\nu(\mathrm{d}z). \tag{A.35}$$

The remaining (and delicate) point is checking that (X_t) has a.s. càdlàg sample paths. First, one may check that for all $n \geq 1$, $(X_t^{(n)})$ is a square-integrable centered martingale w.r.t. time parameter t and for the natural filtration (use Proposition 1.2). The rest of the proof follows the same strategy as in the proof of Proposition 1.5 in Salez's lecture notes on stochastic calculus. By Doob's inequality in L^2 (see Proposition A.3)

$$\begin{aligned} \mathrm{E}\Big(\sup_{0\leq s\leq t} |X_s^{(n+k)} - X_s^{(n)}|^2\Big) &\leq 4 \sup_{0\leq s\leq t} \mathrm{E}\Big(|X_s^{(n+k)} - X_s^{(n)}|^2\Big) \\ &= 4t \int_{\{z\colon \frac{1}{n+k+1} < |z| \leq \frac{1}{n+1}\}} z^2\nu(\mathrm{d}z). \end{aligned} \tag{A.36}$$

Therefore, there exists an increasing sequence of integers $(n_k)_{k\geq 0}$ such that $n_0 = 0$ and

$$\mathrm{E}\Big(\sup_{0\leq s\leq t} |X_s^{(n_k)} - X_s^{(n_{k+1})}|^2\Big) \leq 2^{-k} \times 4t \int_{|z|\leq 1} z^2\nu(\mathrm{d}z). \tag{A.37}$$

‖An alternative proof consists in using [1, Corollary 1.4.6] (canonical Lévy processes)

By Fubini-Tonelli, we get

$$\mathrm{E}\Big(\sum_{k\geq 0}\sup_{0\leq s\leq t}|X_s^{(n_k)}-X_s^{(n_{k+1})}|^2\Big)<\infty. \tag{A.38}$$

This implies that for almost every $\omega\in\Omega$, $(X_s^{(n_k)}(\omega))_{0\leq s\leq t}$ converges uniformly on $[0,t]$ to $(X_s(\omega))_{0\leq s\leq t}$ as $k\to\infty$. Applying Proposition A.2 completes the proof. ∎

Proof of Proposition 4.5. Use Proposition 4.4 and Proposition 2.6 with $f(s,z)=|z|$, $m=\mathrm{d}t\otimes\nu$ and $\mathcal{N}=\mathcal{J}$. ∎

Proof of Proposition 4.6. Let $t>0$. By the right-continuity of X, there are a.s. finitely many jumps with modulus larger than one on $(0,t]$. To deal with the small jumps, we write

$$\mathrm{E}\Big(\sum_{\substack{s\in(0,t]\\|\Delta X_s|\leq 1}}|\Delta X_s|^2\Big)=t\int_{|z|\leq 1}|z|^2\nu(\mathrm{d}z), \tag{A.39}$$

which is finite, since ν is a Lévy measure. ∎

Proof of Theorem 4.2. See [3, Theorem 1.2] and [12, p. 78-9]. ∎

Proof of Proposition 4.7. Admitted. ∎

Proof of Proposition 4.8. To prove (i) use the monotonicity of $t\mapsto X_t$ and the strong Law of Large Numbers. Let us now prove (ii). We first prove the convergence in law. We have

$$\mathrm{E}(e^{-r\frac{X_t}{t}})=e^{-t\phi(r/t)}, \tag{A.40}$$

with

$$t\phi(r/t)=\beta r+\int_{(0,+\infty)}t(1-e^{-rz/t})\nu(\mathrm{d}z). \tag{A.41}$$

The quantity inside the integral converges to 0 as $t\to 0$. Moreover, for all $t\in(0,1)$,

$$t|1-e^{-rz/t}|\leq t\Big(1\wedge\frac{rz}{t}\Big)=1\wedge(rz). \tag{A.42}$$

The right-hand side is integrable w.r.t. ν so, by dominated convergence, we get that $t\phi(r/t)$ converges to βr as $t\to 0$. This proves the convergence in law. We refer to [2, Proposition III 4.8] for a.s. convergence (martingale argument). ∎

Proof of Proposition 5.1. The space $\mathcal{H}^2(T)$ is a subspace of the Hilbert space $L^2(\mathrm{d}t\otimes\nu\otimes\mathrm{P})$ hence it is enough to show that it is closed. If the sequence of predictable and square-integrable functions (F_n) converges to F in $L^2(\mathrm{d}t\otimes\nu\otimes\mathrm{P})$ then it converges $\mathrm{d}t\otimes\nu\otimes\mathrm{P}$-a.e. on a subsequence, so the limit F is predictable. ∎

Proof of Proposition 5.2. Sketch. (i) Adaptedness. Check first that $I_t(F)$ is $\mathcal{F}_t$-measurable when F is a simple predictable function. For a general predictable function F, use that there exists a sequence of simple predictable functions such that $(I_t(F_n))$ converges a.s. to $I_t(F)$. (ii) Martingale property. Prove it first when F is simple. Use that $\widetilde{\mathcal{N}}((u,v]\times A)$ is centered and independent of $\mathcal{F}_u$ $(u<v)$. In the general case, use the approximation by simple functions and the contractive property of conditional expectation in L^2. We refer to [1, Theorem 4.2.3] for details. ∎

Proof of Theorem 5.1. We shall only treat the terms due to the jump component of the Lévy process (the terms in (III)) and hereby assume that $b(t) = \sigma(t) = 0$ for all t. To simplify, we also assume that f only depends on the space variable. Let us first deal with the first term in (III). Suppose that

$$X_t - X_0 = \int_0^t \int_{B^c} K(s,z)\mathcal{N}(\mathrm{d}s, \mathrm{d}z), \qquad B := \{z\colon |z| \le 1\}. \tag{A.43}$$

Since ν is a Lévy measure, we have $\nu(B^c) < \infty$ and we may write

$$X_t - X_0 = \sum_{i \le N_t} K(T_i, Z_i), \tag{A.44}$$

where (N_t) is a Poisson counting process with intensity $\nu(B^c)$ and the Z_i's are i.i.d. with common law $\nu(\cdot \cap B^c)/\nu(B^c)$. Thus (X_t) is piecewise constant w.r.t the intervals $[T_n, T_{n+1})$ (with $n \in \mathbb{N}_0$) and for all $t \in [T_n, T_{n+1})$,

$$\begin{aligned} f(X_t) - f(X_0) &= f(X_{T_n}) - f(X_0) \\ &= \sum_{i \le n} f(X_{T_i}) - f(X_{T_{i-1}}) \\ &= \sum_{i \le n} f(X_{T_i^-} + K(T_i, Z_i)) - f(X_{T_i^-}), \end{aligned} \tag{A.45}$$

which we may rewrite

$$\begin{aligned} f(X_t) - f(X_0) &= \sum_{i \le N_t} f(X_{T_i^-} + K(T_i, Z_i)) - f(X_{T_i^-}) \\ &= \int_0^t \int_{B^c} [f(X_{s^-} + K(s,z)) - f(X_{s^-})]\mathcal{N}(\mathrm{d}s, \mathrm{d}z). \end{aligned} \tag{A.46}$$

Note that the integrand is indeed a predictable process. Let us now explain the second and third terms in (III). To this purpose, suppose that

$$X_t - X_0 = \int_0^t \int_B H(s,z)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z), \qquad B := \{z\colon |z| \le 1\}. \tag{A.47}$$

We approximate this process by

$$X_t^{(\varepsilon)} - X_0 = \int_0^t \int_{B_\varepsilon} H(s,z)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z), \qquad B_\varepsilon := \{z\colon \varepsilon < |z| \le 1\}. \tag{A.48}$$

Using the same argument as above, we may write (for a different Poisson counting process whose law depends on the truncation level $\varepsilon > 0$):

$$X_t^{(\varepsilon)} - X_0 = \sum_{i \le N_t} H(T_i, Z_i) - \int_0^t \int_{B_\varepsilon} H(s,z)\mathrm{d}s\nu(\mathrm{d}z). \tag{A.49}$$

To simplify notations we omitted the dependence of (N_t), (T_n) and (Z_n) on ε. We now proceed as in (A.46) but this time the decomposition reads, when $t \in [T_n, T_{n+1})$:

$$f(X_t^{(\varepsilon)}) - f(X_0) = f(X_t^{(\varepsilon)}) - f(X_{T_n}^{(\varepsilon)}) + \sum_{1 \le i \le n} [f(X_{T_i}^{(\varepsilon)}) - f(X_{T_{i-1}}^{(\varepsilon)})]. \tag{A.50}$$

The expression in the sum above can itself be decomposed as

$$\begin{aligned}&f(X^{(\varepsilon)}_{T_i}) - f(X^{(\varepsilon)}_{T_i^-}) + f(X^{(\varepsilon)}_{T_i^-}) - f(X^{(\varepsilon)}_{T_{i-1}})\\&= f(X^{(\varepsilon)}_{T_i^-} + H(T_i, Z_i)) - f(X^{(\varepsilon)}_{T_i^-}) - \int_{T_{i-1}}^{T_i}\int_{B_\varepsilon} f'(X^{(\varepsilon)}_s)H(s,z)\mathrm{d}s\nu(\mathrm{d}z).\end{aligned} \tag{A.51}$$

We finally obtain

$$\begin{aligned}&f(X^{(\varepsilon)}_t) - f(X_0)\\&= \int_0^t\int_{B_\varepsilon}[f(X^{(\varepsilon)}_{s^-} + H(s,z)) - f(X^{(\varepsilon)}_{s^-})]\mathcal{N}(\mathrm{d}s,\mathrm{d}z) - \int_0^t\int_{B_\varepsilon} f'(X^{(\varepsilon)}_{s^-})H(s,z)\mathrm{d}s\nu(\mathrm{d}z)\\&= \int_0^t\int_{B_\varepsilon}[f(X^{(\varepsilon)}_{s^-} + H(s,z)) - f(X^{(\varepsilon)}_{s^-})]\widetilde{\mathcal{N}}(\mathrm{d}s,\mathrm{d}z)\\&\quad + \int_0^t\int_{B_\varepsilon}[f(X^{(\varepsilon)}_{s^-} + H(s,z)) - f(X^{(\varepsilon)}_{s^-}) - f'(X^{(\varepsilon)}_{s^-})H(s,z))]\mathrm{d}s\nu(\mathrm{d}z).\end{aligned} \tag{A.52}$$

It remains to justify the limit as $\varepsilon \to 0$. To simplify, we first treat the case when H and the derivatives of f are bounded (or alternatively that H and X are bounded). It is then enough to prove that $X^{(\varepsilon)}_s$ converges to X_s in L^2, uniformly in $0 \le s \le t$, as $\varepsilon \to 0$. To this end we argue that $X^{(\varepsilon)}_t$ converges to X_t in L^2 as $\varepsilon \to 0$ (Itô's isometry) and that $(X^{(\varepsilon)}_s - X_s)$ is a martingale w.r.t the time parameter (Proposition 5.2). The uniform convergence follows from Doob's inequality in L^2 (Proposition A.3). The general case follows by a stopping-time argument. ∎

Proof of Theorem 6.1 (Details of the proof without the large jumps). (1) We proceed by induction on n. The statement is true for $n = 0$. Suppose that it is true for $n-1$. Let us first check that the process

$$(s, z, \omega) \mapsto F(Y^{(n-1)}_{s^-}(\omega), z) \quad \text{is predictable.} \tag{A.53}$$

The process $Y^{(n-1)}$ has right-continuous sample paths by assumption, so $(Y^{(n-1)}_{s^-})$ has left-continuous sample paths. Moreover, the latter process is adapted and does not depend on the z-variable, so it is predictable. Since F is (jointly) measurable, (A.53) is proven. The integrability condition

$$\int_0^t\int_{|z|\le 1} \mathrm{E}[F(Y^{(n-1)}_{s^-}, z)^2]\nu(\mathrm{d}z) < \infty, \qquad \forall t \ge 0. \tag{A.54}$$

can be readily checked by using Item (ii) in Assumption 6.1 and the assumption that the process $Y^{(n-1)}$ is a square-integrable martingale. We may now conclude with the help of Proposition 5.2.

(2) For all $s \ge 0$,

$$Y^{(1)}_s - Y^{(0)}_s = \int_0^s\int_{|z|\le 1} F(Y_0, z)\widetilde{\mathcal{N}}(\mathrm{d}u, \mathrm{d}z). \tag{A.55}$$

By the growth condition on F and our assumption on Y_0, we have

$$\mathrm{E}\int_0^s\int_{|z|\le 1} F(Y_0, z)^2\mathrm{d}u\nu(\mathrm{d}z) \le K(1 + \mathrm{E}(Y_0^2))s < \infty \tag{A.56}$$

Moreover, the random function $(s, \omega, z) \mapsto F(Y_0(\omega), z)$ is predictable so it belongs to $\mathcal{H}^2(t)$ for all $t > 0$. Therefore, by Proposition 5.2, the process $(Y_t^{(1)} - Y_t^{(0)})$ is a càdlàg $\mathcal{F}$-adapted square-integrable martingale. By Doob's maximal inequality,

$$y_1(t) \le 4 \sup_{0\le s\le t} \mathrm{E}[(Y_s^{(1)} - Y_s^{(0)})^2] = 4t \int_{|z|\le 1} \mathrm{E}[F(Y_0, z)^2]\nu(\mathrm{d}z). \tag{A.57}$$

With the same line of arguments, one can show that for all $n \in \mathbb{N}$, $(Y_t^{(n+1)} - Y_t^{(n)})$ is a càdlàg $\mathcal{F}$-adapted square-integrable martingale and, by using Doob's maximal inequality and the Lipschitz condition,

$$y_{n+1}(t) \le 4K \int_0^t \mathrm{E}[(Y_{s^-}^{(n)} - Y_{s^-}^{(n-1)})^2]\mathrm{d}s \le 4K \int_0^t y_n(s)\mathrm{d}s. \tag{A.58}$$

Note: we realize that $C_1(t)$ does not depend on t here, but it does when $b \neq 0$.

(3) By induction, we deduce from what precedes that for all $n \in \mathbb{N}$ and $t \ge 0$,

$$y_n(t) \le \frac{(ct)^n}{n!}, \qquad c = 4K[1 + \mathrm{E}(Y_0^2)]. \tag{A.59}$$

Since $\|Y_t^{(n)} - Y_t^{(n-1)}\|_2 \le \sqrt{y_n(t)}$, we get that

$$\sum_{n\ge 1} \|Y_t^{(n)} - Y_t^{(n-1)}\|_2 < \infty, \tag{A.60}$$

which proves that $(Y_t^{(n)})$ is a Cauchy sequence in L^2. Its limit is denoted by Y_t.

(4) By the Chebyshev inequality,

$$\mathrm{P}\Big(\sup_{0\le s\le t} |Y_s^{(n)} - Y_s^{(n-1)}| \ge 2^{-n}\Big) \le 4^n y_n(t) \le \frac{(4ct)^n}{n!}, \tag{A.61}$$

which implies that

$$\sum_{n\ge 1} \mathrm{P}\Big(\sup_{0\le s\le t} |Y_s^{(n)} - Y_s^{(n-1)}| \ge 2^{-n}\Big) < +\infty. \tag{A.62}$$

By the Borel-Cantelli lemma, we obtain that for almost every $\omega \in \Omega$, there exists $n(\omega)$ such that

$$\sup_{0\le s\le t} |Y_s^{(n)}(\omega) - Y_s^{(n-1)}(\omega)| < 2^{-n}, \qquad n \ge n(\omega). \tag{A.63}$$

This implies that for a.e. $\omega \in \Omega$ and all $t > 0$, the sequence $(Y^{(n)}(\omega))_{n\ge 1}$ converges uniformly on $[0, t]$. Therefore, the limit Y is adapted and càdlàg.

(5) For all $t > 0$,

$$Y_t^{(n)} - \widetilde{Y}_t = \int_0^t \int_{|z|\le 1} [F(Y_{s^-}^{(n-1)}, z) - F(Y_{s^-}, z)]\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}z). \tag{A.64}$$

By using the isometry property of the stochastic integral and the Lipschitz condition on F, we get

$$\begin{aligned} \|Y_t^{(n)} - \widetilde{Y}_t\|_2^2 &= \mathrm{E} \int_0^t \int_{|z|\le 1} [F(Y_{s^-}^{(n-1)}, z) - F(Y_{s^-}, z)]^2 \mathrm{d}s\nu(\mathrm{d}z) \\ &\le K \times \mathrm{E} \int_0^t (Y_{s^-}^{(n-1)} - Y_{s^-})^2 \mathrm{d}s. \end{aligned} \tag{A.65}$$

Let us show that the right-hand side converges to zero as n tends to infinity. By using our previous estimates, the convergence of $Y_t^{(n)}$ to Y_t in L^2 and the triangular inequality, we get

$$\|Y_t^{(n)} - Y_t\|_2 \le \sum_{k\ge n} [y_k(t)]^{1/2} \stackrel{n\to\infty}{\longrightarrow} 0, \tag{A.66}$$

and by Fatou's lemma, for all $0 \le s \le t$,

$$\|Y_{s^-}^{(n)} - Y_{s^-}\|_2 \le \sum_{k\ge n} [y_k(t)]^{1/2} \stackrel{n\to\infty}{\longrightarrow} 0. \tag{A.67}$$

By injecting this estimate in (A.65), we see that $Y_t^{(n)}$ converges to $\widetilde{Y}_t$ in L^2. This convergence also holds a.s. on a subsequence. By uniqueness of limits, $\widetilde{Y}_t = Y_t$ a.s.

(6) Let $Y^{[1]}$ and $Y^{[2]}$ be two solutions of the SDE and define

$$\delta(t) = \mathrm{E}\Big[\sup_{0\le s\le t} |Y_s^{[1]} - Y_s^{[2]}|^2\Big]. \tag{A.68}$$

With the same line of arguments as above, we get that

$$\delta(t) \le 4K \int_0^t \delta(s)\mathrm{d}s, \qquad \delta(0) = 0. \tag{A.69}$$

By Gronwall's lemma (see Lemma A.4) we get that $\delta(t) = 0$ for all $t \ge 0$. This completes the proof.

■

Proof of Proposition 7.2. Let us first check that e^{X_t} is integrable for all $t \ge 0$, with

$$\mathrm{E}(e^{X_t}) = \exp\Big(t\Big[b + \frac{1}{2}\sigma^2 + \int (e^z - 1 - z\mathbf{1}_{|z|\le 1})\nu(\mathrm{d}z)\Big]\Big). \tag{A.70}$$

We use the Lévy-Itô decomposition to split X_t in four independent parts and only focus on the jump parts. The large negative jump part ($z < 1$) can be dealt with via Proposition 2.5. The large positive jump ($z > 1$) part can be dealt with via approximation by a non-decreasing sequence of compound Poisson processes. The computation of the exponential moments for compound Poisson processes follows the same line as in Exercise 4. Let us now focus on the small jump part ($|z| \le 1$). From what precedes, we let $(X_{t,n})_{t\ge 0}$ be the truncated Lévy process with triplet $(0, 0, \nu_n)$, where $\nu_n(\mathrm{d}z) = \nu(\mathrm{d}z)\mathbf{1}_{\{|z|>1/n\}}$ and ν is supported by $[-1, 1]$. Thanks to the interpretation in terms of a compound Poisson process, one may explicitely compute:

$$\mathrm{E}(e^{X_{t,n}}) = \exp\Big(t\int (e^z - 1 - z\mathbf{1}_{\{1/n<|z|\le 1\}})\nu(\mathrm{d}z)\Big). \tag{A.71}$$

For every $t \ge 0$, the right-hand side converges to what we want as $n \to \infty$, so it remains to prove that the left-hand side converges to $\mathrm{E}(e^{X_t})$. To this end we remark that $e^{\frac{1}{2}X_{t,n}}$ is a non-negative sub-martingale (in n). By Doob's L^2 inequality for non-negative sub-martingales [14, Section 14.11], we get

$$\mathrm{E}\Big(\sup_{n\le N} e^{X_{t,n}}\Big) \le 4\mathrm{E}(e^{X_{t,N}}). \tag{A.72}$$

Letting $N \to \infty$, we get that $\sup_{n\ge 1} e^{X_{t,n}}$ is integrable. By dominated convergence, we finally get (A.71) from (A.70).

We now conclude. By the property of a Lévy process, we have for all $0 < s < t$,

$$\mathrm{E}[e^{X_t}|\mathcal{F}_s] = e^{X_s}\mathrm{E}[e^{X_t - X_s}|\mathcal{F}_s] = e^{X_s}\mathrm{E}[e^{X_{t-s}}], \tag{A.73}$$

so the process $(e^{X_t})_{t\geq 0}$ is a martingale iff $\mathrm{E}(e^{X_t}) = 1$ for all $t > 0$, and we get the desired result.

Let us notice that the solution above is quite specific to the case when X is a Lévy process, for which exponential moments are computable. Another standard approach, which works beyond this case (for instance when X is a Lévy-type stochastic integral satisfying appropriate assumptions) consists in applying Ito's formula to e^{X_t}, writing it as a Lévy-type stochastic integral with only a (local) martingale component and checking integrability conditions, see for instance [1, Section 5]. ■

Proof of Proposition 7.3 (sketch for one implication). Assume $\sigma_1 = \sigma_2 (=: \sigma)$. Let $0 = t_0 < t_1 < \ldots < t_n = t$. We have

$$\frac{\mathrm{dP}_1^{(t_1,\ldots,t_n)}}{\mathrm{d}t_1\ldots\mathrm{d}t_n}(x_1,\ldots,x_n) = \frac{1}{(2\pi\sigma^2)^{n/2}}\prod_{1\leq i\leq n}\exp\Big(-\frac{[x_i - x_{i-1} - b_1(t_i - t_{i-1})]^2}{2\sigma^2(t_i - t_{i-1})}\Big). \tag{A.74}$$

By taking the logarithm and expanding the square, we see that the only term depending on b_1 is indeed:

$$\frac{b_1}{\sigma^2}x_n - \frac{b_1^2}{2\sigma^2}t. \tag{A.75}$$

■

Proof of Proposition 7.4 (sketch). Let $0 = t_0 < t_1 < \ldots < t_n = t$ and $0 \leq k_1 \leq \ldots \leq k_n$ integers. We have

$$\mathrm{P}_1(N_{t_1} = k_1, \ldots, N_{t_n} = k_n) = \prod_{1\leq i\leq n}\mathrm{P}(\mathcal{P}(\lambda_1(t_i - t_{i-1})) = k_i - k_{i-1}). \tag{A.76}$$

By a straightforward computation, we get

$$\frac{\mathrm{P}_1(N_{t_1} = k_1, \ldots, N_{t_n} = k_n)}{\mathrm{P}_2(N_{t_1} = k_1, \ldots, N_{t_n} = k_n)} = \exp\Big((\lambda_2 - \lambda_1)t + k_n\log(\lambda_1/\lambda_2)\Big), \tag{A.77}$$

from which we conclude. ■

Proof of Proposition 7.5. To simplify, let us treat the case when ν_1 and ν_2 are discrete with finitely many (common) atoms $\{a_1, a_2, \ldots, a_m\}$, that is

$$\nu_1 = \sum_{1\leq i\leq m}\lambda_1^{(i)}\delta_{a_i}, \qquad \nu_2 = \sum_{1\leq i\leq m}\lambda_2^{(i)}\delta_{a_i}. \tag{A.78}$$

By assumption,

$$\nu_1(\mathbb{R}) = \sum_{1\leq i\leq m}\lambda_1^{(i)} < \infty, \qquad \nu_2(\mathbb{R}) = \sum_{1\leq i\leq m}\lambda_2^{(i)} < \infty. \tag{A.79}$$

Moreover, we may write

$$X_t = \sum_{1\leq i\leq m}a_iN_t^{(i)}, \tag{A.80}$$

where the $N^{(i)}$'s are independent Poisson counting processes with respective rates $\lambda_1^{(i)}$ under P_1 and $\lambda_2^{(i)}$ under P_2. For $1 \le i \le m$, we denote by $\mathrm{P}_1^{(i)}$ the law of $N^{(i)}$ under P_1 and $\mathrm{P}_2^{(i)}$ the law of $N^{(i)}$ under P_2. By independence,

$$\mathrm{P}_1 = \mathrm{P}_1^{(1)} \otimes \ldots \otimes \mathrm{P}_1^{(m)}, \qquad \mathrm{P}_2 = \mathrm{P}_2^{(1)} \otimes \ldots \otimes \mathrm{P}_2^{(m)}. \tag{A.81}$$

By Proposition 7.4, we have for $1 \le i \le m$

$$\frac{\mathrm{d}\mathrm{P}_1^{(i)}}{\mathrm{d}\mathrm{P}_2^{(i)}}\Big|_{\mathcal{F}_t} = \exp\Big(\log(\lambda_1^{(i)}/\lambda_2^{(i)}) N_t^{(i)} - (\lambda_1^{(i)} - \lambda_2^{(i)})t \Big). \tag{A.82}$$

By taking the product, we get

$$\prod_{1\le i\le m} \frac{\mathrm{d}\mathrm{P}_1^{(i)}}{\mathrm{d}\mathrm{P}_2^{(i)}}\Big|_{\mathcal{F}_t} = \exp\Big(\sum_{1\le i\le m} \log(\lambda_1^{(i)}/\lambda_2^{(i)}) N_t^{(i)} - \sum_{1\le i\le m} (\lambda_1^{(i)} - \lambda_2^{(i)})t \Big). \tag{A.83}$$

We conclude by observing that

$$\begin{aligned} \sum_{1\le i\le m} \log(\lambda_1^{(i)}/\lambda_2^{(i)}) N_t^{(i)} &= \sum_{0<s\le t} \log(\lambda_1^{(i)}/\lambda_2^{(i)}) \mathbf{1}_{\{\Delta X_s = a_i\}} \\ &= \sum_{\substack{0<s\le t:\\ \Delta X_s \neq 0}} \log\Big(\frac{\mathrm{d}\nu_1}{\mathrm{d}\nu_2}(\Delta X_s)\Big), \end{aligned} \tag{A.84}$$

and

$$\sum_{1\le i\le m} (\lambda_1^{(i)} - \lambda_2^{(i)}) = \nu_1(\mathbb{R}) - \nu_2(\mathbb{R}). \tag{A.85}$$

$\blacksquare$

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